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Topological Invariants for Discrete Group Actions

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Abstract. This Letter is an extensive introduction to a paper by Bertelson and Gromov that proposes a dynamical version of some aspects of Morse theory.

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Introduction

This Letter should be regarded as an extended introduction to a paper by Bertelson and Gromov ([1]) where a 'dynamical' version of the Morse inequality $|\mathrm{Crit}(f)| \geqslant \mathrm{SB}(M) = \sum_i \dim H^i(M; F)$ was introduced, in the spirit of a program due to Gromov. 'Dynamical' refers to the fact that the spaces involved are endowed with an action of a countable discrete group Γ and that the quantities considered are some kind of entropy functions measuring the (exponential) rate of growth of certain topological invariants, such as ranks of certain homology groups associated with the action.

1. First Ingredient: Morse Theory

Let us briefly recall the elements of Morse Theory that will be needed hereafter. Let M be a smooth connected manifold endowed with a smooth function $f\colon M \to \mathbb{R}$. One distinguishes two cases:

- If M is open, then f may have few critical points regardless of the overall topology of M. In fact, an open M always admits a function without critical points at all. (Maps between manifolds without critical points are called submersions. See [5] for existence theorems for such maps.)
- If M is closed, then f must have a minimum number of critical points, prescribed by the topology of M as follows:

$$|\mathrm{Crit}(f)| \geqslant \begin{cases} \mathrm{SB}(M) = \sum\limits_i \dim H^i(M; F), & \text{if all critical points are nondegenerate,} \\ \mathrm{cat(M)} \geqslant \mathrm{cl}(M), & \text{otherwise,} \end{cases}$$

where F is a field, where $\mathrm{cat}(M)$ denotes the category of M, that is the minimum number of contractible (within M) closed subsets necessary to cover M and where $\mathrm{cl}(M)$ denotes the cup length of M, that is

$$\mathrm{cl}(M) = \sup\{k\,; \exists \text{ ring } R \ \& \ \alpha_1, \ldots, \alpha_{k-1} \in H^*(M;R) \text{ with } \alpha_1 \wedge \cdots \wedge \alpha_{k-1} \neq 0\}.$$

We will now provide an explanation for the ('nondegenerate') Morse inequality suitable to our purpose. Suppose, in addition to the previous hypotheses, that M is closed and orientable. If O is an open subset of M, define

$$H^*(O) = \left\{a \in H^*(M;F); \operatorname{supp} a \subset O\right\},$$

where $\operatorname{supp} a \subset O$ ('the class a is supported in O') means that $a|_{O'} = 0$ for some open subset O' with $O \cup O' = M$. Let $c \in \mathbb{R}$, $\delta > 0$ and define

$$\begin{array}{llcl} \varphi_{c,\delta} : H^*\big(f^{-1}(-\infty, c+\delta)\big) \to \mathrm{Hom}[H^*\big(f^{-1}(c-\delta, +\infty)\big) & , & H^*\big(f^{-1}(c-\delta, c+\delta)\big)], \\ \qquad a \qquad\qquad\qquad\qquad\qquad\qquad b & \mapsto & a \wedge b \end{array}$$

$$\beta_{c,\delta} = \operatorname{rank} \varphi_{c,\delta}$$

$$\beta_c = \lim_{\delta \to 0} \beta_{c,\delta}$$

Note. $\beta_{c,\delta}$ is also the 'rank of the set'

$$\left\{a \in H^*\big(f^{-1}(-\infty, c+\delta)\big); \exists\, b \in H^*\big(f^{-1}(c-\delta, +\infty)\big) \text{ with } a \wedge b \neq 0\right\}$$

(meaning the maximal dimension of a subspace contained in the set in question union $\{0\}$).

PROPOSITION (Nondegenerate Morse inequality).

(a) $\sum_c \beta_c = \mathrm{SB}(M)$,

(b) $\beta_c \leqslant |\mathrm{Crit}(f) \cap f^{-1}(c)|$.

Idea of the proof. (a) For all $a \in H^*(M;F)$ define c_a, *the level of* a, as follows: $c_a = \inf\{c\,; \operatorname{supp} a \subset f^{-1}(-\infty, c)\}$. (Observe that c_a coincides with the minimax of f along the isotopy-invariant family $\mathcal{F} = \{O \text{ open}\,; \operatorname{supp} a \subset O\}$.) Thus $a|_{f^{-1}(c_a-\delta,+\infty)} \neq 0$. Hence, by Poincaré duality, there exists a class b in $H^*(f^{-1}(c_a - \delta, +\infty))$ with $a \wedge b \neq 0$. Since this holds for any $\delta > 0$, the class a provides a contribution to β_{c_a}.

This argument is not yet a proof in particular because two independent classes a and a' having the same level could admit a linear combination $\lambda a + \lambda' a'$ having a strictly lower level, that is $c_{\lambda a + \lambda' a'} < c_a$; so that a and a' do not generate a two-dimensional subspace of classes contributing to β_{c_a}.

(b) Let $0 \neq a \in H^*(M;F)$, then c_a is a critical value of f otherwise $f^{-1}(-\infty, c_a + \delta)$ would be isotopic to $f^{-1}(-\infty, c_a - \delta)$ and a would be supported in $f^{-1}(-\infty, c_a - \delta)$.

If $\beta_c \leqslant 1$ for all c, we are done. Otherwise, a precise proof requires showing that if two classes a and a' have same level c and if that level contains only one critical point, then $c_{a''} < c$ for some linear combination $a'' = \lambda a + \lambda' a'$.

We refer to [1] for a more rigorous treatment.

Note. The invariant β_c is thus a measure of the number of 'cohomologically detectable' critical points of f at level c.

2. Second Ingredient: Classical Statistical Mechanics

This section is essentially a rough summary of the first section of Lanford's paper [4].

2.1. HEURISTIC

Classical statistical mechanics investigates macroscopic properties of matters consisting of a very large number of particles on the basis of the behavior of the individual elements (atoms, molecules, etc) of which it is composed. The basic model is thus a large-dimensional phase space endowed with a Hamiltonian function. One would like to find some type of explanation for the empirical observation that the macroscopic behavior of, say, a gas consisting of one type of atoms, despite the fact that it is composed of a huge number of elements, depends only on a few parameters (density and energy, for instance). More so when the number of particles increases. So one expects that, for large systems with a given density, any observable of a certain type achieves approximately the same value at all points of a fixed energy surface. This can be reformulated as follows: the probability distribution of such an observable with respect to normalized Lebesgue measure on a fixed energy surface approaches a delta measure as the number of particles tends to infinity in such a way as to approach a given density.

2.2 DEFINITION OF OBSERVABLES

The observables considered hereafter, called *finite-range observables* (FROs), are those who 'test correlations between particles which are not too far apart'. They are of the following type:

$$f : \coprod_{n=1}^{\infty} (\mathbb{R}^{\nu})^n \to \mathbb{R} : (z_1, \ldots, z_n) = (q_1, p_1, \ldots, q_n, p_n) \mapsto f(z_1, \ldots, z_n),$$

(where $\nu = 2\mu$ is the dimension of the phase space of a single particle) and satisfy the following properties:

(i) *Symmetry*: $f(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) = f(z_1, \ldots, z_n)$ for any permutation σ,

(ii) *Translation – invariance*: $f(q_1 + a, p_1, \ldots, q_n + a, p_n) = f(q_1, \ldots, p_n)$ for any a in $\mathbb{R}^{\mu}$,

(iii) *Finite – range*: there exists $N \geqslant 0$ such that if $|q_i - q'_j| > N$ for all i,j, then

$$f(z_1, \ldots, z_n, z'_1, \ldots, z'_m) = f(z_1, \ldots, z_n) + f(z'_1, \ldots, z'_m).$$

(Two more properties are usually required which do not play an important role in our picture, namely continuity of the functions $f(z_1, ..., z_n)$ and normalization, that is $f(z_1) = 0$.) The smallest number N for which (iii) is satisfied is called the *range* of the observable f.

This definition is motivated by consideration of the potential energy, which should indeed be a function of any number of particles; in other words, a function on $\coprod_{n=1}^{\infty}(\mathbb{R}^{\nu})$. Moreover, in the case where far-apart particles do not interact, as is the case for a gas, it should behave additively on distant clusters of particles. This is expressed by the third property.

EXAMPLE (Finite-range two-body potential)

$$f(z_1, ..., z_n) = \sum_{i \neq j} \Phi(q_i - q_j),$$

where Φ is a continuous function with compact support.

2.3. CONSTRUCTION OF THE ENTROPY

Fix an interaction U, that is a finite range observable, and consider another such observable f. Let $\Lambda_n \subset \mathbb{R}^{\nu}$ be an increasing family of regions with $\cup_n \Lambda_n = \mathbb{R}^{\nu}$ and

$$\frac{n}{\mathrm{Vol}(\Lambda_n)} \approx \rho \quad \text{(the density).}$$

In $(\Lambda_n)^n$, the phase space for n particles in Λ_n, we consider the thickened surface of energy per particles in I_ε (a small interval of size ε)

$$(\Lambda_n)^n \supset \mathcal{S}^n_{I_\varepsilon} = \left\{(z_1, \ldots, z_n)\,; \frac{1}{n} U(z_1, \ldots, z_n) \in I_\varepsilon\right\}.$$

Let $J = (a, b) \subset \mathbb{R}$ be any interval and introduce the quantity

$$v(\Lambda_n, n, f, J) = \mu_{\mathcal{L}eb}\left\{(z_1, \ldots, z_n) \in \mathcal{S}^n_{I_\varepsilon}\,; \frac{1}{n} f(z_1, \ldots, z_n) \in J\right\}.$$

The probability distribution of f/n with respect to the normalized Lebesgue measure on $\mathcal{S}^n_{I_\varepsilon}$ is given by

$$J \to \frac{v(\Lambda_n, n, f, J)}{v(\Lambda_n, n, f, \mathbb{R})} = \frac{v(\Lambda_n, n, f, J)}{\mu_{\mathcal{L}eb(\mathcal{S}^n_{I_\varepsilon})}}.$$

One then proves that the sequence

$$\frac{1}{n} \ln(v(\Lambda_n, n, f, J)), \tag{1}$$

approaches a limit, the asymptotic exponential growth of $v(\Lambda_n, n, f, J)$, as n goes to infinity in such a way that $n/\mathrm{Vol}(\Lambda_n) \to \rho$. It is essentially implied by the lemma below (which is itself a direct consequence of Fubini's theorem). The limit is called the *entropy* of the function f and is denoted by $s(\rho, f, J)$.

LEMMA. *Let f be an observable with finite range N. Consider Λ and Λ' two bounded subsets of $\mathbb{R}^\nu$ at distance at least N from one another. Let $n, n' \in \mathbb{N}_o$ and let $J, J' \subset \mathbb{R}$ be two intervals. Then*

$$v\Big(\Lambda \cup \Lambda', n+n', f, \frac{n}{n+n'}J + \frac{n'}{n+n'}J'\Big) \geqslant v\Big(\Lambda, n, f, J\Big) \times v\Big(\Lambda', n', f, J'\Big). \qquad (2)$$

In other words $v(\Lambda, n, f, J)$ is supermultiplicative with respect to n.

2.4. IMPLICATIONS OF THE LEMMA

On the one hand, this lemma provides existence of the limit of the sequence (1) since it implies that the sequence $\ln v(\Lambda_n, n, f, J)$ is superadditive and, on the other hand, concavity of the entropy, or rather of the function $s(\rho, f, x)$, also called entropy and defined as follows :

$$s(\rho, f, x) = \inf_{J \ni x} s(\rho, f, J) = \inf_{J \ni x} \lim_{n \to \infty} \frac{1}{n} \ln\Big(v(\Lambda, n, f, J)\Big)$$

($s(\rho, f, J)$ can be recovered as the supremum over all x in J of $s(\rho, f, x)$). Indeed, inequality (2) with $n = n'$ and $\Lambda' = \Lambda + D$, where D is sufficiently large for $d(\Lambda, \Lambda') > R$ yields

$$\begin{aligned}&\frac{1}{2n} \ln v\Big(\Lambda \cup \Lambda', n+n', f, \frac{n}{n+n'}J + \frac{n'}{n+n'}J'\Big)\\ &\quad\geqslant \frac{1}{2}\frac{1}{n} \ln v\Big(\Lambda, n, f, J\Big) + \frac{1}{2}\frac{1}{n} \ln v\Big(\Lambda', n', f, J'\Big),\end{aligned}$$

which implies that, after taking limits over appropriate sequences of regions Λ_n,

$$s(\rho, f, \tfrac{1}{2}J + \tfrac{1}{2}J') \geqslant \tfrac{1}{2}s(\rho, f, J) + \tfrac{1}{2}s(\rho, f, J')$$

and that

$$s(\rho, f, \tfrac{1}{2}x + \tfrac{1}{2}x') \geqslant \tfrac{1}{2}s(\rho, f, x) + \tfrac{1}{2}s(\rho, f, x').$$

So concavity is ensured for dyadic rationals and therefore, by a continuity argument, for all real numbers.

Remark. To establish convergence of the sequence (1), one needs to be more specific about the way in which the sets Λ_n become large: the sequence (Λ_n) should increase 'in the sense of Van Hove', that is, for all $r > 0$,

$$\frac{\mu_{\mathcal{L}eb}(\partial_r \Lambda_n)}{\mu_{\mathcal{L}eb}(\Lambda_n)} \underset{n\to\infty}{\longrightarrow} 0,$$

where $\partial_r \Lambda_n$ denotes the set of points lying at a distance less than or equal to r of the boundary of Λ_n. (Such (Λ_n) are called *Foelner sets* in mathematics literature.) Moreover, the sequence needs to be 'approximable by rectangles', meaning that for suitably chosen rectangles $\hat{\Lambda}_n \supset \Lambda_n$, one has $\liminf_{n\to\infty} \mathrm{Vol}(\Lambda_n)/\mathrm{Vol}(\hat{\Lambda}_n) \neq 0$.

2.5. CONSEQUENCE

The function $x \to s(\rho, f, x)$ is continuous (on the interior of the set where it is finite) and achieves its maximum value either never, once, or on a closed interval.

Coming back to our investigation of the probability distribution of $\frac{f}{n}$, let us interpret the case where $s(\rho, f, x)$ achieves its maximum once. Recall that the distribution in question is given by

$$J \to \frac{v(\Lambda_n, n, f, J)}{v(\Lambda_n, n, f, \mathbb{R})} \approx e^{n[s(\rho, f, J) - s(\rho, f, \mathbb{R})]}$$

and thus has the following asymptotic behavior as n tends to infinity:

$$\frac{v(\Lambda_n, n, f, J)}{v(\Lambda_n, n, f, \mathbb{R})} \underset{n\to\infty}{\longrightarrow} \begin{cases} 1 & \text{if } x_o \in \bar{J}, \\ 0 & \text{otherwise.} \end{cases}$$

This makes it reasonable to say that the distribution of f/n approaches a delta measure as n becomes large.

2.6. NOTE ABOUT THE OTHER CASES

For similar reasons, if $s(\rho, f, x)$ does not achieve its supremum value, then f/n approaches a delta measure concentrated around $+\infty$. On the other hand, the case where $s(\rho, f, x)$ achieves its maximum value on an entire interval $[x_o, x'_o]$ does not have a clear interpretation. The distribution of f/n is more and more concentrated in the interval $[x_o, x'_o]$. What goes on there is uncertain. It could still happen that the distribution of f/n approaches ('slowly') a delta measure. This case is sometimes considered as corresponding to a phase transition, see [4, p. 57].

3. Homological Entropy of Discrete Group Actions: A Mix of the First and Second Ingredients

The construction described hereafter can be motivated as belonging to one of Gromov's programs exposed in [2] and briefly described below:

3.1. PHILOSOPHY

Consider some category of spaces $\underline{X}$ and maps as well as some invariant, property or theory of this category denoted Inv. Let Γ be a group. The aim is to extend Inv to a class of Γ-spaces (spaces endowed with an action of Γ) that would include

- $\underline{X}^\Gamma = \mathrm{Map}(\Gamma, \underline{X})$ with the canonical Γ-action,
- 'subshift of finite type' in $\underline{X}^\Gamma$, which are a certain type of Γ-invariant subspaces of $\underline{X}^\Gamma$,
- certain quotients by Γ-invariant equivalence relations, namely $\underline{X}^\Gamma/\mathrm{Fix}(\Gamma)$, in a 'dynamical way', that is, so as to obtain the following equalities:

$$\mathrm{Inv}_\Gamma \underline{X}^\Gamma = \mathrm{Inv}\Big(\underline{X}^\Gamma/\mathrm{Fix}(\Gamma)\Big) = \mathrm{Inv}\underline{X}.$$

EXAMPLES (1) Consider the category of topological spaces and continuous maps. Suppose there is a topological obstruction to the existence of an embedding $\underline{X} \hookrightarrow \underline{Y}$ (for instance, $\underline{X} = \mathbb{R}P^2$ and $\underline{Y} = \mathbb{R}^3$ or even $\underline{X} = \{0, 1\}$ and $\underline{Y} = \{0\}$). Does this obstruction induce a 'dynamical obstruction' to the existence of a Γ-equivariant embedding

$$\underline{X}^\Gamma/\mathrm{Fix}(\Gamma) \hookrightarrow \underline{Y}^\Gamma/\mathrm{Fix}(\Gamma)?$$

Note that an embedding $\underline{X}^\Gamma \hookrightarrow \underline{Y}^\Gamma$ induces an embedding $\underline{X} \simeq \mathrm{Fix}(\Gamma) \subset \underline{X}^\Gamma \hookrightarrow \underline{Y} \simeq \mathrm{Fix}(\Gamma) \subset \underline{Y}^\Gamma$ which explains the presence of the quotient.

(2) In [2], a dynamical version of the topological dimension of a compact metric space is considered. It is denoted by $\dim(X : \Gamma)$. Recall that the topological dimension of $\underline{X}$ is defined as follows:

$$\dim \underline{X} = \lim_{\epsilon \to 0} \dim_\epsilon \underline{X} = \lim_{\epsilon \to 0} \inf\left\{k\ ; \exists\ \varepsilon\text{-embedding } \underline{X} \hookrightarrow [0, 1]^k\right\},$$

where a map $\underline{X} \to [0, 1]^k$ is an ε-embedding provided the diameter of the inverse image of a point is less than or equal to ε. Its dynamical version satisfies

- if $X \hookrightarrow Y$ is a Γ-equivariant embedding, then $\dim(X : \Gamma) \leqslant \dim(Y : \Gamma)$,
- $\dim(\underline{X}^\Gamma : \Gamma) = \dim(\underline{X}^\Gamma/\mathrm{Fix}(\Gamma) : \Gamma) = \dim \underline{X}$.

This yields nonembedding results $\underline{X}^\Gamma/\mathrm{Fix}(\Gamma) \hookrightarrow \underline{Y}^\Gamma/\mathrm{Fix}(\Gamma)$ when $\dim \underline{X} > \dim \underline{Y}$.

The purpose of [1] is to develop a dynamical version of Morse Theory. It is done in the spirit of classical statistical mechanics.

3.2. SETTING

Suppose the following data as given:

- a compact topological space X endowed with an action of a countable (discrete) group Γ:
$$\rho : \Gamma \times X \to X : (\gamma, x) \to \gamma \cdot x,$$
- a left-invariant metric on the group Γ, e.g. the word metric relative to a (finite or not finite) set of generators,
- some continuous function $f : X \to \mathbb{R}$.

MAIN EXAMPLES (Products). The space X is $\mathrm{Map}(\Gamma, M) = M^\Gamma$, the product of Γ copies of a compact manifold M with the product topology and f is some function $M^\Gamma \to \mathbb{R}$, for instance, $f((x_\gamma)_{\gamma\in\Gamma}) = f(x_{\delta_1}, ..., x_{\delta_d})$, where $\{\delta_1, ..., \delta_d\}$ is a set of generators for the group. As a subexample, consider $X = M^{\mathbb{Z}}$ with $f((x_i)_{i\in\mathbb{Z}}) = f_o(x_0, x_1)$.

MAIN IDEA. Define the *(co)homological entropy* of the function f by replacing the Lebesgue measure (of thickened level sets) by the *cohomological measure*, i.e. by

$$\operatorname{rank}\left[\varphi_{c,\delta}: H^*\left(f^{-1}(-\infty, c+\delta)\right)\right.$$
$$\left.\to \operatorname{Hom}\left[H^*\left(f^{-1}(c-\delta, +\infty)\right), H^*\left(f^{-1}(c-\delta, c+\delta)\right)\right]\right]. \tag{3}$$

This is of course motivated by the description of the Morse inequality $|\mathrm{Crit}(f)| \geqslant \mathrm{SB}(M) = \sum_i \dim H^i(M; F)$ presented in Section 1. One wishes to construct a 'dynamical' homological lower bound for the 'dynamical' number of critical points of f (cf. Remark: Dynamical Morse inequality). More precisely, we will translate the construction of Section 2 of the entropy by means of the following dictionary. We first introduce some notations. Let us denote by $F(\Gamma)$ the set of finite subsets of Γ. The cardinality of one of its elements Ω is denoted by $|\Omega|$. Given such a subset Ω, the average of f over Ω is the function

$$f_\Omega: X \to \mathbb{R}: x \mapsto \frac{1}{|\Omega|}\sum_{\gamma\in\Omega} f(\gamma^{-1}x).$$

Classical entropy	*Homological entropy*
• n	• Ω or $\lvert\Omega\rvert$
• $\left(\frac{f}{n}\right)_{n\geqslant 1}$ $\frac{f}{n}: (\mathbb{R}^\nu)^n \to \mathbb{R}$ FRO	• $(f_\Omega)_{\Omega\in F(\Gamma)}$ $f_\Omega: X \to \mathbb{R}$
• $(\Lambda_n)^n$ with $\frac{n}{\mathrm{Vol}(\Lambda_n)} \xrightarrow[[n\to\infty]]{} \rho$	• $A_\Omega = \mathrm{Alg}\langle \oplus_{\gamma\in\Omega}\gamma_* A\rangle$ with $\frac{\lvert\Omega\rvert}{\ln \operatorname{rank} A_\Omega} \xrightarrow[[\lvert\Omega\rvert \to \infty]]{} -\ln$ rank A (cf. Remark about A)
• $v(\Lambda, n, f, (c-\delta, c+\delta))$ $= \mu^n_{\mathcal{L}\mathrm{eb}}\left(\left(\frac{f}{n}\right)^{-1}(c-\delta, c+\delta) \cap (\Lambda_n)^n\right)$	• $\beta_{A,\Omega,c,\delta} = \operatorname{rank}\left[\varphi_{A,\Omega,c,\delta}\right]$ $= \operatorname{rank}\left[\varphi_{A,\Omega,c,\delta}: H^*_{A_\Omega}(f_\Omega^{-1}(-\infty, c+\delta)). \to \ldots\right]$ (cf. (4) below)

Remark about A. Note that the limitation to Λ_n ensured that the Lebesgue measure $v\left(\Lambda, n, f, (c-\delta, c+\delta)\right)$ is finite. Similarly, we restrict our attention to cohomology classes belonging to some finite-dimensional subalgebra $A \subset H^*(X; F)$ to ensure that the cohomological measure is finite. The map $\varphi_{A,\Omega,c,\delta}$ is defined as follows:

$$\varphi_{A,\Omega,c,\delta}: H^*_{A_\Omega}\left(f_\Omega^{-1}(-\infty, c+\delta)\right)$$
$$\to \operatorname{Hom}\left[H^*_{A_\Omega}\left(f^{-1}(c-\delta, +\infty)\right), H^*_{A_\Omega}\left(f^{-1}(c-\delta, c+\delta)\right)\right], \tag{4}$$

where the index A_Ω means that only classes in A_Ω are being considered, that is, $H^*_{A_\Omega} = H^* \cap A_\Omega$.

Since we are interested in the exponential growth of a quantity bounded above by rank A_Ω, the latter should itself have at least exponential growth in order to obtain a nontrivial quantity (i.e. rank $A_\Omega \simeq (\text{rank}\, A)^{|\Omega|}$). This holds in the product case, at least for certain subalgebras, namely those of the type $A = p^*_{\Omega_o} H^*(M^{\Omega_o})$, where $p_{\Omega_o}\colon M^\Gamma \to M^{\Omega_o}$ is the canonical projection, since then $A_\Omega = p^*_{\Omega\cdot\Omega_o} H^*(M^{\Omega\cdot\Omega_o})$ and thus,

$$\text{rank}\, A_\Omega = \left(\text{rank}\ H^*(M)\right)^{\Omega\cdot\Omega_o}$$

grows exponentially with Ω. In order to obtain a nontrivial entropy and in fact a well-defined exponential growth for the cohomological measure, we need to impose a cohomological assumption on the group action. Heuristically speaking, it guarantees richness of the multiplicative structure on cohomology.

ASSUMPTION. There exists a subalgebra $\mathcal{A} \subset H^*(M; F)$ for which any finite-dimensional subalgebra $A \subset \mathcal{A}$ admits a number $N = N(A) \geqslant 0$ such that if $\Omega, \Omega' \in F(\Omega)$ satisfy $d(\Omega, \Omega') \geqslant N(A)$, then the following map is injective :

$$A_\Omega \otimes A_{\Omega'} \hookrightarrow A_{\Omega\cup\Omega'} : a \otimes a' \mapsto a \cup a'. \tag{$*$}$$

DEFINITION. The (co)homological entropy of f is the function

$$s(c) = \sup_A \lim_{\delta\to 0} \lim_{i\to\infty} \frac{1}{|\Omega_i|} \ln \beta_{A,\Omega_i,c,\delta},$$

where $\Omega_1 \subset \Omega_2 \subset \ldots$ is a sequence of finite subsets exhausting Γ.

The limit is guaranteed to exist under some restrictive assumptions on Γ and $(\Omega_i)_{i\geqslant 1}$, namely Γ is a *tileable amenable group* (cf. [1]) and $(\Omega_i)_{i\geqslant 1}$ is an amenable sequence (the boundary of Ω_i is asymptotically negligible compared to the all set Ω_i). Similarly to the thermodynamic entropy, the proof relies essentially on the lemma below, which also yields concavity of s.

LEMMA (Supermultiplicativity of the cohomological measure). *Let $A \subset \mathcal{A}$ be a finite-dimensional algebra. Let $\Omega, \Omega' \in F(\Gamma)$ with $d(\Omega, \Omega') \geqslant N(A)$. Let $c, c' \in \mathbb{R}$. Set* $\alpha = \frac{|\Omega|}{|\Omega\cup\Omega'|}$ (so $1-\alpha = \frac{|\Omega'|}{|\Omega\cup\Omega'|}$). *Then*

$$\beta_{A,\Omega\cup\Omega',\alpha c+(1-\alpha)c',\delta} \geqslant \beta_{A,\Omega,c,\delta} \cdot \beta_{A,\Omega',c',\delta}.$$

Proof. The essential point in summarized in the implication below: consider

- $a, b \in A_\Omega$ with $a \wedge b \neq 0$ and supp $a \subset f_\Omega^{-1}(-\infty, c+\delta)$, supp $b \subset f_\Omega^{-1}(c-\delta, +\infty)$,
- $a', b' \in A_{\Omega'}$ with $a' \wedge b' \neq 0$ and supp $a' \subset f_{\Omega'}^{-1}(-\infty, c'+\delta)$, supp $b' \subset f_{\Omega'}^{-1}(c'-\delta, +\infty)$.

Then

- $(a \wedge a') \wedge (b \wedge b') = (a \wedge b) \wedge (a' \wedge b') \neq 0$ (because assumption ($\star$) above is supposed to hold),

- $\operatorname{supp}(a \wedge a') \subset f_\Omega^{-1}(-\infty, c+\delta) \cap f_{\Omega'}^{-1}(-\infty, c'+\delta)$
  $\subset f_{\Omega\cap\Omega'}^{-1}(-\infty, \alpha c + (1-\alpha)c' + \delta)$,
- $\operatorname{supp}(b \wedge b') \subset f_\Omega^{-1}(c-\delta, +\infty) \cap f_{\Omega'}^{-1}(c'-\delta, +\infty)$
  $\subset f_{\Omega\cap\Omega'}^{-1}(\alpha c + (1-\alpha)c' - \delta, +\infty)$. □

Remark (Dynamical Morse inequality). Of course, in the present context the notion of critical point of a function on X is not defined. But in the case of a product $X = M^{\mathbb{Z}}$ with a two-point function $f((x_i)_{i\in\mathbb{Z}}) = f_o(x_0, x_1)$ (or more generally a function depending on finitely-many variables only), since f_Ω is 'morally' defined on M^Ω (in fact on $M^{\Omega'}$, where Ω' is a small extension of Ω), talking about the critical points of f_Ω makes perfectly good sense. Moreover, the nondegenerate Morse inequality implies that $\beta_{A,\Omega,c,\delta}$ is a lower bound for $|\mathrm{Crit}\,(f_\Omega) \cap f_\Omega^{-1}(c-\delta, c+\delta)|$ (at least when the subalgebra A contains the subalgebra $p^*_{\{0\}}H^*(M;F)$). Hence, if we define $C(c)$ to be the exponential growth of the number of critical points of f_Ω at level c, that is

$$C(c) = \lim_{\delta\to 0}\lim_{i\to\infty}\frac{1}{|\Omega_i|}\ln\left|\mathrm{Crit}(f_{\Omega_i}) \cap f_{\Omega_i}^{-1}(c-\delta, c+\delta)\right|,$$

then we obtain the following inequality: $C(c) \geqslant s(c)$.

PROPOSITION (Nontriviality of the entropy). *In the product case, the function f achieves a strictly positive value.*

The proof is a consequence of Poincaré duality for M. In fact, M^Γ admits some version of Poincaré duality (see [1] Section 11 for a precise statement) that is inherited from M. More generally, any Γ-space X satisfying the assumption $(*)$ above and that version of Poincaré duality will admit a well-defined and nontrivial homological entropy. Projective limits [3] of projective algebraic varieties are examples of such spaces (cf. [1]).

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