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Computable Cross-norm Criterion for Separability

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Abstract. We describe a computable analytical criterion for separability of bipartite mixed states in arbitrary dimension. The criterion stipulates that a certain norm on the state space (the computable cross-norm) is bounded by 1 for separable states. The criterion is shown to be independent of the well-known positive partial transpose (PPT) criterion. In other words, the criterion detects some bound entangled states but fails for some free entangled states.

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1. Introduction

Quantum entanglement is one of the most fascinating features of quantum theory. Its study is an important area of research that has received much attention in the development of quantum information theory. Mathematically, entanglement is a complex property of quantum states defined on tensor product Hilbert spaces. Physically, the role of entanglement in quantum information processing is manifold. Quantum entanglement has been identified as an essential resource in many applications of quantum information processing. In recent years considerable progress has been made towards developing a general theory of quantum entanglement, [1, 3] and references therein. In particular, criteria to decide whether or not a given quantum state is entangled are of high theoretical and practical interest. Such criteria are called *separability criteria*. In this Letter we focus on analytical separability criteria. The case of pure bipartite quantum states (wavefunctions) is well understood and there are several necessary and sufficient separability criteria for them. For bipartite mixed quantum systems a number of important computable criteria have been found. For a review see, e.g., [1]. However, despite all the progress, theoretically our understanding of mixed state entanglement remains incomplete.

A key objective in the theory of entanglement is the complete characterization and classification of entangled states. By now several mathematically necessary and sufficient characterizations for separability of mixed states have been identified, e.g., [5, 12]. One of these characterization in terms of positive maps was shown in

[5] to provide an operational necessary and sufficient condition for separability in dimension 2×2 and 2×3. This condition is the well-known positive partial transpose (PPT) criterion that was first found by Peres [11]. In arbitrary dimension, however, we are lacking a complete set of operational and mutually *independent* analytical separability criteria.

In the present paper we present a powerful computable analytical separability criterion that is *independent* of the PPT criterion. In other words the criterion detects some bound entangled states* but fails for some free entangled states. The criterion stipulates that a certain norm (that we call *the computable cross-norm*) is bounded by 1 for separable states. We shall call this criterion the *computable cross-norm* (CCN) *criterion*. The *cross* property of a norm will be explained below.

The CCN criterion first appeared in the second part of the unpublished preprint [13]. To the best of this authors knowledge it is the first analytical criterion that can be proven to be independent of the PPT criterion while still matching its power and versatility. After [13] appeared, the CCN criterion reappeared in [2] under the name *realignment criterion*. In this reference the criterion and its proof were rewritten using a matrix representation for operators rather than the equivalent Dirac bra–ket notation that was used in [13]. Otherwise the criterion presented in [2] and subsequent results, including examples, are identical to the results in [13]. Based on the CCN criterion the Horodecki family [6] subsequently developed a *linear contractions approach* to characterize entanglement that complements their earlier widely studied positive-maps approach. Moreover, they applied the CCN criterion and its generalization (*index permutation criteria*) to multipartite entanglement, an area where only few systematic results are known, and showed that the CCN criterion detects very subtle forms of multipartite bound entanglement for which previously there was no analytical test known. Recently further properties of the CCN criterion were studied in [14] and it was studied in [7] how the CCN criterion (or more generally linear contraction tests for entanglement) can in principle be implemented experimentally.

Throughout the Letter we adopt the following conventions and notation: all Hilbert spaces are assumed to be complex and finite-dimensional. We further assume that in each Hilbert space $\mathscr{H}$ a canonical real basis has been chosen and thus identify each Hilbert space $\mathscr{H}$ with $\mathbb{C}^d$ where $d=\dim\mathscr{H}$. The canonical real basis of $\mathbb{C}^d$ is denoted by $(|i\rangle)_{i=1}^{d}$. The set of linear operators on $\mathbb{C}^d$ (i.e., $d\times d$ matrices) is denoted by $\mathtt{T}(\mathbb{C}^d)$. A *state* for a d-dimensional quantum system with Hilbert space $\mathbb{C}^d$ is a positive operator on $\mathbb{C}^d$ with trace one (i.e., a density operator). The set of all states on $\mathbb{C}^d$ is denoted by $\mathcal{S}(\mathbb{C}^d)$. *Pure states* are the extreme

*A pure bipartite wavefunction $|\psi\rangle$ in $\mathbb{C}^d\otimes\mathbb{C}^d$ with Schmidt decomposition $|\psi\rangle=\sum_k\sqrt{p_k}|\chi_k\otimes\eta_k\rangle$ is called *maximally entangled* if $p_k=1/d$ for all k. Loosely speaking a *bound entangled* state is an entangled state that cannot be transformed to maximally entangled pure form by local operations assisted by classical communication (LOCC operations). It is known that any entangled state that satisfies the PPT criterion is bound entangled. Entangled states that are not bound entangled are also called *distillable* or *free entangled*. For more details, see [8].

points in $\mathcal{S}(\mathbb{C}^d)$. It is known that pure states are exactly given by the one-dimensional projectors and can thus be identified with unit vectors in $\mathbb{C}^d$ (i.e., wavefunctions). In the present paper we will always identify a wavefunction with its corresponding projection operator. Throughout the Letter $\|\sigma\|_2$ denotes the *Hilbert–Schmidt norm* or the *Frobenius norm* of an operator σ. The Hilbert–Schmidt norm is equal to the sum of the squares of the singular values of σ. The sum of the singular values of σ is called the *trace class norm,* or simply *trace norm,* and is denoted by $\|\sigma\|_1$. The Hilbert–Schmidt inner product of two operators σ, τ is given by $\langle\sigma, \tau\rangle = \mathrm{tr}(\sigma^\dagger\tau)$.

2. Separability of Mixed Quantum States

Let us start with the basic definition [16].

DEFINITION 2.1. Let ρ be a state on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. Then ρ is called *separable* or *disentangled* if it can be written as a convex combination of simple tensor states, i.e., if it can be expressed in the form

$$\rho = \sum_{i=1}^{s} \lambda_i \rho_i^{(A)} \otimes \rho_i^{(B)},$$

where $\lambda_i > 0$, $\sum_i \lambda_i = 1$ and where $\rho_i^{(k)}$ is a state on $\mathbb{C}^{d_k}$ for all i and $k = A, B$. Otherwise ρ is called entangled.

Let for instance $|\psi\rangle$ be a pure bipartite wavefunction in $\mathbb{C}^d \otimes \mathbb{C}^d$ and $|\psi\rangle = \sum_k \sqrt{p_k}|\chi_k \otimes \eta_k\rangle$ be its Schmidt decomposition [4]. Let P_ψ be the projector onto the subspace spanned by $|\psi\rangle$. Then it is easy to see that the state P_ψ is disentangled according to Definition 2.1 if and only if $|\psi\rangle$ is a simple tensor, i.e., if and only if $|\psi\rangle = |a\rangle \otimes |b\rangle$ for some $|a\rangle, |b\rangle \in \mathbb{C}^d$. In general, it is difficult to decide whether or not a given mixed state ρ is entangled. A *necessary separability criterion* is a criterion that is passed by all separable states and violated by some entangled states. Therefore, whenever a state ρ violates a given necessary separability criterion, this indicates that ρ is entangled. We say that such an entangled state is *detected* by the criterion in question.

In [11], Peres obtained a powerful computable necessary separability criterion, the positive partial transpose (PPT) criterion. The PPT criterion stipulates that the partial transpose ρ^{T_2} of any separable quantum state ρ is again a (separable) state.

PROPOSITION 2.2 (Peres). *Let* $\rho \in \mathcal{S}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$ *be a separable state. Then* $\rho^{T_2} \geqslant 0$.

For bipartite quantum systems a number of important and powerful criteria have been found. For a review see, e.g., [1] and references therein.

The following way of speaking is useful in the study of separability criteria.

DEFINITION 2.3. Let (A) and (B) be two necessary separability criteria and denote by $\mathscr{E}(A)$ and $\mathscr{E}(B)$ the set of states detected by (A) and (B) respectively. We say that (A) is *weaker* than (B) if all states that are detected by (A) are also detected by (B), i.e., if $\mathscr{E}(A) \subseteq \mathscr{E}(B)$. In this case we also say that (B) is *stronger* than (A). We say that (A) and (B) are *independent* if (A) is neither weaker nor stronger than (B). Finally we call (A) and (B) *equivalent* if $\mathscr{E}(A) = \mathscr{E}(B)$.

3. The Computable Cross-norm Criterion

Now let us pass to describe the CCN criterion.

DEFINITION 3.1. The *computable cross-norm* $\|\sigma\|_\tau$ of an arbitrary (not necessarily positive) operator σ on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ is defined by

$$\|\sigma\|_\tau := \inf\left\{ \sum_{i=1}^{k} \|\sigma_i^{(A)}\|_2 \|\sigma_i^{(B)}\|_2 \;\middle|\; \sigma = \sum_{i=1}^{k} \sigma_i^{(A)} \otimes \sigma_i^{(B)} \right\}. \tag{1}$$

Here the infimum is taken over all decompositions of σ into a finite sum of simple tensors.

It is important to notice that the simple tensors in the decompositions of σ appearing on the right-hand side of Equation (1) are arbitrary. In particular, if σ is hermitean or positive, we do *not* require the decompositions in Equation (1) to run over hermitean or positive simple tensors only, respectively.

Now the CCN criterion is the following.

CRITERION 3.2. *Let* ρ *be a state on* $\mathbb{C}^d \otimes \mathbb{C}^d$. *The CCN criterion asserts that if* ρ *is separable, then the computable cross-norm of* ρ *is less than or equal to 1. Whenever a state* ρ *satisfies* $\|\rho\|_\tau > 1$, *this signals that* ρ *is entangled.*

Proof. The Hilbert–Schmidt norm is majorized by the computable cross-norm, i.e., $\|x\|_2 \leqslant \|x\|_\tau$ for all $x \in \mathrm{T}(\mathbb{C}^d)$. (This is an immediate consequence of the subadditivity of the Hilbert–Schmidt norm.) Thus it follows from the definition of $\|\cdot\|_\tau$ that $\|\cdot\|_\tau$ satisfies the so-called *subcross property* with respect to the trace norm, which means that we have

$$\|\sigma^{(A)} \otimes \sigma^{(B)}\|_\tau = \|\sigma^{(A)}\|_2 \|\sigma^{(B)}\|_2 \leqslant \|\sigma^{(A)}\|_1 \|\sigma^{(B)}\|_1,$$

for all simple tensors $\sigma^{(A)} \otimes \sigma^{(B)}$. Let $\rho_{\mathrm{sep}} = \sum_{i=1}^{s} \lambda_i \rho_i^{(A)} \otimes \rho_i^{(B)}$ be a separable state. Then by the subadditivity and the subcross property of $\|\cdot\|_\tau$ we find that

$$\|\rho_{\mathrm{sep}}\|_\tau \leqslant \sum_{i=1}^{s} \lambda_i \|\rho_i^{(A)} \otimes \rho_i^{(B)}\|_\tau \leqslant \sum_{i=1}^{s} \lambda_i \|\rho_i^{(A)}\|_1 \|\rho_i^{(B)}\|_1 = 1.$$

□

Next we need to justify the adjective ‘computable’ and explain how the ‘computable’ cross-norm can actually be computed in practice. The key observation is the following representation of the trace class norm. Let $O : \mathscr{H} \to \mathscr{H}$ an arbitrary linear operator on a Hilbert space $\mathscr{H}$. In finite dimensions O can be any linear operator. If $\mathscr{H}$ is an infinite dimensional Hilbert spaces, we need to require that O is of *trace class* [15]. In the present paper we assume that all Hilbert spaces are finite-dimensional. Then

$$\|O\|_1 = \inf \left\{ \sum_{i=1}^{k} \|\psi_i\| \|\phi_i\| \,\middle|\, O = \sum_{i=1}^{k} |\psi_i\rangle\langle\phi_i| \right\}, \tag{2}$$

where the infimum is over all decompositions of O into finite sums of rank one operators. The proof of Equation (2) is easy [15]. By the subadditivity of the norm, we certainly have an inequality in (2). Since the infimum is attained for the singular value decomposition of O (see [15]), it follows that we have equality in (2).

Now we compare the expressions in Equations (1) and (2) and recall that the set of linear operators on a finite-dimensional Hilbert space is itself a Hilbert space when equipped with the Hilbert–Schmidt inner product. It is then obvious that the computable cross norm of ρ can be interpreted as the trace class norm of a certain ‘re-ordered’ operator $\mathfrak{A}(\rho)$. This operator $\mathfrak{A}(\rho)$ is defined as follows. Let $\rho = \sum_k E_k \otimes F_k$ be *any* representation of ρ as finite sum of simple tensors, then $\mathfrak{A}(\rho)$ is defined as

$$\mathfrak{A}(\rho) := \sum_k |E_k\rangle\,\langle F_k^*|, \tag{3}$$

where $*$ denotes complex conjugation. Here we write $|E_k\rangle$ and $|F_k\rangle$ instead of just E_k and F_k respectively to indicate that we now think of the E_k and F_k as elements of the Hilbert–Schmidt Hilbert space $\mathtt{T}(\mathbb{C}^d)$. We remark that it is a consequence of, e.g., Proposition 11.1.8 in [10] that $\mathfrak{A}(\rho)$ is well-defined independently of the particular decomposition of ρ chosen. Let us summarize our main result as a theorem.

THEOREM 3.3. *Let ρ be a state on $\mathbb{C}^d \otimes \mathbb{C}^d$. Then $\|\rho\|_\tau = \|\mathfrak{A}(\rho)\|_1$.*

To check whether the CCN criterion is satisfied by a given density operator ρ reduces to the evaluation of the trace class norm of the operator $\mathfrak{A}(\rho)$. This is completely straightforward using standard linear algebra packages and accordingly the CCN criterion is a computable separability criterion for density operators. From Equation (3) it is a straightforward and trivial exercise to determine the matrix representation for $\mathfrak{A}(\rho)$ in the canonical basis. When $(|i\rangle)_{i=1}^d$ denotes the canonical real basis of $\mathbb{C}^d$, then $(E_{ij} \equiv |i\rangle\langle j|)_{i,j=1}^d$ denotes the canonical basis of the Hilbert–Schmidt space $\mathtt{T}(\mathbb{C}^d)$. For instance, in dimension 2×2 the transformation $\rho \mapsto \mathfrak{A}(\rho)$ corresponds to the following ‘matrix re-ordering’ in the canonical basis (where the matrices are constructed from operators using the respective canonical bases in binary lexicographic ordering)

$$\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \mapsto \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{21} & \rho_{22} \\ \rho_{13} & \rho_{14} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{41} & \rho_{42} \\ \rho_{33} & \rho_{34} & \rho_{43} & \rho_{44} \end{pmatrix}.$$

In higher dimensions analog formulas hold. However, the basis independent 'invariant' formulation of the CCN criterion in terms of the computable cross-norm as given in Equation (1) and Theorem 3.3 is certainly mathematically more elegant. Moreover, in practical calculation it is sometimes awkward to expand a given state into the canonical basis. Rather, it is often simpler to expand a state into a basis different from the canonical basis (for instance in terms of (generalized) Pauli spin matrices as shown for an example in [14]). Therefore the basis independent formulation of the CCN criterion is also more useful in practice since we can expand a given state ρ (and accordingly $\mathfrak{A}(\rho)$) in terms of any suitable set of simple tensors (these sets do not even need to form bases).

4. Examples

Finally we give three examples to demonstrate key features of the CCN criterion. More details and further examples can be found in [13, 14].

EXAMPLE 4.1. Let $|\psi\rangle$ be a pure bipartite wavefunction in $\mathbb{C}^d \otimes \mathbb{C}^d$ and $|\psi\rangle = \sum_k \sqrt{p_k}|\chi_k \otimes \eta_k\rangle$ be its Schmidt decomposition [4]. Let P_ψ be the projector onto the subspace spanned by $|\psi\rangle$. Then

$$\| P_\psi \|_\tau = \left(\sum_k \sqrt{p_k} \right)^2 .$$

Therefore the CCN criterion detects all entangled pure states.

EXAMPLE 4.2 (Werner states). Werner states (first considered in [16]) are mixed quantum states in $\mathtt{T}(\mathbb{C}^d \otimes \mathbb{C}^d)$. They can be parametrized by a real parameter f with $-1 \leqslant f \leqslant 1$ and are given by

$$\rho_f := \frac{1}{d^3 - d}\left((d - f)\mathbb{1} + (df - 1)\mathbb{F}\right),$$

where

$$\mathbb{F} := \sum_{i,j} |i \otimes j\rangle\langle j \otimes i|,$$

where $(|i\rangle)$ is an orthonormal basis of $\mathbb{C}^d$. Werner states are known to be separable if and only if $f \geqslant 0$. A straightforward calculation gives

$$\|\rho_f\|_\tau = \begin{cases} \frac{2}{d} - f & : \quad \text{for } -1 \leqslant f \leqslant \frac{1}{d} \\ f & : \quad \text{for } 1 \geqslant f \geqslant \frac{1}{d} \end{cases} .$$

This proves that for Werner states the CCN criterion is exact if and only if $d = 2$. In higher dimension $d \geqslant 3$ there will always be inseparable Werner states (i.e., those corresponding to $f \in [(2/d) - 1, 0[$) which satisfy the CCN criterion while other inseparable Werner states (i.e., those corresponding to $f \in [-1, (2/d) - 1[$) violate it.

EXAMPLE 4.3 (bound entanglement). Consider $\mathbb{C}^3 \otimes \mathbb{C}^3$ and let $\{|0\rangle, |1\rangle, |2\rangle\}$ be the canonical real basis in $\mathbb{C}^3$. Consider the following family of mixed states defined on $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$\rho_\alpha := \frac{2}{7} \left|\Psi^+_{(3)}\right\rangle\left\langle\Psi^+_{(3)}\right| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-,$$

where we restrict ourselves to the parameter range $2 \leqslant \alpha \leqslant 5$, and where

$$\left|\Psi^+_{(3)}\right\rangle \equiv \frac{1}{\sqrt{3}} \left(|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle\right),$$

$$\sigma_+ \equiv \frac{1}{3} \left(|0\rangle|1\rangle\langle 0|\langle 1| + |1\rangle|2\rangle\langle 1|\langle 2| + |2\rangle|0\rangle\langle 2|\langle 0|\right),$$

$$\sigma_- \equiv \frac{1}{3} \left(|1\rangle|0\rangle\langle 1|\langle 0| + |2\rangle|1\rangle\langle 2|\langle 1| + |0\rangle|2\rangle\langle 0|\langle 2|\right).$$

It is known, see [9], that ρ_α is (i) separable if and only if $2 \leqslant \alpha \leqslant 3$, (ii) bound entangled if and only if $3 < \alpha \leqslant 4$ and (iii) entangled and distillable [17] if and only if $4 < \alpha \leqslant 5$. A straightforward calculation shows that

$$\|\rho_\alpha\|_\tau = \frac{19}{21} + \frac{2}{21}\sqrt{19 - 15\alpha + 3\alpha^2}. \tag{4}$$

It is easy to see that $\|\rho_\alpha\|_\tau \leqslant 1$ if and only if $2 \leqslant \alpha \leqslant 3$, i.e., if and only if ρ_α is separable. This example shows that there are bound entangled states which violate the CCN criterion.

For every new separability criterion it is important to ask how it enters the known implication chains between separability criteria. The examples considered above show that the CCN criterion is neither weaker nor stronger than the PPT criterion. In other words both criteria are independent.

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