



Derived Brackets*

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(Received: 5 January 2004)

Abstract. We survey the many instances of derived bracket construction in differential geometry, Lie algebroid and Courant algebroid theories, and their properties. We recall and compare the constructions of Buttin and of Vinogradov, and we prove that the Vinogradov bracket is the skew-symmetrization of a derived bracket. Odd (resp., even) Poisson brackets on supermanifolds are derived brackets of canonical even (resp., odd) Poisson brackets on their cotangent bundle (resp., parity-reversed cotangent bundle). Lie algebras have analogous properties, and the theory of Lie algebroids unifies the results valid for manifolds on the one hand, and for Lie algebras on the other. We outline the role of derived brackets in the theory of ‘Poisson structures with background’.

Mathematics Subject Classifications (2000). primary, 17B70, 53D17, secondary, 17B66, 17B63, 17D99, 58A50.

Key words. Derived bracket, Courant bracket, Vinogradov bracket, Loday–Leibniz algebra, Courant algebroid, Lie algebroid, odd Poisson supermanifold, Poisson structure with background.

Introduction

On any graded differential Lie algebra, $(\mathfrak{A}, [,], D)$, with bracket of degree n , one can consider the bilinear map,

$$(a, b) \in \mathfrak{A} \times \mathfrak{A} \mapsto (-1)^{n+|a|+1}[Da, b] \in \mathfrak{A},$$

where $|a|$ is the degree of a . This is what is called the *derived bracket* of $[,]$ by D (see [26]). It is not in general a graded Lie bracket because it is not, in general, skew-symmetric. However, it does satisfy the Jacobi identity in the form (1.1) below, therefore \mathfrak{A} with a derived bracket is a graded version of what Loday calls a *Leibniz algebra*, which we prefer to call a *Loday algebra*. Since the derivation D is odd, passing from the original bracket to the derived bracket turns an even (resp., odd) Lie bracket into an odd (resp., even) Loday bracket.

*This Letter is the revised version of a lecture given at the Euroconference, Poisson Geometry, Deformation Quantization and Group Representations (PQR 2003) at the Université Libre de Bruxelles, 18–22 June 2003.

In applications, D will most often be the interior derivation by an odd element, $d \in \mathfrak{A}$, of square 0, $[d, d]=0$. The derived bracket can then be written simply as

$$(a, b) \in \mathfrak{A} \times \mathfrak{A} \mapsto [[a, d], b] \in \mathfrak{A}.$$

Whenever $(\mathfrak{A}, [,]) is Abelian, the derived bracket is a genuine graded Lie bracket, and this property is essential for the applications that we shall describe.$

Instances of derived brackets, though they do not bear that name, appeared in various contexts: in an article on formal noncommutative geometry by Gel'fand, Daletskii and Tsygan [18], in papers by physicists on the BRST quantization, which are too numerous to be exhaustively cited here, see [4], in the 1974 article by Buttin [7], in early work of Vinogradov [55] who introduced a very powerful tool under the unfortunate term of 'lievization', in unpublished papers of Ted Voronov, and certainly in other sources of which I am not aware. The notion was formalized in unpublished notes of Koszul dating from 1990 [33], which he communicated to me in 1994. There followed the article [26] where I placed Koszul's construction in the framework of graded Loday algebras, proved the main properties of the general construction, and gave examples from Poisson geometry and Lie bialgebra theory. Related results were obtained independently by Daletskii and Kushnirevitch [13]. Articles [27] and [28] contain a summary of results and describe applications to gauge Lie algebras in various field theories.

After briefly reviewing the general notion, I shall try to describe enough old and new examples of derived brackets to convince the reader of their ubiquity and importance. The following discussion may seem very formal, but the general results on the derived brackets of Lie brackets briefly recalled in Section 1 of this survey, and more generally of Loday brackets, are powerful tools for proving nontrivial properties of brackets and derivations.

1. Derived Brackets

1.1. LODAY BRACKETS

Loday algebras were introduced (in the ungraded case) by Jean-Louis Loday under the name *Leibniz algebras* [39]. We define a *Loday algebra of degree n* as a graded vector space V over a field R of characteristic $\neq 2$ (or just a module over a commutative ring), equipped with an R -bilinear map, $[,] : V \otimes V \rightarrow V$, satisfying the Jacobi identity in the form,

$$[a, [b, c]] = [[a, b], c] + (-1)^{(n+|a|)(n+|b|)} [b, [a, c]], \quad (1.1)$$

for all a, b and $c \in V$, where $|a|$ denotes the degree of $a \in V$. Whenever the bracket $[,]$ is graded skew-symmetric, a Loday algebra is just a graded Lie algebra. In what follows, we shall often omit the word 'graded'.

1.2. DEFINITION OF DERIVED BRACKETS

In [26] (see also [27, 28]), we defined a general notion of *derived brackets* of Loday brackets, and we proved some simple properties that have far-reaching consequences. This construction turns an even Loday bracket into an odd one, and conversely. Here, for simplicity, we describe this construction in the case of derived brackets of Lie brackets, and, in Theorems 1.1 and 1.2 below, we recall those properties that are most useful in applications.

DEFINITION 1.1. If $(V, [\ , \], D)$ is a graded differential Lie algebra over R with bracket of degree n , we define the bilinear map $[\ , \]_{(D)}: V \otimes V \rightarrow V$ by

$$[a, b]_{(D)} = (-1)^{n+|a|+1}[Da, b], \quad (1.2)$$

for a and $b \in V$, and we call it the derived bracket of $[\ , \]$ by D .

THEOREM 1.1.

- (i) *The derived bracket of a Lie bracket of degree n is a Loday bracket of degree $n + 1$.*
- (ii) *The map D is a morphism of Loday algebras from $(V, [\ , \]_{(D)})$ to $(V, [\ , \])$.*
- (iii) *The map D is a derivation of the Loday bracket $[\ , \]_{(D)}$.*
- (iv) *The restriction of the derived bracket to any Abelian subalgebra, V_0 , of $(V, [\ , \])$, such that $[DV_0, V_0] \subset V_0$, is a Lie bracket of degree $n + 1$.*
- (v) *The bracket $[\ , \]_{(D)}$ induces a Lie bracket of degree $n + 1$ on the quotient space of V by the image of V under D , $V/D(V)$.*

More generally, we can consider the derived bracket by any derivation of odd degree and of square 0.

EXAMPLE. The problem of extending the Poisson bracket of functions into an even bracket on the algebra of all differential forms was a long-standing problem until the mid-nineties. In [26], we proved that such an extension can be easily defined, but only as a Loday bracket. Let P be a Poisson bivector on a smooth manifold M , and let $[\ , \]^P$ be the Koszul bracket of differential forms [32]. The derived bracket of $[\ , \]^P$ by the de Rham differential is an even Loday bracket which extends the Poisson bracket of functions. We observed that one of the brackets defined on symplectic manifolds by Michor in [41], denoted there by $\{ \ , \ }^2$, coincides with this derived bracket. Shortly after that, Grabowski proved that an extension as an even graded Lie bracket can be defined, but this bracket is not a biderivation of the algebra [20].

1.3. THE CASE OF AN INTERIOR DERIVATION

In many applications, the derivation D is an interior derivation of $(V, [\cdot, \cdot])$, $a \mapsto [d, a]$, where d is an element of square 0 in $(V, [\cdot, \cdot])$.

NOTATION. If D is the interior derivation by an element d in the Lie algebra V , we denote the corresponding derived bracket simply by $[\cdot, \cdot]_d$.

THEOREM 1.2. *If D is the interior derivation of $(V, [\cdot, \cdot])$ by an element $d \in V$ such that $|d| + n$ is odd and $[d, d] = 0$, the derived bracket is*

$$[a, b]_d = [[a, d], b], \quad (1.3)$$

for a and $b \in V$. Both $a \mapsto [d, a]$ and $a \mapsto [a, d]$ are morphisms from $(V, [\cdot, \cdot]_d)$ to $(V, [\cdot, \cdot])$.

The proof is obtained by a simple calculation.

EXAMPLE. Let P and $[\cdot, \cdot]^P$ be as in the Example of Section 1.2. Assume that P is nondegenerate, with inverse symplectic form ω . Then the de Rham differential is the interior derivation $[\omega, \cdot]^P$, as was proved in, e.g., [30]. Therefore, in this case, the derived bracket of forms α and β is equal to $[[\alpha, \omega]^P, \beta]^P$.

1.4. SKEW-SYMMETRIZATION OF DERIVED BRACKETS

The skew-symmetrization of the derived bracket $[\cdot, \cdot]_{(D)}$, which we denote by $[\cdot, \cdot]_{(D)}^-$, can be expressed as

$$[a, b]_{(D)}^- = \frac{1}{2} ([a, Db] - (-1)^{n+|a|} [Da, b]), \quad (1.4)$$

while, in the case of an interior derivation, it satisfies

$$[a, b]_d^- = \frac{1}{2} ([[a, d], b] - (-1)^{|b|} [a, [b, d]]). \quad (1.5)$$

In general, this bracket, obtained by skew-symmetrizing a Loday bracket, no longer satisfies the Jacobi identity (1.1): a defect in the Jacobi identity appears, so the skew-symmetrized bracket is not a Lie bracket. We shall give examples below.

1.5. NOTATIONS

When dealing with the geometric objects defined on a smooth manifold or supermanifold, we shall assume all fields to be smooth, and we shall often abbreviate *vector field* and *multivector field* to *vector* and *multivector*, respectively, and *differential form* to *form*. If M is a manifold, we denote the exterior algebra of multivector fields by

$$V^\bullet(M) = \bigoplus_{p \geq 0} V^p(M) = \Gamma \left(\bigoplus_{p \geq 0} \bigwedge^p (TM) \right),$$

so that $V^1(M)$ is the space of vector fields, and we denote the exterior algebra of differential forms by

$$\Omega^\bullet(M) = \bigoplus_{q \geq 0} \Omega^q(M) = \Gamma \left(\bigoplus_{q \geq 0} \bigwedge^q (T^*M) \right),$$

so that $\Omega^1(M)$ is the space of differential 1-forms. (If E is a vector bundle over M , we denote the space of sections of E by ΓE .) Because the Schouten–Nijenhuis bracket of multivectors [42, 43, 51] (or Schouten bracket for short) is the prototypical example of a Gerstenhaber bracket, we also use the term *Schouten algebra* to designate a *Gerstenhaber algebra*, i.e., an associative, graded commutative algebra with an odd Poisson bracket.

Henceforth, the notation $[,]$ stands for the graded commutator of graded endomorphisms of a graded module, unless specified otherwise.

2. The Cartan Formulas and their Generalizations

2.1. THE CARTAN FORMULAS

Everyone is familiar with the formulas to be found in, e.g., Henri Cartan’s celebrated communication presented in Brussels half a century ago [9],

$$[d, d] = 0, \quad [i_x, i_y] = 0, \quad L_x = [i_x, d], \quad [L_x, d] = 0, \quad [L_x, i_y] = i_{[x, y]}. \quad (2.1)$$

Here x and y are vector fields on a manifold, d is the de Rham differential, i_x is the interior product by x and L_x is the Lie derivation by x , each of these being a derivation of the algebra of differential forms, while $[,]$ is the graded commutator of graded endomorphisms of the space of differential forms, but $[x, y]$ in the last formula denotes the Lie bracket of vector fields.

2.2. $(\mathfrak{A}, \mathcal{D})$ -STRUCTURES

Motivated by the extension of the usual differential calculus to the calculus of variations (see [19]), Gel’fand, Daletskii and Tsygan [18], working with multigraded vector spaces, present the general theory of an $(\mathfrak{A}, \mathcal{D})$ -structure, a graded Lie algebra $\mathfrak{A} \oplus \mathfrak{g}_0 \oplus \mathcal{D}$, where \mathfrak{A} is an Abelian Lie algebra, generalizing the space of the i_x ’s, and where \mathcal{D} is a space generated by several commuting elements of square 0, generalizing the one-dimensional vector space generated by the de Rham differential, while \mathfrak{g}_0 generalizes the Lie algebra of Lie derivations. For any $d \in \mathcal{D}$, they let

$$[a, b]_d = [[a, d], b],$$

for $a, b \in \mathfrak{A}$, and they formulate their Theorem 1, stating that this formula defines a graded Lie bracket on \mathfrak{A} , and that the map $a \mapsto [a, d]$ is a morphism of graded Lie algebras, from \mathfrak{A} with bracket $[,]_d$ to \mathfrak{g}_0 with the original Lie bracket. It is

clear that these statements are special cases of the results summarized in Theorems 1.1 and 1.2 above.

In addition, Gel'fand, Daletskii and Tsygan state a property which is equivalent to the compatibility of the brackets $[\cdot, \cdot]_{d_1}$ and $[\cdot, \cdot]_{d_2}$ on \mathfrak{A} , for a pair of commuting differentials d_1 and d_2 , in the sense that the bracket $[\cdot, \cdot]_{d_1 + d_2}$ is a graded Lie bracket on \mathfrak{A} . Since

$$[a, b]_{d_1} + [a, b]_{d_2} = [a, b]_{d_1 + d_2},$$

this compatibility follows from the assumptions $[d_1, d_1] = [d_2, d_2] = [d_1, d_2] = 0$, which imply that $[d_1 + d_2, d_1 + d_2] = 0$.

2.3. THE CARTAN FORMULA FOR MULTIVECTORS

The last of the above-mentioned Cartan equations (2.1), which can be written

$$i_{[x, y]} = [[i_x, d], i_y], \quad (2.2)$$

expresses the fact that *the Lie bracket of vector fields is a derived bracket*.

It is well known [10, 45, 53] that Equation (2.2) is valid more generally for multivectors, when the bracket on the left-hand side is the Schouten–Nijenhuis bracket, showing that *the Schouten–Nijenhuis bracket of multivectors is a derived bracket*.

In the sequel of their paper [18], Gel'fand, Daletskii and Tsygan show that Equation (2.2) is valid in the more general case where x and y are elements of the associative, graded commutative algebra generated by \mathfrak{A} , and the bracket on the left-hand side is extended by the bi-derivation property. This generalizes the preceding statement concerning multivectors on manifolds to the case of $(\mathfrak{A}, \mathcal{D})$ -structures. In [18], further applications are made to the Gerstenhaber algebra structure of the Hochschild cohomology of an associative superalgebra.

2.4. SOME HISTORICAL COMMENTS

The theory of $(\mathfrak{A}, \mathcal{D})$ -structures, which was formulated around 1987, is closely related to that of *complexes over Lie algebras*, developed around 1980 by Gel'fand and Dorfman [19], as an abstract framework for the variational calculus and the theory of integrable systems. In a complex over a Lie algebra, the Cartan formulas (2.1) are taken as the defining properties.

Remark. It is easily seen that there is a complex over a Lie algebra associated to any Lie algebroid, the complex being the algebra of sections of the exterior algebra of the dual vector bundle, equipped with the Lie algebroid differential (see e.g., Section 4.3 below). The same conclusion is valid for a Lie–Rinehart algebra. These are particular cases of complexes over Lie algebras, since no associative multiplication is assumed on the total space of the complex in general.

In a 1986 preprint, Ted Courant and Alan Weinstein introduced the notion of *Dirac structure on a manifold*, by imposing an integrability condition on a field of

Dirac structures at each point of the manifold, defined as totally isotropic subspaces in the direct sum of the tangent and the cotangent bundle. Their study was published in [12]. Later, Courant proved in [11] that this integrability condition amounts to a closure condition under a bracket on $\Gamma(TM \oplus T^*M)$, this bracket that now bears his name being skew-symmetric but not satisfying the Jacobi identity. The theory was later developed by Zhang-Jiu Liu, Weinstein and Ping Xu, who introduced the more general notion of Courant algebroid, and of Dirac structures as integrable subbundles of Courant algebroids [38]. See Section 5 below.

Meanwhile, inspired by the finite-dimensional structures first considered by Courant and Weinstein in 1986, Irene Dorfman introduced [14, 15] a general notion of Dirac structure in the algebraic framework of complexes over Lie algebras. In both cases the motivation was to unify the pre-symplectic and Poisson structures (called *Hamiltonian structures* in the infinite-dimensional case). The equivalence of the definitions in the case of the de Rham complex over the Lie algebra of vector fields on a smooth manifold is not explicit in the literature, but is not hard to prove.

3. The Brackets of Buttin, Vinogradov and Courant

3.1. THE BUTTIN BRACKETS

In [7], an article developed by Pierre Molino from the notes left by Claudette Buttin (1935–1972), we find a study of the differential operators of all orders on the exterior algebra of a module. Buttin calls the order the ‘type’, to distinguish it from the order in the usual sense when the exterior algebra is the algebra of forms on a smooth manifold, and we shall follow this convention. Recall that an endomorphism of a graded associative algebra is called a differential operator of type 0 if it commutes (in the graded sense) with the left multiplication by any element in the algebra, and of type $\leq k$ if its graded commutator with the left multiplication by any element in the algebra is of type $\leq k - 1$.

Buttin first defines a composition law extending the Nijenhuis–Richardson bracket on the space $\bigwedge^\bullet E^* \otimes E$ of vector-valued forms on a module E [44] to a graded Lie bracket on the space of all *multivector-valued forms*, $\bigwedge^\bullet E^* \otimes \bigwedge^\bullet E$.

This bracket is defined by considering the embedding, i , of $\bigwedge^\bullet E^* \otimes \bigwedge^\bullet E$ into the graded Lie algebra of all differential operators on $\bigwedge^\bullet E^*$ equipped with the graded commutator. Let i_x be the interior product of forms by a vector $x \in E$. For a decomposable multivector, $x = x_1 \wedge \dots \wedge x_p \in \bigwedge^p E$, we set $i_x = i_{x_1} \dots i_{x_p}$. The operator i_x is of type p .

DEFINITION 3.1. The *embedding* i of $\bigwedge^\bullet E^* \otimes \bigwedge^\bullet E$ into the vector space of all differential operators on $\bigwedge^\bullet E^*$ is defined on decomposable elements by

$$i_{\xi \otimes x}(\alpha) = \xi \wedge i_x \alpha, \quad (3.1)$$

for $x \in \bigwedge^\bullet E$, ξ and $\alpha \in \bigwedge^\bullet E^*$.

This embedding restricts to the *interior product* of forms by multivectors on the one hand, and to the *exterior product* by forms on the other hand. We shall use this definition in what follows, sometimes reverting to the notation e_ξ instead of i_ξ when ξ is a form acting on forms by exterior product. In this notation, $i_{\xi \otimes x} = e_\xi \circ i_x$. The operator $i_{\xi \otimes x}$ is of degree $|\xi| - |x|$, and of type $|x|$.

We introduce the notation $[\ , \]_{\mathbb{B}}^0$ for Buttin's algebraic bracket, in which her definition becomes

DEFINITION 3.2. For X and $Y \in \bigwedge^\bullet E^* \otimes \bigwedge^\bullet E$, the bracket $[X, Y]_{\mathbb{B}}^0$ is the element in $\bigwedge^\bullet E^* \otimes \bigwedge^\bullet E$ such that $i_{[X, Y]_{\mathbb{B}}^0}$ is the *term of highest type* in $[i_X, i_Y]$.

For $X \in \bigwedge^q E^* \otimes \bigwedge^p E$, $Y \in \bigwedge^{q'} E^* \otimes \bigwedge^{p'} E$, the bracket $[X, Y]_{\mathbb{B}}^0$ is an element of $\bigwedge^{q+q'-1} E^* \otimes \bigwedge^{p+p'-1} E$. We wish to compare this little-known construction with the well-known notion of the *big bracket* [31, 36, 24]. The *big bracket* is the canonical Poisson structure on $\bigwedge^\bullet (E \oplus E^*)$, the even Poisson bracket on the cotangent bundle of the odd supermanifold obtained from E (or E^*) by a change of parity (see Section 4.2). For the properties of the big bracket, see [24, 3]. Here and below, we denote it by $\{ \ , \ }$. The fact that Buttin's bracket, $[\ , \]_{\mathbb{B}}^0$, coincides with the big bracket is a consequence of the following result.

THEOREM 3.1. For any X and $Y \in \bigwedge^\bullet E^* \otimes \bigwedge^\bullet E$, $i_{\{X, Y\}}$ is the *term of highest type* in $[i_X, i_Y]$.

Proof. Both $\{X, Y\}$ and $[i_X, i_Y]$ are 0 when both arguments X and Y belong to $\bigwedge^\bullet E$, or to $\bigwedge^\bullet E^*$. If $X \in E$, $Y \in E^*$, then both $i_{\{X, Y\}}$ and $[i_X, i_Y]$ are multiplication by the scalar $\langle Y, X \rangle$, obtained from the duality of E^* with E . It is now enough to remark that both expressions are derivations with respect to X and Y . On the one hand, we know that the big bracket satisfies, for X, Y and $Z \in \bigwedge^\bullet E^* \otimes \bigwedge^\bullet E$,

$$\{X, Y \wedge Z\} = \{X, Y\} \wedge Z + (-1)^{|X||Y|} Y \wedge \{X, Z\}.$$

On the other hand, it follows from the properties of the graded commutators that

$$[i_X, i_{Y \wedge Z}] = [i_X, i_Y] \circ i_Z + (-1)^{|X||Y|} i_Y \circ [i_X, i_Z].$$

Since the term of highest type of the composition of two differential operators is the composition of the terms of highest type, the theorem follows. \square

COROLLARY 3.1. In all cases, $[\ , \]_{\mathbb{B}}^0$ coincides with the big bracket.

In particular, if $p = p' = 1$, $[i_X, i_Y]$ has only terms of highest type and therefore

$$i_{[X, Y]_{\mathbb{B}}^0} = [i_X, i_Y]. \tag{3.2}$$

This property also follows from the fact that the restriction of the big bracket to the vector-valued 1-forms is equal to the Nijenhuis–Richardson bracket. The same is true of bracket $[\cdot, \cdot]_{\mathbb{B}}^0$ by Corollary 3.1. Therefore, Equation (3.2) reduces to the defining property of the Nijenhuis–Richardson bracket [44]: the embedding i maps the Nijenhuis–Richardson bracket on $\bigwedge^{\bullet} E^* \otimes E$ to the commutator of operators on $\bigwedge^{\bullet} E^*$.

Remark. Contrary to a statement in [7], relation (3.2) does not hold in all generality, as can be shown by the calculation of $[i_{x \wedge y}, e_{\xi}]$, where x and y are vectors, and ξ is a form of degree ≥ 2 . In this case

$$[i_{x \wedge y}, e_{\xi}] = e_{i_y \xi} i_x + (-1)^{|\xi|} e_{i_x \xi} i_y - (-1)^{|\xi|} e_{i_{x \wedge y} \xi},$$

which is the sum of a term of type 1 and a term of type 0. Identifying X with i_X , one can write,

$$[x \wedge y, \xi] = i_y \xi \otimes x + (-1)^{|\xi|} i_x \xi \otimes y - (-1)^{|\xi|} i_{x \wedge y} \xi,$$

while only the first two terms constitute $[x \wedge y, \xi]_{\mathbb{B}}^0$.

Another example is the Buttin algebraic bracket of two bivector-valued 1-forms, which shows that, in this case also, the result is the sum of the term of highest type (which is 3 in this example) and of terms of lower type. Explicitly,

$$\begin{aligned} & [\xi_1 \otimes x_1 \wedge y_1, \xi_2 \otimes x_2 \wedge y_2] \\ &= (-1)^{|\xi_2|} \xi_1 \wedge i_{x_1} \xi_2 \otimes y_1 \wedge x_2 \wedge y_2 + \xi_1 \wedge i_{y_1} \xi_2 \otimes x_1 \wedge x_2 \wedge y_2 - \\ & \quad - (-1)^{(|\xi_1|+1)(|\xi_2|+1)} (\xi_2 \wedge i_{x_2} \xi_1 \otimes y_2 \wedge x_1 \wedge y_1 + (-1)^{|\xi_1|} \xi_2 \wedge i_{y_2} \xi_1 \otimes x_2 \wedge x_1 \wedge y_1) + \\ & \quad + (-1)^{|\xi_2|+1} (\xi_1 \wedge i_{x_1 \wedge y_1} \xi_2 \otimes x_2 \wedge y_2 + (-1)^{|\xi_1|(|\xi_2|+1)} \xi_2 \wedge i_{x_2 \wedge y_2} \xi_1 \otimes x_1 \wedge y_1). \end{aligned}$$

We observe that the expression of the term of highest type coincides with the explicit formula for the big bracket given in [3].

Buttin then considers the case of differential operators on the exterior algebra, $\Omega^{\bullet}(M) = \Gamma(\bigoplus_{q \geq 0} \bigwedge^q(T^*M))$, of all differential forms on a smooth manifold, M . If d is the de Rham differential, and if X and Y are *multivector-valued differential forms* on M , i.e., tensors skew-symmetric in both their contravariant and their covariant indices, one can consider the expression, $[[i_X, d], [i_Y, d]]$. She proves that there exists a differential operator, $\{X, Y\}_{\mathbb{B}}$, called the *generalized differential comitant of the first kind*, such that

$$[\{X, Y\}_{\mathbb{B}}, d] = [[i_X, d], [i_Y, d]], \quad (3.3)$$

which is well-defined when an additional condition is imposed on its symbol. In addition, she shows that, only in certain cases, there exists a tensor $[X, Y]_{\mathbb{B}}$ such that

$$[i_{[X, Y]_{\mathbb{B}}}, d] = [[i_X, d], [i_Y, d]]. \quad (3.4)$$

These cases are

(0) X is a differential form, and in this case, $[X, Y]_{\mathbb{B}}$ is not a true differential concomitant since it does not involve partial derivatives of the components of Y ,

(1) X and Y are multivector fields, and in this case $[,]_{\mathbb{B}}$ is the Schouten–Nijenhuis bracket as it is now usually defined (differing from Lichnerowicz’s definition [37] by a sign),

(2) X and Y are vector-valued differential forms, and in this case $[,]_{\mathbb{B}}$ coincides with the Frölicher–Nijenhuis bracket [17, 16]. In fact, setting $L_X = [i_X, d]$, formula (3.4) becomes

$$L_{[X, Y]_{\mathbb{B}}} = [L_X, L_Y], \quad (3.5)$$

and the Frölicher–Nijenhuis bracket is a solution of this equation. Because, in this case, i_X, i_Y and d are derivations of $\Omega^{\bullet}(M)$, we know that this solution is unique.

The brackets of cases (1) and (2), each extending the Lie bracket of vector fields, are thus seen as particular cases of a more general construction.

Remark. The embedding i can be considered as an embedding of $\bigwedge^{\bullet} E^* \otimes \bigwedge^{\bullet} E$ into the vector space of all differential operators on $\bigwedge^{\bullet} E^* \otimes \bigwedge^{\bullet} E$, defined on decomposable elements by

$$i_{\xi \otimes x}(\eta \otimes y) = i_{\xi \otimes x} \eta \otimes y = \xi \wedge i_x \eta \otimes y, \quad (3.6)$$

for x and $y \in \bigwedge^{\bullet} E$, ξ and $\eta \in \bigwedge^{\bullet} E^*$. Similarly, there is an embedding j of the space $\bigwedge^{\bullet} E \otimes \bigwedge^{\bullet} E^*$ into the space of all differential operators on $\bigwedge^{\bullet} E \otimes \bigwedge^{\bullet} E^*$, defined on decomposable elements by

$$j_{x \otimes \xi}(y \otimes \eta) = x \wedge i_{\xi} y \otimes \eta. \quad (3.7)$$

In an earlier note [6], Buttin had obtained the Schouten–Nijenhuis bracket of multivectors by a construction involving an auxiliary torsionless linear connection, ∇ . For any decomposable element x in $V^p(M)$ such that $x = u \wedge v$, $u \in V^{p-1}(M)$, $v \in V^1(M)$, set $\nabla_x y = u \wedge \nabla_v y$, for all $y \in V^{\bullet}(M)$. To a multivector x of degree p , Buttin associated the derivation of degree $p-1$ of the algebra of multivectors defined by

$$\tilde{x}(y) = \nabla_x y - j_{\nabla_x} y, \quad (3.8)$$

for $y \in V^{\bullet}(M)$, where j is the map defined by (3.7). The Schouten–Nijenhuis bracket of multivectors x and y is then obtained by letting the derivation \tilde{x} act on y .

3.2. THE VINOGRADOV BRACKET

Vinogradov [55, 8] introduced a bilinear operation on the vector space of all graded endomorphisms of the space of differential forms on a smooth manifold. (Actually

his general construction is given for any complex.) If a and b are endomorphisms of the space of differential forms, $\Omega^\bullet(M)$, he sets

$$[a, b]_V = \frac{1}{2} ([a, d], b) - (-1)^{|b|} [a, [b, d]]. \quad (3.9)$$

This bilinear bracket is skew-symmetric but does not satisfy the Jacobi identity. See [55] for the explicit trilinear expression of the defect in the Jacobi identity. The vector space of all multivector-valued forms embeds into this space of endomorphisms, but it is not closed under this bracket. However, the following properties, which will be proved in Section 3.4, are valid.

- (A) The space of multivectors is closed under the Vinogradov bracket. Its restriction to the space of multivectors is a graded Lie bracket, which is the Schouten–Nijenhuis bracket.
- (B) The restriction of the Vinogradov bracket to the space of vector-valued forms is equal to the Frölicher–Nijenhuis bracket, up to a derivation of $\Omega^\bullet(M)$ of the form $[i_Z, d]$, where Z is a vector-valued form.
- (C) The direct sum of the space of vector fields and the space of differential forms is closed under the Vinogradov bracket. The restriction of the bracket $[\ , \]_V$ to this space was not considered by Vinogradov. It is skew-symmetric by definition, but it does not satisfy the Jacobi identity.
 - Case of 1-forms: We shall show in Section 3.4 that, when the Vinogradov bracket is further restricted to the direct sum of the space of vector fields and the space of differential 1-forms, it is nothing other than the bracket of Courant [11].
 - Case of p -forms: In fact, the formula for the Courant bracket (3.25) below also makes sense in the more general case of a vector and a form of arbitrary degree. This was observed by Wade in [57] and independently by Hitchin in [22]. The calculation in Section 3.4 shows that the bracket defined by this formula is the restriction of the Vinogradov bracket.

3.3. UNIFICATION THEOREMS

In Buttin and in Vinogradov, we find two ‘unification theorems’, namely constructions of which both the Schouten–Nijenhuis bracket and the Frölicher–Nijenhuis bracket are, in some sense, particular cases. These constructions are described in different settings: Buttin introduces a skew-symmetric bilinear map from pairs of multivector-valued forms to differential operators on the space, $\Omega^\bullet(M)$, of differential forms, the generalized differential concomitant of the first kind, and she shows that the image of a pair corresponds to a multivector-valued form only in the case of a pair of multivectors or in the case of a pair of vector-valued forms. For his part, Vinogradov constructs a skew-symmetric bracket, which does not satisfy the graded Jacobi identity, defined on pairs of differential operators on

$\Omega^\bullet(M)$, with values in the space of differential operators, which restricts to the Schouten–Nijenhuis bracket on multivectors, and also has the property of restricting to the Frölicher–Nijenhuis bracket on vector-valued forms, but only modulo generalized Lie derivatives.

Using the graded Jacobi identity for the graded commutator, we find the following relation: for any endomorphisms, a and b , of $\Omega^\bullet(M)$,

$$[[a, b]_{\mathbb{V}}, d] = [[a, d], [b, d]]. \quad (3.10)$$

Therefore, whenever $a = i_X, b = i_Y$, for X and Y multivector-valued forms, by (3.3),

$$[[a, b]_{\mathbb{B}}, d] = [[a, b]_{\mathbb{V}}, d] \quad (3.11)$$

and, in particular, if both X and Y are multivectors, or if both are vector-valued forms,

$$[[i_{[X, Y]_{\mathbb{B}}}, d] = [[i_X, i_Y]_{\mathbb{V}}, d]. \quad (3.12)$$

We claim that the situation can be clarified by the consideration of nonskew-symmetric brackets.

3.4. LODAY BRACKETS ON FORMS AND MULTIVECTORS

We shall show that the Vinogradov bracket is the skew-symmetrization of a Loday bracket, which is a derived bracket of the graded commutator of graded endomorphisms. In case (A) of Section 3.2, the derived bracket is skew-symmetric and therefore the Vinogradov bracket coincides with it, while in cases (B) and (C), the derived bracket is not skew-symmetric. The skew-symmetrization then yields the Vinogradov bracket.

Again let $\text{End}(\Omega^\bullet(M))$ be the algebra of graded endomorphisms of $\Omega^\bullet(M)$, and let $[\ , \]$ be the graded commutator. Let d be the de Rham differential. The derived bracket of $a, b \in \text{End}(\Omega^\bullet(M))$ is defined by formula (1.3) of Section 1,

$$[a, b]_d = [[a, d], b]. \quad (3.13)$$

Below we set, for $X \in \Omega^\bullet(M) \otimes V^1(M)$, $L_X = [i_X, d]$, where i_X is defined by (3.6), and is considered as an endomorphism of $\Omega^\bullet(M) \otimes V^1(M)$.

THEOREM 3.2.

- (i) *The derived bracket $[\ , \]_d$ defines an odd Loday algebra structure on $\text{End}(\Omega^\bullet(M))$.*
- (ii) *The space $V^\bullet(M)$ is closed under the derived bracket $[\ , \]_d$. Its restriction to this subspace of $\text{End}(\Omega^\bullet(M))$ is skew-symmetric; it is the Schouten–Nijenhuis bracket.*

(iii) *The algebraic part of the restriction of $[\cdot, \cdot]_{\mathfrak{d}}$ to the subspace $\Omega^{\bullet}(M) \otimes V^1(M)$ of $\text{End}(\Omega^{\bullet}(M))$ is the Frölicher–Nijenhuis bracket. More precisely,*

$$[i_X, i_Y]_{\mathfrak{d}} = i_{[X, Y]_{\text{FN}}} - (-1)^{q(q'-1)} L_{i_Y X}, \quad (3.14)$$

where $[X, Y]_{\text{FN}}$ denotes the Frölicher–Nijenhuis bracket of vector-valued forms $X \in \Omega^q(M) \otimes V^1(M)$ and $Y \in \Omega^{q'}(M) \otimes V^1(M)$.

(iv) *The derived brackets of a vector field x and a differential form ξ are*

$$[x, \xi]_{\mathfrak{d}} = L_x \xi \quad (3.15)$$

and

$$[\xi, x]_{\mathfrak{d}} = -i_x d\xi, \quad (3.16)$$

and the restriction of $[\cdot, \cdot]_{\mathfrak{d}}$ to the direct sum of the space of vector fields and the space of differential forms is given by

$$[x + \xi, y + \eta]_{\mathfrak{d}} = [x, y] + L_x \eta - i_y d\xi, \quad (3.17)$$

for all vector fields x and y , and for all differential forms ξ and η , where $[x, y]$ is the Lie bracket of x and y .

Proof. Part (i) is a corollary of Theorem 1.1 above. Part (ii) is a restatement of the Cartan formula (2.2) for multivector fields.

To prove part (iii), we observe that, for vector-valued forms, X and Y ,

$$[[i_X, i_Y]_{\mathfrak{d}}, \mathfrak{d}] = [i_{[X, Y]_{\text{FN}}}, \mathfrak{d}],$$

because both expressions are equal to $[[i_X, \mathfrak{d}], [i_Y, \mathfrak{d}]]$, the first by the Jacobi identity and the second by definition. This implies that the algebraic parts of $[i_X, i_Y]_{\mathfrak{d}}$ and $i_{[X, Y]_{\text{FN}}}$ are equal. It follows from formula 5.15 of [17] that the Frölicher–Nijenhuis bracket satisfies, for $X = \xi \otimes x$, $Y = \eta \otimes y$,

$$\begin{aligned} [X, Y]_{\text{FN}} &= \xi \wedge \eta \otimes [x, y] + (\xi \wedge L_x \eta + (-1)^{|\xi|} d\xi \wedge i_x \eta) \otimes y - \\ &\quad - (-1)^{|\xi||\eta|} (\eta \wedge L_y \xi + (-1)^{|\eta|} d\eta \wedge i_y \xi) \otimes x. \end{aligned} \quad (3.18)$$

Using $i_{\xi \otimes x} = e_{\xi} \circ i_x$ and $[i_{\xi \otimes x}, \mathfrak{d}] = \xi \wedge L_x + (-1)^{|\xi|} d\xi \wedge i_x$, this formula also implies the expression

$$[X, Y]_{\text{FN}} = \xi \wedge \eta \otimes [x, y] + L_X \eta \otimes y - (-1)^{|\xi||\eta|} L_Y \xi \otimes x. \quad (3.19)$$

to be found in, e.g., [35, 21].

On the other hand, a direct computation shows that

$$\begin{aligned} [i_{\xi \otimes x}, i_{\eta \otimes y}]_{\mathfrak{d}} &= [[i_{\xi \otimes x}, \mathfrak{d}], i_{\eta \otimes y}] \\ &= \xi \wedge \eta \wedge i_{[x, y]} + (\xi \wedge L_x \eta + (-1)^{|\xi|} d\xi \wedge i_x \eta) \wedge i_y + \\ &\quad + (-1)^{|\xi||\eta|+1} \eta \wedge i_y d\xi \wedge i_x + (-1)^{|\xi||\eta|+|\xi|+1} \eta \wedge i_y \xi \wedge L_x. \end{aligned} \quad (3.20)$$

Therefore, we find from (3.18) and (3.20),

$$\begin{aligned} & (-1)^{|\xi||\eta|+|\xi|+1} ([i_{\xi \otimes x}, i_{\eta \otimes y}]_{\mathbf{d}} - i_{[\xi \otimes x, \eta \otimes y]_{\text{FN}}}) \\ & = \eta \wedge i_y \xi \wedge L_x - (-1)^{|\xi|+|\eta|} \mathbf{d}(\eta \wedge i_y \xi) \wedge i_x = [i_{\eta \wedge i_y \xi \otimes x}, \mathbf{d}] = [i_{i_y X}, \mathbf{d}] = L_{i_y X}, \end{aligned}$$

which proves (3.14).

To prove part (iv), we first observe that, on $V^1(M)$, the derived bracket restricts to the Lie bracket of vector fields: this follows from the Cartan formula (2.2).

If ξ is a differential form of degree $|\xi|$,

$$[e_{\xi}, \mathbf{d}] = (-1)^{|\xi|+1} e_{\mathbf{d}\xi}. \quad (3.21)$$

Since any two exterior products by forms commute, the derived bracket vanishes if both arguments are differential forms.

If x is a vector field and ξ a differential form of degree $|\xi|$,

$$[i_x, e_{\xi}]_{\mathbf{d}} = [[i_x, \mathbf{d}], e_{\xi}] = L_x e_{\xi} - e_{\xi} L_x = e_{L_x \xi}, \quad (3.22)$$

and

$$[e_{\xi}, i_x]_{\mathbf{d}} = [[e_{\xi}, \mathbf{d}], i_x] = (-1)^{|\xi|+1} [e_{\mathbf{d}\xi}, i_x] = -e_{i_x \mathbf{d}\xi}. \quad (3.23)$$

Therefore, identifying vectors and forms with their image under the embedding i , we obtain formulas (3.15) and (3.16).

To summarize, $[\ , \]_{\mathbf{d}}$ extends the Lie bracket of vector fields, vanishes on pairs of differential forms and satisfies (3.15) and (3.16), and therefore (3.17) follows. \square

Remark. The space $V^{\bullet}(M) \otimes \Omega^{\bullet}(M) \subset \text{End}(\Omega^{\bullet}(M))$ is not closed under the derived bracket $[\ , \]_{\mathbf{d}}$ unless the manifold is of dimension ≤ 1 . To prove this, let us show that, on manifolds of dimension ≥ 2 , there exist multivector fields, x and y , and differential 1-forms, ξ and η , such that the operator $[[i_{x \otimes \xi}, \mathbf{d}], i_{y \otimes \eta}]$ is not $C^{\infty}(M)$ -linear. For any differential form α ,

$$\begin{aligned} & [[i_{\xi \otimes x}, \mathbf{d}], i_{\eta \otimes y}](\alpha) \\ & = \xi \wedge L_x \mathbf{d}(\eta \wedge i_y \alpha) - (-1)^{|x|+|\xi|} \mathbf{d}\xi \wedge i_x(\eta \wedge i_y \alpha) + \\ & \quad + (-1)^{|x|+|y|+|\xi|+|\eta|} \eta \wedge i_y(\xi \wedge L_x \mathbf{d}\alpha) - (-1)^{|y|+|\eta|} \eta \wedge i_y(\mathbf{d}\xi \wedge i_x \alpha). \end{aligned}$$

If x and y are bivectors, for a 1-form α and a function f ,

$$\begin{aligned} & [[i_{\xi \otimes x}, \mathbf{d}], i_{\eta \otimes y}](f\alpha) - f[[i_{\xi \otimes x}, \mathbf{d}], i_{\eta \otimes y}](\alpha) \\ & = \left(i_y(\xi \wedge i_x(\mathbf{d}f \wedge \mathbf{d}\alpha)) - (-1)^{|x|} (i_x \mathbf{d}\alpha) i_y(\xi \wedge \mathbf{d}f) \right) \eta. \end{aligned}$$

If $\mathbf{d}\alpha = \beta \wedge \gamma$, this expression is

$$(i_x(\beta \wedge \mathbf{d}f) i_y(\xi \wedge \gamma) - i_x(\gamma \wedge \mathbf{d}f) i_y(\xi \wedge \beta)) \eta,$$

which does not vanish in general, as can be proved by using local coordinates.

THEOREM 3.3.

- (i) *The skew-symmetrization of the derived bracket $[\cdot, \cdot]_{\mathfrak{d}}$ is the Vinogradov bracket.*
- (ii) *The skew-symmetrized derived bracket of a vector x and a form ξ is*

$$[x, \xi]_{\mathfrak{d}}^- = L_x \xi - \frac{1}{2} \mathfrak{d} i_x \xi, \tag{3.24}$$

and the restriction of the skew-symmetrization of $[\cdot, \cdot]_{\mathfrak{d}}$ to the direct sum of the space of vector fields and the space of differential forms is given by

$$[x + \xi, y + \eta]_{\mathfrak{d}}^- = [x, y] + L_x \eta - L_y \xi - \frac{1}{2} \mathfrak{d}(i_x \eta - i_y \xi), \tag{3.25}$$

for all vector fields x and y , and all differential forms ξ and η .

- (iii) *The restriction of the skew-symmetrization of $[\cdot, \cdot]_{\mathfrak{d}}$ to the direct sum of the space of vector fields and the space of differential 1-forms is the Courant bracket.*

Proof. Part (i) follows from equations (1.5) and (3.9). Part (ii) follows immediately from part (iv) of Theorem 3.2. On $V^1(M) \oplus \Omega^1(M)$, formula (3.25) is precisely the Courant bracket as defined in [11] and used in [38], proving part (iii). \square

Remark. If we consider the restriction of the skew-symmetrized derived bracket to the vector-valued forms, we find

$$[i_X, i_Y]_{\mathfrak{d}} = [i_X, i_Y]_{\mathfrak{d}}^- = i_{[X, Y]_{FN}} + \frac{1}{2} (-1)^p L_{i_X Y + (-1)^{(p-1)(p'-1)} i_Y X}.$$

It is clear that $\Omega^\bullet(M) \otimes V^1(M)$ is not closed under the derived bracket nor under its skew-symmetrization, because neither $[i_{\xi \otimes x}, i_{\eta \otimes y}]_{\mathfrak{d}}$ nor $[i_{\xi \otimes x}, i_{\eta \otimes y}]_{\mathfrak{d}}^-$ vanishes on functions. In fact,

$$[i_{\xi \otimes x}, i_{\eta \otimes y}]_{\mathfrak{d}} f = \eta \wedge i_y \xi L_x f.$$

The explicit expression (3.17) of the derived bracket for the case of 1-forms appears in Dorfman (see [14,15]), in her study of the properties of Dirac structures on complexes over Lie algebras. However, that expression does not appear as a derived bracket and its properties are not spelled out.

The relationship of the derived bracket with the Vinogradov bracket was shown in [26], but its relationship with the Courant bracket was observed later, independently by myself (in an e-mail letter to Alan Weinstein, 1998), Pavol Ševera and Ping Xu (all unpublished, see [52]). It is the nonskew-symmetric bracket which is now used in the theory of Courant algebroids [48,49,52].

4. Odd and Even Brackets on Supermanifolds

In the interpretation of the brackets of differential geometry in terms of supermanifolds, the odd brackets are obtained as derived brackets of even brackets and

conversely. This was shown in all generality by Voronov in [56]. See also the article by Batalin and Marnelius [4]. The main examples are

- (1) the Schouten–Nijenhuis bracket of multivectors on a manifold, as a derived bracket of the canonical Poisson bracket on the cotangent bundle of the manifold,
- (2) the Poisson bracket of functions on a manifold, M , as a derived bracket of the canonical Schouten–Nijenhuis bracket on the cotangent bundle of M with reversed parity (see [26]),
- (3) the algebraic Schouten bracket on the exterior algebra of a Lie algebra (E, μ) , as the derived bracket of the canonical Poisson structure on $\wedge(E \oplus E^*)$ (see [24]).

All these instances are particular cases of the general construction on Lie algebroids which we shall describe in Section 4.3.

Below, a function on $T^*\mathcal{M}$, where \mathcal{M} is a supermanifold, is called a *Hamiltonian* on \mathcal{M} , and a bracket on the vector space $C^\infty(\mathcal{M})$ is sometimes called a bracket on \mathcal{M} . If $E \rightarrow \mathcal{M}$ is a vector bundle, ΠE denotes the supermanifold obtained by reversing the parity of the fibers.

4.1. ODD AND EVEN POISSON BRACKETS ON SUPERMANIFOLDS ARE DERIVED BRACKETS

The following theorems are due to Voronov [56] (see also [4]).

THEOREM 4.1. *Any odd Poisson bracket on a supermanifold, \mathcal{M} , is a derived bracket of the canonical Poisson bracket $\{ , \}$ on $T^*\mathcal{M}$. More precisely, for any odd Poisson bracket on \mathcal{M} , $[,]$, there exists a quadratic function S on $T^*\mathcal{M}$ such that*

$$[f, g] = \{\{f, S\}, g\}, \quad (4.1)$$

for all $f, g \in C^\infty(\mathcal{M})$. (On the right-hand side of (4.1), f and g are identified with their pull-backs to $T^*\mathcal{M}$, i.e., are considered as functions on the vector bundle $T^*\mathcal{M}$, that are constant on the fibers.)

Formula (4.1) defines a derived bracket $[,] = \{ , \}_S$ which is not only a Loday bracket but is a true Lie bracket, i.e., it is skew-symmetric, because $C^\infty(\mathcal{M})$ is an Abelian subalgebra of the Poisson algebra of $T^*\mathcal{M}$.

In coordinates, if (x^α) are local coordinates on \mathcal{M} and (x^α, p_α) the associated local coordinates on $T^*\mathcal{M}$, then $S = \frac{1}{2} S^{\alpha\beta}(x) p_\alpha p_\beta$, where $S^{\alpha\beta} = [x^\alpha, x^\beta]$.

EXAMPLE. As an exercise, let us illustrate this theorem when $\mathcal{M} = \Pi T^*N$, for N a manifold of dimension n . The Schouten–Nijenhuis bracket of fields of multivectors is a canonically defined odd Poisson bracket, $[,]_{\text{SN}}$, on \mathcal{M} . What is the corresponding quadratic Hamiltonian? We seek its expression in local coordinates.

Let (y^i, \tilde{y}_i) be adapted local coordinates on $\mathcal{M} = \Pi T^*N$ (y^i is even and \tilde{y}_i is odd). Then

$$[y^i, y^j]_{\text{SN}} = 0, \quad [y^i, \tilde{y}_j]_{\text{SN}} = \delta_j^i \quad \text{and} \quad [\tilde{y}_i, \tilde{y}_j]_{\text{SN}} = 0.$$

Let $(y^i, \tilde{y}_i, p_i, \tilde{p}^i)$ be associated local coordinates on $T^*\mathcal{M}$. The canonical Poisson bracket satisfies, $\{y^i, p_j\} = \delta_j^i$, $\{\tilde{y}_i, \tilde{p}^j\} = \delta_j^i$, while all other brackets vanish. The quadratic Hamiltonian $S = -p_i \tilde{p}^i \in C^\infty(T^*(\Pi T^*N))$ satisfies Equation (4.1) above in the form $[f, g]_{\text{SN}} = \{\{f, S\}, g\}$. The Hamiltonian S can be also defined in an invariant way. For each $\varphi \in T^*\mathcal{M}$, $S(\varphi) = -\langle p_{\Pi T^*N}^* \varphi, p_{\Pi T^*N}^*(\kappa \varphi) \rangle$, where $p_{\Pi T^*N}^*$ maps $T^*(\Pi T^*N)$ to ΠT^*N , and $p_{\Pi T^*N}^*$ maps $T^*(\Pi T^*N)$ to ΠT^*N , while κ is the canonical isomorphism [56] from $T^*(\Pi T^*N)$ to $T^*(\Pi T^*N)$.

In the same manner, we can describe every even Poisson bracket on \mathcal{M} as a derived bracket. Let P be the even Poisson bivector defining an even Poisson bracket $\{ , \}$ on \mathcal{M} . Then $\{f, g\} = [[f, P], g]$, where the bracket on the right-hand side is the Schouten–Nijenhuis bracket of multivector fields on \mathcal{M} , and f and g are considered as multivector fields of degree 0. See formula (3.5) of [26] for the case of ordinary manifolds, and [56] for supermanifolds. Since a multivector field on \mathcal{M} is a function on $\Pi T^*\mathcal{M}$, this property can be reformulated in the language of supermanifolds, making it ‘dual’ to Theorem 4.1.

THEOREM 4.2. *Any even Poisson bracket on \mathcal{M} is a derived bracket of the canonical Schouten–Nijenhuis bracket $[\ , \]$ on $\Pi T^*\mathcal{M}$. More precisely, for any even Poisson bracket on \mathcal{M} , $\{ , \}$, there exists a quadratic function P on $\Pi T^*\mathcal{M}$ such that*

$$\{f, g\} = [[f, P], g], \tag{4.2}$$

for all $f, g \in C^\infty(\mathcal{M})$. (On the right-hand side, f and g are identified with their pull-backs to $\Pi T^*\mathcal{M}$, i.e., are considered as functions on the vector bundle $\Pi T^*\mathcal{M}$, that are constant on the fibers.)

The quadratic function P on $\Pi T^*\mathcal{M}$ is nothing but the Poisson bivector giving rise to the given even Poisson bracket. In local coordinates $(x^\alpha, \tilde{p}_\alpha)$ on $\Pi T^*\mathcal{M}$, $P = \frac{1}{2} P^{\alpha\beta}(x) \tilde{p}_\beta \tilde{p}_\alpha$, where $P^{\alpha\beta} = \{x^\alpha, x^\beta\}$.

4.2. DERIVED BRACKETS AND LIE ALGEBRAS

Let E be a finite-dimensional vector space over a field of characteristic 0. For simplicity, we consider the ungraded (purely even) case, but the properties below extend to the case where the vector space E is itself graded, see [56]. The following structures are equivalent:

- a Lie algebra structure on E ,
- a Lie co-algebra structure on E^* ,

- a *linear Schouten structure* on ΠE^* , i.e., a Schouten algebra structure on $C^\infty(\Pi E^*) = \bigwedge^\bullet E$ such that E is closed under the bracket,
- a *linear Poisson structure* on E^* , i.e., a Poisson algebra structure on $C^\infty(E^*)$ such that E is closed under the bracket, i.e., a linear bivector field on E^* ,
- a *linear-quadratic Hamiltonian of Poisson square 0* on ΠE^* , i.e., an element $\mu \in C^\infty(T^*(\Pi E^*)) = C^\infty(\Pi E^* \oplus \Pi E) = \bigwedge^\bullet(E \oplus E^*)$ such that $\mu \in \bigwedge^2 E^* \otimes E$ and $\{\mu, \mu\} = 0$, also denoted by H ,
- a *quadratic homological vector field* on ΠE , i.e., a quadratic differential on $C^\infty(\Pi E) = \bigwedge^\bullet E^*$, often denoted by d , or d_μ , or Q .

The canonical Poisson bracket on $C^\infty(T^*(\Pi E^*)) = \bigwedge^\bullet(E \oplus E^*)$ was first defined in [31] and considered in [36]. It was an essential tool in [24], where we first called it the *big bracket*. Here, we have denoted the big bracket by $\{ , \}$.

The bracket on $C^\infty(\Pi E^*) = \bigwedge^\bullet E$ is called the *algebraic Schouten bracket* of the Lie algebra. The differential on $C^\infty(\Pi E) = \bigwedge^\bullet E^*$, corresponding to the Lie algebra structure on E , is the *Chevalley–Eilenberg differential* on the scalar-valued cochains on E .

Let (e_i) be a basis of E , with coordinates (x^i) , and set $\mu(e_i, e_j) = C_{ij}^k e_k$. Let $(\tilde{\xi}_i)$ be the coordinates in the dual basis on ΠE^* , and let $(\tilde{\xi}_i, \tilde{x}^i)$ be the associated coordinates on $T^*(\Pi E^*) = \Pi E^* \oplus \Pi E$. The Hamiltonian on ΠE^* , which is a function on $T^*(\Pi E^*)$, may be written $H = \frac{1}{2} C_{ij}^k \tilde{\xi}_k \tilde{x}^i \tilde{x}^j$, while the vector field on ΠE may be written $Q = \frac{1}{2} \tilde{x}^j \tilde{x}^i C_{ij}^k \partial / \partial \tilde{x}^k$.

Since Lie algebra structures on E are Poisson (resp., Schouten) structures on E^* (resp., ΠE^*), Theorems 4.1 and 4.2 apply. They take the following form in the case of Lie algebras.

COROLLARY 4.1. *Given a Lie algebra structure $\mu \in \bigwedge^2 E^* \otimes E$ on a vector space, E ,*

- (i) *the Schouten bracket on $C^\infty(\Pi E^*) = \bigwedge^\bullet E$ is given by the derived bracket formula,*

$$[x, y]_\mu = \{\{x, \mu\}, y\}, \quad (4.3)$$

where $\{ , \}$ denotes the canonical Poisson bracket (big bracket) on $T^*(\Pi E^*)$, μ is considered as a Hamiltonian on ΠE^* , i.e., as a function on $T^*(\Pi E^*)$, and x and y are considered as functions on $T^*(\Pi E^*)$ that are constant on ΠE ,

- (ii) *the Poisson bracket of f and g in $C^\infty(E^*)$ is given by the derived bracket formula,*

$$\{f, g\}_\mu = [[f, \mu], g], \quad (4.4)$$

where $[,]$ denotes the canonical Schouten–Nijenhuis bracket of multivector fields on E^* , μ is considered as a bivector field on E^* , and f and g are considered as multivector fields of degree 0 on E^* .

Part (i) of the corollary was observed in [47] as well as in [24], where it was used to prove various properties of Lie bialgebras and Poisson Lie groups. As an application, we now recall how to derive the condition for an r -matrix to define a coboundary Lie bialgebra.

EXAMPLE. Let $\mathfrak{g} = (E, \mu)$ be a Lie algebra. If $r \in \wedge^2 \mathfrak{g}$, by (4.3), the algebraic Schouten bracket, $[r, r]_\mu$, satisfies $[r, r]_\mu = \{\{r, \mu\}, r\}$, where $\{, \}$ denotes the big bracket. Let $d_\mu r$ be the Chevalley–Eilenberg coboundary of r . In order for $d_\mu r$ to be a Lie cobracket on E it is necessary and sufficient that $\{d_\mu r, d_\mu r\} = 0$. Using the relations $d_\mu r = \{\mu, r\}$ and $\{\mu, \mu\} = 0$, the Jacobi identity and (4.3), we obtain

$$\{d_\mu r, d_\mu r\} = \{\{\mu, r\}, \{\mu, r\}\} = \{\mu, \{\{r, \mu\}, r\}\} = \{\mu, [r, r]_\mu\} = d_\mu[r, r]_\mu.$$

Therefore $d_\mu r$ is a Lie cobracket on E if and only if $[r, r]_\mu$ is ad-invariant. Since the Drinfeld bracket, $\langle r, r \rangle$, coincides with the algebraic Schouten bracket up to a factor $-\frac{1}{2}$, the above computation is a short proof of the fact that $(E, \mu, d_\mu r)$ is a coboundary Lie bialgebra if and only if r satisfies the *generalized classical Yang–Baxter equation*, i.e., the ad-invariance of the Drinfeld bracket $\langle r, r \rangle$.

We now state another derived bracket formula in the theory of Lie algebras (see [56]). We can consider $x \in E$ as a constant vector field on ΠE . Let $i : x \in E \mapsto i_x \in V^1(\Pi E) = \text{Der}(C^\infty(\Pi E)) = \text{Der}(\wedge^\bullet E^*)$ be the canonical embedding. With the preceding notations, for x and y in E ,

$$i_{[x, y]_\mu} = [[i_x, d_\mu], i_y], \quad (4.5)$$

where the bracket on the right-hand side is the graded commutator. If, for example, α is a 1-form on E , this formula reduces to

$$(d_\mu \alpha)(x, y) = -\alpha([x, y]_\mu).$$

More generally, for $x \in \wedge^\bullet E$, let $i_x \in \text{End}(\wedge^\bullet E^*)$ be the interior product by x . Formula (4.5) is then valid for x and y in $\wedge^\bullet E$, where the bracket on the left-hand side is the algebraic Schouten bracket.

The axioms of Lie bialgebras and generalizations thereof can be easily formulated in this framework, as shown in [36, 24, 48, 50, 56].

In conclusion, we can state: just as the Lie bracket of vector fields is a derived bracket according to the Cartan relation (2.2), the Lie bracket on any Lie algebra is a derived bracket according to equation (4.5). We observe that the de Rham differential and the Chevalley–Eilenberg cohomology operator play analogous roles. The Lie algebroid framework which we shall now describe unifies these two theories.

4.3. DERIVED BRACKETS AND LIE ALGEBROIDS

The approach to Lie algebroids in terms of supermanifolds is due to Vaintrob [54], and was developed by Roytenberg [48–50] and by Voronov [56]. See also [2] and [46].

Let $A \rightarrow M$ be a vector bundle. A *Lie algebroid structure* on A can be defined in several equivalent ways:

- a *Lie algebroid structure* on A , i.e., a Lie algebra structure on ΓA and a morphism of vector bundles, $\rho: A \rightarrow TM$, called the anchor, satisfying the Leibniz rule,

$$[u, f v] = f[u, v] + (\rho(u)f)v. \quad (4.6)$$

for all u and $v \in \Gamma A$, $f \in C^\infty(M)$,

- a *linear Schouten structure* on ΠA^* , i.e., a Schouten algebra structure on $C^\infty(\Pi A^*) = \Gamma(\wedge^\bullet A^*)$ such that ΓA is closed under the bracket,
- a *linear Poisson structure* on A^* , i.e., a Poisson algebra structure on $C^\infty(A^*)$ such that ΓA is closed under the bracket.

Remark. The fact that the anchor maps the bracket on ΓA to the Lie bracket on ΓTM , which is usually listed in the axioms of a Lie algebroid, is actually a consequence of the Jacobi identity (1.1) for the bracket on ΓA together with the Leibniz rule, as we show by computing $[u, [v, f w]]$ in two ways (see [30]).

A Lie algebroid structure on A can also be defined by a *homological vector field* on ΠA , i.e., a differential on $C^\infty(\Pi A) = \Gamma(\wedge^\bullet A^*)$, the *Lie algebroid differential*, often denoted by Q , or d , or d_A . Let (x^α) be local coordinates on M , let (e_i) be a local basis of ΓA , and let (x^α, y^i) be the corresponding local coordinates on A . Let

$$\rho(e_i) = a_i^\alpha(x) \frac{\partial}{\partial x^\alpha} \quad \text{and} \quad [e_i, e_j] = C_{ij}^k(x) e_k.$$

Then the vector field, Q , on the vector bundle ΠA , equipped with the local coordinates (x^α, \tilde{y}^i) , has the local expression,

$$Q = \tilde{y}^i a_i^\alpha(x) \frac{\partial}{\partial x^\alpha} + \frac{1}{2} \tilde{y}^j \tilde{y}^i C_{ij}^k(x) \frac{\partial}{\partial \tilde{y}^k}.$$

A Lie algebroid structure on A can also be viewed as a *quadratic Hamiltonian* on ΠA^* , of Poisson square 0, denoted by H . If $(x^\alpha, \tilde{\eta}_i)$ are the local coordinates on ΠA^* dual to (x^α, \tilde{y}^i) , and $(x^\alpha, \tilde{\eta}_i, p_\alpha, \tilde{\theta}^i)$ are the associated local coordinates on $T^*(\Pi A^*)$, then

$$H = a_i^\alpha(x) p_\alpha \tilde{\theta}^i + \frac{1}{2} \tilde{\eta}_k C_{ij}^k(x) \tilde{\theta}^j \tilde{\theta}^i.$$

The structure can also be viewed as a *bivector field* on A^* , defining the linear Poisson structure, denoted by P . Its local expression in the local coordinates (x^α, η_i) on A^* is

$$P = a_i^\alpha(x) \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial \eta_i} + \frac{1}{2} \eta_k C_{ij}^k(x) \frac{\partial}{\partial \eta_j} \frac{\partial}{\partial \eta_i}.$$

There are several derived brackets in this theory.

THEOREM 4.3. *Given a Lie algebroid structure on the vector bundle A ,*

- (i) *if H is the Hamiltonian of the Lie algebroid, then the Schouten bracket of u and $v \in C^\infty(\Pi A^*) = \Gamma(\wedge^\bullet A)$ is the derived bracket,*

$$[u, v]_A = \{\{u, H\}, v\}, \quad (4.7)$$

where $\{, \}$ is the canonical Poisson bracket of $T^(\Pi A^*)$, and u and v are considered as functions on $T^*(\Pi A^*)$ that are constant on the fibers,*

- (ii) *if P is the bivector field on A^* of the Lie algebroid, then the Poisson bracket of φ and $\psi \in C^\infty(A^*)$ is the derived bracket,*

$$\{\varphi, \psi\}_A = [[\varphi, P], \psi], \quad (4.8)$$

where $[\ , \]$ is the canonical Schouten–Nijenhuis bracket of multivector fields on A^ , and φ and ψ are considered as multivectors of degree 0.*

In particular, formula (4.7) is valid for u and v sections of A , showing that any Lie algebroid bracket is a derived bracket of a canonical Poisson bracket by a quadratic Hamiltonian.

In terms of endomorphisms of $C^\infty(\Pi A) = \Gamma(\wedge^\bullet A^*)$,

$$i_{[u, v]_A} = [[i_u, d_A], i_v], \quad (4.9)$$

where u and v are functions on ΠA^* , d_A is the Lie algebroid differential, and i is the interior product. When u and v are sections of A , relation (4.9) is an equality of derivations of $\Gamma(\wedge^\bullet A^*)$.

As a particular case of (4.9), we see once more that the Lie algebroid bracket is a derived bracket. Moreover, if $u \in \Gamma A$ and $f \in C^\infty(M)$, $[u, f]_A$ is the function $[[i_u, d_A], f] = \rho(u)f$. Thus, the action of the anchor also appears as a derived bracket.

Also, on a Lie algebroid, A , a Frölicher–Nijenhuis bracket, $[\ , \]_{FN}$, can be defined [16] [21] on vector-valued forms, i.e., sections of $\wedge^\bullet A^* \otimes A$, in such a way that it satisfies

$$[i_{[X, Y]_{FN}}, d_A] = [[i_X, d_A], [i_Y, d_A]]. \quad (4.10)$$

In fact, this generalized Frölicher–Nijenhuis bracket is defined by means of equation (3.18), in which the de Rham differential and the Lie derivation are replaced by their Lie algebroid generalizations.

Obviously, the formulas for brackets on Lie algebroids admit two particular cases of interest:

- Lie algebras. When the base manifold is a point, we recover the case of a Lie algebra. Formula (4.7) reduces to formula (4.3), and formula (4.8) reduces to formula (4.4), while formula (4.9) reduces to (4.5).

- Manifolds. When the Lie algebroid A is a tangent bundle, we recover the case of a manifold. The differential d_A is the de Rham differential. Formula (4.7) reduces to formula (4.1), and formula (4.8) reduces to formula (4.2), while formula (4.9) reduces to the Cartan formula (2.2).

As an application of the general formulas, we mention the case $A = T^*M$, when M is a Poisson manifold with Poisson bivector P . Then it is known (see [30]) that d_A is the Lichnerowicz–Poisson differential, $d_P = [P, \cdot]_{\text{SN}}$, where $[\cdot, \cdot]_{\text{SN}}$ is the Schouten–Nijenhuis bracket of multivectors. If $[\cdot, \cdot]^P$ denotes the Koszul bracket of differential forms, then, by formula (4.9),

$$i_{[\alpha, \beta]^P} = [[i_\alpha, d_P], i_\beta], \quad (4.11)$$

for any forms α and β . This formula was first proved by Krasilsh'chik in [34]. (See also [26].) It appears once more that Poisson structures play a role dual to that of differential structures on manifolds: in this sense, formula (4.11) is dual to the Cartan formula (2.2).

5. Derived Brackets and Courant Algebroids

Courant algebroids were introduced by Liu, Weinstein and Xu in [38], as a generalization of the bracket defined by Courant [11] on the sections of $TM \oplus T^*M$. As we pointed out in Sections 3.2 and 3.4, the Courant bracket is skew-symmetric but does not satisfy the Jacobi identity. The definition of Courant algebroids was later re-formulated [48, 50, 52] in terms of Loday brackets. An in-depth study of the role of derived brackets in the general theory of Courant algebroids can be found in the article by Roytenberg [50], in terms of cubic Hamiltonians on graded supermanifolds, and in the forthcoming study by Alekseev and Xu [1], in terms of Clifford modules and compatible connections. Here we shall be content with exhibiting one more instance of a derived bracket construction.

5.1. COURANT ALGEBROIDS

The vector bundle $TM \oplus T^*M$, for any manifold M , with the field of nondegenerate symmetric bilinear forms defined by the conditions that TM and T^*M be isotropic and $\langle x | \xi \rangle = \langle x, \xi \rangle$, for x a tangent vector and ξ a 1-form at a point in M , with the bracket (3.17) and anchor the projection onto TM , is the prototypical example of a Courant algebroid. As shown in [38], this construction can be generalized to the case of $A \oplus A^*$, where (A, A^*) is a *Lie bialgebroid*, i.e., a pair of Lie algebroids in duality satisfying a compatibility condition [40, 25, 49]: taking into account the bracket of A^* , formula (3.17) can be extended to the sections of $A \oplus A^*$, and together with the sum of the anchors of A and A^* , it defines a Courant algebroid structure on $A \oplus A^*$.

5.2. COURANT BRACKET WITH BACKGROUND AS A DERIVED BRACKET

As an example, let us find an explicit deriving operator for the Courant algebroid associated to a Poisson structure with background in the sense of Ševera and Weinstein [52], i.e., a WZW–Poisson structure in the sense of Klimčík and Strobl [23].

Let ψ be an arbitrary form of odd degree on a manifold M . We consider the operator on $\Omega^\bullet(M)$, $d^\psi = d + e_\psi$. Then $[d^\psi, d^\psi] = e_{d^\psi}$. So, whenever ψ is a *closed* form of odd degree, we can consider the derived bracket on $\text{End}(\Omega^\bullet(M))$, $[\cdot, \cdot]_{d^\psi}$, arising from the graded commutator and the odd interior derivation of square 0 defined by d^ψ . For vector fields x and y ,

$$[i_x, i_y]_{d^\psi} = [[i_x, d^\psi], i_y] = i_{[x, y]} + e_{i_x \wedge y \psi},$$

where $[x, y]$ is the Lie bracket of x and y . The first nontrivial case is when ψ is of degree 3. We see that $V^1(M) \oplus \Omega^1(M)$ is closed under the derived bracket $[\cdot, \cdot]_{d^\psi}$ if and only if ψ is a *form of degree 3*. Therefore, let ψ be a *closed 3-form*. We define

$$[x + \xi, y + \eta]_{d^\psi} = [[x + \xi, d^\psi], y + \eta].$$

for vector fields x and y , and 1-forms ξ and η , and we find the following generalization of (3.17),

$$[x + \xi, y + \eta]_{d^\psi} = [x, y] + L_x \eta - i_y d\xi + i_x \wedge y \psi. \quad (5.1)$$

In particular, the bracket of any two differential 1-forms remains 0, but the bracket of two vector fields has both a component in the space of vector fields and a component in the space of differential 1-forms.

This bracket, together with the field of symmetric bilinear forms recalled above, and, for anchor, the projection onto TM , turn $TM \oplus T^*M$ into a Courant algebroid, called the *Courant algebroid with background ψ* [52]. We have just shown that it is a derived bracket by a modified differential.

PROPOSITION 5.1. *The Courant bracket with background ψ on $TM \oplus T^*M$ is the derived bracket of the commutator of endomorphisms of $\Omega^\bullet(M)$ by $d + e_\psi$.*

5.3. PROPERTIES OF COURANT BRACKETS WITH BACKGROUND

To conclude, we list a few properties of the Courant algebroids with background, following mainly [52]. Now let P be a bivector on M , and let P^\sharp be the mapping from T^*M to TM , defined by $P^\sharp \xi = i_\xi P$. We shall determine the condition for the graph of P^\sharp to be a *ψ -Dirac structure*, i.e., to be maximally isotropic and closed under the bracket $[\cdot, \cdot]_{d^\psi}$. Let ξ and η be 1-forms. Then

$$\begin{aligned}
[P^\sharp\xi + \xi, P^\sharp\eta + \eta]_{\text{d}\psi} &= [P^\sharp\xi, P^\sharp\eta] + L_{P^\sharp\xi}\eta - i_{P^\sharp\eta}\text{d}\xi + i_{P^\sharp\xi\wedge P^\sharp\eta}\psi \\
&= [P^\sharp\xi, P^\sharp\eta] + L_{P^\sharp\xi}\eta - L_{P^\sharp\eta}\xi - \text{d}(P(\xi, \eta)) + i_{P^\sharp\xi\wedge P^\sharp\eta}\psi \\
&= [P^\sharp\xi, P^\sharp\eta] + [\xi, \eta]^P + i_{P^\sharp\xi\wedge P^\sharp\eta}\psi,
\end{aligned}$$

and therefore the condition for the graph of P^\sharp to be closed under the derived bracket is that

$$P^\sharp([\xi, \eta]^P + i_{P^\sharp\xi\wedge P^\sharp\eta}\psi) - [P^\sharp\xi, P^\sharp\eta] = 0.$$

This condition is equivalent to

$$\frac{1}{2}[P, P]_{\text{SN}} = (\wedge^3 P^\sharp)(\psi), \quad (5.2)$$

where $[P, P]_{\text{SN}}$ is the Schouten–Nijenhuis bracket of P with itself. If the graph of P^\sharp is a ψ -Dirac structure, i.e., if condition (5.2) is satisfied, P is called a *Poisson structure with background* ψ .

We set

$$[\xi, \eta]^{P, \psi} = [\xi, \eta]^P + i_{P^\sharp\xi\wedge P^\sharp\eta}\psi. \quad (5.3)$$

Formula (5.3) defines a skew-symmetric bracket on T^*M with anchor P^\sharp . (The Lie bracket of 1-forms defined by a Poisson bivector is recovered as the special case $\psi = 0$.)

The corresponding derivation $d_{P, \psi}$ on $V^\bullet(M)$ coincides with d_P on $C^\infty(M)$, and satisfies

$$(d_{P, \psi}x)(\xi, \eta) = P^\sharp\xi\langle\eta, x\rangle - P^\sharp\eta\langle\xi, x\rangle - \langle[\xi, \eta]^{P, \psi}, x\rangle,$$

for all $x \in V^1(M)$, $\xi, \eta \in \Omega^1(M)$. Let us define $(\wedge^2 P^\sharp)(\psi)$ to be the bivector-valued 1-form such that $(\wedge^2 P^\sharp)(\psi)(x)(\xi, \eta) = \psi(P^\sharp\xi, P^\sharp\eta, x)$, for any vector field x and for any 1-forms ξ and η . Then there is a concise expression (see [52]) for the derivation $d_{P, \psi}$ which we now prove.

PROPOSITION 5.2. *Let $d_P = [P, \cdot]_{\text{SN}}$, where $[\cdot, \cdot]_{\text{SN}}$ is the Schouten–Nijenhuis bracket. The derivation $d_{P, \psi}$ on $V^\bullet(M)$ is*

$$d_{P, \psi} = d_P + i_{(\wedge^2 P^\sharp)(\psi)} \cdot. \quad (5.4)$$

Proof. Writing P for P^\sharp , we compute, for any vector field x , and any 1-forms ξ, η ,

$$\begin{aligned}
(d_{P, \psi}x)(\xi, \eta) + \psi(P\xi, P\eta, x) &= L_{P\xi}\langle\eta, x\rangle - L_{P\eta}\langle\xi, x\rangle - \langle L_{P\xi}\eta, x\rangle + \text{d}\xi(P\eta, x) \\
&= \langle\eta, L_{P\xi}x\rangle - L_x\langle\xi, P\eta\rangle - \langle\xi, L_{P\eta}x\rangle = -\langle\eta, L_xP\xi\rangle - \langle L_x\xi, P\eta\rangle \\
&= -L_x\langle\eta, P\xi\rangle + \langle L_x\eta, P\xi\rangle - \langle L_x\xi, P\eta\rangle = -(L_xP)(\xi, \eta) = (d_P(x))(\xi, \eta).
\end{aligned}$$

Moreover, $i_{(\wedge^2 P^\sharp)(\psi)}$ is a derivation of $V^\bullet(M)$ which vanishes on $V^0(M)$ and coincides with $x \mapsto \psi(P^\sharp \cdot, P^\sharp \cdot, x)$ for $x \in V^1(M)$. We have thus proved formula (5.4).

Computing $[d_{P,\psi}, d_{P,\psi}]$, we see that this derivation vanishes if and only if condition (5.2) is satisfied. Thus, bracket $[\cdot, \cdot]^{P,\psi}$ is a Lie algebroid bracket if and only if P defines a Poisson structure with background ψ . The operator $d_{P,\psi}$ is then the differential of the Lie algebroid $(T^*M, [\cdot, \cdot]^{P,\psi})$. (See [52].)

In addition, the following morphism property of P^\sharp is easily proved [29]. \square

PROPOSITION 5.3. *The relation*

$$P^\sharp[\xi, \eta]^{P,\psi} = [P^\sharp\xi, P^\sharp\eta] \quad (5.5)$$

is equivalent to condition (5.2).

Further properties of the Poisson structures with background and their gauge equivalence are studied in several recent articles, including [5, 29, 49, 52].

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