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GREEN'S FUNCTION FOR AN ELASTIC LAYER WITH TEMPERATURE-DEPENDENT PROPERTIES

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The distributions of stresses and displacements in a thermoelastic layer with temperature-dependent properties are investigated. The problem is considered for the case of antiplane strain state. The boundary planes are assumed to be kept at constant temperatures. The upper boundary plane is free of loading, and the lower plane is loaded by a concentrated force. The solution is found in the form of integrals and the singularities of stresses are determined.

Keywords: temperature, displacements, stresses, elasticity, temperature-dependent properties, concentrated load.

The study of the behavior of stresses in elastic materials with temperature-dependent properties is of importance for many engineering applications. Some elastic materials change their mechanical moduli under the influence of temperature. In these cases, the application of Hookean strain-stress relations is not appropriate to describe stress distributions. The theory of thermoelasticity of materials with temperature-dependent properties seems to be the most adjusted for modeling of the interaction between mechanical and thermal fields. One of the first researchers who considerably developed the theoretical basis for the investigation of elastic bodies with temperature-dependent modulus was J. L. Nowiński (see [1–3] and the monograph [4]). Many experimental results on the determination of the mechanical properties of solids as functions of temperature are presented in the monograph [5] (mainly for metals), as well as in the papers [6–11]. Some theoretical investigations of solid mechanics with temperature-dependent properties can be found in [12–15].

In the present paper, we consider the antiplane strain state for an elastic layer with temperature-dependent properties. The boundary planes are assumed to be kept at given constant temperatures, which leads to a linear temperature distribution in the considered layer. The lower boundary plane is loaded by a concentrated force and the upper boundary plane is free of loading. The shear modulus μ as a function of temperature θ is taken into account in the form of a linear function. The assumption connected with the temperature dependence of the shear modulus leads to the problem of FGM layer in which the material properties continuously depend on the space variables. It can be observed that, in the case of classical thermoelasticity for homogeneous isotropic bodies in the antiplane strain state, the distributions of stresses are independent of temperature, unlike the analyzed problem.

Formulation and Solution of the Problem

Consider an isotropic elastic layer with thickness h. Let (x_1, x_2, x_3) be a Cartesian coordinate system

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such that the planes $x_2 = 0$ and $x_2 = h$ are boundaries of the body, and the $0x_3$-axis is perpendicular to the boundaries. Let the lower and upper boundary planes be kept at given constant temperatures θ_0 and θ_1, respectively.

Moreover, the investigated layer is loaded by forces linearly distributed along the $0x_3$-axis and concentrated forces with intensity P acting in the direction of the $0x_3$-axis. The shear modulus μ is assumed to be a function of temperature θ of the following form:

$$\mu(\theta) = \mu_0(1 - A\theta), \tag{1}$$

where μ_0 and A are constant. The form of the dependence of shear modulus (1) agrees with the experimental results presented in [5].

The assumptions made above lead to the antiplane strain state described by the displacement vector $\boldsymbol{u}(x_1, x_2) = (0, 0, u_3(x_1, x_2))$, and the considered problem is stationary and independent of x_3. The temperature $\theta = \theta(x_1, x_2)$ satisfies the following equation:

$$\frac{\partial^2\theta}{\partial x_1^2} + \frac{\partial^2\theta}{\partial x_2^2} = 0, \qquad x_1 \in R, \qquad x_2 \in (0, h),$$

and boundary conditions

$$\theta(x_1, 0) = \theta_0, \qquad \theta(x_1, h) = \theta_1, \qquad x_1 \in R,$$

causing the distribution of temperature

$$\theta(x_1, x_2) = \frac{(\theta_1 - \theta_0)x_2}{h} + \theta_0, \qquad x_1 \in R, \qquad x_2 \in \langle 0, h \rangle. \tag{2}$$

It follows from Eqs. (1) and (2) that

$$\mu(x_1, x_2) = \mu_0(\alpha_0 + \alpha_1 x_2), \qquad \alpha_0 = 1 - A\theta_0, \qquad \alpha_1 = -\frac{A(\theta_1 - \theta_0)}{h}. \tag{3}$$

The stress state is described by the nonzero components σ_{13} and σ_{23} of the form

$$\sigma_{13}(x_1, x_2) = \mu_0(\alpha_0 + \alpha_1 x_2)\frac{\partial u_3}{\partial x_1}, \qquad \sigma_{23}(x_1, x_2) = \mu_0(\alpha_0 + \alpha_1 x_2)\frac{\partial u_3}{\partial x_2}. \tag{4}$$

The equilibrium equation in the case of stresses given by relations (4) can be written as

$$\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\alpha_1}{\alpha_0 + \alpha_1 x_2}\frac{\partial u_3}{\partial x_2} = 0, \qquad x_1 \in R, \qquad x_2 \in (0, h). \tag{5}$$

The boundary conditions have the form

$$\sigma_{23}(x_1,0) = P\delta(x_1), \qquad \sigma_{23}(x_1,h) = 0, \qquad x_1 \in R, \tag{6}$$

where $\delta(\cdot)$ is the Dirac delta function. We now use the Fourier integral transform [16] with respect to variable x_1 and denote

$$\tilde{u}_3(s,x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_3(x_1,x_2) e^{-isx_1}\, dx_1 \ .$$

Thus, it follows from Eq. (5) that

$$\frac{d^2\tilde{u}_3(s,x_2)}{dx_2^2} + \frac{\alpha_1}{\alpha_0 + \alpha_1 x_2} \frac{d\tilde{u}_3(s,x_2)}{dx_2} - s^2 \tilde{u}_3(s,x_2) = 0 . \tag{7}$$

The linear ordinary second-order differential equation (7) is reduced to the form

$$\frac{d^2\tilde{u}_3}{d\omega^2} + \frac{1}{\omega}\frac{d\tilde{u}_3}{d\omega} - \frac{s^2}{\alpha_1^2}\tilde{u}_3 = 0 \ , \tag{8}$$

where

$$\omega = \alpha_0 + \alpha_1 x_2 . \tag{9}$$

The general solution of Eq. (8) can be written as [17]

$$\tilde{u}_3(s,\omega) = C_1 J_0\left(\frac{i\omega s}{\alpha_1}\right) + C_2 Y_0\left(\frac{i\omega s}{\alpha_1}\right) \tag{10}$$

where C_1 and C_2 are constants and $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and the second kind, respectively. In view of relations [8]

$$J_0(ix) = I_0(x), \qquad Y_0(ix) = K_0(x), \qquad x \in R,$$

where $I_0(\cdot)$ and $K_0(\cdot)$ are modified Bessel functions, and relations (9) and (10), the general solution of equation (7) can be written as follows:

$$\tilde{u}_3(s,x_2) = C_1 I_0\left[\frac{s\omega}{\alpha_1}\right] + C_2 K_0\left[\frac{s\omega}{\alpha_1}\right]. \tag{11}$$

The constants C_1 and C_2 are determined from the boundary conditions (6). By using Eqs. (4), (6), (11) and the following relations [19]:

$$\frac{d}{dz} I_0(z) = I_1(z), \qquad \frac{d}{dz} K_0(z) = -K_1(z)\,, \tag{12}$$

where $I_1(x)$ and $K_1(x)$ are modified Bessel functions, we conclude that a_1 and a_2 must satisfy the following system of algebraic equations:

$$a_1 I_1\left[\frac{s\omega^*}{\alpha_1}\right] - a_2 K_1\left[\frac{s\omega^*}{\alpha_1}\right] = 0, \qquad a_1 I_1\left(\frac{s\alpha_0}{\alpha_1}\right) - a_2 K_1\left(\frac{s\alpha_0}{\alpha_1}\right) = \frac{P}{\sqrt{2\pi} s\mu_0\alpha_0}\,. \tag{13}$$

where $\omega^* = \alpha_0 + \alpha_1 h$.

In view of Eqs. (13) and (11), by applying the inverse Fourier transformation [16], we rewrite the displacement u_3 in the following form:

$$u_3(x_1, x_2) = -\frac{P}{\pi\alpha_0\mu_0}\int_0^\infty \frac{1}{sW}\left\{K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_0\left[\frac{s\omega}{\alpha_1}\right] + I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_0\left[\frac{s\omega}{\alpha_1}\right]\right\}\cos(sx_1)\,ds\,, \tag{14}$$

where

$$W = I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_1\left(\frac{s}{\alpha_1}\alpha_0\right) - K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_1\left(\frac{s\alpha_0}{\alpha_1}\right)..$$

The components of stresses σ_{13} and σ_{23} are computed from relations (4) and (14).

Substituting (14) in (4), we find

$$\sigma_{13}(x_1, x_2) = \frac{P\omega}{\pi\alpha_0}\int_0^\infty \frac{1}{W}\left\{K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_0\left[\frac{s\omega}{\alpha_1}\right] + I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_0\left[\frac{s\omega}{\alpha_1}\right]\right\}\sin(sx_1)\,ds\,, \tag{15}$$

$$\sigma_{23}(x_1, x_2) = \frac{P\omega}{\pi\alpha_0}\int_0^\infty \frac{1}{W}\left\{-K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_1\left[\frac{s\omega}{\alpha_1}\right] + I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_1\left[\frac{s\omega}{\alpha_1}\right]\right\}\cos(sx_1)\,ds\,. \tag{16}$$

Relations (14)–(16) give the fundamental solution (Green's function) for the considered problem in the integral form.

From the viewpoint of mechanics, it is necessary to study the singularities of stresses at the point of action of the concentrated forces. For this purpose, we analyze the asymptotic behavior of the integrand functions in (15) and (16).

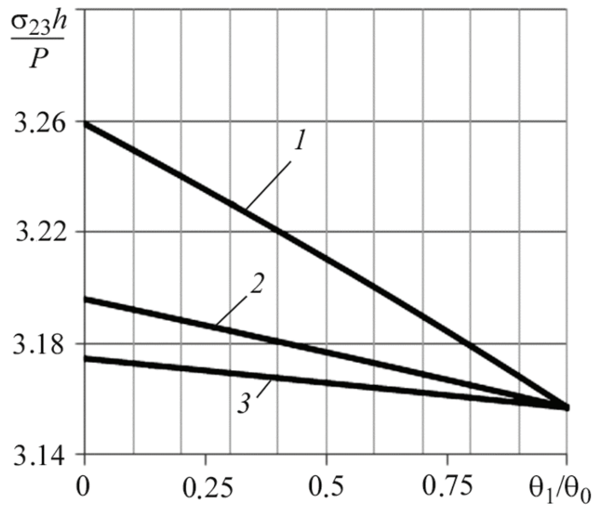

Fig. 1. Dimensionless stresses $\sigma_{23}(\breve{x}_1, \breve{x}_2)h/P$ as a function of the parameter θ_1/θ_0 for $\theta_0 = 819°\text{K}$, $\breve{x}_1 = 0.0$, and $\breve{x}_2 = 0.1$: (1) $A = 0.00051\,\text{K}^{-1}$; (2) $0.00025\,\text{K}^{-1}$; (3) $0.000125\,\text{K}^{-1}$.

Singularities of Stresses

By using relations (see [19])

$$I_\nu(x) \underset{x\to\infty}{\approx} \frac{e^x}{\sqrt{2\pi x}}, \qquad K_\nu(x) \underset{x\to\infty}{\approx} \sqrt{\frac{\pi}{2x}}e^{-x}, \qquad I_0(x) \underset{x\to 0}{\approx} 1,$$

$$I_1(x) \underset{x\to 0}{\approx} \frac{1}{2}x, \qquad K_1(x) \underset{x\to 0}{\approx} \frac{1}{x}, \qquad K_0(x) \underset{x\to 0}{\approx} \ln\frac{2}{x}, \tag{17}$$

and Eq.(14), we conclude that

$$W \underset{s\to 0}{\approx} \frac{\alpha_1 h(2\alpha_0 + \alpha_1 h)}{2\alpha_0(\alpha_0 + \alpha_1 h)} = \text{const}, \qquad W \underset{s\to\infty}{\approx} \frac{\alpha_1}{\sqrt{\alpha_0(\alpha_0 + \alpha_1 h)}} \frac{\sinh(sh)}{s}. \tag{18}$$

Denote

$$L_0 \equiv K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_0\left[\frac{s\omega}{\alpha_1}\right] + I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_0\left[\frac{s\omega}{\alpha_1}\right],$$

$$L_1 \equiv -K_1\left[\frac{s\omega^*}{\alpha_1}\right] I_1\left[\frac{s\omega}{\alpha_1}\right] + I_1\left[\frac{s\omega^*}{\alpha_1}\right] K_1\left[\frac{s\omega}{\alpha_1}\right]. \tag{19}$$

Thus, by using (17) and (18) we obtain

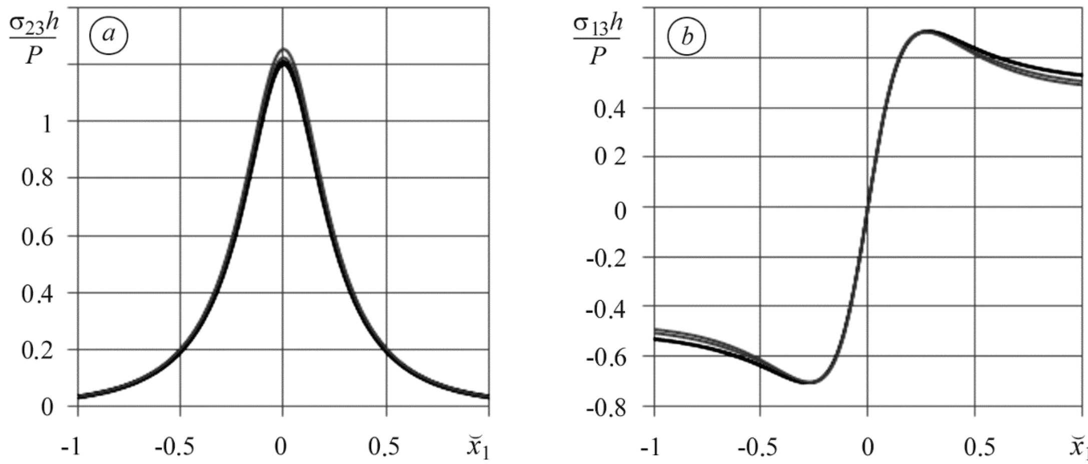

Fig. 2. Dimensionless stresses as a function of the parameter A: (a) $\sigma_{23}h/P$; (b) $\sigma_{13}h/P$: $\theta_0 = 819°\mathrm{K}$, $\theta_1 = 0.5\theta_0$, $\breve{x}_2 = 0.25$.

$$\frac{L_0}{W} \underset{s\to\infty}{\approx} \sqrt{\frac{\alpha_0}{\omega}}\, e^{-sx_2}, \qquad \frac{L_1}{W} \underset{s\to\infty}{\approx} \sqrt{\frac{\alpha_0}{\omega}}\, e^{-sx_2}. \tag{20}$$

In view of Eqs. (15), (16), (19), and (20), the singularities of the stress components can be represented in the form

$$\sigma_{13}(x_1, x_2) = \frac{P}{\pi}\sqrt{\frac{\omega}{\alpha_0}}\,\frac{x_1}{x_1^2 + x_2^2} + 0(1), \qquad \sigma_{23}(x_1, x_2) = \frac{P}{\pi}\sqrt{\frac{\omega}{\alpha_0}}\,\frac{x_2}{x_1^2 + x_2^2} + 0(1). \tag{21}$$

Equation (21) now implies that the order of singularities of the stress components σ_{13} and σ_{23} is the same as in the elastic homogeneous isotropic layer. However, the difference is observed in the coefficients of singularities.

The integrals in expressions (15) and (16) for the stresses σ_{13} and σ_{23} can be found numerically. For this purpose, we use the following dimensionless variables:

$$\breve{x}_1 = \frac{x_1}{h}, \qquad \breve{x}_2 = \frac{x_2}{h}, \qquad \breve{s} = sh,$$

The physical data taken into account are the same as in [20], where the copper material is investigated. In Fig. 1, we observe the influence of the parameter A and the difference between the boundary temperatures θ_0 and θ_1 on the stresses σ_{23}. The dimensionless stresses σ_{23} at the point $\breve{x}_1 = 0.0$, $\breve{x}_2 = 0.1$ as a function of the ratio θ_1/θ_0 are presented in Fig. 1 for three values of the parameter A. It is easy to see that the component of stresses is a linear function of the ratio θ_1/θ_0 and, for θ_1/θ_0, the solutions are reduced to the case of a homogeneous body with constant material properties.

In Fig. 2 the cases $A = 0$ are adequate for the homogeneous elastic body. Small variations of the stresses σ_{23} with respect to the boundary temperatures near the boundary plane loaded by a concentrated force can be observed for the following three values of A: $A = 0.00051\,\mathrm{K}^{-1}$, $A = 0.00025\,\mathrm{K}^{-1}$, and $A = 0$.

The stresses σ_{13} change their sign at $\breve{x}_1 = 0$ (the curve representing σ_{13} is antisymmetric but the curve representing σ_{23} is symmetric). The maximal values of σ_{23} are attained at the point of action of the concentrated force.

CONCLUSIONS

The problem of distribution of stresses in the thermoelastic layer with temperature-dependent properties loaded by a concentrated force in the boundary plane is solved under the conditions of antiplane strain state. It is assumed that the shear modulus is a linear function of temperature. The obtained results for stresses at the point of action of the concentrated force are characterized by the singularity of the same order as in the case of an isotropic homogeneous body with constant material properties. The singularities observed for the two mentioned materials differ by the singularity coefficients. Moreover, we can emphasize that, for the case of ordinary elasticity (when the shear modulus is constant), the boundary temperatures affect the stresses σ_{13} and σ_{23}. In the considered problem of the layer with temperature-dependent properties, the temperature is coupled with the displacement u_3.

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