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ELASTIC HALF SPACE WITH LAMINATED COATING OF PERIODIC STRUCTURE UNDER THE ACTION OF HERTZ'S PRESSURE

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We consider an axially symmetric problem of the theory of elasticity for a nonuniform half space loaded by Hertz's pressure. The half space consists of a uniform base and a system of two periodically deposited elastic layers. The solution of the problem of the theory of elasticity for the nonuniform coating is compared with the solution of the problem in which this coating is simulated by a homogenized uniform layer.

Keywords: stresses, laminated coating of periodic structure, Hertz's pressure.

In the mechanics of contact interaction, much attention is now given to coatings used for the improvement of the tribological characteristics of friction couples. Thus, the coatings formed by periodically deposited elastic layers are now extensively investigated [7, 8] parallel with the uniform coatings [1–3] and nonuniform coatings whose mechanical properties are described by continuous functions of the distance from the surface [4–6]. In the analysis of the stressed state, the researchers, as a rule, focus their attention on the evaluation of the tensile and Huber–Mises stresses described by the second invariant of the deviator of the stress tensor. In modeling laminated half spaces or coatings with periodic structures, it is customary to use two different approaches. The first of these approaches takes into account the structure and treats the layers as separate elastic media. The second approach is based on the analysis of a homogenized uniform coating whose mechanical properties are determined on the basis of the mechanical and geometric characteristics of the strip of periodicity [9–11]. The solutions obtained for the laminated half space are compared in [12–14]. It is shown that if the ratio of the thickness of the strip of periodicity to the characteristic size of the region of loading is smaller than 0.1, then the nonuniform half space with periodic structure can be modeled by a homogenized half space. The analyzed coating, unlike the laminated half space, contains finitely many slips of periodicity. Hence, it is necessary to show how these approximations affect the difference between the solutions.

In the present work, we consider an axially symmetric problem of the theory of elasticity of elastic half space with laminated coating of periodic structure loaded by Hertz's pressure. Since some components of the stress tensor can be noticeably different in different layers of the strip of periodicity, we choose the method of homogenization [10–11], which enables us to take this feature into account. We analyze the difference between the vertical displacements of points on the surface of the nonuniform half space and the distributions of tensile and Huber–Mises stresses caused by the use of two different models of nonuniform coatings.

Statement of the Problem

Suppose that Hertz's pressure

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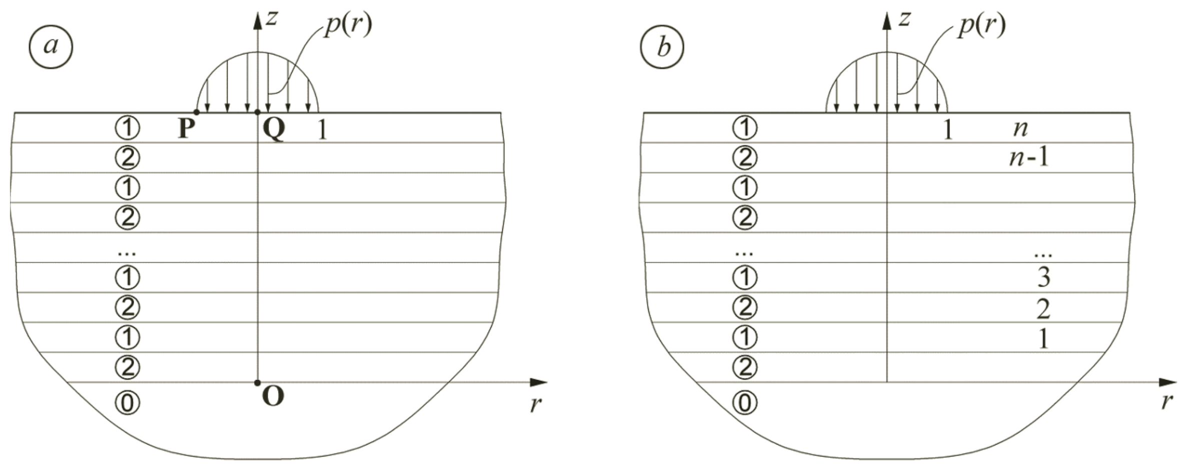

Fig. 1. Schematic diagrams of the problems of the theory of elasticity for half spaces covered with a laminated coating of periodic structure (a) or with a stack of layers (b).

$$p(r) = p_0\sqrt{1-r^2}, \qquad r \le 1, \tag{1}$$

is applied to a circle of radius a on the surface $z = h = H/a$ of the nonuniform elastic half space (Fig. 1a). Here, r, ψ, and z are dimensionless cylindrical coordinates (relative to the linear size a) and H is the thickness of the coating.

The nonuniform half space is formed by the homogeneous isotropic half space with Young's modulus E_0 and Poisson's ratio μ_0 and a system of two periodically deposited elastic layers with thicknesses l_1 and l_2 ($l = l_1 + l_2$ is the thickness of the strip of periodicity), Young's moduli E_1 and E_2, and Poisson's ratios μ_1 and μ_2, respectively. Assume that the conditions of perfect mechanical contact are realized between the layers of the coating and between the coating and the base.

Homogenized Uniform Coating

The analyzed problem of the theory of elasticity is reduced to the solution of differential equations [14]

$$\Psi^{(1)}_{i,rr} + r^{-1}\Psi^{(1)}_{i,r} + \gamma_i^2\Psi^{(1)}_{i,zz} = 0, \qquad i = 1,2, \tag{2}$$

$$d_0\Delta\boldsymbol{u}^{(0)} + \mathrm{grad\,div}\boldsymbol{u}^{(0)} = 0 \tag{3}$$

with boundary conditions on the surface of the nonuniform half space

$$\sigma^{(1)}_{zz}(r,h) = -p(r)H(1-r), \qquad \sigma^{(1)}_{rz}(r,h) = 0, \tag{4}$$

conditions of perfect mechanical contact between the coating and the uniform half space

$$\mathbf{u}^{(1)}(r,0) = \mathbf{u}^{(0)}(r,0), \qquad \sigma^{(1)}(r,0)\cdot\mathbf{n} = \sigma^{(0)}(r,0)\cdot\mathbf{n}, \tag{5}$$

and conditions imposed at infinity

$$\boldsymbol{u}^{(j)}(r,z) \to 0, \qquad r^2 + z^2 \to \infty, \qquad j = 1,2, \tag{6}$$

where $\mathbf{u}^{(j)}$ is the dimensionless vector of elastic displacements (relative to the parameter a), $\sigma^{(j)}$ is the stress tensor, the superscript $j = 0$ marks the parameters and functions of state in the uniform half space, $j = 1$ corresponds to the homogenized coating, $\Psi_i^{(1)}$, $i = 1,2$ are elastic potentials, γ_i, $i = 1,2$ are the roots of the characteristic equation [14], $d_0 = 1/(1-2\mu_0)$, $H(r)$ is the Heaviside function, $\mathbf{n} = (0, 0, 1)$, and the subscripts placed after commas denote the derivatives with respect to the corresponding variables.

We seek the components of the vector of displacements and stress tensor in the coating in the form

$$u_r^{(1)} = \Psi_{1,r}^{(1)} + \Psi_{2,r}^{(1)}, \qquad u_z^{(1)} = \kappa_1 \Psi_{1,z}^{(1)} + \kappa_2 \Psi_{2,z}^{(1)}; \tag{7}$$

$$\sigma_{rr}^{(1),k} = K_k u_{r,r}^{(1)} + L_k r^{-1} u_r^{(1)} + M_k u_{z,z}^{(1)}, \qquad k = 1,2, \tag{8}$$

$$\sigma_{\psi\psi}^{(1),k} = L_k u_{r,r}^{(1)} + K_k r^{-1} u_r^{(1)} + M_k u_{z,z}^{(1)}, \qquad k = 1,2, \tag{9}$$

$$\sigma_{zz}^{(1)} = A_3 u_{r,r}^{(1)} + A_3 r^{-1} u_r^{(1)} + A_4 u_{z,z}^{(1)}, \qquad \sigma_{rz}^{(1)} = A_5 (u_{r,z}^{(1)} + u_{z,r}^{(1)}), \tag{10}$$

where

$$L_k = \lambda_k - h_k \lambda_k \frac{[\lambda]}{\bar{\lambda} + 2\bar{G}}, \qquad K_k = L_k + 2G_k, \qquad M_k = \lambda_k - h_k \lambda_k \frac{[\lambda] + 2[G]}{\bar{\lambda} + 2\bar{G}}, \qquad h_1 = 1, \quad h_2 = -\frac{l_1}{l_2},$$

λ_k and G_k are the Lamé coefficients, the subscript k corresponds to the number of the layer in the strip of periodicity, and the mechanical constants A_3, A_4, A_5, κ_1, κ_2, $[\lambda]$, $[G]$, $\bar{\lambda}$, and $\bar{G}$ can be found by using the well-known relations [14].

Since the algorithm of solution of the analyzed problem is similar to the algorithm presented in [14], we restrict ourselves to its brief description. By applying the Hankel integral transformation to Eqs. (2), (3) and relations (7)–(10), we find the Hankel transforms of the components of the vector of displacements and stress tensor satisfying the boundary conditions (6). The obtained relations contain six unknown functions of the parameter of integral transformation. To determine these functions, we use the boundary conditions (4) and (5) and arrive at a system of six linear algebraic equations. Since only one equation of this system is nonuniform and its right-hand side is equal to the Hankel transform of load (1), the solution of the system of equations is proportional to this transform. Returning in these transforms to the space of originals, we arrive at the relationships (in the form of integrals) between the displacements and stresses, on the one hand, and the applied load, on the other hand. The integrals at internal points of the nonuniform half space $(z < h)$ are taken with the help of the Gaussian quadrature. On the surface $z = h$, we take into account the asymptotic behavior of the solution of the system of equations obtained as the parameter of the integral transformation tends to infinity. The integrals in which the integrands are replaced by their asymptotics are taken analytically. To find the remaining integrals, we apply the Gaussian quadrature.

Modeling of the Nonuniform Coating by a Stack of Layers

The layers of the nonuniform coating are enumerated from the bottom upward (starting from the layer in immediate contact with the uniform base) (Fig. 1b). Their mechanical properties are described by Young's moduli $E^{(j)}$ $[E^{(2k)} = E_1,\ E^{(2k-1)} = E_2,\ k = 1,\dots,n/2]$ and Poisson's ratios $\mu^{(j)}$ $[\mu^{(2k)} = \mu_1,\ \mu^{(2k-1)} = \mu_2,$ $k = 1,\dots,n/2]$. Here, the superscript j corresponds to the number of layer in the stack and n is an even number corresponding to the total number of layers in the stack. The general solution of equations of the theory of elasticity in the layers of the stack and in the uniform half space $z \le 0$ can be found in [6]. By using this solution, we arrive at the following relations for the components of the vector of displacements and stress tensor in the layers of the stack:

$$u_k^{(j)} = \int_0^\infty U_k^{(j)}(s,z)\tilde{p}(s)J_k(sr)\,ds, \qquad j = 1,\dots,n, \qquad k = r,z, \tag{11}$$

$$\sigma_{rr}^{(j)} = \int_0^\infty S_{r1}^{(j)}(s,z)\tilde{p}(s)J_0(sr)s\,ds - r^{-1}\int_0^\infty S_{r2}^{(j)}(s,z)\tilde{p}(s)J_1(sr)\,ds, \qquad j = 1,\dots,n, \tag{12}$$

$$\sigma_{\psi\psi}^{(j)} = \int_0^\infty (S_{r1}^{(j)}(s,z) - S_{r2}^{(j)}(s,z))\tilde{p}(s)J_0(sr)\,sds + r^{-1}\int_0^\infty S_{r2}^{(j)}(s,z)\tilde{p}(s)J_1(sr)\,ds, \qquad j = 1,\dots,n, \tag{13}$$

$$\sigma_{kz}^{(j)}(r,z) = -\int_0^\infty S_{kz}^{(j)}(s,z)\,\tilde{p}(s)\,J_k(sr)\,sds, \qquad j = 1,\dots,n, \qquad k = r,z, \tag{14}$$

$$2U_z^{(j)}(s,z) = D_j s_j a_{4j-3}^{(p)}(s) + D_j c_j a_{4j-2}^{(p)}(s) + 2ss_j a_{4j-1}^{(p)}(s) + 2sc_j a_{4j}^{(p)}(s),$$

$$2U_r^{(j)}(s,z) = \frac{G^{(n)}S_{r2}^{(j)}(s,z)}{G^{(j)}} = 2sc_j a_{4j-1}^{(p)}(s) + 2ss_j a_{4j}^{(p)}(s)$$

$$+ ((2+d_j)s_j + D_j c_j)a_{4j-3}^{(p)}(s) + ((2+d_j)c_j + D_j s_j)a_{4j-2}^{(p)}(s),$$

$$\frac{G^{(n)}S_{r1}^{(j)}(s,z)}{G^{(j)}} = 2U_r^{(j)}(s,z) + (d_j - 1)s_j a_{4j-3}^{(p)}(s) + (d_j - 1)c_j a_{4j-2}^{(p)}(s),$$

$$\frac{G^{(n)}S_{zz}^{(j)}(s,z)}{G^{(j)}} = 2U_r^{(j)}(s,z) - (1+d_j)s_j a_{4j-3}^{(p)}(s) - (1+d_j)c_j a_{4j-2}^{(p)}(s),$$

$$\frac{G^{(n)}S_{rz}^{(j)}(s,z)}{G^{(j)}} = 2U_z^{(j)}(s,z) + (1+d_j)c_j a_{4j-3}^{(p)}(s) + (1+d_j)s_j a_{4j-2}^{(p)}(s),$$

where $\tilde{p}(s)$ is the Hankel transform of load (1), the functions $a_i^{(p)}(s)$, $i = -1,\dots,4n$, are the solutions of the system of linear algebraic equations

$$\sum_{j=1}^{4n+2} A_{ij}^{(2)} a_{j-2}^{(p)} = \delta_{i,4n+1}, \qquad i = 1, 2, \ldots, 4n+2, \tag{15}$$

$$d = \frac{1}{1-2\mu^{(j)}}, \quad c_j = \cosh\left(s(h_j - z)\right), \quad s_j = \sinh\left(s(h_j - z)\right), \quad D_j = d_j s(h_j - z),$$

h_j is the z-coordinate of the point of intersection of the upper plane bounding the layer of the stack with the z-axis, $G^{(j)} = E^{(j)}/2/(1+\mu^{(j)})$, $J_r(sr) = J_1(sr)$, $J_z(sr) = J_0(sr)$, $J_0(sr)$ and $J_1(sr)$ are the Bessel functions, and δ_{ij} is the Kronecker symbol.

The nonzero coefficients of system (15) are found by using the well-known relations from [6]. As in the previous section, we take integrals (11)–(14) at internal points of the nonuniform half space $(z < h)$ with the help of the Gaussian quadrature. On the surface $z = h$, we take into account the asymptotic behavior of the functions $a_{4n-2}^{(p)}(s)$ and $a_{4n}^{(p)}(s)$ as $s \to \infty$:

$$\lim_{s\to\infty} a_{4n-2}^{(p)}(s) = -(1-2\mu^{(n)}) \quad \text{and} \quad \lim_{s\to\infty} a_{4n}^{(p)}(s) = -(1-\mu^{(n)}).$$

Analysis of the Results

Estimating the original relations, we conclude that the distributions of displacements and stresses in the problem of homogenized coating depend on seven dimensionless parameters: the thickness of the coating h, the ratios of Young's moduli E_1/E_0 and E_2/E_0, Poisson's ratios μ_0, μ_1, and μ_2, and the ratio of the thicknesses of layers in the strip of periodicity l_1/l_2. Similar distributions for the nonuniform coating additionally depend on the number of layers in the stack n. To decrease the number of input parameters, we assume that the mechanical properties of one layer in the strip of periodicity coincide with the mechanical properties of the base and that Poisson's ratios and the thicknesses of all layers in the stack are identical. We also assume that $\mu_0 = 0.25$, E_1/E_0 (or E_2/E_0) = 2, 4, or 8, $h = 0.2, 0.4$, or 0.8, and $n = 10, 20$, or 40. The results of calculations demonstrate that **P**, **O**, and **Q** (see Fig. 1a) are characteristic points of the analyzed problem. We compare the obtained solutions by using the following parameters: the displacements u_z at the points **P** and **Q**, the first principal stress σ_1 at the points **P** and **O**, and the second invariant of the deviator of stress tensor J_2 at points of the z-axis and at the point **P**.

The values computed for the homogenized coating are presented in the columns marked "hom." in Table 1. In the columns with $n = 10, 20$, and 40, we present the solutions obtained for the nonuniform coatings with the indicated numbers of layers. The first number in the entries corresponds to the parameters $E_1/E_0 = 8$ and $E_2/E_0 = 1$, whereas the second number (without brackets) corresponds to $E_1/E_0 = 1$ and $E_2/E_0 = 8$. The relative deviations of the displacement $u_z(\mathbf{Q})E_0/p_0$ from the corresponding displacement obtained as a result of modeling of the nonuniform coating by the homogenized layer are presented in the parentheses (in %). We see that these values are in good agreement even in the case where the Young's moduli of the layers in the strip of periodicity are strongly different. If we consider the stacks containing at least ten layers, then the indicated deviations depend not on the number of layers but on the ratio δ of the thickness of the slip of periodicity to the radius of the circle of loading. For the same values of δ, larger deviations are observed for thicker coatings.

In finding the stresses, we assume that $E_1/E_0 > 1$ but $E_2/E_0 = 1$. In Figs. 2 and 3, the rhombs mark the numerical results obtained for the nonuniform laminated coating, whereas the solid lines correspond to the homogenized uniform coating.

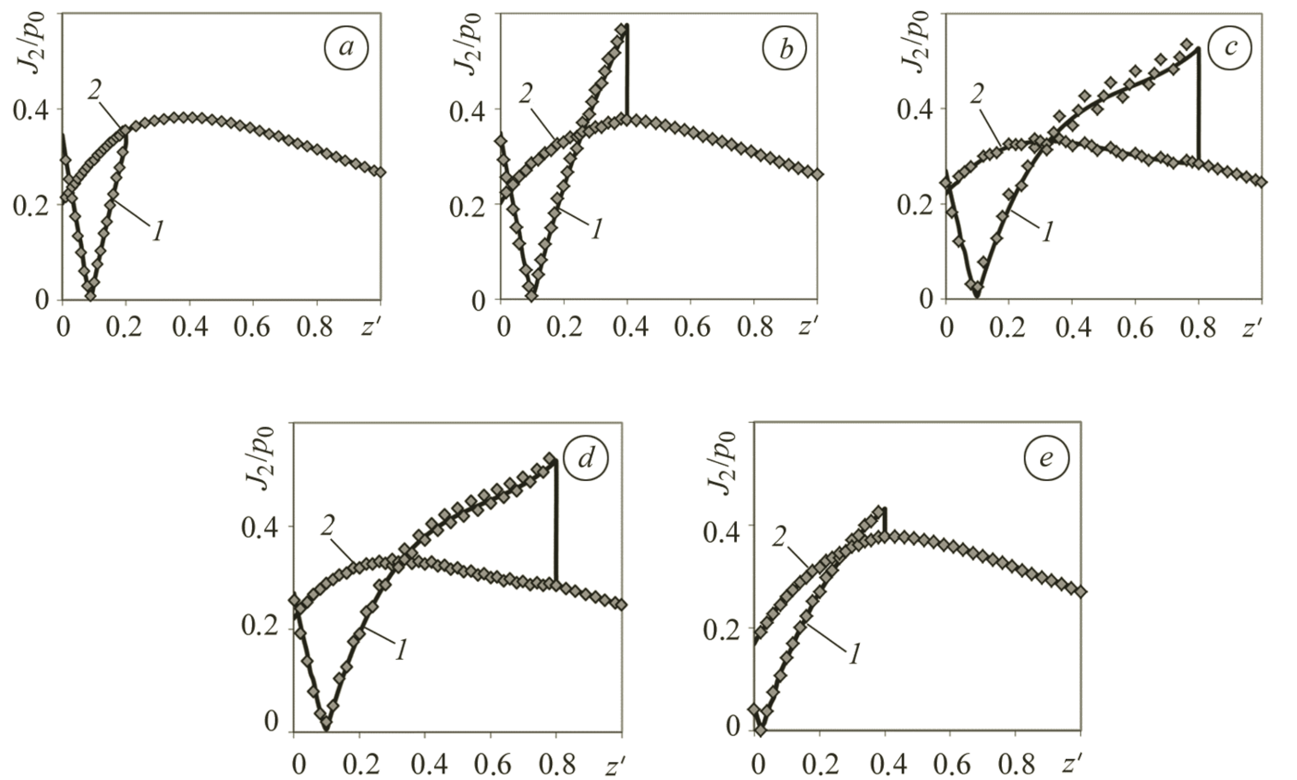

Fig. 2. Distributions of the values of J_2 along the z-axis ($z' = h - z$) for the parameters: (a) $E_1/E_0 = 4$, $h = 0.2$, $n = 20$; (b) $E_1/E_0 = 4$, $h = 0.4$, $n = 20$; (c) $E_1/E_0 = 4$, $h = 0.8$, $n = 20$; (d) $E_1/E_0 = 4$, $h = 0.8$, $n = 40$; (e) $E_1/E_0 = 2$, $h = 0.4$, $n = 20$.

Table 1. Dependence of the Displacement u_z at the Points Q and P on the Parameter h

h	$u_z(\mathbf{Q})E_0/p_0$				$u_z(\mathbf{P})E_0/p_0$			
	hom.	$n = 10$	$n = 20$	$n = 40$	hom.	$n = 10$	$n = 20$	$n = 40$
0.2	–1.3354	–1.3316 (0.28) –1.3398 (–0.33)	–1.3334 (0.15) –1.3375 (0.16)	–1.3344 (0.08) –1.3364 (–0.08)	–0.7043	–0.7044 –0.7043	–0.7044 –0.7043	–0.7044 –0.7043
0.4	–1.2306	–1.2236 (0.57) –1.2400 (–0.76)	–1.2267 (0.32) –1.2350 (–0.36)	–1.2288 (0.15) –1.2327 (–0.17)	–0.6814	–0.6818 –0.6815	–0.6817 –0.6813	–0.6816 –0.6813
0.8	–1.0747	–1.0580 (1.55) –1.0960 (–1.98)	–1.0655 (0.86) –1.0851 (–1.01)	–1.0699 (0.45) –1.0798 (–0.47)	–0.6240	–0.6251 –0.6250	–0.6245 –0.6240	–0.6243 –0.6238

Curves 1 and 2 correspond to the stresses acting in the layers with larger and smaller Young's moduli, respectively. It should be emphasized that, in the case of homogenized coating, we do not know which layer of the slip of periodicity is located at the analyzed point of the coating. Hence, the stressed state at every point of the coating is described by the two stress tensors given by relations (8)–(10).

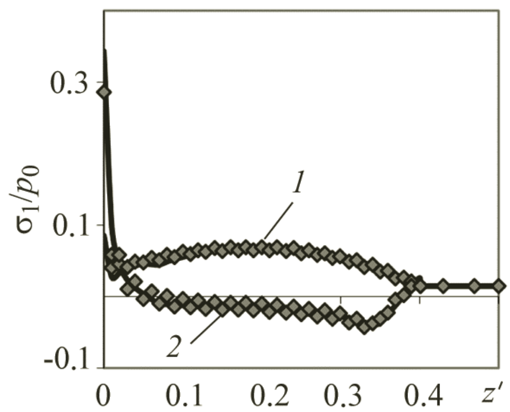

Fig. 3. Distribution of the first principal stress along the line $r = 1$ $(z' = h - z)$; $E_1/E_0 = 4$, $E_2/E_0 = 1$, $h = 0.4$, $n = 40$.

Depending on the analyzed layer in the strip of periodicity, one of the tensors given by these relations corresponds to the stress tensor in the nonuniform coating (see Figs. 2 and 3). In the uniform base, curves 1 and 2 coincide.

Comparing the stresses obtained in both analyzed problems, we conclude that, only in the case of stresses acting in the uniform base, we get deviations comparable with the deviations of displacements. In the layers of the coating, the deviations of stresses vary from 1–5% $(E_1/E_0 \le 4,\ n = 20)$ up to 10–20% for the stresses σ_1 acting on the boundary of the region of loading. The indicated deviations strongly depend on the gradient of the analyzed parameter in the investigated layer of the slip of periodicity, which explains the following observations: in the layers with lower Young's modulus, the deviations are much smaller (Figs. 2 and 3) and the maximum deviations are observed at the point **P** (Figs. 1a and 3). As could be expected, the agreement between the solutions improves as the number of layers in the coating increases (Fig. 2d) or the ratio of the Young's moduli of layers in the slip of periodicity decreases (Fig. 2e).

By analogy with the case of uniform half space, the principal stress σ_1 is positive near the points of the unloaded surface $(z = h,\ r \ge 1)$. Its highest value in this domain is attained at the point **P**. Tensile stresses may appear in the layers with higher Young's modulus on the interface of the coating and the base. The highest value of these stresses is attained at the point **O**. As a rule, the tensile stresses at the point **P** are higher than at the point **O**. However, as the parameter E_1/E_0 and the thickness of the coating increase, the global maximum may appear at the point **O**.

The distribution of the parameter J_2 in the layers with higher Young's modulus is characterized by the presence of three local maxima at the points **P**, **Q**, and **O**. With the exception of thin coatings with significant difference between Young's moduli $(h = 0.2,\ E_1/E_0 = 4)$, the global maximum is attained on the boundary of the coating and the base.

CONCLUSIONS

It is shown that the solution of the axially symmetric problem of the theory of elasticity for the elastic half space with laminated coating of periodic structure loaded by Hertz's pressure is in good agreement with the solution of the problem in which the coating is modeled by a homogenized uniform layer. The smallest deviations are obtained in finding the displacements and stresses in the uniform base. This is a strong argument in favor of the application of the method of homogenization for the evaluation of the actual contact pressure in the problems of indentation of punches into nonuniform half spaces of this sort. Although the deviations of stresses in the coating are much higher than the deviations of displacements, we observe very good qualitative agreement of the solutions. Except the boundary of the region of loading, where it is necessary to perform additional calculations with regard for the structure of the coating, the stresses obtained for the homogenized coating can be used to estimate the strength of laminated coatings with periodic structure.

The character of distributions of the first principal stress and the Huber–Mises stresses is the same as for the uniform half space covered with a uniform layer with different mechanical characteristics.

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