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AXIALLY SYMMETRIC CONTACT PROBLEM OF PRESSING OF AN ABSOLUTELY RIGID BALL INTO AN ELASTIC HALF SPACE WITH INHOMOGENEOUS COATING

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We consider an axially symmetric contact problem of pressing of an absolutely rigid ball into an inhomogeneous half space formed by a homogeneous base and an inhomogeneous surface layer. The Poisson's ratio of the layer is constant and its Young modulus is an exponential function of the distance from the surface of the half space. The solution of the problem of the theory of elasticity with continuous dependence of the Young modulus on the coordinate is compared with the solution of the problem in which the inhomogeneous layer is replaced with a package of homogeneous layers.

Keywords: axially symmetric contact problem, elastic half space, inhomogeneous coating.

The problems of loading of an inhomogeneous elastic half space whose mechanical properties depend on the distance from its surface z and the corresponding contact problems were studied within the framework of the theory of elasticity as early as in the 1950–70s [1–11]. New procedures were proposed for the construction of the general solution of equations of the theory of elasticity for some functional dependences of mechanical properties on the coordinate z and, in particular, for the case where Poisson's ratio is constant and the behavior of the shear or Young's modulus is described by a power or exponential function.

The advances in contemporary engineering connected with the extensive application of elastic surface layers aimed at the improvement of the tribological characteristics of friction couples renewed the interest of the researchers to this class of problems [12–22]. In the last decade, significant attention is focused on the contact problem of pressing of an absolutely rigid die into an elastic inhomogeneous half space formed by a homogeneous isotropic elastic half space and an inhomogeneous surface layer whose mechanical properties vary over its thickness. For the power (or exponential) law of variation of Young's modulus, it is customary to solve the corresponding partial differential equations with variable coefficients by using the finite-element method or the available analytic approaches [12–17, 22]. Some analytic methods for the construction of solutions in the case where the Lamé constants are (to a certain extent) arbitrary functions of the coordinate z can be found in [9, 10, 18]. However, the proposed approaches are connected with fairly complicated mathematical transformations whose practical realization often depends on the type of the analyzed functional dependence. Thus, in particular, in the examples considered in [18], Poisson's ratio is regarded as constant. Moreover, the authors restricted themselves to the analysis of the distribution of contact pressure. The numerical analyses of stresses acting in the surface layer require additional independent mathematical investigations.

Parallel with the application of analytic methods for the solution of partial differential equations, inhomogeneous layers are also modeled by using an approach according to which the layer is replaced with a package of homogeneous or inhomogeneous layers. For the first time, this approach was proposed at the end of the 1960s [7]. However, the computer equipment available at that time significantly restricted the practical significance of

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the proposed algorithm. The development of contemporary computers and tremendous increase in their computation power drastically changed the existing situation. Thus, in [19–21], the plane and axially symmetric contact problems of pressing of an absolutely rigid die into a homogeneous elastic half space covered with an inhomogeneous layer are studied by replacing this layer with a package of layers. The Poisson's ratio of each layer in the package is constant, whereas the shear modulus varies according to a linear law. At the same time, the problems of heat conduction [23, 24] and the contact problems of the theory of elasticity [25–28] were studied for a half space (layer) formed by two periodically located homogeneous isotropic layers. It is reasonable to apply the method developed in [25–27] to the solution of contact problems for the elastic half space whose mechanical properties are described above. The realization of the proposed approach is simpler as compared with the approach presented in [19–21]. Moreover, our approach is more universal and applicable to a broad class of contact problems of the theory of elasticity and thermoelasticity. As a possible drawback of this approach, we can mention the fact that, for the surface layer whose Poisson's ratio is constant, the method presented in [25–27] requires the analysis of a larger number of layers in the package. However, the numerical analyses carried out in [25–28] enable one to conclude that the results obtained for more than 40 layers become stable and, hence, the corresponding computations can be realized on a medium-class personal computer.

In what follows, we study the axially symmetric contact problem of pressing of an absolutely rigid ball into an inhomogeneous half space formed by a homogeneous half space and a surface layer whose Poisson's ratio is constant and Young's modulus depends on the coordinate z as a natural exponential function.

We consider two approaches to the solution of this problem:

(i) the variation of Young's modulus across the thickness of the surface layer is regarded as continuous;

(ii) the inhomogeneous layer is modeled by a package of homogeneous isotropic layers.

The difference between the obtained solutions is analyzed. We also study the dependence of the contact parameters and stresses acting in the inhomogeneous half space on the thickness of the layer and the ratio of the Young's modulus of the surface of inhomogeneous half space to the Young's modulus of the base.

Statement of the Problem

Consider the problem of contact interaction of an absolutely rigid ball with an inhomogeneous half space (Fig. 1).

The inhomogeneous half space is formed by an isotropic homogeneous half space with Young's modulus E_0 and Poisson's ratio μ and an inhomogeneous layer whose Poisson's ratio is constant and equal to μ. The dependence of Young's modulus on the dimensionless coordinate z' related to the radius of the contact zone a is described by the function

$$E(z') = E_0 \exp(\beta z'), \quad \beta = \frac{1}{h} \ln\left(\frac{E_1}{E_0}\right), \quad 0 \le z' \le h,$$

where E_1 is Young's modulus on the surface of the inhomogeneous half space, $h = H/a$, H is the thickness of the surface layer, r, φ, and z' are dimensionless cylindrical coordinates, $z = h - z'$, $r = \breve{r}/a$, $z' = \breve{z}'/a$, and $\breve{r}$, $\breve{z}'$ are dimensional coordinates.

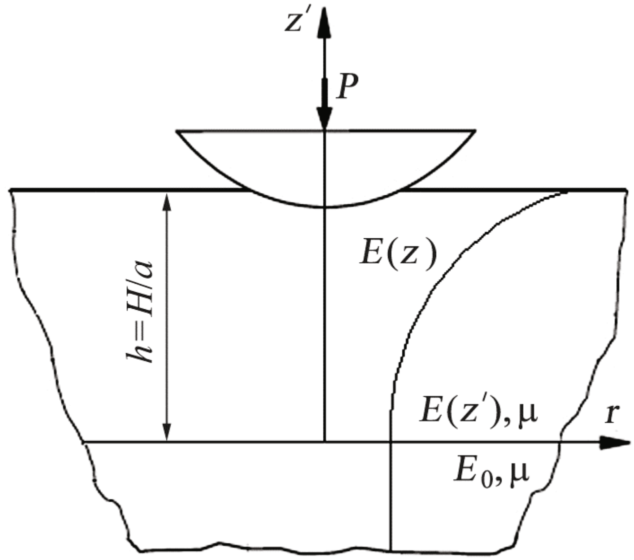

Fig. 1. Schematic diagram of the contact problem for a half space covered with an inhomogeneous layer.

Assume that the radius of the ball is much larger than the radius of the contact zone. This enables us to approximate the surface of the ball in the contact zone by the surface of a circular paraboloid [29]:

$$z(r, \varphi) = -\frac{ar^2}{2R}, \qquad r < 1, \qquad -\pi < \varphi \le \pi,$$

where R is the radius of the ball. The effect of tangential stresses is neglected.

The Case of Continuous Variation of the Mechanical Properties

In the space of Hankel transforms, the general solution of equations of the theory of elasticity in displacements for the description of the indicated behavior of the mechanical properties takes the form [30, 31]

$$\bar{u}_r^{(1)}(s, z') = -d_1 \sum_{i=1}^{4} \frac{m_i^2 - s^2 - \beta^2}{\beta^2 s} a_i(s) \exp(m_i z'), \quad 0 \le z' \le h,$$

$$\tilde{u}_z^{(1)}(s, z') = d_1 \sum_{i=1}^{4} \frac{m_i^2 - s^2 - \iota^2 \beta^2}{\beta^2 m_i} a_i(s) \exp(m_i z'), \quad 0 \le z' \le h,$$

$$2s\bar{u}_r^{(0)}(s, z') = -\big((2 + d_0) a_{-1}(s) + d_0 s z' a_{-1}(s) + 2a_0(s) s\big) \exp(sz'), \quad z' \le 0, \tag{1}$$

$$2\tilde{u}_z^{(0)}(s, z') = \big(d_0 z' a_{-1}(s) + 2a_0(s)\big) \exp(sz'), \quad z' \le 0, \tag{2}$$

where $\mathbf{u}^{(j)}$ is the dimensionless vector of elastic displacements related to the radius of the contact zone, $\bar{u}_r^{(j)}$ and $\tilde{u}_z^{(j)}$ are, respectively, the first- and zero-order Hankel transforms, the superscripts $j = 0$ and $j = 1$ correspond to the parameters and functions of state in the homogeneous half space and in the inhomogeneous layer, respectively, $a_i(s)$, $i = -1, \ldots, 4$, are unknown functions of the parameter $s \in [0, \infty)$ of the integral transformation,

$$\iota^2 = \frac{\mu}{1-\mu}, \qquad d_1 = \frac{1-\mu}{1-2\mu}, \qquad d_0 = \frac{1}{1-2\mu},$$

and m_i, $i = 1, \ldots, 4$ are the roots of the characteristic equation [32]

$$(m^2 - s^2)^2 + 2m\beta(m^2 - s^2) + \beta^2(m^2 + \iota^2 s^2) = 0\,.$$

To determine the unknown functions $a_i(s)$, $i = -1, \ldots, 4$, we use the boundary conditions

$$\sigma_{z'z'}^{(1)}(r,h) = -p(r)H(1-r), \qquad \sigma_{rz'}^{(1)}(r,h) = 0\,, \qquad u_r^{(0)}(r,0) = u_r^{(1)}(r,0)\,,$$

$$u_{z'}^{(0)}(r,0) = u_{z'}^{(1)}(r,0), \qquad \sigma_{rz'}^{(0)}(r,0) = \sigma_{rz'}^{(1)}(r,0), \qquad \sigma_{z'z'}^{(0)}(r,0) = \sigma_{z'z'}^{(1)}(r,0)\,,$$

where $\sigma^{(j)}$ are the stress tensors in the homogeneous half space $(j = 0)$ and in the inhomogeneous layer $(j = 1)$, $p(r)$ is the unknown contact pressure, and $H(r)$ is the Heaviside function.

We determine the unknown radius of the contact zone and the distribution of contact pressure by satisfying the boundary condition of existence of the common contact surface

$$u_{z'}^{(1)}(r,h) = \frac{ar^2}{2R} - U_0, \qquad r < 1\,,$$

and the balance condition for the ball

$$2\pi a^2 \int_0^1 rp(r)dr = P\,, \tag{3}$$

where U_0 is a dimensionless vertical displacement of the ball related to the radius of the contact zone and P is a pressing force.

Modeling of the Surface Layer by a Package of Layers

We split the surface layer into n layers of the same thickness $h' = h/n$ (Fig. 2).

It is assumed that all layers are homogeneous. Their mechanical properties are described by their Young's moduli and Poisson's ratios:

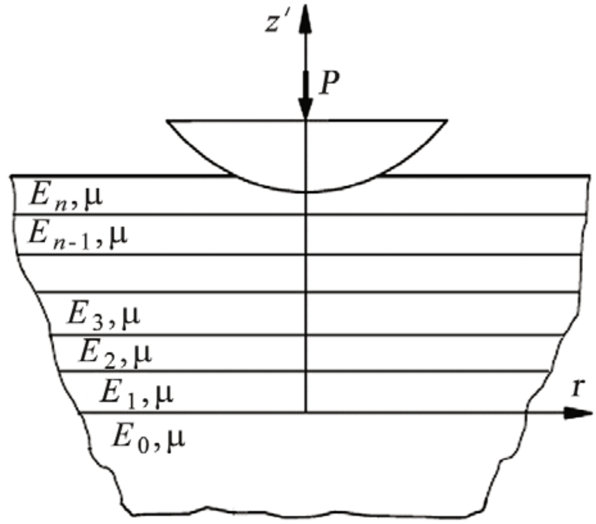

Fig. 2. Schematic diagram of the contact problem for a half space covered with a system of layers.

$$E_j = \frac{1}{h'} \int_{(j-1)h'}^{jh'} E(z')\,dz' = \frac{2E_0 \exp((j-0.5)\beta h')\sinh(0.5\beta h')}{\beta h'},$$

$$\mu_j = \frac{1}{h'} \int_{(j-1)h'}^{jh'} \mu\,dz' = \mu, \quad j = 1, 2, \ldots, n,$$

where the value of the parameter j corresponds to the number of a layer in the package. The layers are numbered upward from the bottom layer (operating in direct contact with the elastic homogeneous half space) (Fig. 2).

In the space of Hankel transforms, the general solution of equations of the theory of elasticity in layers of the package takes the form [27]:

$$\begin{aligned} 2s\bar{u}_r^{(j)}(s,z') = {} & \left\{(2+d_j)\sinh(s(h_j-z')) + d_j s(h_j-z')\cosh(s(h_j-z'))\right\} a_{4j-3}(s) \\ & + \left\{(2+d_j)\cosh(s(h_j-z')) + d_j s(h_j-z')\sinh(s(h_j-z'))\right\} a_{4j-2}(s) \\ & + 2s\cosh(s(h_j-z'))a_{4j-1}(s) + 2s\sinh(s(h_j-z'))\,a_{4j}(s), \qquad j = 1, \ldots, n, \end{aligned} \tag{4}$$

$$\begin{aligned} 2\tilde{u}_z^{(j)}(s,z') = {} & d_j(h_j-z')\sinh(s(h_j-z'))a_{4j-3}(s) \\ & + d_j(h_j-z')\cosh(s(h_j-z'))\,a_{4j-2}(s) + 2\sinh(s(h_j-z'))a_{4j-1}(s) \\ & + 2\cosh(s(h_j-z'))a_{4j}(s), \qquad j = 1, \ldots, n, \end{aligned} \tag{5}$$

where $d_j = 1/(1-2\mu_j)$, $h_j = jh'$, and $\mu_j = \mu$, $j = 0,1,\ldots,n$.

The general solution of equations of the theory of elasticity in the homogeneous half space $z' \le 0$ is given by relations (1) and (2). Relations (1), (2), (4), and (5) include $4n+2$ unknown functions of the parameter of the integral transformation $a_i(s)$, $i = -1,0,\ldots,4n$. For their determination, it is necessary to satisfy the boundary conditions

$$\sigma_{z'z'}^{(n)}(r,h) = -p(r)H(1-r), \quad \sigma_{rz'}^{(n)}(r,h) = 0\,, \tag{6}$$

$$u_r^{(i-1)}(r,h_{i-1}) = u_r^{(i)}(r,h_{i-1}), \quad u_{z'}^{(i-1)}(r,h_{i-1}) = u_{z'}^{(i)}(r,h_{i-1}), \quad i = 1,\ldots,n\,, \tag{7}$$

$$\sigma_{rz'}^{(i-1)}(r,h_{i-1}) = \sigma_{rz'}^{(i)}(r,h_{i-1}), \quad \sigma_{z'z'}^{(i-1)}(r,h_{i-1}) = \sigma_{z'z'}^{(i)}(r,h_{i-1}), \quad i = 1,\ldots,n\,. \tag{8}$$

The unknown radius of the contact zone and the distribution of contact pressure are determined from the boundary condition of existence of the common contact surface

$$u_{z'}^{(n)}(r,h) = \frac{ar^2}{2R} - U_0, \quad r < 1\,, \tag{9}$$

and the balance condition (3) for the ball.

The algorithm used for the solution of the axially symmetric contact problem for a layered half space is described in detail in [27]. For this reason, we now present only its brief description. By using relations (1), (2), (4), and (5), we determine the Hankel transforms of the components of stress tensor $\sigma_{rz'}$ and $\sigma_{z'z'}$. Satisfying the boundary conditions (6)–(8), we arrive at a system of $4n+2$ linear algebraic equations for the unknown functions $a_i(s)$, $i = -1,0,\ldots,4n$. Since only one equation in this system is inhomogeneous and its right-hand side is equal to $G_n^{-1}\tilde{p}(s)$ (G_n is the shear modulus of the upper layer in the package and $\tilde{p}(s)$ is the Hankel transform of contact pressure), the solution of the system of equations takes the form

$$a_i(s) = G_n^{-1}\tilde{p}(s)a_i^*(s)\,, \tag{10}$$

where $a_i^*(s)$ is the solution of the system of equations for the functions $a_i(s)$ on the right-hand side of which the function $G_n^{-1}\tilde{p}(s)$ is replaced by a constant equal to 1. Returning to the space of originals in relations (4) and (5) and using formula (10), we arrive at the integral relations for the components of the vector of elastic displacements and stress tensor, on the one hand, and the level of contact pressure, on the other hand.

Substituting the relation

$$u_{z'}^{(n)}(r,h) = G_n^{-1}\int_0^\infty s a_{4n}^*(s)\tilde{p}(s)J_0(sr)\,ds$$

in the differentiated boundary condition (9), we get an integral equation. Together with the balance condition (3), this equation is used for the determination of the unknown contact pressure and the radius of the contact zone. It is solved by the collocation method. In this case, the contact pressure is approximated by the formula

$$p(r) = \sum_{k=1}^{m} p_k \sqrt{l_k^2 - r^2}\, H(l_k - r)\,,$$

where l_k, $k = 1,\ldots,m$, are points from the interval $(0,1]$ chosen in exactly the same way as in [27, 31]. The collocation points are chosen in the form $r_k = (l_{k-1} + l_k)/2$, $k = 1,\ldots,m$, $l_0 = 0$.

To take the integrals encountered in reducing the integral equation to a system of linear algebraic equations, we use the asymptotic behavior of the function $\overset{*}{a}_{4n}(s)$ as $s \to \infty$. The integrals in which this function is replaced by its asymptotics are taken in the analytic form. The remaining integrals are computed by using the Gaussian quadrature.

In a similar way, we take the integrals in the expressions for the stresses σ_{rr} and $\sigma_{\varphi\varphi}$ acting on the surface of the inhomogeneous half space. The stresses inside the inhomogeneous half space are found with the help of the Gaussian quadratures.

The algorithm used for the solution of the contact problem for a medium with continuous variation of the mechanical properties is similar to the algorithm described above. The only difference is that the formulas used to find the components of the vector of elastic displacements in the surface layer are different.

Analysis of the Results

The analysis of the basic relations presented in [27] enables us to conclude that the solution of the posed contact problem of modeling of the inhomogeneous layer by a package of layers depends on four dimensionless parameters: the thickness of the layer h, the ratio of the Young modulus of the surface of the inhomogeneous half space to the Young modulus of the base E_1/E_0, Poisson's ratio μ, and the number of layers in the package n. The solution of the contact problem for the inhomogeneous layer obtained with regard for the continuous dependence of the mechanical properties on the coordinate depends on the first three indicated parameters. To compare the solutions obtained in these cases, we use the following dimensionless quantities: the ratio of the radius of the contact zone for the problem with inhomogeneous layer to the radius of the contact zone for the problem without surface layer a/a_H $\{a_H^3 = 3PR(1-\mu^2)/(4E_0)$ [29]$\}$, the ratio of the level of contact pressure at the center of the contact zone to the mean contact pressure $p(0)/p_0$, and the ratio of the level of radial stresses to the mean contact pressure σ_{rr}/p_0 computed at the center of the contact area $(r = 0,\ z' = h)$ and on its boundary $(r = 1,\ z' = h)$. It is assumed that $\mu = 1/3$, $E_1/E_0 = 0.125$ or 8, $h = 0.2, 0.4$, or 0.8.

The quantities computed for the contact problem with continuous variation of mechanical properties in the surface layer are presented in the row of Table 1 with $n = \infty$. In the same row, we also present the admissible relative errors of calculations. Since the tensile stresses $\sigma_{rr}(1,h)$ are several times lower than the compressive stresses $\sigma_{rr}(0,h)$, the error made in finding the dimensionless parameter $\sigma_{rr}(1,h)/p_0$ is equal to 0.5%. The relative error of finding these quantities (determined by modeling the surface layer by a package of n homogeneous layers) is presented in the rows with $n = 10, 20$, and 40. It is easy to see that, even for ten layers in the package, the parameter $p(0)/p_0$ is computed with an accuracy that does not exceed 0.26%. As the number of layers becomes twice larger, the error becomes almost twice lower. For 40 layers in the package and $E_1/E_0 \le 8$, the errors of finding the radius of the contact zone and the radial stresses at its center do not exceed 1 and 2%, respectively. Somewhat higher relative errors are attained in computing the radial stresses on the boundary of the contact zone. Clearly, this is explained by the fact that the level of tensile stresses is, in this case, several times lower than the level of compressive stresses in the central part of the zone.

Table 1. Dependences of the Radius of Contact Zone, Contact Pressure at Its Center, Radial Stresses in the Central Part of the Zone and on Its Boundary, and the Errors of Their Evaluation Obtained by Modeling the Inhomogeneous Layer by a Package of n Layers on the Parameters E_1/E_0 and h

E_1/E_0	h	n	a/a_H	$p(0)/p_0$	$\sigma_{rr}(0,h)/p_0$	$\sigma_{rr}(1,h)/p_0$
0.125	0.2	10	3.51%	< 0.25%	0.79%	9.09%
		20	1.77%	< 0.25%	0.44%	4.33%
		40	0.9%	< 0.25%	0.28%	2.04%
		∞	1.187 ± 0.25%	1.734 ± 0.25%	–0.948 ± 0.25%	0.0506 ± 0.5%
	0.4	10	3.32%	< 0.25%	0.59%	9.11%
		20	1.58%	< 0.25%	< 0.25%	4.63%
		40	0.72%	< 0.25%	< 0.25%	2.44%
		∞	1.304 ± 0.25%	1.764 ± 0.25%	–0.944 ± 0.25%	0.0706 ± 0.5%
	0.8	10	3.3%	< 0.25%	0.48%	8.44%
		20	1.63%	< 0.25%	< 0.25%	4.34%
		40	0.78%	< 0.25%	< 0.25%	2.7%
		∞	1.468 ± 0.25%	1.725 ± 0.25%	–0.915 ± 0.25%	0.1185 ± 0.5%
8	0.2	10	–3.36%	< 0.25%	–7.35%	–2.14%
		20	–1.74%	< 0.25%	–3.89%	–1.9%
		40	–0.91%	< 0.25%	–1.91%	–1.6%
		∞	0.903 ± 0.25%	1.302 ± 0.25%	–2.916 ± 0.25%	0.1683 ± 0.5%
	0.4	10	–3.21%	0.252%	–7.21%	17.53%
		20	–1.60%	< 0.25%	–3.78%	6.49%
		40	–0.77%	< 0.25%	–1.93%	2.08%
		∞	0.840 ± 0.25%	1.192 ± 0.25%	–2.817 ± 0.25%	0.0422 ± 0.5%
	0.8	10	–3.25%	< 0.25%	–7%	–17.39%
		20	–1.65%	< 0.25%	–3.72%	–7.95%
		40	–0.83%	< 0.25%	–1.91%	–2.77%
		∞	0.750 ± 0.25%	1.172 ± 0.25%	–2.674 ± 0.25%	–0.1334 ± 0.5%

The results of calculations presented in Figs. 3–6 also reveal good agreement between the solutions obtained by using the analyzed two models of the surface layer. In this figures, the rhombi correspond to the solution of the problem with continuous variation of the mechanical properties.

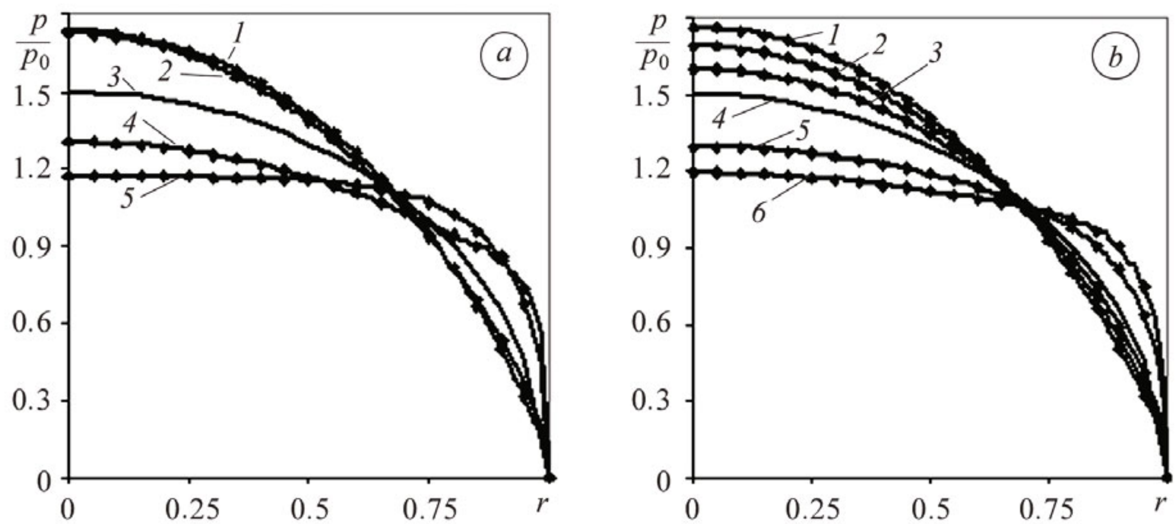

Fig. 3. Dependences of the distribution of contact pressure on the parameters E_1/E_0 and h ($\mu = 1/3$, $n = 20$); (a): (1) $E_1/E_0 = 0.125$, $h = 0.2$; (2) $E_1/E_0 = 0.125$, $h = 0.8$; (3) homogeneous half space; (4) $E_1/E_0 = 8$, $h = 0.2$; (5) $E_1/E_0 = 8$, $h = 0.8$; (b) ($h = 0.4$): (1) $E_1/E_0 = 0.125$, (2) 0.25, (3) 0.5, (4) homogeneous half space, (5) $E_1/E_0 = 4$, (6) 8.

The continuous lines correspond to the results obtained for a package formed by 20 homogeneous layers. As follows from Figs. 4 and 5, the maximum absolute error of evaluation of stresses is observed in the central part of the contact zone.

We observe the same character of redistribution of contact pressure (Fig. 3) as in the case of a homogeneous half space covered with a homogeneous layer with different mechanical properties [33, 34]. If the Young modulus of the inhomogeneous layer increases in the direction of the surface of the inhomogeneous half space ($E_1/E_0 > 1$), then the distribution of contact pressure is leveled. If $E_1/E_0 < 1$, then the ratio of the maximum contact pressure to the mean pressure is larger than for the Hertz distribution obtained as the solution of the contact problem for a homogeneous half space.

As in the classical Hertz problem [29], the zone of tensile stresses is formed in the vicinity of points of the unloaded surface of the inhomogeneous half space (Fig. 4). If $E_1/E_0 < 1$, then the maximum tensile stresses are formed on the boundary of the contact zone. They are lower than for the Hertz problem. For $E_1/E_0 > 1$, the maximum tensile stresses are recorded on the unloaded surface of the inhomogeneous half space. The level of these stresses is much higher than for the Hertz problem.

The numerical results show that the highest value of the second invariant of the deviator of stress tensor

$$J_2 = \frac{\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}}{\sqrt{6}},$$

where σ_1, σ_2, and σ_3 are the principal stresses, is attained, as in the Hertz problem, on the z'-axis, where the following equality is true:

$$\tau_{\max} = \frac{\sqrt{3}J_2}{2} = \frac{\sigma_1 - \sigma_3}{3} = \frac{|\sigma_{rr} - \sigma_{zz}|}{2}.$$

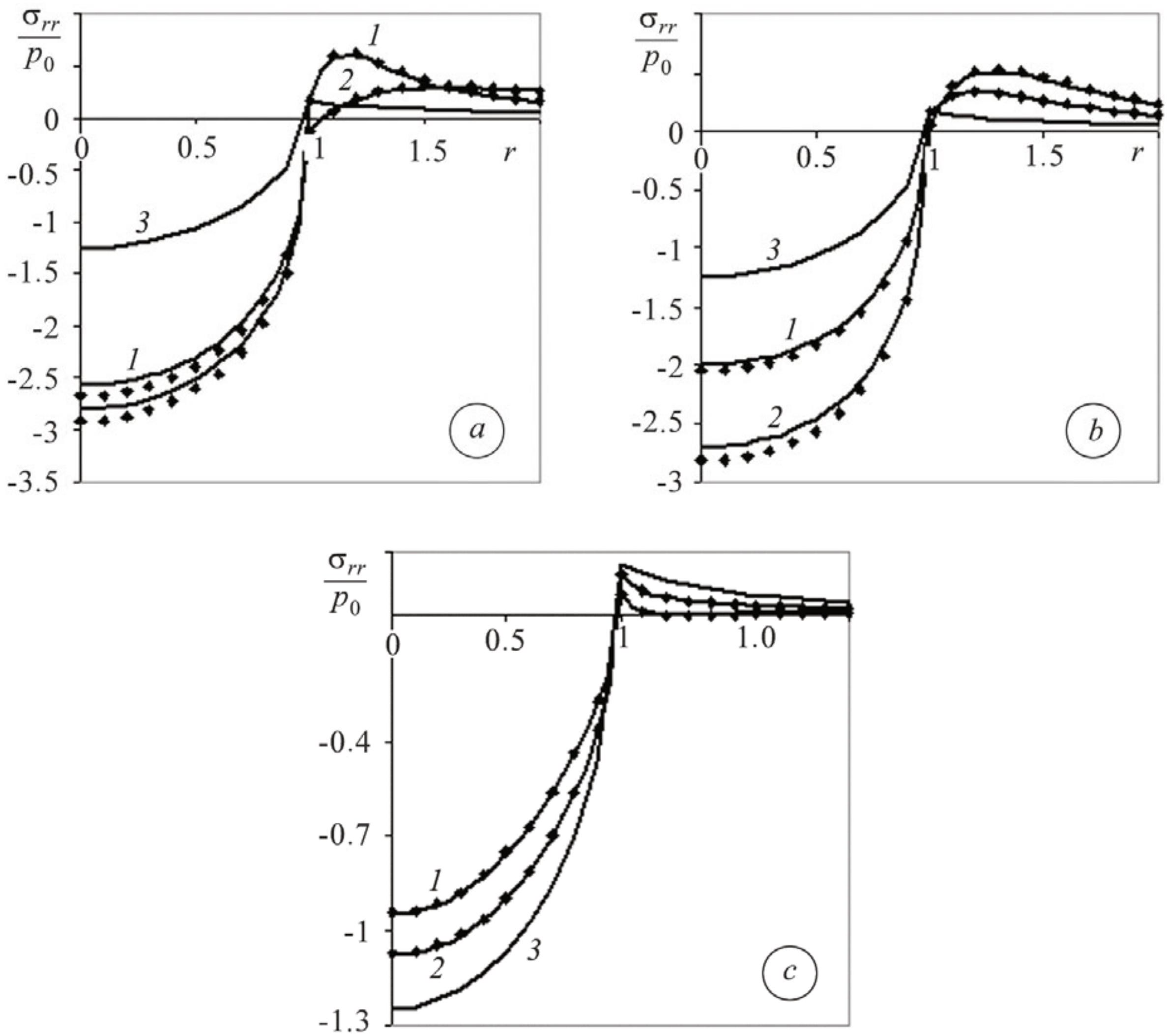

Fig. 4. Dependences of the radial stresses on the surface of the inhomogeneous half space on the parameters E_1/E_0 and h [$\mu = 1/3$, $n = 20$, curve (3) corresponds to the homogeneous half space]; (a) ($E_1/E_0 = 8$): (1) $h = 0.2$, (2) $h = 0.8$; (b) ($h = 0.4$): (1) $E_1/E_0 = 4$, (2) $E_1/E_0 = 8$; (c) ($h = 0.4$): (1) $E_1/E_0 = 0.125$, (2) $E_1/E_0 = 0.5$.

This means that the point of maximum of the parameter J_2 is, at the same time, the point of maximum of the maximum tangential stresses.

For $E_1/E_0 >> 1$, we can expect that the difference between the stresses σ_{rr} and σ_{zz} at the center of the contact zone should be significant. As a result, the highest value of the parameter J_2 (or $\tau_{\max}$) is observed just in this region (Fig. 7a). As the ratio E_1/E_0 becomes lower (in what follows, we assume that $E_1/E_0 > 1$), the indicated difference decreases. The maximum value of the parameter J_2 (or $\tau_{\max}$) is attained in the inhomogeneous layer at an insignificant distance from the interface of the layer and the base (Fig. 7b). In the middle part of the inhomogeneous layer, we detect a well-pronounced minimum of these parameters, as described for the homogeneous surface layer in [35, 36]. For $E_1/E_0 < 1$, the highest values of the parameter J_2 in the inhomogeneous layer are attained (Figs. 7c, d) on the surface of the inhomogeneous half space $(h = 0.2)$, in the interface $(h = 0.4)$, and in the middle part of the inhomogeneous layer $(h = 0.8)$.

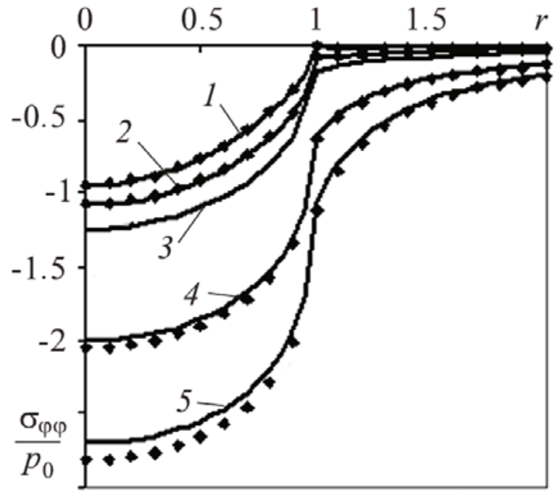

Fig. 5. Dependences of the hoop stresses acting on the surface of the inhomogeneous half space on the parameters E_1/E_0 and h ($\mu = 1/3$, $n = 20$, $h = 0.4$): (1) $E_1/E_0 = 0.125$, (2) 0.5, (3) homogeneous half space, (4) $E_1/E_0 = 4$, (5) 8.

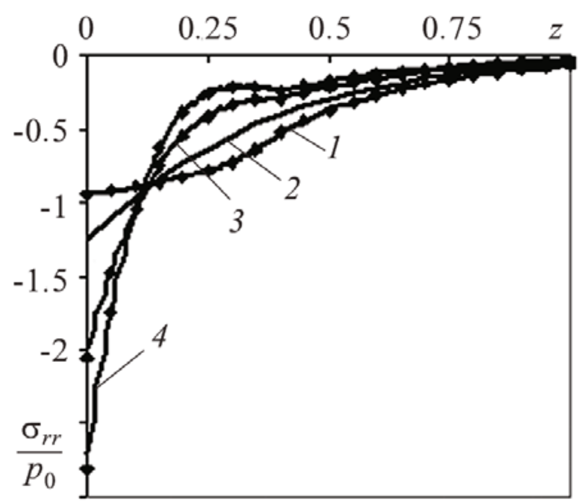

Fig. 6. Dependences of radial stresses formed on the z'-axis ($z = h - z'$) on the parameters E_1/E_0 and h ($\mu = 1/3$, $n = 20$, $h = 0.4$): (1) $E_1/E_0 = 0.125$, (2) homogeneous half space, (3) $E_1/E_0 = 4$, (4) (8).

It is worth noting that the results of calculations carried out for the inhomogeneous layer whose thickness is much larger than the radius of the contact zone are in good agreement with the results obtained in [22], where the authors restricted themselves to the analysis of the problem for $h = 5$.

CONCLUSIONS

It is shown that the solution of the contact problem for a package of 20–40 homogeneous layers is in good agreement with the solution of the problem for the surface layer whose Poisson's ratio is constant and the dependence of Young's modulus on the coordinate z' is described by a natural exponential function.

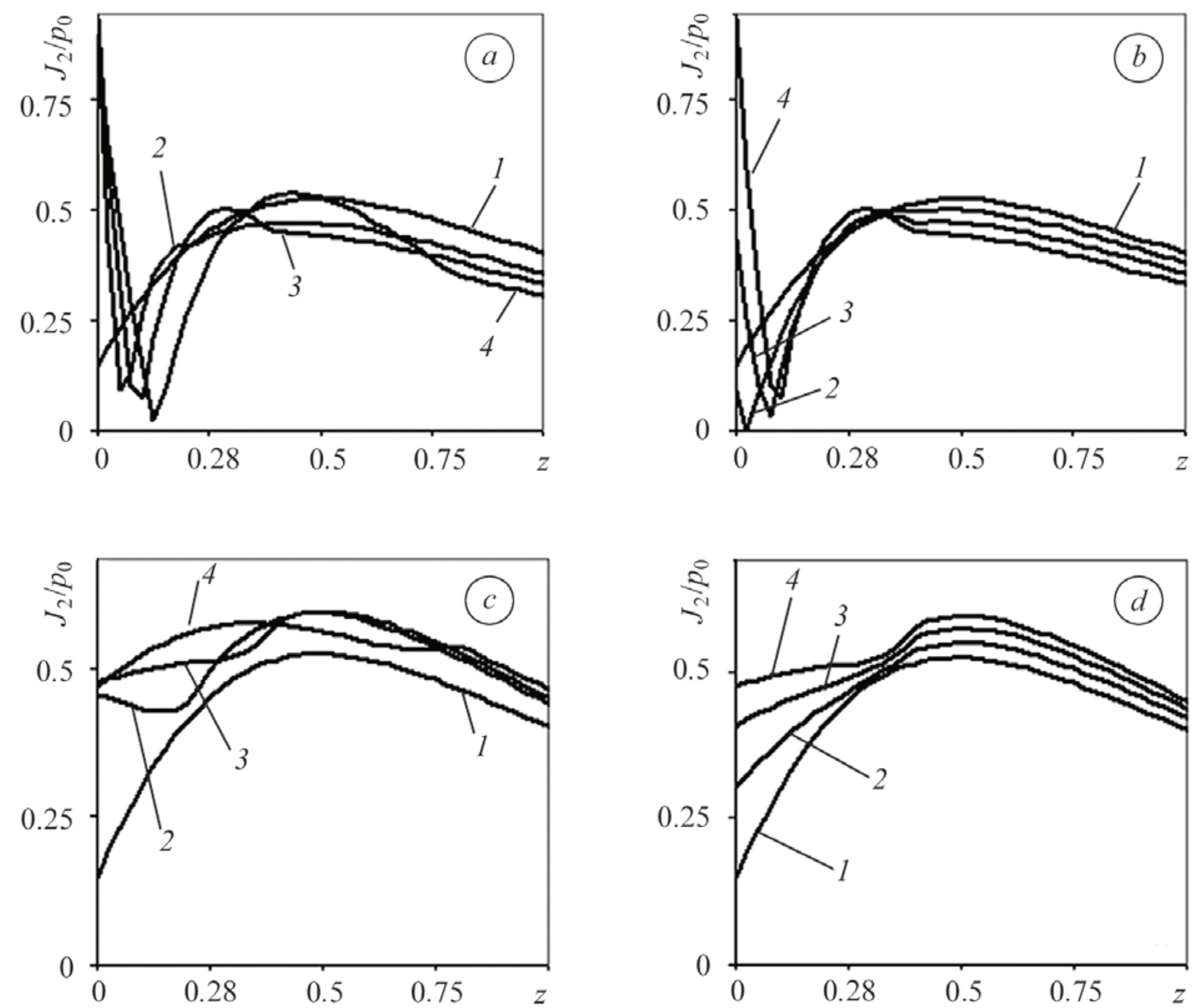

Fig. 7. Dependences of the distributions of the parameter J_2 along the z'-axis $(z = h - z')$ on the parameters E_1/E_0 and h ($\mu = 1/3$; curves 1 correspond to the homogeneous half space); (a) ($E_1/E_0 = 8$): (2) $h = 0.2$, (3) 0.4, (4) 0.8; (b) ($h = 0.4$): (2) $E_1/E_0 = 2$, (3) 4, (4) 8; (c) ($E_1/E_0 = 0.125$): (2) $h = 0.2$, (3) 0.4, (4) 0.8; (d) ($h = 0.4$): (2) $E_1/E_0 = 0.5$, (3) 0.25, (4) 0.125.

This is a strong argument for the possibility of modeling of the surface layer with continuous variation of mechanical properties by a package of homogeneous layers.

The character of the redistribution of contact pressure and the second invariant of the deviator of stress tensor caused by the application of the surface layer with the above-mentioned mechanical properties is similar to the redistribution of these parameters for the homogeneous half space covered with a homogeneous layer with different mechanical properties [33–36]. In the analyzed problem, unlike the contact problem discussed in [33–36], the radial and hoop stresses formed on the interface of the layer and the base are always compressive.

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