



# A non-local fractional stress–strain gradient theory

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Received: 6 March 2019 / Accepted: 10 July 2019 / Published online: 18 July 2019  
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**Abstract** A generalized non-local stress–strain gradient theory is presented using fractional calculus. The proposed theory includes as a special case: the classical theory; the non-local strain gradient theory; the Eringen non-local theory; the strain gradient theory; the general Eringen non-local theory; and the general strain gradient theory. This new formulation is therefore more comprehensive and more complete to model physical phenomena. Its application has been shown in free vibration, buckling and bending of simply supported (S–S) nano-beams. The non-linear governing equations have been solved by the Galerkin method. Furthermore the effects of different (additional) model parameters like: the length scale parameter; the non-local parameter; and different orders (integer and non-integer) of strain and stress gradients have been shown.

**Keywords** Non-local stress–strain gradient theory · Conformable derivative · Vibration · Bending · Buckling

## 1 Introduction

Many experimental results (Wong et al. 1997; Jing et al. 2006; Agrawal et al. 2008; Li et al. 2003; Sadeghian et al. 2009) and atomistic simulations (Zhu et al. 2006; Diao et al. 2006; Cao and Chen 2007; Olsson et al. 2007) have shown significant small-scale effects in nano-structures such as nano-rods, nano-beams and nano-plates. Forasmuch, as the theoretical continuum models are more cost effective than experimental and atomistic methods, therefore several non-classical continuum theories involving additional material length scale parameters have been developed.

Among these theories [the non-local elasticity theory (Eringen 1972, 1983), the strain gradient theory (Mindlin 1964, 1965; Lazopoulos 2009; Rahimi et al. 2018a), the non-local strain gradient theory (Lim et al. 2015), the consistent couple stress theory (Hadjesfandiari and Dargush 2015), the modified strain gradient theories (Liebold and Müller 2015)] the non-local strain gradient one, combines both of the classical non-local elasticity and the strain gradient theories and results in a higher-order non-local strain gradient theory. The non-local elasticity and the strain gradient theories are devoted to different aspects of materials

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and structures at small scale. The non-local elasticity theory does not include the non-locality of higher-order stresses while the strain gradient theory considers local higher-order strain gradients without non-local effects. Note that in the aforementioned theories, the integer gradients of stresses and strains exist, and therefore herein the important question is raised “*what effect will result if the non-integer gradients are considered?*”. This states the central point of this paper, namely to present that the non-integer gradients of stresses and strains make the modeling more flexible to mimic experimentally observed physical phenomena.

Fractional calculus is a branch of mathematical analysis, related to real or complex numbers, dealing with differential and integral operators of non-integer orders. Fractional derivatives have played a significant role in engineering in recent years (Ray et al. 2014; da Graça Marcos et al. 2008; Sapora et al. 2017; Faraji Oskouie et al. 2018; Yang 2012; Hilfer 2000). In many theoretical investigations, it has been extended to solid mechanics and has been shown that many physical systems can be represented more accurately using fractional operators. Challamel et al. (2013) introduced a general form of the Eringen non-local elasticity theory by using the Caputo fractional definition and also presented an optimized fractional derivative model. Moreover, they concluded that the model showed a perfect matching with the dispersive wave properties of the Born–Kármán model of lattice dynamics and was better than the Eringen non-local elasticity theory. Tarasov and Aifantis (2014) studied some fractional gradient elasticity models using the Caputo and the Riesz fractional derivatives definitions. Moreover, they proposed a new extended elasticity model which can describe elasticity of materials with fractional non-locality, memory and fractality (Tarasov and Aifantis 2015). Malara and Spanos (2017) discussed an approximated method to determine the non-linear response of a plate based on the fractional calculus and the results has been assessed versus Monte Carlo data. Sumelka et al. (2015) presented the space-fractional non-local Euler–Bernoulli beam theory. They showed that the theory provides better approximation for the experimental Young’s modulus values. Rahimi et al. (2017a) presented a non-integer non-local model using conformable derivative definition and investigated its application to static instability of nano-beams under

electrostatic force. As in previously mentioned papers they showed that the non-integer model is in a better agreement to the experimental data than the classical Eringen non-local theory and classical (local) theory. Failla et al. (2013) presented a two-dimensional foundation model using a mechanically based non-local elasticity theory in form of fractional calculus. Carpinteri et al. (2014) investigated a spatial fractional model for materials whose non-local stress is defined as fractional integral of the strain field. D’Elia and Gunzburger (2013) analyzed a non-local diffusion operator having as special cases the fractional Laplacian and fractional differential operators that arise in several applications. More recently, Evgrafov and Bellido studied the case when the Eringen non-local theory is in general ill-posed in the case of smooth kernels; moreover, they have also considered the case of singular, non-smooth kernels.

In this paper, a non-local fractional stress–strain gradient theory has been investigated, using conformable derivative. This formulation includes two new free parameters, namely the fractional parameters which control the stresses and the strains gradients orders in the constitutive relation, respectively. As an illustrative example, free vibration, bending and buckling of S–S nano-beam have been studied and the meaning of the fractional parameters beside the non-local and the length scale parameters has been shown. Note that the values of the fractional parameters considered were  $1 < \alpha \leq 2$ . It should be pointed out that the presented conformal non-local model includes the classical theory, the non-local strain gradient theory (Lim et al. 2015), the Eringen non-local theory (Eringen 1972, 1983), the strain gradient theory (Mindlin 1964, 1965; Lazopoulos 2009; Rahimi et al. 2018a), the general Eringen non-local theory (Rahimi et al. 2017a) and the general strain gradient theory therefore it combines the flexibility and power of the previous ones.

## 2 Mathematical construction of the non-local fractional stress–strain gradient theory

According to the stress and strain gradient theories the constitutive equation for the beam is (Wong et al. 1997):

$$\left[1 - (e_0 a)^2 \nabla^2\right] t_{xx} = E(1 - l^2 \nabla^2) \varepsilon_{xx} \quad (1)$$

where  $t_{xx}$  is the stress field that contains not only the non-local elastic stress field, but also the strain gradient field,  $\varepsilon_{xx}$  is the strain,  $e_0a$  denote the non-local parameter (where  $e_0a$  is a material constant, and  $a$  is the internal characteristic lengths (e.g. lattice parameter, granular size)),  $E$  is the Young’s modulus and  $l$  is the material length scale parameter.

Now, let us consider the general form of Eq. (1) using the non-integer order calculus as bellow:

$$[1 - (e_0a)^{\alpha_1} \nabla^{\alpha_1}] t_{xx} = E(1 - l^{\alpha_2} \nabla^{\alpha_2}) \varepsilon_{xx}$$

$$\nabla^{\alpha_i} = \frac{\partial^{\alpha_i}}{\partial x^{\alpha_i}} \quad i = 1, 2 \tag{2}$$

where  $\alpha_1$  and  $\alpha_2$  are the fractional parameters and they control the gradient orders in the constitutive relation and can take integer or non-integer values. In should be emphasized that in the case of  $\alpha_1 = 2$  Eq. (2) reduces to the classical strain gradient theory, and in the case of  $l = 0$  it takes the form of the general Eringen theory (Jing et al. 2006; Agrawal et al. 2008):

$$[1 - (e_0a)^{\alpha_1} \nabla^{\alpha_1}] t_{xx} = E \varepsilon_{xx} \tag{3}$$

Note that as it is shown in Fig. 1, the general Eringen theory (Eq. 3) is a subset of non-local fractional stress–strain gradient theory (Eq. 2).

All the relations between the developed fractional stress–strain gradient theory and other theories have been shown in Fig. 1. As it can be seen, six theories (the classical theory, the Eringen non-local theory, the strain gradient theory, the non-local strain gradient theory, the general Eringen non-local theory, and the general strain gradient theory) can be obtained from the proposed formulation, which is the property that makes the overall formulation to be a powerful phenomenological model of physical phenomena.

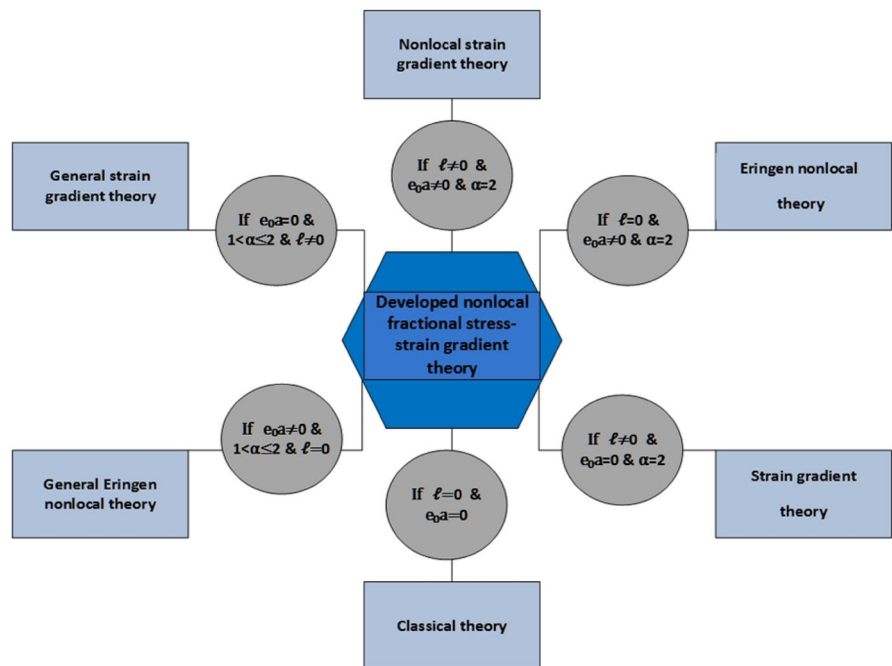
### 3 Mathematical modeling

In this section the application of the developed theory in the nano-scale is presented.

#### 3.1 Conformable derivative definition

Let  $f, g: [0, \infty) \rightarrow \mathbb{R}$  and  $x, y > 0$  then the conformable derivative definition is (Rahimi et al. 2017a):

**Fig. 1** The summary of relations between the developed non-local fractional stress–strain gradient theory and other theories



$$\begin{aligned}
 f_x^\alpha(x, y) &= \frac{d^\alpha f(x, y)}{dx^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f^{[\alpha]-1}(x + \varepsilon x^{[\alpha]-\alpha}, y) - f^{[\alpha]-1}(x, y)}{\varepsilon} \\
 f_y^\alpha(x, y) &= \frac{d^\alpha f(x, y)}{dy^\alpha} = \lim_{\varepsilon \rightarrow 0} \frac{f^{[\alpha]-1}(x, y + \varepsilon y^{[\alpha]-\alpha}) - f^{[\alpha]-1}(x, y)}{\varepsilon}
 \end{aligned}
 \tag{4a}$$

where  $\alpha \in (n, n + 1]$ ,  $f$  is  $(n + 1)$ -differentiable at  $x > 0$  and  $[\alpha]$  is the smallest integer greater than or equal to  $\alpha$ . In the case of  $\alpha = n$  it reduces to classic form:

$$\begin{aligned}
 f_x^n(x, y) &= \frac{d^n f(x, y)}{dx^n} = \lim_{\varepsilon \rightarrow 0} \frac{f^{n-1}(x + \varepsilon, y) - f^{n-1}(x, y)}{\varepsilon} \\
 f_y^n(x, y) &= \frac{d^n f(x, y)}{dy^n} = \lim_{\varepsilon \rightarrow 0} \frac{f^{n-1}(x, y + \varepsilon) - f^{n-1}(x, y)}{\varepsilon}
 \end{aligned}
 \tag{4b}$$

This definition makes the modeling more flexible than classical derivative as one can use both integer and non-integer derivatives order. Herein, it gives us the possibility of studying the effects of non-integer strain and stress derivatives in the constitutive relation.

### 3.2 Mathematical modeling of motion of a nano-beam

We assume that the displacement fields of S–S nano-beam (of Euler–Bernoulli type) (cf. Figure 2) obtain the form as bellow:

$$\begin{aligned}
 u_1(x, z, t) &= -z \frac{\partial w(x, t)}{\partial x} \\
 u_2(x, z, t) &= 0 \\
 u_3(x, z, t) &= w(x, t),
 \end{aligned}
 \tag{5}$$

where  $u_1$ ,  $u_2$  and  $u_3$  are displacement in  $x$ ,  $y$  and  $z$  directions respectively,  $u$  and  $w$  are axial and transverse displacements of middle axis. Therefore the only non-zero linear strain is

$$\varepsilon_{xx} = -z \frac{\partial^2 w(x, t)}{\partial x^2}.
 \tag{6}$$

The Hamilton’s principle for the analyzed system is



**Fig. 2** A schematic view of a S–S nano beam

$$\int_{t_1}^{t_2} (\delta K - \delta U + \delta W) = 0,
 \tag{7}$$

where the virtual strain energy is defined as

$$\begin{aligned}
 \delta U &= \int_V (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{xx}^\dagger \nabla \delta \varepsilon_{xx}) dV = \left( \int_V (\sigma_{xx} - \nabla \sigma_{xx}^\dagger \delta \varepsilon_{xx}) dV \right) \\
 &+ \int_A (\sigma_{xx}^\dagger \delta \varepsilon_{xx}) dA \Big|_0^L = \left( \int_V (t_{xx} \delta \varepsilon_{xx}) dV \right) + \int_A (\sigma_{xx}^\dagger \delta \varepsilon_{xx}) dA \Big|_0^L
 \end{aligned}
 \tag{8}$$

where

$$\begin{aligned}
 \sigma_{xx} &= \int_0^L E \beta_2(x, x', e_2 a) \varepsilon'_{xx, x}(x') dx', \\
 \sigma_{xx}^\dagger &= I^2 \int_0^L E \beta_1(x, x', e_1 a) \varepsilon'_{xx, x}(x') dx', \quad e_2 a = e_1 a = e_0 a
 \end{aligned}
 \tag{9}$$

and  $\beta_1(x, x', e_1 a)$  and  $\beta_2(x, x', e_2 a)$  are functions for the classical stress tensor and the strain gradient stress tensor, respectively. The virtual kinetic energy has the form

$$\delta K = \int_0^L \rho A \frac{\partial w}{\partial t} \delta \left( \frac{\partial w}{\partial t} \right) dx,
 \tag{10}$$

and the virtual potential energy of external loads is expressed as

$$\delta W = - \int_0^L q \delta w dx.
 \tag{11}$$

Next, by substituting Eqs. (8), (10) and (11) in Eq. (7) and considering the bending moment  $M$  and the axial force  $N$  defined as

$$M = \int_A z t_{xx} dA, \quad N = \int_A t_{xx} dA,
 \tag{12}$$

where  $A$  is the section area of the nano-beam, we have

$$\frac{\partial^2 M}{\partial x^2} - N \frac{\partial^2 w}{\partial x^2} + \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - \rho A \frac{\partial^2 w}{\partial t^2} = q.
 \tag{13}$$

Taking now Eq. (2) and multiplying it by  $z dA$  and applying integration through the beam length from 0 to  $L$  leads to

$$[1 - (e_0 a)^{\alpha_1} \nabla^{\alpha_1}] M = -EI(1 - I^{\alpha_2} \nabla^{\alpha_2}) \frac{\partial^2 w}{\partial x^2}. \tag{14}$$

Finally, by using the conformable derivative definition and considering  $1 < \alpha_1, \alpha_2 \leq 2$ , Eq. (14) will be:

$$M - (e_0 a)^{\alpha_1} x^{2-\alpha_1} \frac{\partial^2 M}{\partial x^2} = EI \left( \frac{\partial^2 w}{\partial x^2} - I^{\alpha_2} x^{2-\alpha_2} \frac{\partial^4 w}{\partial x^4} \right). \tag{15}$$

In the last step, to obtain the general form of Euler–Bernoulli beam equation of motion one takes the second derivative of Eq. (15), the second derivative of  $M$  from Eq. (13), and put it to Eq. (15), hence

$$\begin{aligned} EI \left[ \frac{\partial^4 w}{\partial x^4} \right] - EI I^{\alpha_2} \left[ (2 - \alpha_2)(1 - \alpha_2) x^{-\alpha_2} x^{-\alpha_2} \frac{\partial^4 w}{\partial x^4} \right. \\ \left. + 2(2 - \alpha_2) x^{1-\alpha_2} \frac{\partial^5 w}{\partial x^5} + x^{2-\alpha_2} \frac{\partial^6 w}{\partial x^6} \right] \\ + N \left( \frac{\partial^2 w}{\partial x^2} \right) - N (e_0 a)^{\alpha_1} \left\{ (2 - \alpha_1)(1 - \alpha_1) x^{-\alpha_1} \frac{\partial^2 w}{\partial x^2} \right. \\ \left. + 2(2 - \alpha_1) x^{1-\alpha_1} \frac{\partial^3 w}{\partial x^3} + x^{2-\alpha_1} \frac{\partial^4 w}{\partial x^4} \right\} \\ + \left[ \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - \rho A \frac{\partial^2 w}{\partial t^2} \right] - (e_0 a)^{\alpha_1} \\ \left[ (2 - \alpha_1)(1 - \alpha_1) x^{-\alpha_1} \left( \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - \rho A \frac{\partial^2 w}{\partial t^2} \right) \right. \\ \left. + 2(2 - \alpha_1) x^{1-\alpha_1} \left( \rho I \frac{\partial^5 w}{\partial x^3 \partial t^2} - \rho A \frac{\partial^3 w}{\partial t^2 \partial x} \right) + x^{2-\alpha_1} \right. \\ \left. \left( \rho I \frac{\partial^6 w}{\partial x^4 \partial t^2} - \rho A \frac{\partial^4 w}{\partial t^2 \partial x^2} \right) \right] \\ = [1 + (e_0 a)^{\alpha_1} (2 - \alpha_1)(1 - \alpha_1) x^{-\alpha_1}] q \\ + (e_0 a)^{\alpha_1} 2(2 - \alpha_1) x^{1-\alpha_1} \frac{dq}{dx} + (e_0 a)^{\alpha_1} x^{2-\alpha_1} \frac{d^2 q}{dx^2}. \tag{16} \end{aligned}$$

For convenience, the following non-dimensional parameters are used:

$$\begin{aligned} \hat{w} = \frac{w}{L}, \quad \hat{x} = \frac{x}{L}, \quad \hat{t} = \frac{t}{t^*}, \quad t^* = \sqrt{\frac{\rho A L^4}{EI}}, \\ \tau = \frac{e_0 a}{L}, \quad \zeta = \frac{l}{L}, \tag{17} \end{aligned}$$

therefore, the non-dimensional form of Eq. (16) is

$$\begin{aligned} \left[ \left( \frac{\partial^4 \hat{w}}{\partial \hat{x}^4} \right) - \zeta^{\alpha_2} \left\{ (2 - \alpha_2)(1 - \alpha_2) \hat{x}^{-\alpha_2} \frac{\partial^4 \hat{w}}{\partial \hat{x}^4} \right. \right. \\ \left. \left. + 2(2 - \alpha_2) \hat{x}^{1-\alpha_2} \frac{\partial^5 \hat{w}}{\partial \hat{x}^5} + \hat{x}^{2-\alpha_2} \frac{\partial^6 \hat{w}}{\partial \hat{x}^6} \right\} \right] \\ + \kappa_1 \left[ \left( \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} \right) - \tau^{\alpha_1} \left\{ (2 - \alpha_1)(1 - \alpha_1) \hat{x}^{-\alpha_1} \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} \right. \right. \\ \left. \left. + 2(2 - \alpha_1) \hat{x}^{1-\alpha_1} \frac{\partial^3 \hat{w}}{\partial \hat{x}^3} + \hat{x}^{2-\alpha_1} \frac{\partial^4 \hat{w}}{\partial \hat{x}^4} \right\} \right] \\ + \kappa_2 \left[ \left\{ \frac{I}{L} \frac{\partial^4 \hat{w}}{\partial \hat{x}^2 \partial \hat{t}^2} - AL \frac{\partial^2 \hat{w}}{\partial \hat{t}^2} \right\} \right. \\ \left. - \tau^{\alpha_1} \left\{ (2 - \alpha_1)(1 - \alpha_1) \hat{x}^{-\alpha_1} \left( AL \frac{\partial^2 \hat{w}}{\partial \hat{t}^2} - \frac{I}{L} \frac{\partial^4 \hat{w}}{\partial \hat{x}^2 \partial \hat{t}^2} \right) \right. \right. \\ \left. \left. + 2(2 - \alpha_1) \hat{x}^{1-\alpha_1} \left( AL \frac{\partial^3 \hat{w}}{\partial \hat{t}^2 \partial \hat{x}} - \frac{I}{L} \frac{\partial^5 \hat{w}}{\partial \hat{x}^3 \partial \hat{t}^2} \right) \right. \right. \\ \left. \left. + \hat{x}^{2-\alpha_1} \left( AL \frac{\partial^4 \hat{w}}{\partial \hat{t}^2 \partial \hat{x}^2} - \frac{I}{L} \frac{\partial^6 \hat{w}}{\partial \hat{x}^4 \partial \hat{t}^2} \right) \right\} \right] \\ = \kappa_3 q + \kappa_4 [(2 - \alpha_1)(1 - \alpha_1) \hat{x}^{-\alpha_1} q \\ + 2(2 - \alpha_1) \hat{x}^{1-\alpha_1} \frac{dq}{d\hat{x}} + \hat{x}^{2-\alpha_1} \frac{d^2 q}{d\hat{x}^2}], \tag{18} \end{aligned}$$

where

$$\kappa_1 = \frac{NL^2}{EI}, \quad \kappa_2 = \frac{\rho L^3}{(t^*)^2 EI}, \quad \kappa_3 = \frac{L^3}{EI}, \quad \kappa_4 = \frac{\tau^\alpha L^3}{EI}.$$

### 4 Numerical solution

In this section the numerical solution of bending, free vibration and buckling of nano-beams structures have been shown.

#### 4.1 Galerkin residual method

The presented numerical solution for different configurations of Eq. (18) which include the conformal derivatives is less difficult compared to the application of fractional derivatives which have integral form like the Caputo, the Riemann–Liouville or the Grunwald–Letnikov. For the latter, many advanced numerical methods have been elaborated to present approximate solutions (Shah et al. 2017; Al-Smadi et al. 2017; Bhrawy and Alofi 2013; Secer et al. 2013; Rahimkhani et al. 2017). Herein, due to the final form, the

governing conformal equations have been solved applying the classical Galerkin residual method (Rashidi et al. 2018; Rahimi et al. 2018b).

### 4.2 Free vibration

Based on the Galerkin method the approximate solution for dynamic system is

$$w(\hat{x}, \hat{t}) \approx \sum_{i=1}^n \varphi_i(\hat{x}) \cdot q_i(\hat{t}), \tag{19}$$

where  $\varphi_i(x)$  and  $q_i(t)$  are the mode shapes and a time dependent functions to be determined, respectively. Herein  $\varphi(x)$  is selected as the  $i$ -th undamped linear mode shape of the straight nano-beam. Substituting Eq. (19) into Eq. (18), multiplying the outcome by  $\varphi_j(\hat{x})$ , using the orthogonality property of mode shapes, and integrating the outcome from 0 to 1 leads to

$$\ddot{q} + \eta_1 q = 0, \tag{20}$$

where

$$\begin{aligned} \eta_1 = \frac{\mathfrak{S}_2}{\mathfrak{S}_1} \mathfrak{S}_2 = & \int_0^1 \left\{ \varphi_i^{(4)}(\hat{x}) - \zeta^{\alpha_2} \left[ (2 - \alpha_2)(1 - \alpha_2) \hat{x}^{-\alpha_2} \varphi_i^{(4)}(\hat{x}) \right. \right. \\ & \left. \left. + 2(2 - \alpha_2) \hat{x}^{1-\alpha_2} \varphi_i^{(5)}(\hat{x}) + \hat{x}^{2-\alpha_2} \varphi_i^{(6)}(\hat{x}) \right] \right\} \varphi_j(\hat{x}) d\hat{x} \\ \mathfrak{S}_1 = & -\kappa_2 \int_0^1 \left\{ \left[ \frac{I}{L} \varphi_i''(\hat{x}) - AL\varphi_i(\hat{x}) \right] \right. \\ & + \tau^{\alpha_1} \left[ (2 - \alpha_1)(1 - \alpha_1) \hat{x}^{-\alpha_1} \left( \frac{I}{L} \varphi_i''(\hat{x}) - AL\varphi_i(\hat{x}) \right) \right. \\ & + 2(2 - \alpha_1) \hat{x}^{1-\alpha_1} \left( \frac{I}{L} \varphi_i'''(\hat{x}) - AL\varphi_i'(\hat{x}) \right) \\ & \left. \left. + \hat{x}^{2-\alpha_1} \left( \frac{I}{L} \varphi_i^{(4)}(\hat{x}) - AL\varphi_i''(\hat{x}) \right) \right] \right\} \varphi_j(\hat{x}) d\hat{x}. \end{aligned} \tag{21}$$

### 4.3 Bending

Based on the Galerkin method the approximate solution for static system is

$$w(\hat{x}) \approx \sum_{i=1}^n a_i \cdot \varphi_i(\hat{x}). \tag{22}$$

The equation for statics is obtained by neglecting the inertia terms, and axial force in Eq. (18). Next substituting Eq. (22) into reduced Eq. (18),

multiplying result by  $\varphi_j(\hat{x})$  as a weight function and then integrating the outcome from 0 to 1, leads to a set of linear algebraic equations

$$F_j = \sum_{i=1}^n k_{ij} a_i, \quad i = 1, 2, \dots, n, \tag{23}$$

where

$$\begin{aligned} k_{ij} = & \int_0^1 \left( \varphi_i^{(4)}(\hat{x}) - \zeta^{\alpha_2} \left[ (2 - \alpha_2)(1 - \alpha_2) \hat{x}^{-\alpha_2} \varphi_i^{(4)}(\hat{x}) \right. \right. \\ & \left. \left. + 2(2 - \alpha_2) \hat{x}^{1-\alpha_2} \varphi_i^{(5)}(\hat{x}) + \hat{x}^{2-\alpha_2} \varphi_i^{(6)}(\hat{x}) \right] \right) \varphi_j(\hat{x}) d\hat{x} \\ F_j = & \int_0^1 \left[ \kappa_3 q + \kappa_4 \left[ (2 - \alpha_1)(1 - \alpha_1) \hat{x}^{-\alpha_1} q \right. \right. \\ & \left. \left. + 2(2 - \alpha_1) \hat{x}^{1-\alpha_1} \frac{dq}{d\hat{x}} + \hat{x}^{2-\alpha_1} \frac{d^2q}{d\hat{x}^2} \right] \right] \varphi_j(\hat{x}) d\hat{x}. \end{aligned}$$

### 4.4 Buckling

Equations for buckling analysis are obtained by neglecting the inertia and transverse force terms in Eq. (18), and next by substituting Eq. (22) into it, together with multiplying the result by  $\varphi_j(\hat{x})$  and then integrating the outcome from 0 to 1 leads to

$$N_{Cr} = \frac{P_1}{P_2}, \tag{24}$$

where

$$\begin{aligned} P_1 = & \int_0^1 \left( \varphi_i^{(4)}(\hat{x}) - \zeta^{\alpha_2} \left\{ (2 - \alpha_2)(1 - \alpha_2) \hat{x}^{-\alpha_2} (\varphi_i^{(4)}(\hat{x}) \right. \right. \\ & \left. \left. + 2(2 - \alpha_2) \hat{x}^{1-\alpha_2} (\varphi_i^{(5)}(\hat{x}) + \hat{x}^{2-\alpha_2} (\varphi_i^{(6)}(\hat{x})) \right\} \right) \varphi_j(\hat{x}) d\hat{x} \\ P_2 = & k_1 \int_0^1 \left[ \varphi_i''(\hat{x}) - \tau^{\alpha_1} \left\{ (2 - \alpha_1)(1 - \alpha_1) \hat{x}^{-\alpha_1} \varphi_i''(\hat{x}) \right. \right. \\ & \left. \left. + 2(2 - \alpha_1) \hat{x}^{1-\alpha_1} \varphi_i'''(\hat{x}) + \hat{x}^{2-\alpha_1} \varphi_i^{(4)}(\hat{x}) \right\} \right] \varphi_j(\hat{x}) d\hat{x}. \end{aligned}$$

## 5 Results

The effects of the different parameters (the fractional, the length scale and the non-local parameters) on free

**Table 1** A comparison of the obtained non-dimensional natural frequencies based on the Eringen non-local theory where  $\tau = 0$  and  $\alpha = 2$ . ( $L = 10$  nm,  $E = 30 \times 10^6$  Pa,  $\vartheta = 0.3$ ,  $L/h = 100$ )

Non-local parameter $(e_0a)^2$ [nm] <sup>2</sup>	Rahimi et al. (2017b)	Reddy (2007)	Aydogdu (2009)	Present
0	9.8696	9.8696	9.8696	9.8696
0.5	9.6347	9.6347	–	9.6347
1	9.4159	9.4159	9.4124	9.4159
1.5	9.2113	9.2113	–	9.2113
2	9.0195	9.0195	9.0133	9.0195

**Table 2** A comparison of the obtained non-dimensional natural frequencies based on the non-local gradient theory ( $L = 10$  nm,  $\vartheta = 0.3$ ,  $L/h = 10$ )

Non-local parameter $(e_0a)^2$ [nm] <sup>2</sup>	Length scale parameter ( $\ell/L$ )	Lu (Lu et al. 2017)	Present
0	0	9.8293	9.8293
	0.5	9.9498	9.9498
	1	10.3029	10.3029
0.5	0	9.7102	9.7102
	0.5	9.8293	9.8293
	1	10.1781	10.1781
1	0	9.3774	9.3774
	0.5	9.4924	9.4924
	1	9.8293	9.8293

**Table 3** A comparison of the obtained non-dimensional natural frequencies based on the fractional non-local theory where  $\tau = 0$  ( $L = 10$  nm,  $E = 30 \times 10^6$  Pa,  $\vartheta = 0.3$ ,  $L/h = 100$ )

Non-local parameter $(e_0a)^2$ [nm] <sup>2</sup>	Rahimi et al. (2017b)		Present	
	$\alpha_1 = 1.2$	$\alpha_1 = 1.6$	$\alpha_1 = 1.2$	$\alpha_1 = 1.6$
0	9.8696	9.8696	9.8696	9.8696
0.5	8.8227	9.3206	8.8227	9.3206
1	8.3984	8.9681	8.3984	8.9681
1.5	8.0964	8.6828	8.0964	8.6828
2	7.8579	8.4394	7.8579	8.4394

vibration, bending and critical buckling load of nano-beams structures have been illustrated below. Firstly, the validation of the results has been shown in Tables 1, 2, 3, 4, 5 and 6 are compared with Reddy (2007), Rahimi et al. (2017b), Aydogdu (2009), Khaniki (Khaniki et al. 2018), Lu (Lu et al. 2017) and Li and Hu (2015) based on the Eringen non-local theory, classical theory, strain gradient theory, non-local strain gradient theory and the fractional non-

local theory. As it can be seen, the outcomes are in good agreement with those published in literature.

In following discussion, let us assume for convenience that the derivative orders  $\alpha_1$  and  $\alpha_2$  are equal  $\alpha$ . As mentioned above, this theory consists of four free parameters that make it more flexible. In Tables 7, 8 and 9, the effects of different parameters are shown. Recall, that although any interval ( $n < \alpha \leq n + 1$  in which  $n$  is positive integer number) of the fractional

**Table 4** A comparison of the obtained (static) non-dimensional center deflection where  $\tau = 0$  and  $\alpha = 2$  ( $L = 10$  nm,  $E = 30 \times 10^6$  Pa,  $\vartheta = 0.3$ ,  $L/h = 100$  and  $\hat{w} = wEI/qL^4$ )

Non-local parameter $(e_0a)^2$ [nm <sup>2</sup> ]	Reddy (2007)	Aydogdu (2009)	Present
0	0.0131	0.0131	0.0130
0.5	0.0138	–	0.0137
1	0.0145	0.0144	0.0144
1.5	0.0152	–	0.0151
2	0.0158	0.0158	0.0157

**Table 5** A comparison of obtained non-dimensional buckling load where  $\tau = 0$  and  $\alpha = 2$  ( $L = 10$  nm,  $E = 30 \times 10^6$  Pa,  $\vartheta = 0.3$ ,  $L/h = 100$ )

Non-local parameter $(e_0a)^2$ [nm <sup>2</sup> ]	Reddy (2007)	Aydogdu (2009)	Present
0	9.8696	9.8696	9.8696
0.5	9.4055	–	9.4055
1	8.9830	8.9830	8.9830
1.5	8.5969	–	8.5969
2	8.2426	8.2426	8.2426

**Table 6** A comparison of obtained non-dimensional buckling load where  $\tau = 0$  and  $\alpha = 2$  ( $L = 10$  nm,  $\vartheta = 0.3$ ,  $L/h = 100$ )

Non-local parameter $e_0a/L$	Length scale parameter $l/L$	Khaniki (Khaniki et al. 2018)	Li (Li and Hu 2015) (10 nodes)	Present
0	0	9.8696	9.8696	9.8696
	0.25	15.9577	15.9577	15.9577
	0.5	34.2219	34.2219	34.2219
0.5	0	2.8464	2.8464	2.8464
	0.25	4.6022	4.6022	4.6022
	0.5	9.8696	9.8696	9.8696

order can be assumed, herein it is considered  $1 < \alpha \leq 2$ . The values of the non-local parameter and the length scale parameter are considered based on those existing in the literature (Rahimi et al. 2017b; Li and Hu 2015; Reddy 2007; Li et al. 2016).

As it can be seen in Table 7, decreasing of the fractional parameter causes decrease in the non-dimensional natural frequency, and as the value of the length scale parameter increases it causes the opposite effect. On the other hand, for a constant beam length and constant fractional parameter, the increase of  $e_0a$  causes decrease in the non-dimensional natural frequency. The variation of the non-dimensional natural frequency versus the fractional parameter

and the length scale parameter has been illustrated in more details in Fig. 3. The non-local parameter is chosen  $e_0a = 1$  nm.

In Table 8 different values of the non-dimensional static maximum center deflection are presented when the fractional, the length scale and the non-local parameters have different values. It is visible in Table 8 (and also in Fig. 4) that decrease of the fractional parameter decreases the beam stiffness. On the other hand, keeping the fractional and the length scale parameters constant, growth of the non-local parameter rises down the stiffness. Moreover, increase of the length scale at first decrease and then increase



**Table 7** A non-dimensional natural frequencies of S–S nano-beam ( $E = 169 \times 10^9$  Pa,  $\nu = 0.3$ ,  $L/h = 10$ )

L (nm)	Non-local parameter ( $e_0 a$ ) nm	Length scale parameter ( $\ell$ ) nm	Fractional parameter ( $\alpha$ )				
			1.2	1.4	1.6	1.8	2
100	1	0.1	9.7283	9.7818	9.8071	9.8190	9.8245
		0.2	9.7371	9.7850	9.8083	9.8194	9.8246
		0.3	9.7470	9.7890	9.8099	9.8200	9.8249
		0.4	9.7575	9.7935	9.8118	9.8208	9.8252
		0.5	9.7685	9.7986	9.8141	9.8218	9.8256
	3	0.1	9.4501	9.6071	9.7004	9.7547	9.7859
		0.2	9.4587	9.6103	9.7015	9.7551	9.7861
		0.3	9.4683	9.6142	9.7031	9.7557	9.7863
		0.4	9.4785	9.6187	9.7050	9.7565	9.7867
		0.5	9.4892	9.6236	9.7072	9.7575	9.7871
	5	0.1	9.1567	9.3887	9.5438	9.6450	9.7102
		0.2	9.1651	9.3918	9.5449	9.6454	9.7104
		0.3	9.1743	9.3956	9.5464	9.6461	9.7106
		0.4	9.1842	9.4000	9.5483	9.6468	9.7110
		0.5	9.1946	9.4048	9.5505	9.6478	9.7114
50	1	0.1	9.6021	9.7055	9.7627	9.7936	9.8101
		0.2	9.6222	9.7140	9.7662	9.7950	9.8107
		0.3	9.6444	9.7244	9.7709	9.7972	9.8117
		0.4	9.6681	9.7363	9.7768	9.8000	9.8130
		0.5	9.6930	9.7494	9.7835	9.8034	9.8148
	3	0.1	9.0192	9.2738	9.4535	9.5766	9.6594
		0.2	9.0380	9.2819	9.4569	9.5780	9.6599
		0.3	9.0589	9.2919	9.4615	9.5801	9.6609
		0.4	9.0811	9.3032	9.4672	9.5828	9.6622
		0.5	9.1045	9.3158	9.4737	9.5862	9.6639
	5	0.1	8.4621	8.7798	9.0331	9.2291	9.3776
		0.2	8.4798	8.7875	9.0364	9.2305	9.3781
		0.3	8.4994	8.7969	9.0408	9.2325	9.3791
		0.4	8.5202	8.8077	9.0462	9.2351	9.3804
		0.5	8.5422	8.8195	9.0524	9.2384	9.3820

the stiffness slightly when both the length and the non-local parameters are constant.

In Table 9 the non-dimensional critical buckling load of S–S nano-beam is presented. It is clear that when the length scale and the non-local parameters are constant smaller values of the fractional parameters leads to smaller values of non-dimensional critical buckling load. Moreover, when the fractional parameter is constant, growth of the length scale and the non-local parameters causes the increase and decrease of

the non-dimensional critical buckling load, respectively. These effects are shown in Fig. 5 also.

### 6 Conclusion

In this paper, the non-local fractional stress–strain gradient theory was investigated using fractional calculus. This new formulation has two additional free parameters versus the classical gradient theory which are called fractional parameters. These

**Table 8** A non-dimensional maximum center deflection of S–S nano-beams ( $E = 169 \times 10^9$  Pa,  $\vartheta = 0.3$ ,  $L/h = 10$ )

L (nm)	Non-local parameter ( $e_0 a$ ) nm	Length scale parameter ( $\ell$ ) nm	Fractional parameter ( $\alpha$ )				
			1.2	1.4	1.6	1.8	2
100	1	0.1	1.3097	1.3033	1.3015	1.3013	1.3021
		0.2	1.3074	1.3025	1.3012	1.3012	1.3021
		0.3	1.3048	1.3015	1.3008	1.3010	1.3020
		0.4	1.3021	1.3003	1.3003	1.3008	1.3020
		0.5	1.2992	1.2990	1.2997	1.3005	1.3018
	3	0.1	1.3354	1.3100	1.2995	1.2968	1.3032
		0.2	1.3331	1.3092	1.2992	1.2967	1.3032
		0.3	1.3305	1.3081	1.2988	1.2965	1.3031
		0.4	1.3277	1.3069	1.2983	1.2963	1.3030
		0.5	1.3248	1.3056	1.2977	1.2960	1.3029
	5	0.1	1.3652	1.3189	1.2965	1.2889	1.3053
		0.2	1.3628	1.3180	1.2962	1.2888	1.3052
		0.3	1.3601	1.3170	1.2958	1.2887	1.3052
		0.4	1.3572	1.3158	1.2953	1.2884	1.3051
		0.5	1.3543	1.3144	1.2947	1.2882	1.3050
50	1	0.1	1.4468	1.3464	1.2852	1.2564	1.3150
		0.2	1.4409	1.3441	1.2843	1.2560	1.3148
		0.3	1.4345	1.3412	1.2830	1.2554	1.3146
		0.4	1.4276	1.3380	1.2815	1.2546	1.3142
		0.5	1.4205	1.3344	1.2797	1.2537	1.3138
	3	0.1	1.3786	1.3230	1.2944	1.2837	1.3067
		0.2	1.3730	1.3208	1.2934	1.2834	1.3065
		0.3	1.3669	1.3180	1.2922	1.2828	1.3063
		0.4	1.3604	1.3149	1.2907	1.2820	1.3059
		0.5	1.3536	1.3114	1.2889	1.2811	1.3055
	5	0.1	1.3195	1.3055	1.3003	1.2994	1.3025
		0.2	1.3142	1.3033	1.2994	1.2990	1.3023
		0.3	1.3084	1.3006	1.2982	1.2984	1.3021
		0.4	1.3022	1.2975	1.2967	1.2977	1.3018
		0.5	1.2958	1.2941	1.2950	1.2968	1.3013

parameters control the strain and stress gradient orders in the constitutive relation and can be integer or non-integer. The proposed theory includes the classical theory, the non-local strain gradient theory, the Eringen non-local theory, the strain gradient theory, the general Eringen non-local theory and the general strain gradient theory.

The proposed theory is illustrated by the analysis of free vibration, bending and buckling of S–S nano-beam structures. The influence of all free parameters (the fractional parameters, the length scale parameter

and the non-local parameters) was discussed in details. The specific solutions were obtained by the Galerkin residual method and the following results were achieved:

- Decreasing of the fractional parameter decreases the non-dimensional natural frequency (for higher values of the length scale parameter such effect is less pronounced).
- When the beam length and fractional parameter are constant, increase of the non-local parameter

**Table 9** The non-dimensional critical buckling load of S–S nano-beam ( $E = 169 \times 10^9$  Pa,  $\vartheta = 0.3$ ,  $L/h = 10$ )

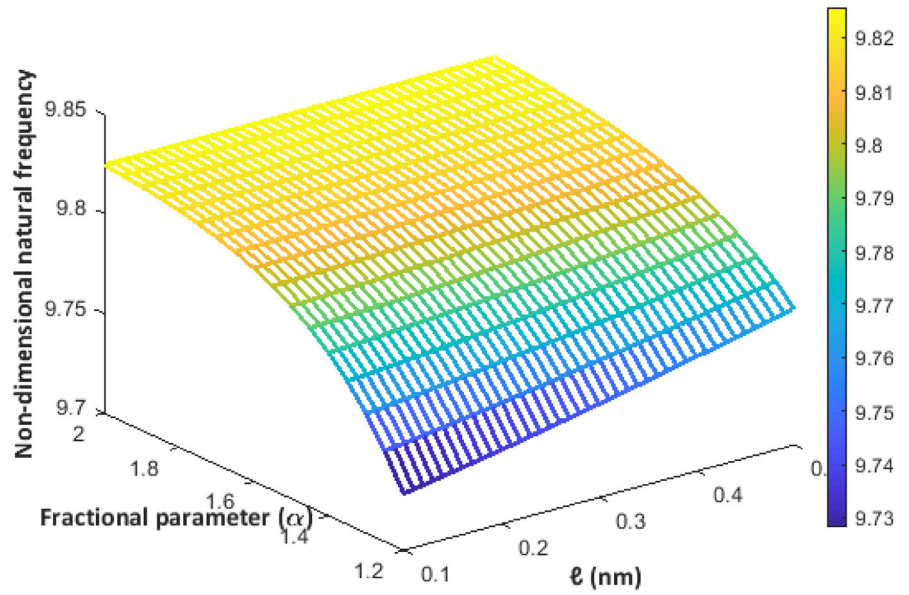
L (nm)	Non-local parameter ( $e_0 a$ ) nm	Length scale parameter ( $\ell$ ) nm	Fractional parameter ( $\alpha$ )				
			1.2	1.4	1.6	1.8	2
100	1	0.1	9.6678	9.7745	9.8252	9.8490	9.8600
		0.2	9.6854	9.7810	9.8276	9.8498	9.8603
		0.3	9.7050	9.7889	9.8307	9.8510	9.8607
		0.4	9.7259	9.7980	9.8346	9.8527	9.8614
		0.5	9.7480	9.8081	9.8391	9.8546	9.8623
	3	0.1	9.1229	9.4285	9.6125	9.7203	9.7828
		0.2	9.1395	9.4348	9.6147	9.7211	9.7831
		0.3	9.1580	9.4425	9.6178	9.7224	9.7836
		0.4	9.1777	9.4512	9.6216	9.7240	9.7843
		0.5	9.1985	9.4610	9.6260	9.7259	9.7851
	5	0.1	8.5652	9.0046	9.3046	9.5031	9.6320
		0.2	8.5808	9.0106	9.3068	9.5039	9.6323
		0.3	8.5982	9.0179	9.3098	9.5051	9.6328
		0.4	8.6167	9.0263	9.3135	9.5067	9.6335
		0.5	8.6362	9.0356	9.3177	9.5086	9.6343
50	1	0.1	7.3150	7.8746	8.3356	8.7012	8.9834
		0.2	7.3456	7.8884	8.3415	8.7037	8.9844
		0.3	7.3795	7.9053	8.3497	8.7075	8.9862
		0.4	7.4159	7.9246	8.3596	8.7126	8.9887
		0.5	7.4541	7.9460	8.3712	8.7187	8.9919
	3	0.1	8.3098	8.7857	9.1294	9.3686	9.5313
		0.2	8.3445	8.8010	9.1360	9.3714	9.5325
		0.3	8.3831	8.8199	9.1449	9.3755	9.5344
		0.4	8.4244	8.8415	9.1558	9.3809	9.5370
		0.5	8.4678	8.8654	9.1685	9.3875	9.5404
	5	0.1	7.3150	7.8746	8.3356	8.7012	8.9834
		0.2	7.3456	7.8884	8.3415	8.7037	8.9844
		0.3	7.3795	7.9053	8.3497	8.7075	8.9862
		0.4	7.4159	7.9246	8.3596	8.7126	8.9887
		0.5	7.4541	7.9460	8.3712	8.7187	8.9919

causes decrease in the non-dimensional natural frequency.

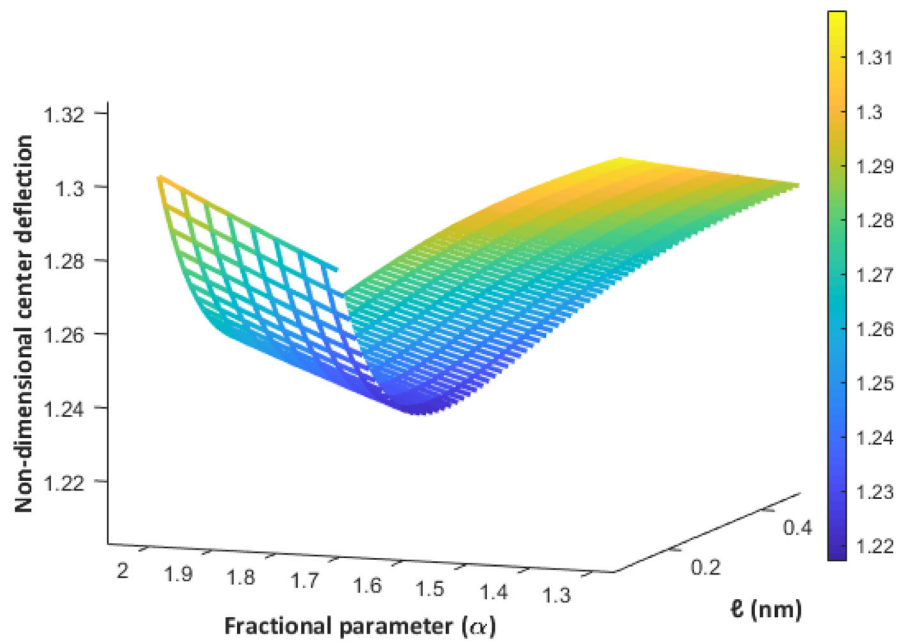
- Decrease of the fractional parameter decreases the beam stiffness.
- When the fractional and the length scale parameters are constant, growth of the non-local parameter rises down the stiffness.
- When the fractional and non-local parameters are constant, growth of the length scale parameter rises down and then rises up the stiffness with a slight slope.

- When the length scale and the non-local parameters are constant smaller values of the fractional parameters leads to smaller values of non-dimensional critical buckling load.
- When the fractional parameter is constant, growth of the length scale and the non-local parameters cause increase and decrease of the non-dimensional critical buckling load, respectively.

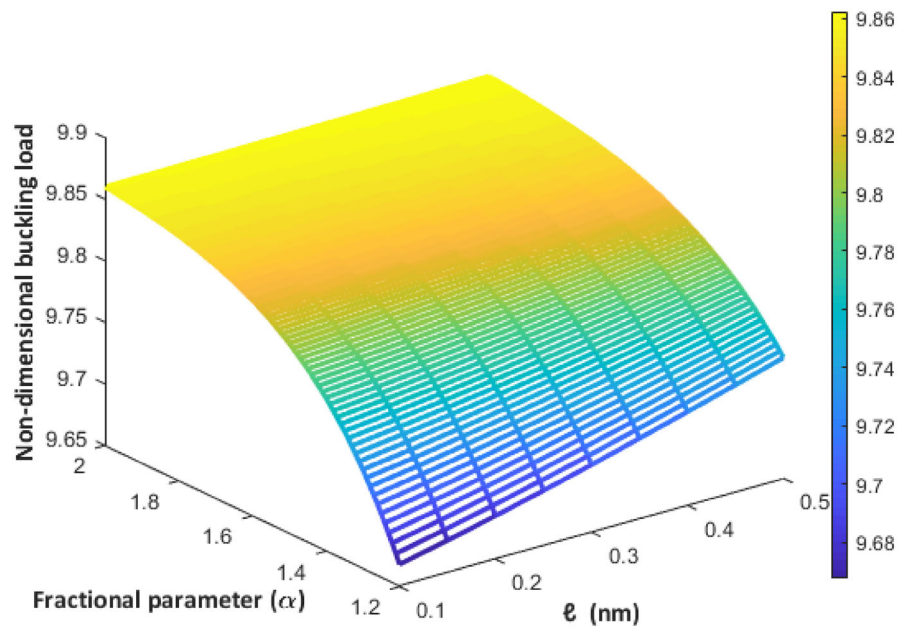
**Fig. 3** The influence of the length scale and the fractional parameters on the non-dimensional fundamental natural frequency when  $e_0a = 1$  nm



**Fig. 4** The influence of the length scale and the fractional parameters on the non-dimensional center deflection when  $e_0a = 1$  nm



**Fig. 5** The influence of the length scale and the fractional parameters on the critical non-dimensional buckling load when  $e_0a = 1$  nm



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