

Thermo-electrical buckling of piezoelectric functionally graded material Timoshenko beams

Y. Kiani · M. Rezaei · S. Taheri · M. R. Eslami

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Abstract In this article, buckling analysis of functionally graded material (FGM) beams with or without surface-bonded piezoelectric layers subjected to both thermal loading and constant voltage is studied. Thermal and mechanical properties of FGM layer is assumed to follow the power law distribution in thickness direction, except Poisson's ratio which is considered constant. The Timoshenko beam theory and nonlinear strain-displacement relations are used to obtain the governing equations of piezoelectric FGM beam. Beam is assumed under three types of thermal loading and various types of boundary conditions. For each case of boundary conditions, existence of bifurcation-type buckling is examined and for each case of thermal loading and boundary conditions, closed-form solutions are obtained which are easily usable for engineers and designers. The effects of the applied actuator voltage, beam geometry, boundary conditions,

and power law index of FGM beam on critical buckling temperature difference are examined.

Keywords Thermal buckling · Piezoelectric layers · Bifurcation buckling · Timoshenko beam theory

1 Introduction

A survey in literature reveals the existence of wealth investigations on the analysis of functionally graded material (FGM) beams. Sankar (2001) developed an elasticity solution for the FGM beams, when material properties are graded exponentially across the thickness. Following the Euler–Bernoulli beam theory to derive the equilibrium equations, he found that, the Euler beam theory is valid for long and slender beams with slowly varying transverse loading. Free vibration analysis of simply-supported FGM beams is reported by Aydoglu and Taskin (2007). They used both exponential and power law form of material properties distribution to derive the governing equations. Their study includes four types of displacement fields namely; the classical beam theory, the first order theory, and the parabolic and exponential shear deformation beam theories. They concluded that, in comparison with the classical beam theory, other three types of displacement fields accurately predict the natural frequencies. Sina et al. (2009) presented

Y. Kiani · M. R. Eslami (✉)
Mechanical Engineering Department, Amirkabir
University of Technology, Tehran, Iran
e-mail: eslami@aut.ac.ir

Y. Kiani
e-mail: y.kiani@aut.ac.ir

M. Rezaei · S. Taheri
Mechanical Engineering Department, Islamic Azad
University, South Tehran branch, Tehran, Iran
e-mail: me_re13@yahoo.com

S. Taheri
e-mail: saeedtaheri12@yahoo.com

an analytical method to predict the natural frequencies of FGM beams based on the Timoshenko beam theory. They derived five time-dependent differential equations as the equilibrium equations of the beam based on Hamilton's principle and presented their results for three types of boundary conditions. An analytical solution to solve the linear bending of functionally graded anisotropic cantilever beam under the simultaneous action of thermal and uniformly distributed loads is reported by Huang et al. (2007). The static and free vibration analysis of layered FGM beams based on a third order shear deformation beam theory is developed by Kapuria et al. (2008). A two nodes finite element method is adopted to solve the coupled ordinary differential equations. Kang and Li (2010) presented explicit expressions for deflection and rotation functions of an FGM cantilever beam subjected to an end moment. It is concluded that, considering the large deflections, an FGM beam can bear larger applied load than a homogeneous beam. Nirmala et al. (2006) obtained analytical expressions for thermo-elastic analysis of three layered beams, when the middle layer is made of FGMs. An exact closed-form solution of the dynamic coupled thermoelastic response of functionally graded Timoshenko beams is reported by Abbasi et al. (2010). The coupled equations are solved using the finite Fourier transformation method combined with Laplace transform.

The growth of the FGM have necessitated more researches on the thermo-mechanical instability of beams, as a major solid structure. A shooting method based solution to solve the highly-coupled nonlinear ordinary differential equations which give the post-buckling paths of FGM Timoshenko beams is carried out by Li et al. (2006). They found that, the post-buckling deflection of the FGM beam increases with the increase of the power law index for a given temperature. Rastgo et al. (2005) analyzed the buckling problem of FGM curved beams when two edges are immovable simply supported and beam is under linear temperature loading. They established the governing equations based on the classical beam theory and linear composition of material properties and obtained the critical buckling temperatures by means of eigenvalue solution based on Galerkin method. The problem of thermally induced instability of functionally graded Euler beam with various boundary conditions is carried out by Kiani and

Eslami (2010). In their study, the solution for thermal buckling load is obtained using the eigen value solution of stability equations. Their study concluded that the buckling temperature of FGM beam depends upon the modulus of elasticity of ceramic and steel, but for the isotropic homogeneous beam, the buckling temperature is independent of the modulus of elasticity. Ma and Lee (2011) discussed the nonlinear behavior of FGM beams under in-plane thermal loading by means of first order shear deformation theory of beams. The derivation of the equations is based on the concept of neutral surface, and the numerical shooting method is used to solve the coupled nonlinear equations. Their study concluded that when a clamped-clamped FGM beam is subjected to uniform thermal loading, it follows the bifurcation-type buckling while the simply-supported beams do not. Anandrao et al. (2010) presented post-buckling equilibrium path of an axially immovable FGM beam based of Euler beam theory, using both Ritz formulation and finite elements method.

Smart materials are another class of advanced materials which are used widely in engineering. As a branch of smart materials, piezoelectric materials are used extensively due to their applications in controlling the deformation, vibration, and instability of solid structures. Many studies are reported on behavior of structures integrated with the piezoelectric layers. Kapuria et al. (2004) developed an efficient coupled zigzag theory for electro-thermal stress analysis of hybrid piezoelectric beams. Control and stability analysis of a composite beam with piezoelectric layers subjected to axial periodic compressive loads is reported by Chen et al. (2002). In their study, by employing the Euler beam theory and nonlinear strain-displacement, Hamiltons principle is used to obtain the dynamic equation of the beams integrated with the piezoelectric layers.

Piezoelectric FGM structures have the advantages of FGM and piezoelectric materials linked together. Bian et al. (2006) presented an exact solution based on the state space formulation to study the functionally graded beams integrated with surface piezoelectric actuators and sensors. Alibeigloo (2010) reported an analytical solution for thermo-elasticity analysis of FGM beams integrated with piezoelectric layers. By assuming the simply-supported edge conditions, he used the state space method in conjunction with the Fourier series in longitudinal direction to obtain exact

solutions. An analytical method for deflection control of FGM beams containing two piezoelectric layers is reported by Gharib et al. (2008). Following the Timoshenko beam theory and simple power form of material property distribution, three coupled ordinary differential equations are derived as equilibrium equations of FGM beam integrated with orthotropic piezoelectric layers. Vibration of thermal post-buckled FGM beams with surface-bonded piezoelectric layers subjected to both thermal and electrical loading is carried out by Li et al. (2009).

Also, some works are reported on buckling and post-buckling of FGM plate and shell integrated with piezoelectric layers. Among these work, Liew et al. (2003) presented post-buckling of piezoelectric FGM plates subject to the thermo-electro-mechanical loading. They used a semi-analytical iteration to determine the post-buckling response of the plate. Mirzavand and Eslami (2007) obtained closed-form solutions for the critical buckling temperatures of simply-supported piezoelectric FGM cylindrical shells based on the higher order displacement field. Their study includes three types of temperature loading combined with constant applied voltage. Thermo-electro-mechanical buckling and post-buckling of FGM plates with piezoelectric actuators is reported by Shen (2005a, b) and Shen and Noda (2007) based on the singular perturbation method.

The present article deals with the instability problem of piezoelectric FGM beams which are subjected to thermal loading and applied constant voltage. Three types of thermal loading and five types of boundary conditions are assumed for the beam. Based on the Timoshenko beam theory and power law assumption for property distribution, the equilibrium and stability equations for the beam are derived and the eigenvalue solution is carried out to obtain the critical buckling temperature. The novelty of the present work is to obtain closed-form solutions for critical buckling temperature difference of FGM beams integrated with the piezoelectric layers.

2 Functionally graded beams

Consider a three-layered beam, where the middle layer is made of FGMs and the other layers are made of piezoelectric materials. The bottom surface of the FGM beam is metal and the top surface is ceramic.

The region between the two surfaces comprises material with different mix ratios of ceramic and metal. Using the Voigt rule as the rule of mixture (Suresh and Mortensen 1998) and a power law distribution for volume fraction (Praveen and Reddy 1998), the non-homogeneous material property variation of the FGM beam is expressed by

$$P(z) = P_m + P_{cm} \left(\frac{1}{2} + \frac{z}{h} \right)^k \quad (1)$$

Here, $P_{cm} = P_c - P_m$ and P_m and P_c are the corresponding properties of the metal and ceramic, respectively. In this analysis the material properties, such as Young's modulus $E(z)$, coefficient of thermal expansion $\alpha(z)$, and thermal conductivity $K(z)$ are expressed by this equation, whereas Poisson's ratio ν is assumed to be constant across the FGM beam thickness (Mirzavand and Eslami 2007). Also, z is the thickness coordinate measured from the middle surface of the beam ($-h/2 \leq z \leq h/2$) and k is the power law index which has the value equal or greater than zero. The value of k equal to zero represents a fully ceramic beam.

3 Governing equations

Consider a beam with rectangular cross section, made of an FGM substrate of thickness h , width b , length L , and piezoelectric films of thickness h_a that are perfectly bonded on its top and bottom surfaces as actuators. The rectangular Cartesian coordinates is used such that the x axis is along the length of the beam on its middle surface and z is measured from the middle surface and is positive upward, as shown in Fig. 1. The analysis is based on the Timoshenko beam theory with the following displacement field (Aydoglu 2007)

$$\begin{aligned} \bar{u}(x, z) &= u + z\varphi \\ \bar{w}(x, z) &= w \end{aligned} \quad (2)$$

where $\bar{u}(x, z)$ and $\bar{w}(x, z)$ are displacements of an arbitrary point of the beam along the x and z -directions, respectively. Here, u and w are the displacement components of middle surface and φ is the rotation of the beam cross-section, which are functions of x only. The strain-displacement relations for the beam are given in the form

$$\begin{aligned}\varepsilon_{xx} &= \frac{\partial \bar{u}}{\partial x} + \frac{1}{2} \left(\frac{\partial \bar{w}}{\partial x} \right)^2 \\ \gamma_{xz} &= \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\end{aligned}\quad (3)$$

where ε_{xx} and γ_{xz} are the axial and shear strains. Substituting Eq. 2 into Eq. 3 gives

$$\begin{aligned}\varepsilon_{xx} &= \frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 + z \frac{d\varphi}{dx} \\ \gamma_{xz} &= \varphi + \frac{dw}{dx}\end{aligned}\quad (4)$$

If the applied voltage V_0 is across the thickness, then the electrical field is generated only in the z -direction and is denoted by E_z , which is equal to (Mirzavand and Eslami 2007; Shen and Noda 2007)

$$E_z = \frac{V_0}{h_a} \quad (5)$$

We also assume that FGM layer is under thermal loading. For such a case, the constitutive equation for this layer is given by

$$\begin{aligned}\sigma_{xx} &= E(\varepsilon_{xx} - \alpha(T - T_0)) \\ \sigma_{xz} &= \frac{E}{2(1 + \nu)} \gamma_{xz}\end{aligned}\quad (6)$$

and for the piezoelectric layers (Mirzavand and Eslami 2007)

$$\begin{aligned}\sigma_{xx}^a &= E_a(\varepsilon_{xx} - d_{31}E_z), \quad D_z = E_a d_{31} \varepsilon_{xx} + k_{33} E_z \\ \sigma_{xz}^a &= \frac{E_a}{2(1 + \nu_a)} \gamma_{xz}\end{aligned}\quad (7)$$

In the above equations, σ_{xx} and σ_{xz} are the axial and shear stresses through the FGM beam and σ_{xx}^a and σ_{xz}^a are the axial and shear stresses through the piezoelectric layers. Also, ν and ν_a are Poisson's ratios for FGM beam and piezoelectric layers, respectively. T_0 is the reference temperature, and T is the temperature distribution through the beam. Here, E_a , D_z , d_{31} , and k_{33} are the elasticity modulus, electric displacement, piezoelectric constant, and the dielectric permittivity coefficient for the piezoelectric layers, respectively. Equations 4–7 are combined to give the axial and shear stresses in terms of the middle surface displacements as

$$\begin{aligned}\sigma_{xx} &= E \left(\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 + z \frac{d\varphi}{dx} - \alpha(T - T_0) \right) \\ \sigma_{xx}^a &= E_a \left(\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 + z \frac{d\varphi}{dx} - \frac{V_0}{h_a} d_{31} \right) \\ \sigma_{xz} &= \frac{E}{2(1 + \nu)} \left(\varphi + \frac{dw}{dx} \right) \\ \sigma_{xz}^a &= \frac{E_a}{2(1 + \nu_a)} \left(\varphi + \frac{dw}{dx} \right)\end{aligned}\quad (8)$$

The force and moment per unit length of the beam expressed in terms of the stress through the thickness, according to the Timoshenko beam theory, are

$$\begin{aligned}N_x &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} dz + \int_{\frac{h}{2}}^{\frac{h}{2}+h_a} \sigma_{xx}^a dz + \int_{-\frac{h}{2}-h_a}^{-\frac{h}{2}} \sigma_{xx}^a dz \\ M_x &= \int_{-\frac{h}{2}}^{\frac{h}{2}} z \sigma_{xx} dz + \int_{\frac{h}{2}}^{\frac{h}{2}+h_a} z \sigma_{xx}^a dz + \int_{-\frac{h}{2}-h_a}^{-\frac{h}{2}} z \sigma_{xx}^a dz \\ Q_{xz} &= K_s \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} dz + \int_{\frac{h}{2}}^{\frac{h}{2}+h_a} \sigma_{xz}^a dz + \int_{-\frac{h}{2}-h_a}^{-\frac{h}{2}} \sigma_{xz}^a dz \right)\end{aligned}\quad (9)$$

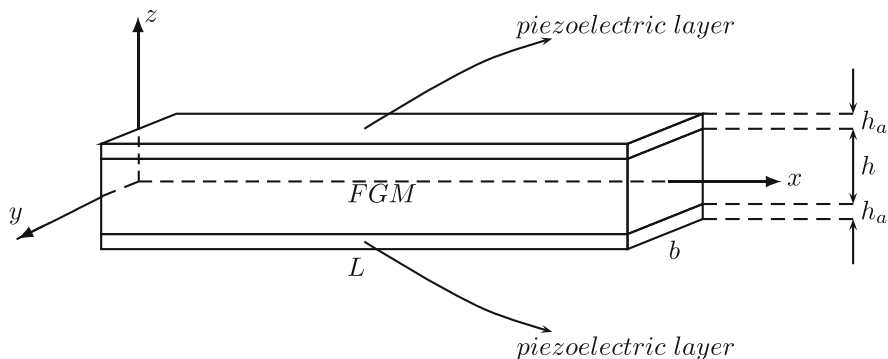
Here K_s is the correction shear factor. Its exact value, however, have to be evaluated through complicated integrals, but the value $(5/6)$ and $(\pi^2/12)$ are used widely as its approximations. The value of $(K_s = \pi^2/12)$ is adopted in the present work.

Using Eqs. 8 and 9 and noting that u and w are only functions of x , N_x , M_x and Q_{xz} are obtained as

$$\begin{aligned}N_x &= (E_1 + 2h_a E_a) \left(\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right) \\ &\quad + E_2 \frac{d\varphi}{dx} - N_x^T - 2V_0 E_a d_{31} \\ M_x &= E_2 \left(\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right) + (E_3 + H E_a) \frac{d\varphi}{dx} - M_x^T \\ Q_{xz} &= K_s \left(\frac{E_1}{2(1 + \nu)} + \frac{h_a E_a}{1 + \nu_a} \right) \left(\varphi + \frac{dw}{dx} \right)\end{aligned}\quad (10)$$

where H , E_1 , E_2 , and E_3 are constants and N_x^T and M_x^T are the thermal force and thermal moment resultants, which are calculated using the following relations

Fig. 1 Coordinate system and geometry of FGM beam integrated with piezoelectric layers



$$\begin{aligned}
 H &= \frac{2}{3} \left(h_a^3 + \frac{3}{2} h_a^2 h + \frac{3}{4} h_a h^2 \right) \\
 E_1 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) dz = h \left(E_m + \frac{E_{cm}}{k+1} \right) \\
 E_2 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} z E(z) dz = h^2 E_{cm} \left(\frac{1}{k+2} - \frac{1}{2k+2} \right) \\
 E_3 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 E(z) dz \\
 &= h^3 \left(\frac{1}{12} E_m + E_{cm} \left(\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4k+4} \right) \right) \\
 N_x^T &= \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \alpha(z) (T - T_0) dz \\
 M_x^T &= \int_{-\frac{h}{2}}^{\frac{h}{2}} z E(z) \alpha(z) (T - T_0) dz
 \end{aligned}
 \tag{11}$$

The total potential energy U for a piezoelectric FGM beam under thermal loads is defined as the sum of total potential energies for piezoelectric layers U_p and the potential energy of FGM beam U_b as

$$\begin{aligned}
 U_p &= \frac{1}{2} \int_x \int_y \int_z [\sigma_{xx}^a \epsilon_{xx} + K_s \sigma_{xz}^a \gamma_{xz} - E_z D_z] dz dy dx \\
 U_b &= \frac{1}{2} \int_x \int_y \int_z [\sigma_{xx} (\epsilon_{xx} - \alpha(T - T_0)) + K_s \sigma_{xz} \gamma_{xz}] \\
 &\quad \times dz dy dx
 \end{aligned}
 \tag{12}$$

where in definition of U_b , $z \in [-\frac{h}{2}, \frac{h}{2}]$, and in definition of U_p , $z \in [-\frac{h}{2} - h_a, -\frac{h}{2}] \cup [\frac{h}{2}, h_a + \frac{h}{2}]$. To derive

the equilibrium equations, the variational approach may be used. Assume that the total functional of U is F . In this case, Euler's equations are expressed by

$$\begin{aligned}
 \frac{\partial F}{\partial u} - \frac{d}{dx} \left(\frac{\partial F}{\partial (\frac{du}{dx})} \right) &= 0 \\
 \frac{\partial F}{\partial \varphi} - \frac{d}{dx} \left(\frac{\partial F}{\partial (\frac{d\varphi}{dx})} \right) &= 0 \\
 \frac{\partial F}{\partial w} - \frac{d}{dx} \left(\frac{\partial F}{\partial (\frac{dw}{dx})} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial (\frac{d^2w}{dx^2})} \right) &= 0
 \end{aligned}
 \tag{13}$$

The equilibrium equations for a piezoelectric FGM beam using Eqs. 12 and 13 become

$$\begin{aligned}
 \frac{dN_x}{dx} &= 0 \\
 \frac{dM_x}{dx} - Q_{xz} &= 0 \\
 \frac{dQ_{xz}}{dx} + N_x \frac{d^2w}{dx^2} &= 0
 \end{aligned}
 \tag{14}$$

4 Existence of bifurcation type buckling

Consider a beam made of FGMs with piezoelectric layers subjected to the transverse temperature distribution and applied actuator voltage. When the axial deformation is prevented in the beam, an applied thermal loading may produce an axial load. Only perfectly flat pre-buckling configurations are considered in the present work, which lead to bifurcation type buckling. Now, based on Eq. 10, in the prebuckling state, when beam is completely undeformed, the generated pre-buckling force through the beam is equal to

$$N_{x0} = -N_x^T - 2V_0 E_a d_{31}
 \tag{15}$$

Here a subscript 0 is adopted to indicate the pre-buckling state deformation. Also according to Eq. 10,

an extra moment is produced through the beam which is equal to

$$M_{x0} = -M_x^T \tag{16}$$

In general, this extra moment may cause deformation through the beam, except when it is vanished for some especial types of thermal loading or when boundary conditions are capable of handling the extra moments. The clamped and roller boundary conditions are capable of supplying the extra moments on the boundaries, while the simply-supported edge does not. Therefore, the $C-C$ and $C-R$ piezoelectric FGM Timoshenko beams remain undeformed prior to buckling, while for the other types of beams with at least one simply supported edge, beam commence to deflect. Also, symmetrically mid-plane beam remains undeformed when it is subjected to uniform temperature rise, because thermal moment vanishes through the beam. Therefore, bifurcation type buckling exists for $C-C$ and $C-R$ piezoelectric FGM beams subjected to arbitrary transverse thermal loading and constant voltage. The same is true for the beams with isotropic homogeneous core subjected to the combined action of uniform temperature rise and constant voltage.

5 Stability equations

To derive the stability equations, the adjacent-equilibrium criterion is used. Assume that the equilibrium state of a beam is defined in terms of the displacement components u_0 , w_0 , and φ_0 . The displacement components of a neighboring stable state differ by u_1 , w_1 , and φ_1 with respect to the equilibrium position. Thus, the total displacements of a neighboring state are (Brush and Almonth 1975)

$$\begin{aligned} u &= u_0 + u_1 \\ w &= w_0 + w_1 \\ \varphi &= \varphi_0 + \varphi_1 \end{aligned} \tag{17}$$

Similar to the displacements, the force and moment resultants of a neighboring state may be related to the state of equilibrium as

$$\begin{aligned} N_x &= N_{x0} + N_{x1} \\ M_x &= M_{x0} + M_{x1} \\ Q_{xz} &= Q_{xz0} + Q_{xz1} \end{aligned} \tag{18}$$

Here, stress resultants with subscript 1 represent the linear parts of the force and moment resultant

increments corresponding to u_1 , φ_1 , and w_1 . The stability equations may be obtained by substituting Eqs. 17 and 18 in Eq. 14. Upon substitution, the terms in the resulting equations with subscript 0 satisfy the equilibrium conditions and therefore drop out of the equations. Also, the non-linear terms with subscript 1 are ignored because they are small compared to the linear terms. The remaining terms form the stability equations as

$$\begin{aligned} \frac{dN_{x1}}{dx} &= 0 \\ \frac{dM_{x1}}{dx} - Q_{xz1} &= 0 \\ \frac{dQ_{xz1}}{dx} + N_{x0} \frac{d^2w_1}{dx^2} &= 0 \end{aligned} \tag{19}$$

Using Eqs. 10, 17, and 18, the stress resultants with subscript 1 may be defined by

$$\begin{aligned} N_{x1} &= (E_1 + 2h_a E_a) \frac{du_1}{dx} + E_2 \frac{d\varphi_1}{dx} \\ M_{x1} &= E_2 \frac{du_1}{dx} + (E_3 + HE_a) \frac{d\varphi_1}{dx} \\ Q_{xz1} &= K_s \left(\frac{E_1}{2(1+\nu)} + \frac{h_a E_a}{1+\nu_a} \right) \left(\varphi_1 + \frac{dw_1}{dx} \right) \end{aligned} \tag{20}$$

Recalling Eq. 15, combining Eqs. 17–19 by eliminating u_1 and φ_1 , provides an ordinary differential equation in terms of w_1 , which is the stability equation of piezoelectric FGM beam under thermal loading and constant voltage

$$\frac{d^4w_1}{dx^4} + \mu^2 \frac{d^2w_1}{dx^2} = 0 \tag{21}$$

with

$$\begin{aligned} \mu^2 &= \\ &= \frac{(E_1 + 2h_a E_a)(N_x^T + 2V_0 E_a d_{31})}{((E_3 + HE_a)(E_1 + 2h_a E_a) - E_2^2) \left(1 - \frac{N_x^T + 2V_0 E_a d_{31}}{K_s \left(\frac{E_1}{2(1+\nu)} + \frac{h_a E_a}{1+\nu_a} \right)} \right)} \end{aligned} \tag{22}$$

When the temperature distribution in the beam is a function of thickness direction only, the parameter μ is constant. In this case, the exact solution of Eq. 21 is

$$w_1(x) = C_1 \sin(\mu x) + C_2 \cos(\mu x) + C_3 x + C_4 \tag{23}$$

Using Eqs. 19, 20, and 23 the expressions for u_1 , φ_1 , N_{x1} , M_{x1} , and Q_{xz1} become

$$\begin{aligned}
 \varphi_1(x) &= -S(\mu)(C_1 \cos(\mu x) - C_2 \sin(\mu x)) - C_3 \\
 u_1(x) &= \frac{E_2}{E_1 + 2h_a E_a} S(\mu)(C_1 \cos(\mu x) \\
 &\quad - C_2 \sin(\mu x)) + C_5 x + C_6 \\
 M_{x1}(x) &= \mu S(\mu) \frac{((E_3 + HE_a)(E_1 + 2h_a E_a) - E_2^2)}{E_1 + 2h_a E_a} \\
 &\quad \times (C_1 \sin(\mu x) + C_2 \cos(\mu x)) + E_2 C_5 \\
 Q_{xz1}(x) &= K_s \left(\frac{E_1}{2(1 + \nu)} + \frac{h_a E_a}{1 + \nu_a} \right) (\mu - S(\mu)) \\
 &\quad \times (C_1 \cos(\mu x) - C_2 \sin(\mu x)) \\
 N_{x1}(x) &= (E_1 + 2h_a E_a) C_5
 \end{aligned} \tag{24}$$

with

$$S(\mu) = \frac{\mu}{1 + \mu^2 \frac{(E_3 + HE_a)(E_1 + 2h_a E_a) - E_2^2}{K_s(E_1 + 2h_a E_a) \left(\frac{E_1}{2(1 + \nu)} + \frac{h_a E_a}{1 + \nu_a} \right)}} \tag{25}$$

The constants of integration C_1 to C_6 are obtained using the boundary conditions of the beam. The parameter μ must be minimized to find the minimum value of N_x^T associated with the thermal buckling load.

Five types of boundary conditions are assumed for the piezoelectric FGM beam. Boundary conditions in each case are listed in Table 1. Let us consider a beam with both edges clamped. The edge conditions of the clamped-clamped beam are (Li et al. 2006; Khdeir 2001)

$$\begin{aligned}
 u_1(0) = w_1(0) = \varphi_1(0) = u_1(L) = w_1(L) \\
 = \varphi_1(L) = 0
 \end{aligned} \tag{26}$$

Using Eqs. 24–26, the constants C_1 to C_6 have to satisfy the system of equations

Table 1 Boundary conditions for piezoelectric FGM Timoshenko beams

Edge supports	B.Cs at $x = 0$	B.Cs at $x = L$
C–C	$u_1 = w_1 = \varphi_1 = 0$	$u_1 = w_1 = \varphi_1 = 0$
S–S	$w_1 = u_1 = M_{x1} = 0$	$w_1 = u_1 = M_{x1} = 0$
C–S	$u_1 = w_1 = \varphi_1 = 0$	$w_1 = u_1 = M_{x1} = 0$
C–R	$u_1 = w_1 = \varphi_1 = 0$	$u_1 = \varphi_1 = Q_{xz1} + N_{x0} \frac{dw_1}{dx} = 0$
S–R	$w_1 = u_1 = M_{x1} = 0$	$u_1 = \varphi_1 = Q_{xz1} + N_{x0} \frac{dw_1}{dx} = 0$

C indicates clamped, S shows simply-supported, and R is used for roller edge

$$\begin{bmatrix}
 0 & 1 & 0 & 1 & 0 & 0 \\
 \sin(\mu L) & \cos(\mu L) & L & 1 & 0 & 0 \\
 -S(\mu) & 0 & -1 & 0 & 0 & 0 \\
 -S(\mu) \cos(\mu L) & S(\mu) \sin(\mu L) & -1 & 0 & 0 & 0 \\
 \frac{E_2}{E_1 + 2h_a E_a} S(\mu) & 0 & 0 & 0 & 0 & 1 \\
 \frac{E_2}{E_1 + 2h_a E_a} S(\mu) \cos(\mu L) & -\frac{E_2}{E_1 + 2h_a E_a} S(\mu) \sin(\mu L) & 0 & 0 & L & 1
 \end{bmatrix}
 \times \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{27}$$

To have a non-trivial solution, the determinant of coefficient matrix must be set to zero, which yields

$$\mu L(2 - 2 \cos(\mu L) - S(\mu)L \sin(\mu L)) = 0 \tag{28}$$

The smallest positive value of μ which satisfies Eq. 29 is ($\mu_{\min} = (6.28319/L)$). It can be seen easily that for the other types of boundary conditions, except C–S case, the nontrivial solution leads to an exact parameter for μ . Using an approximate solution given in Wang et al. (2004) for the critical axial force of C–S beams, the critical thermal force for a piezoelectric FGM Timoshenko beam with arbitrary boundary conditions can be expressed as below

$$\begin{aligned}
 N_{xcr}^T &= \frac{r((E_3 + HE_a)(E_1 + 2h_a E_a) - E_2^2)}{L^2 \left\{ (E_1 + 2h_a E_a) + \frac{s((E_3 + HE_a)(E_1 + 2h_a E_a) - E_2^2)}{K_s \left(\frac{E_1}{1 + \nu} + \frac{2h_a E_a}{1 + \nu_a} \right) L^2} \right\}} \\
 &\quad - 2V_0 E_a d_{31}
 \end{aligned} \tag{29}$$

where r and s are constants depending upon the boundary conditions and are listed in Table 2.

Neglecting the term produced by shear deformation, gives the critical thermal load of piezoelectric FGM beams based on the Euler–Bernoulli beam theory which is equal to

$$N_{xcr}^T = \frac{r}{L^2} \left(E_3 + HE_a - \frac{E_2^2}{E_1 + 2h_a E_a} \right) - 2V_0 E_a d_{31} \tag{30}$$

Also, Eq. 29 may be reduced to thermal buckling force of an FGM beam without piezoelectric layers, when both h_a and V_0 tend to zero. In this case, N_{xcr}^T becomes

Table 2 Constants of Eq. 29 which are related to boundary conditions

Parameter	C–C	S–S	C–S	S–R	C–R
<i>r</i>	39.47842	9.86960	20.19077	2.46740	9.86960
<i>s</i>	78.95684	19.73920	44.41969	4.93480	19.73920

$$N_{xcr}^T = \frac{\frac{r}{L^2} \left(E_3 - \frac{E_2^2}{E_1} \right)}{1 + s \frac{1+\nu}{K_s L^2} \left(\frac{E_3}{E_1} - \left(\frac{E_2}{E_1} \right)^2 \right)} \tag{31}$$

6 Types of thermal loading

6.1 Uniform temperature rise

Consider a beam at reference temperature T_0 . In such a case, the uniform temperature may be raised to $T_0 + \Delta T$ such that the beam buckles. Substituting $T = T_0 + \Delta T$ into Eq. 11 gives

$$N_x^T = \Delta T h \left(E_m \alpha_m + \frac{E_{cm} \alpha_m + E_m \alpha_{cm}}{k + 1} + \frac{E_{cm} \alpha_{cm}}{2k + 1} \right) \tag{32}$$

Using Eqs. 29 and 32, the critical buckling temperature ΔT_{cr} is expressed in the form

$$\Delta T_{cr} = \frac{r \left((E_3 + HE_a)(E_1 + 2h_a E_a) - E_2^2 \right)}{L^2 Ph \left\{ (E_1 + 2h_a E_a) + \frac{s \left((E_3 + HE_a)(E_1 + 2h_a E_a) - E_2^2 \right)}{K_s L^2 \left(\frac{E_1}{1+\nu} + \frac{2h_a E_a}{1+\nu_a} \right)} \right\} - \frac{2V_0 E_a d_{31}}{Ph}} \tag{33}$$

with

$$P = \left(E_m \alpha_m + \frac{E_m \alpha_{cm} + E_{cm} \alpha_m}{k + 1} + \frac{E_{cm} \alpha_{cm}}{2k + 1} \right) \tag{34}$$

6.2 Linear temperature through the thickness

Consider a thin FGM beam which the temperature in ceramic-rich and metal-rich surfaces are T_c and T_m , respectively. The temperature distribution for the given boundary conditions is obtained by solving the heat conduction equation along the beam thickness. If the beam thickness is thin enough, the temperature distribution is approximated linear through the thickness. So the temperature as a function of thickness coordinate z can be written in the form

$$T = T_m + (T_c - T_m) \left(\frac{1}{2} + \frac{z}{h} \right) \tag{35}$$

Substituting Eq. 35 into Eq. 11 gives the thermal force as

$$N_x^T = h(T_m - T_0) \left(E_m \alpha_m + \frac{E_{cm} \alpha_m + E_m \alpha_{cm}}{k + 1} + \frac{E_{cm} \alpha_{cm}}{2k + 1} \right) + h \Delta T \left(\frac{E_m \alpha_m}{2} + \frac{E_{cm} \alpha_m + E_m \alpha_{cm}}{k + 2} + \frac{E_{cm} \alpha_{cm}}{2k + 2} \right) \tag{36}$$

where $\Delta T = T_c - T_m$. Combining Eqs. 29 and 36 gives the final form for the critical buckling temperature difference under linear thermal loading through the thickness as

$$\Delta T_{cr} = \frac{r \left((E_3 + HE_a)(E_1 + 2h_a E_a) - E_2^2 \right)}{L^2 Q h \left\{ (E_1 + 2h_a E_a) + \frac{s \left((E_3 + HE_a)(E_1 + 2h_a E_a) - E_2^2 \right)}{K_s L^2 \left(\frac{E_1}{1+\nu} + \frac{2h_a E_a}{1+\nu_a} \right)} \right\} - \frac{2V_0 E_a d_{31} + Ph(T_m - T_0)}{Qh}} \tag{37}$$

with

$$Q = \left(\frac{E_m \alpha_m}{2} + \frac{E_m \alpha_{cm} + E_{cm} \alpha_m}{k + 2} + \frac{E_{cm} \alpha_{cm}}{2k + 2} \right) \tag{38}$$

6.3 Non-linear temperature through the thickness

Assume an FGM beam where the temperature in the ceramic-rich and metal-rich surfaces are T_c and T_m , respectively. The governing equation for the steady-state one-dimensional heat conduction equation, in the absence of heat generation, becomes

$$\frac{d}{dz} \left(K(z) \frac{dT}{dz} \right) = 0$$

$$T \left(\frac{h}{2} \right) = T_c, \quad T \left(-\frac{h}{2} \right) = T_m \tag{39}$$

where $K(z)$ is given in Eq. 1. The solution of Eq. 39 may be obtained by means of polynomial series. Taking the first seven terms of the series, we have

$$T = T_m + \frac{(T_c - T_m)}{D} \left[\sum_{i=0}^5 \frac{(-1)^i}{ik + 1} \left(\frac{K_{cm}}{K_m} \right)^i \left(\frac{1}{2} + \frac{z}{h} \right)^{ik+1} \right] \tag{40}$$

with

$$D = \sum_{i=0}^5 \frac{(-1)^i}{ik + 1} \left(\frac{K_{cm}}{K_m} \right)^i \tag{41}$$

Evaluating N_x^T and solving for ΔT gives the critical bucking temperature difference as

$$\Delta T_{cr} = \frac{r((E_3 + HE_a)(E_1 + 2h_a E_a) - E_2^2)}{L^2 Rh \left\{ (E_1 + 2h_a E_a) + \frac{s((E_3 + HE_a)(E_1 + 2h_a E_a) - E_2^2)}{K_s L^2 \left(\frac{E_1}{1+\nu} + \frac{2h_a E_a}{1+\nu_a} \right)} \right\}} - \frac{2V_0 E_a d_{31} + Ph(T_m - T_0)}{Rh} \tag{42}$$

with

$$R = \frac{1}{D} \left\{ E_m \alpha_m \sum_{i=0}^5 \frac{(-1)^i}{(ik + 1)(ik + 2)} \left(\frac{K_{cm}}{K_m} \right)^i + (E_{cm} \alpha_m + E_m \alpha_{cm}) \sum_{i=0}^5 \frac{(-1)^i}{(ik + 1)(ik + k + 2)} \left(\frac{K_{cm}}{K_m} \right)^i + E_{cm} \alpha_{cm} \sum_{i=0}^5 \frac{(-1)^i}{(ik + 1)(ik + 2k + 2)} \left(\frac{K_{cm}}{K_m} \right)^i \right\} \tag{43}$$

7 Result and discussions

Consider a piezoelectric FGM beam. The combination of materials consists of aluminum and alumina for the FGM substrate and PZT-5A for piezoelectric layers. The actuator layer thickness is $h_a = 0.001$ m. Young’s modulus, coefficient of thermal expansion, and conductivity for aluminum are $E_m = 70$ GPa, $\alpha_m = 23 \times 10^{-6}/^\circ\text{C}$ and $K_m = 204$ W/m °K, and for alumina are $E_c = 380$ GPa, $\alpha_c = 7.4 \times 10^{-6}/^\circ\text{C}$ and $K_c = 10.4$ W/m °K, respectively. The PZT-5A properties are $E_a = 63$ GPa, $\nu_a = 0.3$ and $d_{31} = 2.54 \times 10^{-10}$ m/V (Mirzavand and Eslami 2007).

To validate the results, the effect of shear on critical buckling temperature of an isotropic homogeneous beam without piezoelectric layers is plotted in Fig. 2. For this purpose, the results are compared between the Euler and Timoshenko beam theories. The beam is under the uniform temperature rise. Non-dimensional critical buckling temperature is

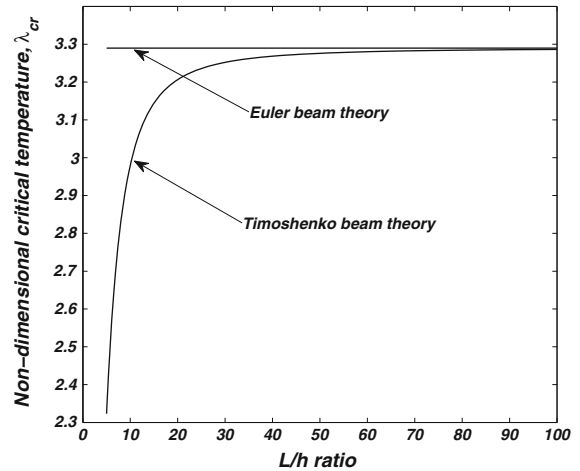


Fig. 2 Effect of shear deformation on non-dimensional critical temperature

defined by $\lambda_{cr} = \alpha \Delta T_{cr}^{Uni} (L/h)^2$. For a C–C beam, based on Euler beam theory this parameter is equal to $\lambda_{cr} = 3.289$, which is similar with the one stated in Li et al. (2006). It is apparent that, the critical buckling temperature for beams with L/h ratio more than 50 is identical between the two theories. But, for L/h ratio less than 50 the difference between the two theories become larger, and it will become more different for L/h values less than 20. Same graph is reported in Li et al. (2006).

Figure 3 depicts the critical bucking temperature difference versus h for a piezoelectric/ceramic/piezoelectric beam for various types of boundary conditions subjected to the uniform temperature rise, when power law index is chosen $k = 0$. It is apparent that by increasing h , ΔT_{cr} gets larger. Also, the critical buckling temperature for the S–S and C–R types of boundary conditions are identical and lower than C–C and C–S beams, but larger than the value related to the S–R beams.

The influence of beam geometry on ΔT_{cr} , for various power law indices, when the applying voltage is $V_0 = +200$ V, is illustrated in Fig. 4, for the uniform temperature rise and C–R boundary conditions. As the thickness increases, the critical buckling temperature increases. Also it may be concluded that the critical temperature for the given constituents decreases for $k < 2$, then increases for $2 < k < 10$ and finally decreases for $k > 10$.

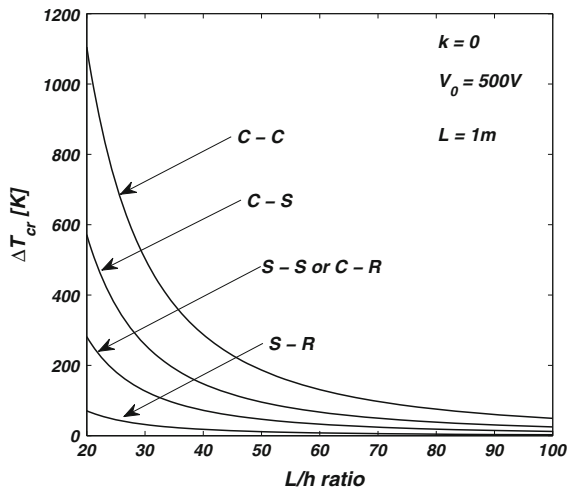


Fig. 3 Boundary conditions effect on ΔT_{cr}

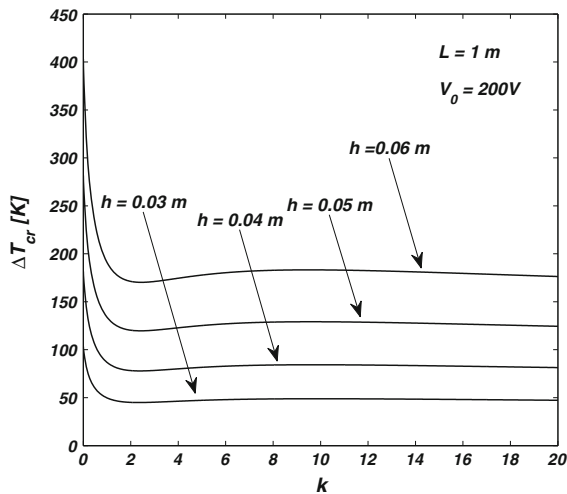


Fig. 4 Influence of power law index on critical buckling temperature difference

The buckling temperature difference ΔT_{cr} for a C–C piezoelectric FGM beam ($L = 0.25$ m, $h = 0.01$ m) that is subjected to uniform temperature rise and constant voltage is calculated and presented in Table 3. Five electric loading cases are considered $V_0 = 0, \pm 200, \pm 500$ V. Here, $V_0 = 0$ V denotes a grounding condition. The results show that, for this type of piezoelectric layer, the critical buckling temperature difference decreases with the increase of the applied voltage. The changes are, however,

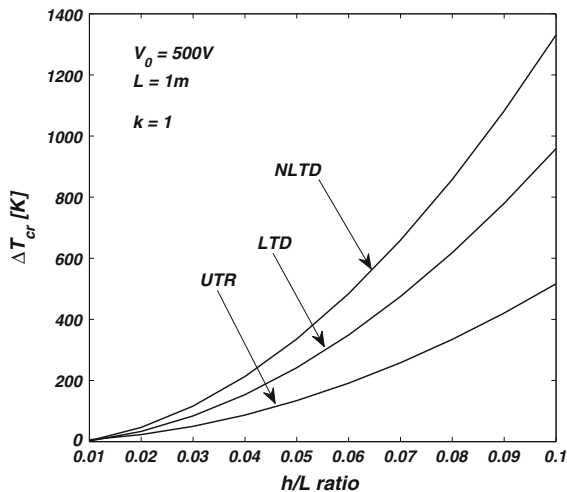
small. It should be mentioned that increasing or decreasing the critical temperature difference by applying voltage in comparison with the grounding condition depends on both the sign of applied voltage and the sign of piezoelectric constant. For the piezoelectric layers used in this study, the piezoelectric constant d_{31} is positive, and it can be seen that the critical buckling temperature decreases by increasing the voltage. For a piezoelectric material, when the piezoelectric constant is negative (such as PZT-4 or PZT-5H), when voltage increases, the critical buckling temperature difference gets larger.

Figure 5 shows a comparison between the critical buckling temperature difference of three loading cases (uniform UTR, nonlinear NLTD, and linear LTD temperature distributions across the thickness) for C–R piezoelectric FGM beam when $k = 1$ and $V_0 = 500$ V. We assume that the temperature rises 5°C in the metal-rich surface of the beam. The critical buckling temperature difference of UTR is the lowest and the NLTD is the highest curve.

The critical buckling temperature difference of an FGM beam, without piezoelectric layers, for the linear and nonlinear cases of temperature distribution through the thickness is shown in Tables 4 and 5, respectively. Results are tabulated for both Euler and Timoshenko beam theories. In these cases, the C–R boundary condition is assumed for the beam. Results are listed for various values of k and L/h ratio. It is apparent that for the linear and nonlinear temperature distributions, the critical buckling temperature difference of the FGM beams decreases with increasing L/h ratio, where this decrease slows down with the increase of L/h ratio. Also, the Euler beam theory over predicts the critical buckling temperature of the thick beams, but it is in good agreement with the values related to the Timoshenko beam theory for thin beams. Note that the case of LTD, as an approximate solution of the heat conduction equation (39) across the thickness, underrates the buckling temperature differences. The exception is for the case of reducing the FGM middle layer to a homogeneous one, where the exact solution of the heat conduction equation (39) is also linear. For both cases, after a swift decrease for $k < 2$, ΔT_{cr} follows non-significant changes for $k > 2$.

Table 3 Effect of applied voltage on buckling temperature difference of $C-C$ piezo-FGM beams subjected to uniform temperature rise ($L = 0.25$ m, $h = 0.01$ m)

V_0 (V)		$k = 0$	$k = 0.5$	$k = 1$	$k = 2$	$k = 5$
500	Euler	796.606	478.876	413.243	389.489	420.050
	Timoshenko	782.473	470.930	406.483	382.738	411.168
200	Euler	796.948	479.174	413.561	389.851	420.491
	Timoshenko	782.814	471.229	406.801	383.100	411.609
0	Euler	797.175	479.373	413.773	390.092	420.785
	Timoshenko	783.042	471.428	407.013	383.341	411.903
-200	Euler	797.403	479.572	413.985	390.333	421.079
	Timoshenko	783.270	471.627	407.225	383.582	412.197
-500	Euler	796.606	478.876	413.243	389.489	420.050
	Timoshenko	783.611	471.925	407.544	383.944	412.638
Without piezo layers	Euler	711.323	403.017	330.461	292.974	302.254
	Timoshenko	699.680	397.133	325.896	288.874	297.174

**Fig. 5** A comparison between different types of thermal loading

8 Conclusion

In this article, the equilibrium and stability equations for functionally graded beams with or without piezoelectric layers based on the Timoshenko beam theory are obtained by minimizing the total energy of beam, with assumption of power law composition for the constituent materials. To assure the existence of bifurcation type buckling, pre-buckling analysis of the beam is included. The buckling analysis of such beams under different types of thermal loads is

investigated. Closed form solutions for the critical buckling temperature differences of beams are presented. The followings are concluded:

1. The critical buckling temperature of piezoelectric FGM beam is higher than the piezoelectric metallic beam, but lower than the piezoelectric ceramic one.
2. The critical buckling temperature difference can be controlled by applying suitable voltage on the actuator layers. The effect of this control voltage is small.
3. For functionally graded beams under uniform temperature rise, by increasing the power law index k , the critical buckling temperature decreases for $k < 2$, then increases for $2 < k < 10$, and finally decreases for $k > 10$.
4. Increasing the beam thickness increases the critical buckling temperature difference of the piezoelectric FGM beam.
5. For a piezoelectric FGM beam, the solution of the heat conduction equation results in a nonlinear temperature distribution across the beam thickness. The resulting buckling temperature is greater than that obtained under the assumption of linear temperature distribution.
6. The Timoshenko beam theory predicts lower values for the critical buckling temperature of beams. For beams with $L/h < 20$, the Euler–Bernoulli beam theory over-predicts the buckling temperatures.

Table 4 Critical temperature difference (in [K]) for $C-R$ FGM beams in the case of linear temperature distribution through the thickness

L/h		$k = 0$	$k = 0.5$	$k = 1$	$k = 2$	$k = 5$	$k = 10$
10	Euler	2212.88	1249.79	959.01	797.03	804.31	851.48
	Timoshenko	2156.55	1221.29	938.27	779.55	783.16	826.30
15	Euler	977.95	549.90	421.02	349.34	352.69	373.51
	Timoshenko	966.66	544.20	416.87	345.85	348.45	368.46
20	Euler	545.72	304.95	232.72	192.66	195.62	206.22
	Timoshenko	542.13	303.13	231.40	191.54	193.27	204.62
30	Euler	236.99	129.97	98.22	80.73	81.72	86.73
	Timoshenko	236.28	129.61	97.96	80.51	81.45	86.41
40	Euler	128.93	68.73	51.14	41.56	42.20	44.91
	Timoshenko	128.70	68.62	51.06	41.49	42.12	44.81
50	Euler	78.91	40.39	29.36	23.43	23.91	25.55
	Timoshenko	78.82	40.34	29.32	23.40	23.88	25.51
75	Euler	29.51	12.39	7.84	5.52	5.84	6.43
	Timoshenko	29.50	12.38	7.83	5.52	5.84	6.43

Table 5 Critical temperature difference (in [K]) for $C-R$ FGM beams in the case of non-linear temperature distribution through the thickness

L/h		$k = 0$	$k = 0.5$	$k = 1$	$k = 2$	$k = 5$	$k = 10$
10	Euler	2212.88	1647.78	1331.19	1093.12	1005.90	984.13
	Timoshenko	2156.55	1610.20	1302.40	1069.15	979.45	955.03
15	Euler	977.95	725.02	584.41	479.12	441.09	431.70
	Timoshenko	966.66	717.50	578.65	474.33	435.79	425.86
20	Euler	545.72	402.05	323.04	264.23	243.40	238.35
	Timoshenko	542.13	399.66	321.21	262.70	241.72	236.49
30	Euler	236.99	171.36	136.34	110.73	102.20	100.25
	Timoshenko	236.28	170.89	135.98	110.43	101.86	99.88
40	Euler	128.93	90.62	71.00	57.00	52.77	51.90
	Timoshenko	128.70	90.47	70.88	56.91	52.67	51.79
50	Euler	78.91	53.25	40.75	32.13	29.90	29.53
	Timoshenko	78.82	53.20	40.70	32.10	29.86	29.49
75	Euler	29.51	16.34	10.88	7.58	7.31	7.44
	Timoshenko	29.50	16.33	10.87	7.57	7.30	7.43

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