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The ascending chain condition on principal right ideals for semigroup constructions

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Abstract

We call a semigroup *$\mathcal{R}$-noetherian* if it satisfies the ascending chain condition on principal right ideals, or, equivalently, the ascending chain condition on $\mathcal{R}$-classes. We investigate the behaviour of the property of being $\mathcal{R}$-noetherian under the following standard semigroup-theoretic constructions: semidirect products, Schützenberger products, free products, Rees matrix semigroups, Brandt extensions, Bruck–Reilly extensions and semilattices of semigroups.

Keywords Semigroup · Principal right ideal · Ascending chain condition

Mathematics Subject Classification 20M10 · 20M12

1 Introduction

A *finiteness condition* for a class of universal algebras is a property that is satisfied by at least all finite members of that class. The study of finiteness properties was pioneered by Noether in the early 20th century in the context of ascending chain conditions on rings [14], and has become an established theme in many algebraic disciplines. The main motivation is to develop a better insight into the structure of the objects of study, and, in particular, to get a sense of how different they are to finite objects.

This article is concerned with the class of semigroups and the finiteness condition of satisfying the ascending chain condition on principal right ideals. We call semigroups satisfying this condition *$\mathcal{R}$-noetherian*,[1] owing to the fact that it is equivalent to the ascending chain condition on $\mathcal{R}$-classes ($\mathcal{R}$ is one of the five Green's relations on a semigroup).

The property of being $\mathcal{R}$-noetherian has a natural analogue in ring theory: the ascending chain condition on principal right ideals of rings. Indeed, the study of $\mathcal{R}$-noetherian semigroups was initiated in a paper [8] investigating the ascending chain condition on principal

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[1] $\mathcal{R}$-noetherian semigroups are also known in the literature as 'ACCPR-semigroups' or 'semigroups satisfying ACCPR'.

ideals of rings of generalised power series. The article [10] built on this work by characterising the ascending chain on principal right (and left) ideals for the more general class of skew generalised power series rings $R[[S, \omega]]$ (with coefficients in a ring R and exponents in a strictly totally ordered monoid S), and here again the property of S being $\mathcal{R}$-noetherian is crucial [10, Theorem 3.3]. This work motivated the paper [18], which considers the ascending chain conditions on principal right and left ideals of semidirect product of semigroups and makes a connection with the corresponding properties for rings of skew generalised power series.

A stronger condition than that of being $\mathcal{R}$-noetherian is the property of satisfying the ascending chain condition on *all* right ideals; we call semigroups satisfying this condition *weakly right noetherian*.[2] Such semigroups have received significant attention; see for instance [1, 5, 7, 11]. The property of being weakly right noetherian can be characterised in terms of principal right ideals: a semigroup S is weakly right noetherian if and only if it is $\mathcal{R}$-noetherian and contains no infinite antichain of principal right ideals (or, equivalently, S contains no infinite strictly ascending chain or infinite antichain of $\mathcal{R}$-classes) [11, Theorem 3.2]. Both the properties of being $\mathcal{R}$-noetherian and being weakly right noetherian were considered in the author's recent article [12], of which the main purpose was to study the relationship between a semigroup and its one-sided ideals with respect to each of these properties.

The purpose of the present paper is to investigate the behaviour of the property of being $\mathcal{R}$-noetherian under various semigroup-theoretic contructions. (Of course, our results will also have left-right duals for the property of being $\mathcal{L}$-noetherian.) After introducing the necessary preliminary material in Sect. 2, we consider semidirect products in Sect. 3, Schützenberger products in Sect. 4, free products in Sect. 5, Rees matrix semigroups and Brandt extensions in Sect. 6, Bruck–Reilly extensions in Sect. 7, and semilattices of semigroups in Sect. 8.

2 Preliminaries

Throughout this section, S will denote a semigroup. We denote by S^1 the monoid obtained from S by adjoining an identity if necessary (if S is already a monoid, then $S^1 = S$), and we denote by S^0 the semigroup with zero obtained by adjoining a zero if necessary.

A subset $I \subseteq S$ is said to be a *right ideal* of S if $IS \subseteq I$. Left ideals are defined dually, and an *ideal* of S is a subset that is both a right ideal and a left ideal.

A right (resp. left) ideal I of S is said to be *generated by* $X \subseteq I$ if $I = XS^1$ (resp. $I = S^1X$). A right (resp. left) ideal is said to be *finitely generated* if it can be generated by a finite set, and *principal* if it can be generated by a single element.

Principal (one-sided) ideals determine the five Green's relations on a semigroup: $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$. In this paper we are only concerned with the relation $\mathcal{R}$. Green's preorder $\leq_{\mathcal{R}}$ on S is given by

$$a \leq_{\mathcal{R}} b \iff aS^1 \subseteq bS^1,$$

and this leads to the relation $\mathcal{R}$:

$$a \,\mathcal{R}\, b \iff [a \leq_{\mathcal{R}} b \text{ and } b \leq_{\mathcal{R}} a] \iff aS^1 = bS^1.$$

[2] In the literature, 'right noetherian' is the standard name given to semigroups that satisfy the ascending chain condition on right congruences. However, weakly right noetherian semigroups have occasionally been termed 'right noetherian', while right noetherian semigroups have been called 'strongly right noetherian'.

When we need to distinguish between Green's relation $\mathcal{R}$ on different semigroups, we will write the semigroup as a subscript, i.e. $\mathcal{R}_S$ for $\mathcal{R}$ on S. For convenience, we will write $\leq_S$ rather than $\leq_{\mathcal{R}_S}$, and $a <_S b$ if $a \leq_S b$ but $(a, b) \notin \mathcal{R}_S$.

Green's pre-order $\leq_{\mathcal{R}}$ induces a partial order on the set of $\mathcal{R}$-classes of S. We note that the poset of $\mathcal{R}$-classes of S is isomorphic to the poset of principal right ideals of S (under containment).

A poset P is said to satisfy the *ascending chain condition* if every ascending chain

$$a_1 \leq a_2 \leq \cdots$$

of elements of P eventually stabilises. We say that S is *$\mathcal{R}$-noetherian* if its poset of principal right ideals satisfies the ascending chain condition. The following result provides a number of equivalent formulations for a semigroup to be $\mathcal{R}$-noetherian.

Proposition 2.1 *([12, Proposition 2.3]) The following are equivalent for a semigroup S:*

(1) S is $\mathcal{R}$-noetherian;
(2) every non-empty set of principal right ideals of S contains a maximal element;
(3) the poset of $\mathcal{R}$-classes of S satisfies the ascending chain condition;
(4) every non-empty set of $\mathcal{R}$-classes of S contains a maximal element;
(5) S contains no infinite strictly ascending chain of elements under the $\mathcal{R}$-preorder.

Corollary 2.2 *([12, Corollary 2.10]) Any semigroup S (with zero) that is a union of (0-)minimal right ideals is $\mathcal{R}$-noetherian. In particular, all completely (0-)simple semigroups and all null semigroups are $\mathcal{R}$-noetherian.*

For any non-empty set X, recall that the *free semigroup* on X, denoted by X^+, is the set of all words over X, and the *free monoid* on X, denoted by X^*, is X^+ with an identity adjoined. Clearly, for any $u, v \in X^+$, we have $uX^* \subseteq vX^*$ if and only if v is a subword of u. We deduce that:

Proposition 2.3 *For any non-empty set X, both X^+ and X^* are $\mathcal{R}$-noetherian.*

Since every semigroup is the quotient of a free semigroup, and there certainly exist semigroups that are not $\mathcal{R}$-noetherian, the property of being $\mathcal{R}$-noetherian is not closed under quotients. It turns out, however, that this property is closed under Rees quotients:

Lemma 2.4 *([12, Corollary 3.4]) Let S be a semigroup and let I be an ideal of S. If S is $\mathcal{R}$-noetherian then so is S/I.*

The property of being $\mathcal{R}$-noetherian is also inherited by one-sided ideals:

Proposition 2.5 *([12, Corollary 3.2]) Let S be a semigroup and let I be a right/left/two-sided ideal of S. If S is $\mathcal{R}$-noetherian then so is I.*

Let T be a subsemigroup of S. We say that T is *$\mathcal{R}$-preserving* (in S) if the $\mathcal{R}_T$-preorder is the restriction of the $\mathcal{R}_S$-preorder to T; that is,

$$\leq_T \quad = \quad \leq_S \cap (T \times T).$$

The property of being $\mathcal{R}$-noetherian is inherited by $\mathcal{R}$-preserving subsemigroups:

Proposition 2.6 *Let S be a semigroup and let T be an $\mathcal{R}$-preserving subsemigroup of S. If S is $\mathcal{R}$-noetherian then so is T.*

Proof Consider an ascending chain

$$a_1 \leq_T a_2 \leq_T \cdots$$

in T. Then clearly we have an ascending chain

$$a_1 \leq_S a_2 \leq_S \cdots$$

in S. Since S is $\mathcal{R}$-noetherian, there exists $N \in \mathbb{N}$ such that $a_n \, \mathcal{R}_S \, a_N$ for all $n \geq N$. Then, since T is $\mathcal{R}$-preserving in S, we have $a_n \, \mathcal{R}_T \, a_N$ for all $n \geq N$. Hence T is $\mathcal{R}$-noetherian. □

A semigroup T is called *regular* if for every $a \in T$ there exists $b \in T$ such that $aba = a$. It is well known that regular subsemigroups are $\mathcal{R}$-preserving; see [11, paragraph before Corollary 4.7] for a proof. Thus we have:

Corollary 2.7 *Let S be a semigroup with a regular subsemigroup T. If S is $\mathcal{R}$-noetherian then so is T.*

A subsemigroup T of S is called *right unitary* (in S) if it satisfies the following condition: for all $a \in T$ and $b \in S$, if $ab \in T$ then $b \in T$.

Clearly a right unitary subsemigroup is $\mathcal{R}$-preserving, so by Proposition 2.6 we have:

Corollary 2.8 *Let S be a semigroup and let T be a right unitary subsemigroup of S. If S is $\mathcal{R}$-noetherian then so is T.*

If the complement of a subsemigroup is a left ideal, then the subsemigroup is right unitary, so we have:

Corollary 2.9 *Let S be a semigroup with a subsemigroup T such that $S \setminus T$ is a left ideal of S. If S is $\mathcal{R}$-noetherian then so is T.*

We now a introduce a key notion for this paper.

Definition 2.10 Let S be a semigroup and let $a \in S$. We say that $b \in S$ is a *local right identity* of a if $a = ab$.

Clearly in a monoid or regular semigroup, every element has a local right identity. On the other hand, any left cancellative, idempotent-free semigroup (e.g. a free semigroup) has *no* element with a local right identity. Note that a semigroup in which no element has a local right identity is $\mathcal{R}$-trivial, i.e. $\mathcal{R}$ is the identity relation.

Proposition 2.11 *Let S and T be semigroups with a map $\theta : S \to T$ such that $(aS)\theta \subseteq (a\theta)T$ for each $a \in S$. If T is $\mathcal{R}$-noetherian and has no element with a local right identity, then S is $\mathcal{R}$-noetherian and has no element with a local right identity.*

Proof It is clear if S had an element with a local right identity then so would T, so S has no element with a local right identity. By Proposition 2.1, to prove that S is $\mathcal{R}$-noetherian it suffices to show that it contains no infinite strictly ascending chain of elements under the $\mathcal{R}$-preorder. So, consider an ascending chain

$$a_1 \leq_S a_2 \leq_S \cdots$$

in S. Then, for each $i \in \mathbb{N}$, we have $a_i \in a_{i+1}S^1$. Therefore, by assumption, we have $a_i\theta \in (a_{i+1}S^1)\theta \subseteq (a_{i+1}\theta)T^1$. Thus, we have an ascending chain

$$a_1\theta \leq_T a_2\theta \leq_T \cdots$$

in T. Since T is $\mathcal{R}$-noetherian and $\mathcal{R}$-trivial, there exists $N \in \mathbb{N}$ such that $a_n\theta = a_N\theta$ for all $n \geq N$. For each $n \geq N$, we cannot have $a_N \in a_nS$, for then we would have

$$a_N\theta \in (a_nS)\theta \subseteq (a_n\theta)T = (a_N\theta)T,$$

contradicting the fact that T has no element with a local right identity. Thus $a_n = a_N$ for all $n \geq N$. Hence S is $\mathcal{R}$-noetherian. □

Corollary 2.12 *Let S and T be semigroups with a homomorphism $\theta\colon S \to T$. If T is $\mathcal{R}$-noetherian and has no element with a local right identity, then S is $\mathcal{R}$-noetherian and has no element with a local right identity.*

Proposition 2.13 *[[12, Proposition 3.9]] Let S be a semigroup, let I be an ideal of S, and suppose that every element of I has a local right identity in I. Then S is $\mathcal{R}$-noetherian if and only if both I and S/I are $\mathcal{R}$-noetherian.*

3 Semidirect products

Let S and T be semigroups, and let $\varphi\colon T \to \mathrm{End}(S)$ be a homomorphism, where $\mathrm{End}(S)$ denotes the monoid of all endomorphisms of S. The image of an element $t \in T$ under φ will be denoted by φ_t. We write φ_t on the left of its argument; i.e. $\varphi_t(s)$ for $s \in S$. The *semidirect product of S and T with respect to φ*, denoted by $S \rtimes_\varphi T$, is the semigroup with underlying set $S \times T$ and multiplication given by

$$(s, t)(s', t') = (s\varphi_t(s'), tt').$$

Note that the direct product $S \times T$ is the semidirect product $S \rtimes_\varphi T$ where $\varphi_t = id_S$ for all $t \in T$.

The property of being $\mathcal{R}$-noetherian was considered for semidirect products in [18]. Several partial characterisations were obtained for a semidirect product $S \rtimes_\varphi T$ to be $\mathcal{R}$-noetherian. For instance, if both S and T contain at least one idempotent and φ_t is surjective for every $t \in T$, then $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian if and only if S and T are $\mathcal{R}$-noetherian [18, Theorem 3.13].

The purpose of this section is to provide necessary and sufficient conditions for a semidirect product to be $\mathcal{R}$-noetherian. To this end, we first prove a few lemmas.

Lemma 3.1 *If $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian, then at least one of S and T is $\mathcal{R}$-noetherian.*

Proof Suppose for a contradiction that neither S nor T are $\mathcal{R}$-noetherian. Then there exist infinite strictly ascending chains

$$a_1 <_S a_2 <_S \cdots \quad \text{and} \quad b_1 <_T b_2 <_T \cdots$$

in S and T, respectively. Then, for each $i \in \mathbb{N}$, there exist $s_i \in S$ and $t_i \in T$ such that $a_i = a_{i+1}s_i$ and $b_i = b_{i+1}t_i$. Observe that $b_1 = b_{i+1}t_i \dots t_1$ for each $i \in \mathbb{N}$. We have

$$\begin{aligned}\varphi_{b_1}(a_i) = \varphi_{b_1}(a_{i+1}s_i) = \varphi_{b_1}(a_{i+1})\varphi_{b_1}(s_i) &= \varphi_{b_1}(a_{i+1})\varphi_{b_{i+1}t_i\dots t_1}(s_i)\\ &= \varphi_{b_1}(a_{i+1})\varphi_{b_{i+1}}(\varphi_{t_i\dots t_1}(s_i)).\end{aligned}$$

Therefore, we have

$$(\varphi_{b_1}(a_i), b_i) = (\varphi_{b_1}(a_{i+1}), b_{i+1})(\varphi_{t_i\dots t_1}(s_i), t_i).$$

Thus, letting $U = S \rtimes_\varphi T$, we have an ascending chain

$$(\varphi_{b_1}(a_1), b_1) \leq_U (\varphi_{b_1}(a_2), b_2) \leq_U \cdots$$

in U. Since U is $\mathcal{R}$-noetherian, there exists $N \in \mathbb{N}$ such that $(\varphi_{b_1}(a_n), b_n) \mathcal{R}_U (\varphi_{b_1}(a_N), b_N)$ for all $n \geq N$. In particular, there exists $(s, t) \in U$ such that $(\varphi_{b_1}(a_{N+1}), b_{N+1}) = (\varphi_{b_1}(a_N), b_N)(s, t)$. But then $b_{N+1} = b_N t$, contradicting that $b_N <_T b_{N+1}$. □

Lemma 3.2 *Suppose that S has an element with a local right identity. If $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian, then T is $\mathcal{R}$-noetherian.*

Proof Suppose for a contradiction that T is not $\mathcal{R}$-noetherian. Then there exists an infinite strictly ascending chain

$$b_1 <_T b_2 <_T \cdots$$

in T. Then, for each $i \in \mathbb{N}$, there exists $t_i \in T$ such that $b_i = b_{i+1}t_i$. Let $a \in S$ have a local right identity $s \in S$, so that $a = as$. We have

$$\varphi_{b_1}(a) = \varphi_{b_1}(as) = \varphi_{b_1}(a)\varphi_{b_1}(s) = \varphi_{b_1}(a)\varphi_{b_{i+1}}(\varphi_{t_i \ldots t_1}(s)),$$

and hence

$$(\varphi_{b_1}(a), b_i) = (\varphi_{b_1}(a), b_{i+1})(\varphi_{t_i \ldots t_1}(s), t_i).$$

The final part of the proof is essentially the same as that of Lemma 3.1. □

Lemma 3.3 *If either S or T is $\mathcal{R}$-noetherian and has no element with a local right identity, then $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian.*

Proof We claim that the projection map $\pi_S \colon S \rtimes_\varphi T \to S$ satisfies the condition of Proposition 2.11. Indeed, for any $(a, b), (s, t) \in S \rtimes_\varphi T$, we have

$$\big((a, b)(s, t)\big)\pi_S = (a\varphi_b(s), bt)\pi_S = a\varphi_b(s) = \big((a, b)\pi_S\big)\varphi_b(s) \in \big((a, b)\pi_S\big)S,$$

as required. It is clear that the projection map $\pi_T \colon S \rtimes_\varphi T \to T$ is a homomorphism. Hence, the result follows from Proposition 2.11 and Corollary 2.12. □

Lemma 3.4 *If $a, a' \in S$ and $b, b' \in T$ with $b \in b'T$, then $a\big(\varphi_b(S)\big)^1 \subseteq a'\big(\varphi_{b'}(S)\big)^1$ if and only if $a \in a'\big(\varphi_{b'}(S)\big)^1$. Moreover, if $b \,\mathcal{R}_T\, b'$ then $\varphi_b(S) = \varphi_{b'}(S)$.*

Proof If $a\big(\varphi_b(S)\big)^1 \subseteq a'\big(\varphi_{b'}(S)\big)^1$, then clearly $a \in a'\big(\varphi_{b'}(S)\big)^1$. Conversely, suppose that $a \in a'\big(\varphi_{b'}(S)\big)^1$. There exists $t \in T$ such that $b = b't$. Thus, we have

$$\varphi_b(S) = \varphi_{b't}(S) \subseteq \varphi_{b'}(\varphi_t(S)) \subseteq \varphi_{b'}(S),$$

and hence $a(\varphi_b(S))^1 \subseteq a'(\varphi_{b'}(S))^1$.

Now, if $b \,\mathcal{R}_T\, b'$, a similar argument as above proves that $\varphi_{b'}(S) \subseteq \varphi_b(S)$, and hence $\varphi_b(S) = \varphi_{b'}(S)$. □

Before stating the main result of this section, we first introduce the following definition.

Definition 3.5 Let S and T be semigroups, and let $\varphi \colon T \to \mathrm{End}(S)$ be a homomorphism. A *φ-chain* in S is an ascending chain of the form

$$a_1\big(\varphi_{b_1}(S)\big)^1 \subseteq a_2\big(\varphi_{b_2}(S)\big)^1 \subseteq a_3\big(\varphi_{b_3}(S)\big)^1 \subseteq \cdots$$

where $a_i \in S$, $b_i \in T$ and $b_i \in b_{i+1}T$ for all $i \geq 1$.

Theorem 3.6 *Let S and T be semigroups, and let $\varphi\colon T \to \mathrm{End}(S)$ be a homomorphism. Then $S \ltimes_{\varphi} T$ is $\mathcal{R}$-noetherian if and only if either:*

(1) S is $\mathcal{R}$-noetherian and has no element with a local right identity; or
(2) every φ-chain in S eventually stabilises and T is $\mathcal{R}$-noetherian.

Proof Let $U = S \ltimes_{\varphi} T$.

($\Rightarrow$) Suppose that (1) does not hold. Then either S is not $\mathcal{R}$-noetherian or S has an element with a local right identity. In the former case, T is $\mathcal{R}$-noetherian by Lemma 3.1, and in the latter case, T is $\mathcal{R}$-noetherian by Lemma 3.2.

Now suppose for a contradiction that there exists an infinite φ-chain

$$a_1\big(\varphi_{b_1}(S)\big)^1 \subsetneq a_2\big(\varphi_{b_2}(S)\big)^1 \subsetneq \cdots$$

in S. Then, for each $i \in \mathbb{N}$, there exists $t_i \in T$ such that $b_i = b_{i+1}t_i$. Thus, we have an ascending chain

$$b_1 \leq_T b_2 \leq_T \cdots$$

in T. Since T is $\mathcal{R}$-noetherian, there exists $N \in \mathbb{N}$ such that $b_n \,\mathcal{R}_T\, b_N$ for all $n \geq N$. Then, by Lemma 3.4, we have $\varphi_{b_n}(S) = \varphi_{b_N}(S)$ for all $n \geq N$. Therefore, we have an infinite φ-chain

$$a_N\big(\varphi_{b_N}(S)\big)^1 \subsetneq a_{N+1}\big(\varphi_{b_N}(S)\big)^1 \subsetneq \cdots.$$

It follows that for each $n \geq N$ there exists $s_n \in S$ such that $a_n = a_{n+1}\varphi_{b_N}(s_n)$. Also, since $b_N \,\mathcal{R}_T\, b_{N+1}$ and $b_N \in b_{N+1}T$, there exists $t \in T$ such that $b_N = b_N t$. Hence, we have $(a_n, b_N) = (a_{n+1}, b_N)(s_n, t)$ for each $n \geq N$. Thus, we have an ascending chain

$$(a_N, b_N) \leq_U (a_{N+1}, b_N) \leq_U \cdots$$

in U. Since U is $\mathcal{R}$-noetherian, there exists $N' \geq N$ such that $(a_n, b_N)\,\mathcal{R}_U\,(a_{N'}, b_N)$ for all $n \geq N'$. In particular, there exists $(s, t') \in U$ such that $(a_{N'+1}, b_N) = (a_{N'}, b_N)(s, t')$. Then $a_{N'+1} = a_{N'}\varphi_{b_N}(s)$. But then

$$a_{N'}\big(\varphi_{b_{N'}}(S)\big)^1 = a_{N'}\big(\varphi_{b_N}(S)\big)^1 = a_{N'+1}\big(\varphi_{b_N}(S)\big)^1 = a_{N'+1}\big(\varphi_{b_{N'+1}}(S)\big)^1,$$

and we have a contradiction.

($\Leftarrow$) If (1) holds, then U is $\mathcal{R}$-noetherian by Lemma 3.3. Assume then that (2) holds. Consider an ascending chain

$$(a_1, b_1) \leq_U (a_2, b_2) \leq_U \cdots$$

in U. We may assume without loss of generality that $(a_i, b_i) \in (a_{i+1}, b_{i+1})U$ for each $i \in \mathbb{N}$. Then, for each $i \in \mathbb{N}$, there exists $(s_i, t_i) \in U$ such that $(a_i, b_i) = (a_{i+1}, b_{i+1})(s_i, t_i)$. Then $a_i = a_{i+1}\varphi_{b_{i+1}}(s_i)$ and $b_i = b_{i+1}t_i$. Therefore, we have a φ-chain

$$a_1\big(\varphi_{b_1}(S)\big)^1 \subseteq a_2\big(\varphi_{b_2}(S)\big)^1 \subseteq \cdots$$

in S, and an ascending chain

$$b_1 \leq_T b_2 \leq_T \cdots$$

in T. Since every φ-chain eventually stabilises and T is $\mathcal{R}$-noetherian, there exists $N \in \mathbb{N}$ such that $a_n\big(\varphi_{b_n}(S)\big)^1 = a_N\big(\varphi_{b_N}(S)\big)^1$ and $b_n \,\mathcal{R}_T\, b_N$ for all $n \geq N$. Therefore, by Lemma 3.4, for each $n \geq N$ we have $a_n \in a_N\big(\varphi_{b_N}(S)\big)^1$ and $\varphi_{b_n}(S) = \varphi_{b_N}(S)$. Since $a_N = a_{N+1}\varphi_{b_{N+1}}(s_N)$, we have

$$a_n \in a_{N+1}\varphi_{b_{N+1}}(S)\big(\varphi_{b_N}(S)\big)^1 = a_{N+1}\varphi_{b_{N+1}}(S)\big(\varphi_{b_{N+1}}(S)\big)^1 = a_{N+1}\varphi_{b_{N+1}}(S).$$

Thus, for each $n \geq N+1$, there exist $u_n \in S$ and $v_n \in T$ such that $a_n = a_{N+1}\varphi_{b_{N+1}}(u_n)$ and $b_n = b_{N+1}v_n$. It follows that $(a_n, b_n) = (a_{N+1}, b_{N+1})(u_n, v_n)$, and hence we have $(a_n, b_n)\,\mathcal{R}_U\,(a_{N+1}, b_{N+1})$. This completes the proof. □

Corollary 3.7 *Let S be an $\mathcal{R}$-noetherian semigroup whose $\mathcal{R}$-classes are all finite, let T be a semigroup, and let $\varphi\colon T \to \mathrm{End}(S)$ be a homomorphism. Then $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian if and only if either S has no element with a local right identity or T is $\mathcal{R}$-noetherian.*

Proof By Theorem 3.6, it suffices to prove that if T is $\mathcal{R}$-noetherian then every φ-chain in S eventually stablises. So, let T be $\mathcal{R}$-noetherian and suppose for a contradiction that there exists an infinite φ-chain

$$a_1\big(\varphi_{b_1}(S)\big)^1 \subsetneq a_2\big(\varphi_{b_2}(S)\big)^1 \subsetneq \cdots$$

in S. Then we have ascending chains

$$a_1 \leq_S a_2 \leq_S \cdots \quad \text{and} \quad b_1 \leq_T b_2 \leq_T \cdots$$

in S and T, respectively. Since S and T are $\mathcal{R}$-noetherian, there exists $N \in \mathbb{N}$ such that $a_n\,\mathcal{R}_S\,a_N$ and $b_n\,\mathcal{R}_T\,b_N$ for all $n \geq N$. Then, by Lemma 3.4, we have $\varphi_{b_n}(S) = \varphi_{b_N}(S)$ for all $n \geq N$. Therefore, we have an infinite φ-chain

$$a_N\big(\varphi_{b_N}(S)\big)^1 \subsetneq a_{N+1}\big(\varphi_{b_N}(S)\big)^1 \subsetneq \cdots .$$

But then $a_m \neq a_n$ for all $m, n \geq N$ with $m \neq n$, contradicting the fact that the $\mathcal{R}_S$-class of a_N is finite. □

Certainly every finite semigroup has an element with a local right identity, so we deduce:

Corollary 3.8 *Let S be a finite semigroup, let T be a semigroup, and let $\varphi\colon T \to \mathrm{End}(S)$ be a homomorphism. Then $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian if and only if T is $\mathcal{R}$-noetherian.*

The following example demonstrates that for a semidirect product $S \rtimes_\varphi T$ to be $\mathcal{R}$-noetherian, the semigroup S need not be $\mathcal{R}$-noetherian, even in the case that T is trivial.

Example 3.9 Let S be any monoid with identity 1, let $T = \{e\}$ be the trivial semigroup, and let $\varphi\colon T \to \mathrm{End}(S)$ be given by defining φ_e to be the constant map c_1 on 1. Then $(a, e)(a', e) = (a, e)$ for all $a, a' \in S$, so that $S \rtimes_\varphi T$ is a left zero semigroup, and is hence $\mathcal{R}$-noetherian by Corollary 2.2. (Alternatively, for any $a, a' \in S$, we clearly have $a \in a'\big(\varphi_e(S)\big)^1$ if and only if $a = a'$, so certainly every φ-chain in S eventually stabilises. Since T is trivially $\mathcal{R}$-noetherian, it then follows from Theorem 3.6 that $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian.)

We now show that a semidirect product may *not* be $\mathcal{R}$-noetherian even if both its semidirect factors are $\mathcal{R}$-noetherian.

Example 3.10 Let S be the disjoint union of (a copy of) the free monogenic monoid $\{x\}^*$ and a set $\{a_i : i \in \mathbb{Z}\}$. Define a multiplication on S, extending that on $\{x\}^*$, by

$$a_i x^n = a_{i-n} \quad \text{and} \quad x^n a_j = a_i a_j = a_j$$

for all $n \in \mathbb{N}_0$ and $i, j \in \mathbb{Z}$. It is straightforward to show that S is a monoid under this multiplication. Clearly $I = \{a_i : i \in \mathbb{Z}\}$ is an ideal of S and a right zero semigroup; in particular, every element of I has a local right identity in I. Since I and $S/I \cong \mathbb{N}^0$ are $\mathcal{R}$-noetherian, we have that S is $\mathcal{R}$-noetherian by Proposition 2.13. Hence S^0 is $\mathcal{R}$-noetherian. Now let $T = \{e\}$ be the trivial semigroup, and let $\varphi\colon T \to \mathrm{End}(S^0)$ be given by

$$\varphi_e(s) = \begin{cases} s & \text{if } s \in \{x^n : n \in \mathbb{N}_0\}, \\ 0 & \text{if } s \in \{a_i : i \in \mathbb{Z}\} \cup \{0\}. \end{cases}$$

For each $i \geq 0$, we have $a_i = a_{i+1}x = a_{i+1}\varphi_e(x)$, so that $a_i\big(\varphi_e(S)\big)^1 \subseteq a_{i+1}\big(\varphi_e(S)\big)^1$. For any $s \in S^0$ we have

$$a_i\varphi_e(s) \in a_i(\{x_n : n \in \mathbb{N}_0\} \cup \{0\}) \subseteq \{a_k : k \leq i\} \cup \{0\}.$$

Thus, we have an infinite φ-chain

$$a_1\big(\varphi_e(S)\big)^1 \subsetneq a_2\big(\varphi_e(S)\big)^1 \subsetneq \cdots .$$

Hence, by Theorem 3.6, $S^0 \rtimes_\varphi T$ is not $\mathcal{R}$-noetherian.

For a semigroup S, we denote by $\mathrm{SEnd}(S)$ the monoid of all surjective endomorphisms of S. We shall use Theorem 3.6 to deduce necessary and sufficient conditions for $S \rtimes_\varphi T$ to be $\mathcal{R}$-noetherian in the case that $\varphi_t \in \mathrm{SEnd}(S)$ for every $t \in T$. First we prove another lemma.

Lemma 3.11 *Suppose that there exists an element $b \in T$ such that b has a local right identity and $\varphi_b \in \mathrm{SEnd}(S)$. If $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian, then S is $\mathcal{R}$-noetherian.*

Proof Suppose for a contradiction that S is not $\mathcal{R}$-noetherian. Then there exists an infinite strictly ascending chain

$$a_1 <_S a_2 <_S \cdots$$

in S. Then, for each $i \in \mathbb{N}$, there exists $s_i \in S$ such that $a_i = a_{i+1}s_i$. Since φ_b is surjective, for each $i \in \mathbb{N}$ there exists $s_i' \in S$ such that $\varphi_b(s_i') = s_i$. Let t' be a local right identity of b, so that $b = bt'$. Then $(a_i, b) = (a_{i+1}, b)(s_i', t')$. Thus, letting $U = S \rtimes_\varphi T$, we have an ascending chain

$$(a_1, b) \leq_U (a_2, b) \leq_U \cdots$$

in U. Since U is $\mathcal{R}$-noetherian, there exists $N \in \mathbb{N}$ such that $(a_n, b)\,\mathcal{R}_U\,(a_N, b)$ for all $n \geq N$. In particular, there exists $(s, t) \in U$ such that $(a_{N+1}, b) = (a_N, b)(s, t)$. But then $a_{N+1} = a_N\varphi_b(s)$, contradicting that $a_N <_T a_{N+1}$. □

Theorem 3.12 *Let S and T be semigroups, and let $\varphi\colon T \to \mathrm{SEnd}(S)$ be a homomorphism. Then $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian if and only if one of the following holds:*

(1) both S and T are $\mathcal{R}$-noetherian;
(2) S is $\mathcal{R}$-noetherian and has no element with a local right identity;
(3) T is $\mathcal{R}$-noetherian and has no element with a local right identity.

Proof ($\Rightarrow$) Assume that (2) and (3) do not hold. Then T has an element, say b, with a local right identity. Since $\varphi_b \in \mathrm{SEnd}(S)$, we have that S is $\mathcal{R}$-noetherian by Lemma 3.11. Also, it follows from Theorem 3.6 that T is $\mathcal{R}$-noetherian. Thus (1) holds.

($\Leftarrow$) If (2) or (3) holds, then $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian by Lemma 3.3. Assume then that (1) holds. Since $\varphi_b(S) = S$ for all $b \in T$, every φ-chain in S is an ascending chain of principal right ideals of S, and hence must eventually stabilise as S is $\mathcal{R}$-noetherian. Therefore, since T is $\mathcal{R}$-noetherian, we have that $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian by Theorem 3.6. □

Corollary 3.13 *[18, Theorem 3.13] Let S and T be semigroups with idempotents, and let $\varphi\colon T \to \mathrm{SEnd}(S)$ be a homomorphism. Then $S \rtimes_\varphi T$ is $\mathcal{R}$-noetherian if and only if S and T are $\mathcal{R}$-noetherian.*

Corollary 3.14 *Let S and T be semigroups. Then the direct product $S \times T$ is $\mathcal{R}$-noetherian if and only if one of the following holds:*

(1) both S and T are $\mathcal{R}$-noetherian;
(2) S is $\mathcal{R}$-noetherian and has no element with a local right identity;
(3) T is $\mathcal{R}$-noetherian and has no element with a local right identity.

4 Schützenberger products

The Schützenberger product of semigroups was introduced by Schützenberger in [17] in relation to the study of finite aperiodic monoids. It has since found many other useful applications in semigroup theory; see [9, 15] for instance.

For any set X, let $\mathcal{P}_f(X)$ denote the set of all finite subsets of X. Let S and T be semigroups. For $s \in S, t \in T$ and $P \in \mathcal{P}_f(S \times T)$, we define

$$sP = \{(sp, q) : (p, q) \in P\} \quad \text{and} \quad Pt = \{(p, qt) : (p, q) \in P\}.$$

The *Schützenberger product* of S and T, denoted by $S \diamondsuit T$, is the semigroup with universe $S \times \mathcal{P}_f(S \times T) \times T$ and multiplication given by

$$(s_1, P_1, t_1)(s_2, P_2, t_2) = (s_1 s_2, s_1 P_2 \cup P_1 t_2, t_1 t_2).$$

Observe that the direct product $S \times T$ embeds into $S \diamondsuit T$ via $(s, t) \mapsto (s, \emptyset, t)$. Unlike for $S \times T$, the multiplication in $S \diamondsuit T$ is asymmetrical, and hence $S \diamondsuit T$ is not in general isomorphic to $T \diamondsuit S$. The main theme of this section is the relationship between the Schützenberger product and the direct product with regard to being $\mathcal{R}$-noetherian. We begin with the following lemma.

Lemma 4.1 *Let S and T be semigroups. If $S \diamondsuit T$ is $\mathcal{R}$-noetherian, then so is the direct product $S \times T$.*

Proof Notice that $S \times \{\emptyset\} \times T \cong S \times T$ and that $S \times \{\emptyset\} \times T$ is a subsemigroup of $S \diamondsuit T$ such that $(S \diamondsuit T) \setminus (S \times \{\emptyset\} \times T)$ is an ideal of $S \diamondsuit T$. Therefore, by Corollary 2.9, if $S \diamondsuit T$ is $\mathcal{R}$-noetherian then is $S \times T$. ◻

Letting $\{1\}$ denote the trivial group, it is clear that $\{1\} \diamondsuit T$ is isomorphic to the semigroup $\mathcal{P}_f(T) \times T$ with multiplication given by

$$(P_1, t_1)(P_2, t_2) = (P_1 t_2 \cup P_2, t_1 t_2),$$

where $Pt = \{pt : p \in P\}$ for $P \in \mathcal{P}_f(T)$ and $t \in T$.

Lemma 4.2 *Let S and T be semigroups, and suppose that there exists an element $a \in S$ that has a local right identity in S. If $S \diamondsuit T$ is $\mathcal{R}$-noetherian, then $\{1\} \diamondsuit T$ is $\mathcal{R}$-noetherian.*

Proof We prove the contrapositive. Let $U = \mathcal{P}_f(T) \times T \cong \{1\} \diamondsuit T$ and let $V = S \diamondsuit T$. Assume that U is not $\mathcal{R}$-noetherian. Then there exists an infinite strictly ascending chain

$$(P_1, b_1) <_U (P_2, b_2) <_U \cdots$$

in U. Then, for each $i \in \mathbb{N}$, there exists $(Q_i, t_i) \in U$ such that $(P_i, b_i) = (P_{i+1}, b_{i+1})(Q_i, t_i)$. Let s be a local right identity of a (so $as = a$), and for each $i \in \mathbb{N}$ let $P_i' = \{a\} \times P_i$ and $Q_i' = \{s\} \times Q_i$. We then have

$$(a, P_i', b_i) = (a, P_{i+1}', b_{i+1})(s, Q_i', t_i).$$

Suppose for a contradiction that there exists $i \in \mathbb{N}$ such that $(a, P_{i+1}', b_{i+1}) \in (a, P_i', b_i)V^1$. Then, either $(a, P_{i+1}', b_{i+1}) = (a, P_i', b_i)$ or there exists $(s', Q, t') \in V$ such that

$$(a, P_{i+1}', b_{i+1}) = (a, P_i', b_i)(s', Q, t') = (as', P_i't' \cup aQ, b_it').$$

But then either $(P_{i+1}, b_{i+1}) = (P_i, b_i)$ or, letting π_T denote the projection map $S \times T \to T$, we have

$$(P_{i+1}, b_{i+1}) = (P_i, b_i)(Q\pi_T, t')$$

so that $(P_{i+1}, b_{i+1}) \in (P_i, b_i)U^1$, a contradiction. Thus, we have an infinite strictly ascending chain

$$(a, P_1', b_1) <_V (a, P_2', b_2) <_V \cdots$$

in V. Hence V is not $\mathcal{R}$-noetherian. □

We now provide an example of an $\mathcal{R}$-noetherian semigroup T such that $\{1\} \Diamond T$ is not $\mathcal{R}$-noetherian. In particular, the converse of Lemma 4.1 does not hold (since $\{1\} \times T \cong T$).

Example 4.3 Let T be the monoid S from Example 3.10; that is,

$$T = \{x^n : n \in \mathbb{N}_0\} \sqcup \{a_i : i \in \mathbb{Z}\}$$

with product given by

$$x^m x^n = x^{m+n}, \quad a_i x^n = a_{i-n} \quad \text{and} \quad x^n a_j = a_i a_j = a_j.$$

As shown in Example 3.10, T is $\mathcal{R}$-noetherian. We prove that $U = \mathcal{P}_f(T) \times T \cong \{1\} \Diamond T$ is not $\mathcal{R}$-noetherian.

Consider $i \in \mathbb{N}$. We have $(\{a_i\}, a_{i+1}) = (\{a_{i+1}\}, a_{i+2})(\emptyset, x)$. Suppose for a contradiction that $(\{a_{i+1}\}, a_{i+2}) = (\{a_i\}, a_{i+1})(P, t)$ for some $(P, t) \in U$. Then $a_{i+1} = a_i t$ and $a_{i+2} = a_{i+1}t$. We cannot have $t \in \{x\}^*$, for then we would have $a_i t \in \{a_j : j \leq i\}$, and we cannot have $t \in \{a_j : j \in \mathbb{Z}\}$, for then we would have $a_i t = a_{i+1} t = t$. Thus, no such t exists, and we have the desired contradiction. It follows that we have an infinite ascending chain

$$(\{a_1\}, a_2) <_U (\{a_2\}, a_3) <_U \cdots$$

in U, so U is not $\mathcal{R}$-noetherian.

In the remainder of this section, we explore situations in which $S \Diamond T$ being $\mathcal{R}$-noetherian is equivalent to $S \times T$ being $\mathcal{R}$-noetherian. This turns out to be the case when T is finite or cancellative.

Since S and T are homomorphic images of $S \Diamond T$, by Proposition 2.11 we have:

Lemma 4.4 *Let S and T be semigroups. If either of S and T is $\mathcal{R}$-noetherian and has no element with a local right identity, then $S \Diamond T$ is $\mathcal{R}$-noetherian.*

Lemmas 4.1 and 4.4 and Corollary 3.14 yield:

Proposition 4.5 *Let S and T be semigroups where S (resp. T) is not $\mathcal{R}$-noetherian. Then the following are equivalent:*

(1) $S \Diamond T$ is $\mathcal{R}$-noetherian;
(2) $S \times T$ is $\mathcal{R}$-noetherian;
(3) T (resp. S) is $\mathcal{R}$-noetherian and has no element with a local right identity.

Theorem 4.6 *Let S and T be semigroups where T is finite. Then the following are equivalent:*

(1) $S \Diamond T$ is $\mathcal{R}$-noetherian;
(2) $S \times T$ is $\mathcal{R}$-noetherian;
(3) S is $\mathcal{R}$-noetherian.

Proof (1)⇒(2) is Lemma 4.1, and (2)⇒(3) follows from Corollary 3.14.

(3)⇒(1). Let $U = S \Diamond T$, and suppose for a contradiction that U is not $\mathcal{R}$-noetherian. Then there exists an infinite strictly ascending chain

$$(a_1, P_1, b_1) <_U (a_2, P_2, b_2) <_U \cdots$$

in U. For each $i \in \mathbb{N}$, there exists $(s_i, Q_i, t_i) \in U$ such that

$$(a_i, P_i, b_i) = (a_{i+1}, P_{i+1}, b_{i+1})(s_i, Q_i, t_i) = (a_{i+1}s_i, P_{i+1}t_i \cup a_{i+1}Q_i, b_{i+1}t_i),$$

so $a_i = a_{i+1}s_i$, $P_i = P_{i+1}t_i \cup a_{i+1}Q_i$ and $b_i = b_{i+1}t_i$. Thus, we have an ascending chain

$$a_1 \leq_S a_2 \leq_S \cdots$$

in S. Observe that for each $i \in \mathbb{N}$ we have

$$P_i\pi_S = P_{i+1}\pi_S \cup a_{i+1}Q_i\pi_S.$$

It follows that

$$P_1\pi_S \supseteq P_2\pi_S \supseteq \cdots .$$

Since S is $\mathcal{R}$-noetherian, and there does not exist any infinite strictly descending chain of finite subsets of S, there exists $N \in \mathbb{N}$ such that $a_n \mathcal{R}_S a_N$ and $P_n\pi_S = P_N\pi_S$ for all $n \geq N$. Then, for each $n \geq N$, we have $P_n \subseteq P_N\pi_S \times T$. Since P_N and T are finite (and hence $P_N\pi_S \times T$ is finite), it follows that there exist $m, n \geq N$ with $m \leq n-2$ such that $P_m = P_n$ and $b_m = b_n$. Since $a_m \mathcal{R}_S a_n$ and $a_m \in a_n S$, there exists $s' \in S$ such that $a_n = a_m s'$. Then we have

$$\begin{aligned}(a_{m+1}, P_{m+1}, b_{m+1})(s_m s', Q_m, t_m) &= (a_{m+1}s_m s', P_{m+1}t_m \cup a_{m+1}Q_m, b_{m+1}t_m)\\ &= (a_m s', P_m, b_m)\\ &= (a_n, P_n, b_n).\end{aligned}$$

But this contradicts that $(a_{m+1}, P_{m+1}, b_{m+1}) <_U (a_n, P_n, b_n)$. Hence U is $\mathcal{R}$-noetherian. □

Theorem 4.7 *Let S and T be semigroups where T is cancellative. Then $S \Diamond T$ is $\mathcal{R}$-noetherian if and only if $S \times T$ is $\mathcal{R}$-noetherian.*

Proof (⇒). This follows immediately from Lemma 4.1.

(⇐) Let $U = S \Diamond T$, and suppose for a contradiction that U is not $\mathcal{R}$-noetherian. Then, by Lemma 4.4 and Corollary 3.14, both S and T are $\mathcal{R}$-noetherian and have elements with local right identities. It is straightforward to show that a cancellative semigroup has an element with a local right identity if and only it is a monoid; thus T is a monoid.

Now, there exists an infinite strictly ascending chain

$$(a_1, P_1, b_1) <_U (a_2, P_2, b_2) <_U \cdots$$

in U. For each $i \in \mathbb{N}$, there exists $(s_i, Q_i, t_i) \in U$ such that

$$(a_i, P_i, b_i) = (a_{i+1}, P_{i+1}, b_{i+1})(s_i, Q_i, t_i) = (a_{i+1}s_i, P_{i+1}t_i \cup a_{i+1}Q_i, b_{i+1}t_i),$$

so $a_i = a_{i+1}s_i$, $P_i = P_{i+1}t_i \cup a_{i+1}Q_i$ and $b_i = b_{i+1}t_i$. Thus, we have ascending chains

$$a_1 \leq_S a_2 \leq_S \cdots \quad \text{and} \quad b_1 \leq_T b_2 \leq_T \cdots$$

in S and T, respectively. Since S is $\mathcal{R}$-noetherian, there exists some $N \in \mathbb{N}$ such that $a_n \mathcal{R}_S a_N$ for all $n \geq N$.

Now, it follows from the cancellativity of T that the maps

$$P_{i+1} \to P_{i+1}t_i, (x, y) \mapsto (x, yt_i) \ (i \in \mathbb{N})$$

are bijections. Using the fact that $P_{i+1}t_i \subseteq P_i$, we deduce that we have a chain

$$|P_1| \geq |P_2| \geq \cdots$$

of non-negative integers. This chain must eventually stabilise; we may assume without loss of generality that $|P_n| = |P_N|$ for all $n \geq N$. Then $P_n = P_{n+1}t_n$ for all $n \geq N$. In particular, we have

$$P_N\pi_S = P_{N+1}\pi_S = \cdots .$$

Let $|P_N| = m$, and for all $n \geq N$ let

$$P_n = \{(u_1, v_{n,1}), \ldots, (u_m, v_{n,m})\}$$

such that $v_{n,j} = v_{n+1,j}t_n$ for each $j \in \{1, \ldots, m\}$. Then we have ascending chains

$$\begin{array}{lcl} b_N & \leq_T & b_{N+1} \leq_T \cdots \\ v_{N,1} & \leq_T & v_{N+1,1} \leq_T \cdots \\ v_{N,2} & \leq_T & v_{N+1,2} \leq_T \cdots \\ & \vdots & \\ v_{N,m} & \leq_T & v_{N+1,m} \leq_T \cdots \end{array}$$

Since T is $\mathcal{R}$-noetherian, the above chains must all eventually stabilise. So, there exists $N' \geq N$ such that, for all $n \geq N'$, we have $b_n \mathcal{R}_T b_{N'}$ and $v_{n,j} \mathcal{R}_T v_{N',j}$ for each $j \in \{1, \ldots, m\}$. Consider $n \geq N'$. There exists $s'_n \in S$ such that $a_{n+1} = a_n s'_n$. Also, there exists $x_n \in T$ such that $b_{n+1} = b_n x_n$, and for each $j \in \{1, \ldots, m\}$ there exists $x_{n,j} \in T$ such that $v_{n+1,j} = v_{n,j}x_{n,j}$. Then $b_n = b_n x_n t_n$, and $v_{n,j} = v_{n,j}x_{n,j}t_n$ for each $j \in \{1, \ldots, m\}$. Since T is cancellative, it follows that $x_n t_n = 1$ and $x_{n,j}t_n = 1$ for each $j \in \{1, \ldots, m\}$. By cancellativity again we have

$$x_n = x_{n,1} = \cdots = x_{n,m}.$$

It follows that $(u_{n,j}, v_{n+1,j}) = (u_{n,j}, v_{n,j}x_n)$ for each $j \in \{1, \ldots, m\}$, and hence $P_{n+1} = P_n x_n$. Thus, we have

$$(a_{n+1}, P_{n+1}, b_{n+1}) = (a_n, P_n, b_n)(s'_n, \emptyset, x_n).$$

But this contradicts the assumption that $(a_n, P_n, b_n) <_U (a_{n+1}, P_{n+1}, b_{n+1})$. Hence U is $\mathcal{R}$-noetherian. □

By Theorem 4.7 and Corollary 3.14, we have:

Corollary 4.8 *Let S be a semigroup and let G be a group. Then the following are equivalent:*

(1) $S \Diamond G$ is $\mathcal{R}$-noetherian;
(2) $S \times G$ is $\mathcal{R}$-noetherian;
(3) S is $\mathcal{R}$-noetherian.

5 Free products

Let S_i $(i \in I)$ be a collection of pairwise disjoint semigroups. Let S be the set of all finite non-empty sequences $(a_1, \dots, a_m)$ where $a_j \in \bigcup_{i \in I} S_i$ $(1 \leq j \leq m)$ and each a_k belongs to a different S_i to that of a_{k+1} $(1 \leq k \leq m-1)$. Define a multiplication on S as follows:

$$(a_1, \dots, a_m)(b_1, \dots, b_n) = \begin{cases} (a_1, \dots, a_m, b_1, \dots, b_n) & \text{if } a_m \in S_i, b_1 \in S_j \text{ where } i \neq j, \\ (a_1, \dots, a_m b_1, \dots, b_n) & \text{if } a_m, b_1 \in S_i \text{ for some } i \in I. \end{cases}$$

It is straightforward to verify that this multiplication is associative. The semigroup S under this multiplication is called the *semigroup free product* of S_i $(i \in I)$ and is denoted by $\prod * \{S_i : i \in I\}$.

Now suppose that the semigroups S_i $(i \in I)$ are monoids with identities 1_i, respectively. Let ρ be the congruence on $\prod * \{S_i : i \in I\}$ generated by

$$\{(1_i, 1_j) : i, j \in I, i \neq j\},$$

and denote the ρ-class $\{1_i : i \in I\}$ by 1. The *monoid free product* of S_i $(i \in I)$, denoted by $\prod *_1 \{S_i : i \in I\}$, is the monoid $\prod * \{S_i : i \in I\}/\rho$ with identity 1.

We note that the monoid free product of groups coincides with the group free product [6, p. 266].

The following result provides necessary and sufficient conditions for a semigroup free product to be $\mathcal{R}$-noetherian.

Theorem 5.1 *Let S_i $(i \in I)$ be a collection of pairwise disjoint semigroups. Then the semigroup free product $\prod * \{S_i : i \in I\}$ is $\mathcal{R}$-noetherian if and only if each S_i $(i \in I)$ is $\mathcal{R}$-noetherian.*

Proof Let $S = \prod * \{S_i : i \in I\}$. Notice that each S_i embeds into S via $a \mapsto (a)$; we shall identify S_i with its image under this mapping. We denote the 'length' of $u \in S$ by $|u|$, i.e. if $u = (a_1, \dots, a_m)$ then $|u| = m$. Oberve that for any $u, v \in S$ we have $|uv| \in \{|u| + |v| - 1, |u| + |v|\}$.

($\Rightarrow$) We claim that each S_i is an $\mathcal{R}$-preserving subsemigroup of S, and is hence $\mathcal{R}$-noetherian by Proposition 2.6. Indeed, let $a \leq_S b$ where $a, b \in S_i$. Either $a = b$ or $a = bs$ for some $s \in S$. In the latter case, we must have $s \in S_i$, for otherwise $|bs| = |b| + |s| > 1 = |a|$. Thus $a \leq_{S_i} b$, as required.

($\Leftarrow$) Consider an ascending chain

$$u_1 \leq_S u_2 \leq_S \cdots$$

in S. Then, for each $n \in \mathbb{N}$, there exists $v_n \in S^1$ such that $u_n = u_{n+1} v_n$. Then

$$|u_1| \geq |u_2| \geq \cdots.$$

Hence, there exists $N \in \mathbb{N}$ such that $|u_n| = |u_N|$ for all $n \geq N$. Let $m = |U_N|$. It follows from the definition of the multiplication in S that the u_n $(n \geq N)$ have the same first $m-1$ terms, and that there exists $i \in I$ such that the m-th term of each u_n belongs to S_i and $v_n \in S_i^1$ $(n \geq N)$. Thus, for each $n \geq N$, we let $u_n = (a_1 \ldots, a_{m-1}, b_n)$ where $b_n \in S_i$. Then $b_n = b_{n+1} v_n \in b_{n+1} S_i^1$. Thus, we have an ascending chain

$$b_N \leq_{S_i} b_{N+1} \leq_{S_i} \cdots$$

in S_i. Since S_i is $\mathcal{R}$-noetherian, there exists $N' \geq N$ such that $b_n \, \mathcal{R}_{S_i} \, b_{N'}$ for all $n \geq N'$. Therefore, for each $n \geq N$ there exists $s_n \in S_i^1$ such that $b_n = b_{N'} s_n$. If $s_n = 1$, then $b_n = b_{N'}$ and hence $u_n = u_{N'}$. Otherwise, if $s_n \in S_i$, we have

$$u_n = (a_1 \ldots, a_{m-1}, b_{N'} s_n) = (a_1 \ldots, a_{m-1}, b_{N'})(s_n) = u_{N'}(s_n).$$

Thus, we have $u_n \, \mathcal{R}_S \, u_{N'}$ for all $n \geq N'$. Hence S is $\mathcal{R}$-noetherian. □

We now turn our attention to the monoid free product. First, we make some observations regarding this construction.

Consider a monoid free product $\prod *_1 \{S_i : i \in I\}$. We may view the non-identity elements of $\prod *_1 \{S_i : i \in I\}$ as sequences $(a_1, \ldots, a_n) \in \prod * \{S_i : i \in I\}$ where each a_i belongs to some $S_i \setminus \{1_i\}$ [6, p. 266]. More precisely, with ρ as given above, in each non-identity ρ-class there exists a unique sequence that contains no elements from $\{1_i : i \in I\}$; we call this sequence *reduced*. Thus, we identify the non-identity elements of S with their corresponding reduced sequences.

Now, consider a reduced squence $u = (a_1, \cdots, a_n) \in S$. Letting $a_n \in S_i$, observe that if a_n is not right invertible in S_i, then for any $v \in S \setminus \{1\}$ we have $|uv| \in \{|u|+|v|-1, |u|+|v|\}$. It follows that, if a_n is not right invertible in S_i, u is of minimal length in its $\mathcal{R}$-class (i.e. $|u| = \min\{|w| : u \, \mathcal{R} \, w\}$). In fact, the converse also holds. Indeed, if a_n is right invertible, then there exists $s \in S_i$ such that $a_n s = 1_i$. Then $us = (a_1, \ldots, a_{n-1})$, and of course $(a_1, \ldots, a_{n-1})(a_n) = u$, so $a_n \, \mathcal{R} \, (a_1, \ldots, a_{n-1})$. Hence, u is not of minimal length in its $\mathcal{R}$-class.

Theorem 5.2 *Let S_i $(i \in I)$ be a collection of pairwise disjoint monoids. Then the monoid free product $\prod *_1 \{S_i : i \in I\}$ is $\mathcal{R}$-noetherian if and only if each S_i $(i \in I)$ is $\mathcal{R}$-noetherian.*

Proof The proof is essentially the same as that of Theorem 5.1. The only difference is that, in ($\Leftarrow$), we stipulate that each u_i is of minimal length in its $\mathcal{R}$-class, and it then follows that $|u_1| \geq |u_2| \geq \cdots$. □

6 Rees matrix semigroups

Let S be a semigroup, let I and J be non-empty index sets, and let $P = (p_{ji})$ be a $J \times I$ matrix with entries from S. The set $I \times S \times J$ becomes a semigroup under the multiplication given by

$$(i, s, j)(k, t, l) = (i, s p_{jk} t, l),$$

and is called the *Rees matrix semigroup over S with respect to P*. We denote this semigroup by $\mathcal{M}(S; I, J; P)$.

We note that Rees matrix semigroups over groups are precisely the completely simple semigroups (i.e. semigroups with no proper ideals that possess minimal left and right ideals) [6, Theorem 3.3.1].

We now state the main result of this section, providing necessary and sufficient conditions for a Rees matrix semigroup to be $\mathcal{R}$-noetherian.

Theorem 6.1 *Let $T = \mathcal{M}(S; I, J; P)$ be a Rees matrix semigroup. Let U denote the right ideal $\{p_{j,i} : j \in J, i \in I\}S$ of S. Then T is $\mathcal{R}$-noetherian if and only if every ascending chain*

$$a_1U^1 \subseteq a_2U^1 \subseteq \cdots ,$$

where $a_i \in S$, eventually stabilises.

Proof We prove the contrapositive for both directions.

($\Rightarrow$) Suppose that there exists an infinite strictly ascending chain

$$a_1U^1 \subsetneq a_2U^1 \subsetneq \cdots ,$$

where $a_i \in S$. Fix $i \in I$ and $j_1 \in J$. For each $n \in \mathbb{N}$ there exists $i_{n+1} \in I$, $j_{n+1} \in J$ and $s_n \in S$ such that $a_n = a_{n+1}p_{j_{n+1},i_{n+1}}s_n$. Then

$$(i, a_n, j_n) = (i, a_{n+1}, j_{n+1})(i_{n+1}, s_n, j_n),$$

so $(i, a_n, j_n) \leq_T (i, a_{n+1}, j_{n+1})$. Suppose for a contradiction that $(i, a_n, j_n)\,\mathcal{R}_T\,(i, a_{n+1}, j_{n+1})$. We cannot have $(i, a_n, j_n) = (i, a_{n+1}, j_{n+1})$, for then $a_n = a_{n+1}$. Therefore, there exist some $k \in I$ and $s \in S$ such that $(i, a_{n+1}, j_{n+1}) = (i, a_n, j_n)(k, s, j_{n+1})$. But then $a_{n+1} = a_n(p_{j_n,k}s) \in a_nU$, so that $a_nU^1 = a_{n+1}U^1$, a contradiction. Thus, we have an infinite strictly ascending chain

$$(i, a_1, j_1) <_T (i, a_2, j_2) <_T \cdots$$

in T, and hence T is not $\mathcal{R}$-noetherian.

($\Leftarrow$) Suppose that T is not $\mathcal{R}$-noetherian. Then there exists an infinite strictly ascending chain

$$(i_1, a_1, j_1) <_T (i_2, a_2, j_2) <_T \cdots$$

in T. Letting $i = i_1$, we have

$$i = i_1 = i_2 = \cdots .$$

For each $n \in \mathbb{N}$ there exist $k_n \in I$ and $s_n \in S$ such that

$$(i, a_n, j_n) = (i, a_{n+1}, j_{n+1})(k_n, s_n, j_n).$$

Thus $a_n = a_{n+1}p_{j_{n+1},k_n}s_n \in a_{n+1}U$, so $a_nU^1 \subseteq a_{n+1}U^1$. We cannot have $a_nU^1 = a_{n+2}U^1$. Indeed, if we did, then there would exist $u \in U^1$ such that $a_{n+2} = a_nu$, and hence $a_{n+2} = a_{n+1}(p_{j_{n+1},k_n}s_nu)$. But then

$$(i, a_{n+2}, j_{n+2}) = (i, a_{n+1}, j_{n+1})(k_n, s_nu, j_{n+2}) \in (i, a_{n+1}, j_{n+1})T,$$

contradicting the fact that $(i, a_{n+1}, j_{n+1}) <_T (i, a_{n+2}, j_{n+2})$. Thus, we have an infinite strictly ascending chain

$$a_1U^1 \subsetneq a_3U^1 \subsetneq a_5U^1 \subsetneq \cdots ,$$

as desired. □

Corollary 6.2 *Let $T = \mathcal{M}(S; I, J; P)$ be a Rees matrix semigroup. If S is $\mathcal{R}$-noetherian then so is T.*

Proof Let U be as given in the statement of Theorem 6.1. Suppose for a contradiction that there exists an infinite strictly ascending chain

$$a_1U^1 \subsetneq a_2U^1 \subsetneq \cdots$$

where $a_i \in S$. Then clearly we have an ascending chain

$$a_1 \leq_S a_2 \leq_S \cdots .$$

Since S is $\mathcal{R}$-noetherian, there exists $N \in \mathbb{N}$ such that $a_nS^1 = a_NS^1$ for all $n \geq N$. But then

$$a_{N+2} \in a_NS \subseteq a_{N+1}US \subseteq a_{N+1}U,$$

contradicting the fact that $a_{N+1}U^1 \subsetneq a_{N+2}U^1$. Hence, by Theorem 6.1, T is $\mathcal{R}$-noetherian. □

The converse of Corollary 6.2 does not hold, as demonstrated by the following example.

Example 6.3 Let S be a semigroup with 0 that is not $\mathcal{R}$-noetherian. Let P be the 1×1 matrix whose entry is 0, and let $T = \mathcal{M}(S; \{1\}, \{1\}; P)$. For any $s, s' \in S$ we have $(1, s, 1)(1, s', 1) = (1, s0s', 1) = (1, 0, 1)$, and clearly $(1, 0, 1)$ is a zero element in T, so T is a null semigroup. Hence, by Corollary 2.2, T is $\mathcal{R}$-noetherian.

Corollary 6.4 *Let $T = \mathcal{M}(S; I, J; P)$ be a Rees matrix semigroup such that every element of S has a local right identity in $U = \{p_{j,i} : j \in J, i \in I\}S$. Then T is $\mathcal{R}$-noetherian if and only if S is $\mathcal{R}$-noetherian.*

Proof Consider any $a \in S$. By assumption, we have $a \in aU$, so $aS^1 \subseteq aUS^1 \subseteq aU^1$. Clearly $aU^1 \subseteq aS^1$, so $aS^1 = aU^1$. The result now follows readily from Theorem 6.1. □

Corollary 6.5 *Let $T = \mathcal{M}(S; I, J; P)$ be a Rees matrix semigroup where S is a monoid, and suppose that there exist $i \in I$ and $j \in J$ such that $p_{j,i}$ is right invertible. Then T is $\mathcal{R}$-noetherian if and only if S is $\mathcal{R}$-noetherian.*

Proof We have $1_S \in p_{j,i}S$, and 1_S is obviously a local right identity of every element of S. Hence, by Corollary 6.4, S is $\mathcal{R}$-noetherian. □

We now consider a variant of the Rees matrix construction. Let S be a semigroup with zero 0, let I and J be non-empty index sets, and let $P = (p_{ji})$ be a $J \times I$ matrix with entries from S. Let $T' = \mathcal{M}(S; I, J; P)$, and let T denote the Rees quotient T'/Q, where Q is the ideal $I \times \{0\} \times J$ of T'. The semigroup T is called the *Rees matrix semigroup with zero over S with respect to P*, and is denoted by $\mathcal{M}^0(S; I, J; P)$.

Rees matrix semigroups with zero over groups are precisely the completely 0-simple semigroups [6, Theorem 3.2.3].

Corollary 6.6 *Let $T = \mathcal{M}^0(S; I, J; P)$ be a Rees matrix semigroup with zero. Let U denote the right ideal $\{p_{j,i} : j \in J, i \in I\}S$ of S. Then the following are equivalent:*

(1) T is $\mathcal{R}$-noetherian;
(2) $\mathcal{M}(S; I, J; P)$ is $\mathcal{R}$-noetherian;
(3) every ascending chain

$$a_1U^1 \subseteq a_2U^1 \subseteq \cdots ,$$

where $a_i \in S$, eventually stabilises.

Proof (1)⇔(2). Let $T' = \mathcal{M}(S; I, J; P)$ and $Q = I \times \{0\} \times J$, so that $T = T'/Q$. Since Q is $\mathcal{R}$-noetherian and every element of Q has a local right identity in Q, it follows from Proposition 2.13 that T is $\mathcal{R}$-noetherian if and only if T' is $\mathcal{R}$-noetherian.

(2) and (3) are equivalent by Theorem 6.1. □

Related to the Rees matrix with zero construction is that of the Brandt extension, defined as follows. Let S be a semigroup and let I be a non-empty set. The *Brandt extension* of S by I, denote by $\mathcal{B}(S, I)$, is the semigroup with universe $(I \times S \times I) \cup \{0\}$ and multiplication given by

$$(i, s, j)(k, t, l) = \begin{cases} (i, st, l) & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

and $0x = x0 = 0$ for all $x \in (I \times S \times I) \cup \{0\}$. Notice that if S is a monoid, then $\mathcal{B}(S, I)$ is isomorphic to $\mathcal{M}^0(S; I, I; P)$ where P is the $I \times I$ identity matrix. Brandt extensions of groups are precisely the completely 0-simple inverse semigroups [6, Theorem 5.1.8].

Theorem 6.7 *Let S be a semigroup and let I be a non-empty set. Then $\mathcal{B}(S, I)$ is $\mathcal{R}$-noetherian if and only if S is $\mathcal{R}$-noetherian.*

Proof (⇒) It is straightforward to show that, for any $i \in I$, S is isomorphic the subsemigroup $S_i = \{i\} \times S \times \{i\}$ of $\mathcal{B}(S, I)$, and that S_i is right unitary in $\mathcal{B}(S, I)$. Hence, S is $\mathcal{R}$-noetherian by Corollary 2.8.

(⇐) Letting $T = \mathcal{B}(S^1, I)$, we have $T \cong \mathcal{M}^0(S^1; I, I; P)$ where P is the $I \times I$ identity matrix. Since S is $\mathcal{R}$-noetherian, we have that $\mathcal{M}(S^1; I, I; P)$ is $\mathcal{R}$-noetherian by Corollary 6.2. Hence, by Corollary 6.6, T is $\mathcal{R}$-noetherian. Since $\mathcal{B}(S, I)$ is an ideal of T, it is also $\mathcal{R}$-noetherian by Proposition 2.5. □

7 Bruck–Reilly extensions

Let M be a monoid with identity 1_M, and let $\theta\colon M \to M$ be an endomorphism. We define a binary operation on the set $\mathbb{N}_0 \times M \times \mathbb{N}_0$ by

$$(i, a, j)(p, b, q) = (i - j + t, (a\theta^{t-j})(b\theta^{t-p}), q - p + t),$$

where $t = \max(j, p)$ and θ^0 denotes the identity map on M. With this operation the set $\mathbb{N}_0 \times M \times \mathbb{N}_0$ is a monoid with identity $(0, 1_M, 0)$. It is denoted by $BR(M, \theta)$ and called the *Bruck–Reilly extension of M determined by θ*.

Special instances of this construction were introduced by Bruck [2] and Reilly [16], after whom it is named, and it was given in its general form by Munn in [13].

Theorem 7.1 *Let M be a monoid and let $\theta\colon M \to M$ be a monoid homomorphism. Then $BR(M, \theta)$ is $\mathcal{R}$-noetherian if and only if M is $\mathcal{R}$-noetherian.*

Proof (⇒) Let $N = BR(M, \theta)$. It is straightforward to show that M is isomorphic to the submonoid $\{0\} \times M \times \{0\}$ of N, and that this submonoid is right unitary in N. Hence, M is $\mathcal{R}$-noetherian by Corollary 2.8.

(⇐) Consider an ascending chain

$$u_1 \leq_N u_2 \leq_N \cdots$$

in N, where $u_k = (i_k, a_k, j_k)$. Then for each $k \in \mathbb{N}$ there exists (p_k, m_k, q_k) such that $u_k = u_{k+1}(p_k, m_k, q_k)$. Letting $t_k = \max(j_{k+1}, p_k)$, we have

$$i_k = i_{k+1} - j_{k+1} + t_k.$$

Since $t_k \geq j_{k+1}$, it follows that $i_k \geq i_{k+1}$. Thus we have

$$i_1 \geq i_2 \geq \cdots ,$$

and hence there exists $N \in \mathbb{N}$ such that $i_N = i_{N+1} = \cdots$. Let $i = i_N$. Then for $k \geq N$ we have $i = i - j_{k+1} + t_k$, so $t_k = j_{k+1}$. It follows that, for each $k \geq N$, we have

$$a_k = a_{k+1}(m_k \theta^{j_{k+1} - p_k}) \in a_{k+1} M.$$

Hence, we have an ascending chain

$$a_N \leq_M a_{N+1} \leq_M \cdots$$

in M. Since M is $\mathcal{R}$-noetherian, there exists $N' \geq N$ such that $a_p \, \mathcal{R}_M \, a_{N'}$ for all $p \geq N'$. Therefore, for each $p \geq n$ there exists $m'_p \in M$ such that $a_p = a_{N'} m'_p$. Then

$$u_p = (i, a_p, j_p) = (i, a_{N'}, j_{N'})(j_{N'}, m'_p, j_p) = u_{N'}(j_{N'}, m'_p, j_p) \in u_{N'} N.$$

We conclude that $u_p \, \mathcal{R}_N \, u_{N'}$ for all $p \geq N'$. This completes the proof. □

Corollary 7.2 *Every semigroup S that is $\mathcal{R}$-noetherian embeds into a simple semigroup that is $\mathcal{R}$-noetherian.*

Proof Let $\theta \colon S^1 \to S^1$ be the endomorphism given by $s\theta = 1$ for all $s \in S^1$, and let $M = BR(S^1, \theta)$. Then M is simple by [6, Proposition 5.6.6(1)]. The monoid S^1 is $\mathcal{R}$-noetherian since S is, and hence M is $\mathcal{R}$-noetherian by Theorem 7.1. We have already observed that S^1 is isomorphic to $\{0\} \times S^1 \times \{0\} \subseteq M$, and clearly S embeds into S^1, so we conclude that S embeds into M. □

8 Semilattices of semigroups

Let Y be a semilattice and let $(S_\alpha)_{\alpha \in Y}$ be a family of disjoint semigroups, indexed by Y. If $S = \bigcup_{\alpha \in Y} S_\alpha$ is a semigroup such that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$, then S is called a *semilattice of semigroups*, and we denote it by $S = \mathcal{S}(Y, S_\alpha)$. If, additionally, each S_α is a monoid, we call S a *semilattice of monoids*.

Now let $S = \bigcup_{\alpha \in Y} S_\alpha$, and suppose that for each $\alpha, \beta \in Y$ with $\alpha \geq \beta$ there exists a homomorphism $\phi_{\alpha,\beta} \colon S_\alpha \to S_\beta$. Furthermore, assume that:

- for each $\alpha \in Y$, the homomorphism $\phi_{\alpha,\alpha}$ is the identity map on S_α;
- for each $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$, we have $\phi_{\alpha,\beta} \, \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$.

For $a \in S_\alpha$ and $b \in S_\beta$, we define

$$ab = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}).$$

With this multiplication, S is a semilattice of semigroups. In this case we call S a *strong semilattice of semigroups* and denote it by $S = \mathcal{S}(Y, S_\alpha, \phi_{\alpha,\beta})$.

We begin by investigating the behaviour of the property of being $\mathcal{R}$-noetherian in the general setting of semilattices of semigroups. Note that a semilattice is $\mathcal{R}$-noetherian if and only if it satisfies the ascending chain condition on elements under its partial order.

Definition 8.1 Let $S = \mathcal{S}(Y, S_\alpha)$. We say that a chain

$$\alpha_1 \leq \alpha_2 \leq \cdots$$

in Y is *$\mathcal{R}$-witnessed* (with respect to S) if there exist $a_i \in S_{\alpha_i}$ $(i \in \mathbb{N})$ such that

$$a_1 \leq_S a_2 \leq_S \cdots .$$

We note that if $S = \mathcal{S}(Y, S_\alpha)$ and Y is $\mathcal{R}$-noetherian, then certainly every $\mathcal{R}$-witnessed chain in Y with respect to S eventually stabilises. It turns out, however, that the converse does not hold in general.

Lemma 8.2 *Let $S = \mathcal{S}(Y, S_\alpha)$. If S is $\mathcal{R}$-noetherian, then every $\mathcal{R}$-witnessed chain in Y eventually stabilises and each S_α is $\mathcal{R}$-noetherian.*

Proof Consider an $\mathcal{R}$-witnessed chain

$$\alpha_1 \leq \alpha_2 \leq \cdots$$

in Y. Then there exist $a_i \in S_{\alpha_i}$ $(i \in \mathbb{N})$ such that

$$a_1 \leq_S a_2 \leq_S \cdots .$$

Since S is $\mathcal{R}$-noetherian, there exists $n \in \mathbb{N}$ such that $a_n \mathrel{\mathcal{R}_S} a_N$ for all $n \geq N$. This implies that $\alpha_n = \alpha_N$ for all $n \geq N$, as required.

Now let $\alpha \in Y$, and let $I = \bigcup_{\beta \leq \alpha} S_\beta$. Then I is an ideal of S, so it is $\mathcal{R}$-noetherian by Proposition 2.5. Since $I \setminus S_\alpha$ is an ideal of I, it follows from Corollary 2.9 that S_α is $\mathcal{R}$-noetherian. □

The following example shows that $\mathcal{S}(Y, S_\alpha)$ may not be $\mathcal{R}$-noetherian even if Y is finite and each S_α is $\mathcal{R}$-noetherian. In particular, the converse of Lemma 8.2 does not hold.

Example 8.3 Let S be the disjoint union of (a copy of) the free monogenic semigroup $\{x\}^+$ and a set $N = \{a_i : i \in \mathbb{Z}\} \cup \{0\}$. Define a multiplication on S, extending that on $\{x\}^+$, by

$$x^i a_j = a_j x^i = a_{j-i} \text{ and } x^i 0 = 0x^i = uv = 0 \ (i \in \mathbb{N}, j \in Z, u, v \in N).$$

This multiplication turns N into a null semigroup. It is straightforward to show that, under this multiplication, S is a semilattice of semigroups with structure semilattice $Y = \{1 > 0\}$ and corresponding components $\{x\}^+$ and N. Certainly $\{x\}^+$ and N are $\mathcal{R}$-noetherian. On the other hand, it is easy to see that we have an infinite strictly ascending chain

$$a_0 <_S a_1 <_S \cdots$$

in S, so that S is not $\mathcal{R}$-noetherian.

The following result provides a condition under which the converse of Lemma 8.2 *does* hold. For this, recall that a semigroup is *weakly right noetherian* if it satisfies the ascending chain condition on right ideals.

Proposition 8.4 *Let $S = \mathcal{S}(Y, S_\alpha)$, and suppose that for each $\alpha \in Y$ the semigroup S_α contains no infinite antichain of $\mathcal{R}_{S_\alpha}$-classes. Then the following are equivalent:*

(1) S is $\mathcal{R}$-noetherian;
(2) every $\mathcal{R}$-witnessed chain in Y eventually stabilises and each S_α is $\mathcal{R}$-noetherian;
(3) every $\mathcal{R}$-witnessed chain in Y eventually stabilises and each S_α is weakly right noetherian.

Proof (1)⇒(2) follows from Lemma 8.2.

(2)⇒(1). Suppose that every $\mathcal{R}$-witnessed chain in Y eventually stabilises but that S is not $\mathcal{R}$-noetherian. We need to prove that some S_α is not $\mathcal{R}$-noetherian.

Since S is not $\mathcal{R}$-noetherian, there exists an infinite strictly ascending chain

$$a_1 <_S a_2 <_S \cdots$$

in S. Let $a_i \in S_{\alpha_i}$, and for $i < j$ let $b_{i,j} \in S_{\beta_{i,j}}$ be such that $a_i = a_j b_{i,j}$; then $\alpha_i = \alpha_j \beta_{i,j}$. Now, we have an $\mathcal{R}$-witnessed chain

$$\alpha_1 \leq_Y \alpha_2 \leq_Y \cdots$$

in Y. By assumption, there exists $N \in \mathbb{N}$ such that $\alpha_n = \alpha_N$ for all $n \geq N$. Then, letting $\alpha = \alpha_N$, we have $\alpha\beta_{i,j} = \alpha$ for all $i, j \in \mathbb{N}$ with $N \leq i < j$.

Consider the set $\{a_n : n \geq N\}$ of elements of S_α. By assumption, this set cannot form an infinite antichain under the $\mathcal{R}$-preorder on S_α. Also, we cannot have $a_j \leq_{S_\alpha} a_i$ for any $N \leq i < j$, since this would contradict the fact that $a_i <_S a_j$. It follows that there exist $i_1, j_1 \geq N$ with $i_1 < j_1$ such that $a_{i_1} <_{S_\alpha} a_{j_1}$. Now, either $i_1 = N$ or

$$a_N = a_{i_1} b_{N,i_1} \in a_{j_1}(S_\alpha b_{N,i_1}) \subseteq a_{j_1} S_\alpha,$$

so $a_N <_{S_\alpha} a_{j_1}$. Now consider the infinite set $\{a_n : n \geq j_1\}$. By the same argument as above, there exists $j_2 > j_1$ such that $a_{j_1} <_{S_\alpha} a_{j_2}$. Continuing in this way, we obtain an infinite strictly ascending chain

$$a_N <_{S_\alpha} a_{j_1} <_{S_\alpha} a_{j_2} <_{S_\alpha} \cdots$$

in S_α, as required.

(2)⇔(3) follows from the fact that a semigroup is weakly right noetherian if and only if it is $\mathcal{R}$-noetherian and contains no infinite antichain of $\mathcal{R}$-classes [11, Theorem 3.2]. □

The converse of Lemma 8.2 also holds in the case that each S_α is $\mathcal{R}$-preserving in S.

Theorem 8.5 *Let $S = \mathcal{S}(Y, S_\alpha)$ where each S_α is $\mathcal{R}$-preserving in S. Then S is $\mathcal{R}$-noetherian if and only if every $\mathcal{R}$-witnessed chain in Y eventually stabilises and each S_α is $\mathcal{R}$-noetherian.*

Proof The forward implication follows immediately from Lemma 8.2 For the converse, consider an ascending chain

$$a_1 \leq_S a_2 \leq_S \cdots$$

in S. Let $a_i \in S_{\alpha_i}$. Then

$$\alpha_1 \leq_Y \alpha_2 \leq_Y \cdots$$

is an $\mathcal{R}$-witnessed chain in Y. By assumption, there exists $N \in \mathbb{N}$ such that $\alpha_n = \alpha_N$ for all $n \geq N$. Let $\alpha = \alpha_N$. Then $a_n \in S_\alpha$ for all $n \geq N$. Since S_α is $\mathcal{R}$-preserving in S, we have

$$a_N \leq_{S_\alpha} a_{N+1} \leq_{S_\alpha} \cdots .$$

Since S_α is $\mathcal{R}$-noetherian, there exists $N' \geq N$ such that $a_n \,\mathcal{R}_{S_\alpha}\, a_{N'}$ for all $n \geq N'$. Then $a_n \,\mathcal{R}_S\, a_{N'}$ for all $n \geq N'$. This completes the proof. □

In what follows we consider certain situations where we have $S = \mathcal{S}(Y, S_\alpha)$ with all the S_α being $\mathcal{R}$-preserving in S. The first such situation is where every S_α has the property that each element has a local right identity (in S_α). Recall that this holds if each S_α is a monoid or regular semigroup.

Proposition 8.6 *Let $S = \mathcal{S}(Y, S_\alpha)$ where, for each $\alpha \in Y$, every element of S_α has a local right identity in S_α. Then S is $\mathcal{R}$-noetherian if and only if every $\mathcal{R}$-witnessed chain in Y eventually stabilises and each S_α is $\mathcal{R}$-noetherian.*

Proof We show that each S_α is $\mathcal{R}$-preserving, and the result then follows from Theorem 8.5. So, let $\alpha \in Y$, and let $a, b \in S_\alpha$ be such that $a \leq_S b$. Then $a = bs$ for some $s \in S^1$. If $s = 1$ then $a = b$. Suppose that $s \in S$. Then $s \in S_\beta$ for some $\beta \in Y$, and we have $\alpha\beta = \alpha$. Let $c \in S_\alpha$ be a local right identity of b, so that $b = bc$. Then we have

$$a = bs = (bc)s = b(cs) \in bS_\alpha,$$

so $a \leq_{S_\alpha} b$, as required. □

A semigroup is called *completely regular* if it is a union of groups. A semigroup is completely regular if and only if it is a semilattice of completely simple semigroups [6, Theorem 4.1.3]. Completely simple semigroups are certainly $\mathcal{R}$-noetherian by Corollary 2.2. Thus, by Proposition 8.6, we have:

Corollary 8.7 *Let S be a completely regular semigroup, and let $S = \mathcal{S}(Y, S_\alpha)$ be its decomposition into a semilattice of completely simple semigroups. Then S is $\mathcal{R}$-noetherian if and only if every $\mathcal{R}$-witnessed chain in Y eventually stabilises.*

The *free semilattice* on a non-empty set X, which we denote by F_X, is defined as the set of all finite non-empty subsets of X under the operation of union. Clearly, for any $U, V \in F_X$, we have $U \leq_{F_X} V$ if and only if $V \subseteq U$. It follows that F_X is $\mathcal{R}$-noetherian. Both the free band on X and free completely regular semigroup on X are semilattices of semigroups where the structure semilattice is F_X; see [6, p. 120] and [3, Corollary 4.3], respectively. Therefore, by Corollary 8.7, we have:

Corollary 8.8 *Let X be a non-empty set. Then the following semigroups are $\mathcal{R}$-noetherian: the free semilattice on X, the free band on X, and the free completely regular semigroup on X.*

Corollary 8.9 *Let $S = \mathcal{S}(Y, S_\alpha)$ be a semilattice of monoids such that $1_\alpha 1_\beta = 1_{\alpha\beta}$ for all $\alpha, \beta \in Y$ (where 1_α denotes the identity of S_α). Then S is $\mathcal{R}$-noetherian if and only if Y and all S_α are $\mathcal{R}$-noetherian.*

Proof Given Proposition 8.6, it suffices to show that every ascending chain of elements of Y is $\mathcal{R}$-witnessed with respect to S. This follows from the fact that if $\alpha \leq \beta$ then $1_\alpha = 1_{\beta\alpha} = 1_\beta 1_\alpha$, and hence $1_\alpha \leq 1_\beta$.

A *Clifford semigroup* is an inverse completely regular semigroup, or, equivalently, a (strong) semilattice of groups. By Corollary 8.9 we have:

Corollary 8.10 *Let S be a Clifford semigroup, and let $S = \mathcal{S}(Y, G_\alpha)$ be its decompostion into a semilattice of groups. Then S is $\mathcal{R}$-noetherian if and only if Y is $\mathcal{R}$-noetherian.*

We now turn our attention to strong semilattices of semigroups.

Proposition 8.11 *Let $S = \mathcal{S}(Y, S_\alpha, \phi_{\alpha,\beta})$. Then S is $\mathcal{R}$-noetherian if and only if every $\mathcal{R}$-witnessed chain in Y eventually stabilises and each S_α is $\mathcal{R}$-noetherian.*

Proof We show that each S_α is $\mathcal{R}$-preserving, and the result then follows from Theorem 8.5. So, let $\alpha \in Y$, and let $a, b \in S_\alpha$ be such that $a \leq_S b$. Then $a = bs$ for some $s \in S^1$. If $s = 1$ then $a = b$. Suppose that $s \in S$. Then $s \in S_\beta$ for some $\beta \in Y$, and we have $\alpha\beta = \alpha$. Thus, we have $a = b(s\phi_{\beta,\alpha}) \in bS_\alpha$, and hence $a \leq_{S_\alpha} b$, as required. □

Corollary 8.12 *Let $S = \mathcal{S}(Y, S_\alpha, \phi_{\alpha,\beta})$ where, for each $\alpha \in Y$, S_α has no element with a local right identity. Then S is $\mathcal{R}$-noetherian if and only if each S_α is $\mathcal{R}$-noetherian.*

Proof Given Proposition 8.11, it suffices to prove that if each S_α is $\mathcal{R}$-noetherian then every $\mathcal{R}$-witnessed chain in Y eventually stabilises. So, consider an $\mathcal{R}$-witnessed chain

$$\alpha_1 \leq_Y \alpha_2 \leq_Y \cdots .$$

Then there exist $a_i \in S_{\alpha_i}$ $(i \in \mathbb{N})$ such that

$$a_1 \leq_S a_2 \leq_S \cdots .$$

By the definition of the product in S, for each $i \in \mathbb{N}$ we have either $a_i = a_{i+1}$ or $a_i = (a_{i+1}\phi_{\alpha_{i+1},\alpha_i})s_i$ for some $s_i \in S_{\alpha_i}$. If $a_i = a_{i+1}$ then $a_i\phi_{\alpha_i,\alpha_1} = a_{i+1}\phi_{\alpha_{i+1},\alpha_1}$, and if $a_i = (a_{i+1}\phi_{\alpha_{i+1},\alpha_i})s_i$ then

$$a_i\phi_{\alpha_i,\alpha_1} = (a_{i+1}\phi_{\alpha_{i+1},\alpha_1})(s_i\phi_{\alpha_i,\alpha_1}).$$

Thus, letting $T = S_{\alpha_1}$, we have an ascending chain

$$a_1 = a_1\phi_{\alpha_1,\alpha_1} \leq_T a_2\phi_{\alpha_2,\alpha_1} \leq_T \cdots$$

in T. Since T is $\mathcal{R}$-trivial (as it has no element with a local right identity) and $\mathcal{R}$-noetherian, there exists $N \in \mathbb{N}$ such that $a_n\phi_{\alpha_n,\alpha_1} = a_N\phi_{\alpha_N,\alpha_1}$ for all $n \geq N$. We claim that $\alpha_n = \alpha_N$ for all $n \geq N$. Indeed, suppose not. Then there exists $n \geq N$ such that $\alpha_n \neq \alpha_{n+1}$, and hence $a_n \neq a_{n+1}$. Then we have

$$a_n\phi_{\alpha_n,\alpha_1} = (a_{n+1}\phi_{\alpha_{n+1},\alpha_1})(s_n\phi_{\alpha_n,\alpha_1}).$$

But then

$$a_N\phi_{\alpha_N,\alpha_1} = (a_N\phi_{\alpha_N,\alpha_1})(s_n\phi_{\alpha_n,\alpha_1}) \in (a_N\phi_{\alpha_N,\alpha_1})T,$$

contradicting the fact that T has no element with a local right identity. This completes the proof. □

We conclude this section with an example of a strong semilattice of semigroups that is $\mathcal{R}$-noetherian but whose structure semilattice is not $\mathcal{R}$-noetherian.

Example 8.13 Let Y be the semilattice $(\mathbb{N}, \min)$. Then Y is not $\mathcal{R}$-noetherian. For each $i \in \mathbb{N}$, let S_i be the semilattice $\{i\} \times \mathbb{N}$ with multiplication

$$(i, j)(i, k) = \big(i, \max(j, k)\big) \text{ for all } j, k \in \mathbb{N}.$$

Then each S_i is isomorphic to $(\mathbb{N}, \max)$, which is certainly $\mathcal{R}$-noetherian. For each $i, j \in \mathbb{N}$ with $i \geq j$, define a map

$$\phi_{i,j}\colon S_i \to S_j,\ (i, n) \mapsto (j, n + i - j).$$

For any $m, n \in \mathbb{N}$, we have

$$\big((i, m)(i, n)\big)\phi_{i,j} = \big(i, \max(m, n)\big)\phi_{i,j} = \big(j, \max(m, n) + i - j\big)$$

$$= \big(j, \max(m+i-j, n+i-j)\big) = (j, m+i-j)(j, n+i-j)$$
$$= \big((j,m)\phi_{i,j}\big)\big((j,n)\phi_{i,j}\big),$$

so $\phi_{i,j}$ is a homomorphism. Let S be the strong semilattice of semilattices $\mathcal{S}(Y, S_i, \phi_{i,j})$. Then $S = \mathbb{N} \times \mathbb{N}$, and for any $(i,m), (j,n) \in S$ we have

$$\begin{aligned}(i,m)(j,n) &= \big((i,m)\phi_{i,\min(i,j)}\big)\big((j,n)\phi_{j,\min(i,j)}\big)\\ &= \big(\min(i,j), m+i-\min(i,j)\big)\big(\min(i,j), n+j-\min(i,j)\big)\\ &= \big(\min(i,j), \max(m+i-\min(i,j), n+j-\min(i,j))\big).\end{aligned}$$

It is easy to see that this product is commutative. Therefore, since each element of S is an idempotent, we have that S is a semilattice. We claim that S is $\mathcal{R}$-noetherian. So, consider an ascending chain

$$(i_1, n_1) \leq_S (i_2, n_2) \leq_S \cdots$$

in S. Since S is a semilattice, for each $j \in \mathbb{N}$ we have $(i_j, n_j) = (i_{j+1}, n_{j+1})(i_j, n_j)$. Thus, we have $i_j = \min(i_{j+1}, i_j)$ and $n_j = \max(i_{j+1} + n_{j+1} - i_j, n_j)$. It follows that

$$i_1 \leq i_2 \leq \cdots \quad \text{and} \quad n_1 \geq n_2 \geq \cdots,$$

and if $i_j < i_{j+1}$ then $n_j > n_{j+1}$. We conclude that there exists $N \in \mathbb{N}$ such that $i_m = i_N$ and $n_m = n_N$ for all $m \geq N$, and hence $(i_m, n_m) = (i_N, n_N)$ for all $m \geq N$, as required.

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