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Singular value and norm inequalities for products and sums of matrices

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Abstract
In this paper, we give singular value and norm inequalities involving convex functions of positive semidefinite matrices. Our results generalize some known inequalities for the spectral norm and for the Schatten p-norms for $p \geq 1$.

Keywords Singular value · Spectral norm · Schatten p-norm · Unitarily invariant norm · Positive semidefinite matrix · Contraction · Convex function · Inequality

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1 Introduction

Let $\mathbb{M}_n\mathbb{C})$ be the C^*-algebra of all $n \times n$ complex matrices. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be positive semidefinite if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$. The absolute value of $A \in \mathbb{M}_n(\mathbb{C})$, denoted by $|A|$, is the unique positive semidefinite square root of the matrix A^*A, that is, $|A| = (A^*A)^{1/2}$. The singular values of $A \in \mathbb{M}_n(\mathbb{C})$, denoted by $s_1(A), s_2(A), \ldots, s_n(A)$, are the eigenvalues of $|A|$ arranged in decreasing order and repeated according to multiplicity. In fact, it can be seen that $s_j(A) = s_j(|A|) = s_j(A^*)$ for $j = 1, 2, \ldots, n$.

The spectral norm, denoted by $\|\cdot\|$, is a matrix norm defined on $\mathbb{M}_n(\mathbb{C})$ by $\|A\| = \max_{\|x\|=1} \|Ax\|$ for $A \in \mathbb{M}_n(\mathbb{C})$. Moreover, for $p \geq 1$, the Schatten p-norms, denoted by $\|A\|_p$, are also matrix norms defined on $\mathbb{M}_n(\mathbb{C})$ by $\|A\|_p = (\mathrm{tr}|A|^p)^{1/p}$ for $A \in \mathbb{M}_n(\mathbb{C})$,

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where tr$(\cdot)$ denotes the usual trace functional. In fact, it can be seen that $\|A\| = s_1(A)$ and $\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{1/p}$ for $A \in \mathbb{M}_n(\mathbb{C})$.

A matrix norm $|||\cdot|||$ on $\mathbb{M}_n(\mathbb{C})$ is said to be unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n(\mathbb{C})$ and for all unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. Typical examples of unitarily invariant norms, that we are interested in, are the spectral norm and the Schatten p-norms for $p \geq 1$.

For $A, B \in \mathbb{M}_n(\mathbb{C})$, let $A \oplus B$ be the direct sum of A and B, that is, the matrix given by $A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Note that $\left|\left|\left|\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}\right|\right|\right| = |||A \oplus A^*||| = |||A \oplus A|||$, and that $|||A||| \leq |||B|||$ is equivalent to $|||A \oplus A||| \leq |||B \oplus B|||$ for all unitarily invariant norms. By convenience, for $A \in \mathbb{M}_n(\mathbb{C})$ and $B \in \mathbb{M}_{2n}(\mathbb{C})$, by the inequality $|||A||| \leq |||B|||$ we mean that $|||A \oplus 0||| \leq |||B|||$. It is evident that $\|A \oplus B\| = \max(\|A\|, \|B\|)$ and $\|A \oplus B\|_p^p = \|A\|_p^p + \|B\|_p^p$, for $p \geq 1$, $s_j(A \oplus 0) = s_j(A)$ for $j = 1, \ldots, n$, and $s_j(A \oplus 0) = 0$ for $j = n+1, \ldots, 2n$. For other basic properties of unitarily invariant norms and singular values, we refer to [6, 11].

In [1], Al-Natoor, Benzamia, and Kittaneh proved that if $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$\|AB - BA\| \leq \|A\| \, \|B\| + \frac{1}{2} \left\|A^*B - BA^*\right\|, \tag{1.1}$$

which refines the inequality

$$\|AB - BA\| \leq 2\|A\|\|B\|.$$

Related to inequality (1.1), Al-Natoor and Kittaneh [4] have proved that if $A, B, C \in \mathbb{M}_n(\mathbb{C})$, then

$$\|AB + BC\| \leq \max(\|A\|, \|C\|) \|B\| + \frac{1}{2} \left\|A^*B + BC^*\right\|.$$

In particular, letting $C = A$, we have

$$\|AB + BA\| \leq \|A\| \, \|B\| + \frac{1}{2} \left\|A^*B + BA^*\right\|. \tag{1.2}$$

In [15], Zhan proved that if $A, B \in M_n(\mathbb{C})$ are positive semidefinite, then

$$|||A - B||| \leq |||A \oplus B||| \tag{1.3}$$

for all unitarily invariant norms.

A generalization of inequality (1.3) was given by Kittaneh [14]. This generalization asserts that if $A, B, X \in \mathbb{M}_n(\mathbb{C})$ are such that A and B are positive semidefinite, then

$$|||AX - XB||| \leq \|X\| \, |||A \oplus B||| \tag{1.4}$$

for all unitarily invariant norms.

In [3], Al-Natoor and Kittaneh gave a refinement of inequality (1.4). This refinement asserts that if $A, B, X \in \mathbb{M}_n(\mathbb{C})$ are such that A and B are positive semidefinite, then

$$|||AX - XB||| \leq \sqrt{|||A \oplus B||| \;\; |||(X^*AX) \oplus (XBX^*)|||} \tag{1.5}$$

for all unitarily invariant norms.

Applying inequality (1.5) for the spectral norm and the Schatten p -norms, for $p \geq 1$, we have

$$\|AX - XB\| \leq \sqrt{\max\left(\|A\|, \|B\|\right) \max\left(\|X^*AX\|, \|XBX^*\|\right)} \tag{1.6}$$

and

$$\|AX - XB\|_p \le \sqrt[2p]{\left(\|A\|_p^p + \|B\|_p^p\right) \ \left(\|X^*AX\|_p^p + \|XBX^*\|_p^p\right)}. \tag{1.7}$$

Also, in the same paper, Al-Natoor and Kittaneh gave a generalization of the inequality

$$\|\|A + B\|\| \le \|\|A \oplus B\|\| + \left\|\left|\left(A^{1/2}B^{1/2}\right) \oplus \left(A^{1/2}B^{1/2}\right)\right|\right\|$$

for all unitarily invariant norms, which is due to Kittaneh [13]. This generalization asserts that if $A, B, X \in \mathbb{M}_n(\mathbb{C})$ are such that A and B are positive semidefinite, then

$$\|\|AX + XB\|\| \le \|X\| \, \|\|A \oplus B\|\| + \left\|\left|\left(A^{1/2}XB^{1/2}\right) \oplus \left(A^{1/2}XB^{1/2}\right)\right|\right\| \tag{1.8}$$

for all unitarily invariant norms.

Applying inequality (1.8) for the spectral norm and the Schatten p-norms, for $p \ge 1$, we have

$$\|AX + XB\| \le \|X\| \max \ (\|A\|, \|B\|) + \left\|A^{1/2}XB^{1/2}\right\| \tag{1.9}$$

and

$$\|AX + XB\|_p \le \|X\| \left(\|A\|_p^p + \|B\|_p^p\right)^{1/p} + 2^{1/p} \left\|A^{1/2}XB^{1/2}\right\|_p. \tag{1.10}$$

In Sect. 2 of this paper, we introduce singular value inequalities for functions of matrices, and applications of our results are given. In Section 3, we give generalizations of inequalities (1.1) and (1.5)–(1.10).

2 Singular value and norm inequalities for matrices

We start with the following theorem, which is based on three lemmas. The first lemma can be found in [2], the second follows directly from the definition of convex functions, while the third can be found in [12].

Lemma 2.1 *If $A, B, X \in \mathbb{M}_n(\mathbb{C})$ are such that A and B are nonzero and X is positive semidefinite, then*

$$s_j\left(AXB^*\right) \le \frac{1}{2}\left\|\frac{|A|^2}{\|A\|^2} + \frac{|B|^2}{\|B\|^2}\right\| \|A\| \, \|B\| \, s_j(X)$$

for $j = 1, 2, \ldots, n$.

Lemma 2.2 *If $f : [0, \infty) \to \mathbb{R}$ is convex with $f(0) = 0$, then $f(\alpha a) \le \alpha f(a)$ for all $a \ge 0$ and $0 \le \alpha \le 1$.*

Lemma 2.3 *If $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$, then*

$$\left\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right\| \le \left\|\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right\|.$$

In our next results, I stands for the identity matrix in $\mathbb{M}_n(\mathbb{C})$.

Theorem 2.4 *Let $A, B, C, D, X, Y \in \mathbb{M}_n(\mathbb{C})$ be such that A, B, C, and D are nonzero with $AA^* + CC^* \le I$, $BB^* + DD^* \le I$, and X, Y are positive semidefinite. If f is a nonnegative convex function on $[0, \infty)$ with $f(0) = 0$, then*

$$s_j\left(f\left(\left|AXB^* + CYD^*\right|\right)\right) \le \frac{1}{2}\alpha\sqrt{ab} \, s_j\left(f(X) \oplus f(Y)\right) \tag{2.1}$$

for $j = 1, 2, \ldots, n$, *where*

$$\alpha = \frac{\left\| \frac{|A|^2}{a} + \frac{|B|^2}{b} \right\| + \left\| \frac{|C|^2}{a} + \frac{|D|^2}{b} \right\| + \sqrt{\left(\left\| \frac{|A|^2}{a} + \frac{|B|^2}{b} \right\| - \left\| \frac{|C|^2}{a} + \frac{|D|^2}{b} \right\| \right)^2 + 4 \left\| \frac{A^*C}{a} + \frac{B^*D}{b} \right\|^2}}{2},$$

$a = \|AA^* + CC^*\|$, *and* $b = \|BB^* + DD^*\|$. *In particular,*

$$\begin{aligned} & s_j\left(f\left(\left|AXB^* + BYA^*\right|\right)\right) \\ & \quad \leq \left(\left\||A|^2 + |B|^2\right\| + \left\|A^*B + B^*A\right\|\right) \ s_j\left(f\left(X\right) \oplus f\left(Y\right)\right). \end{aligned} \tag{2.2}$$

Proof Let $R = \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}$, $S = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$, and $T = \begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}$. Then

$$\begin{aligned} & s_j\left(f\left(\left|AXB^* + CYD^*\right|\right)\right) \\ & \quad = f\left(s_j\left(AXB^* + CYD^*\right)\right) \\ & \quad = f\left(s_j\left(RST^*\right)\right) \\ & \quad \leq f\left(\frac{1}{2}\left\|\frac{|R|^2}{\|R\|^2} + \frac{|T|^2}{\|T\|^2}\right\| \|R\| \, \|T\| \, s_j\left(S\right)\right) \text{ (by Lemma 2.1)} \\ & \quad \leq \frac{1}{2}\left\|\frac{|R|^2}{\|R\|^2} + \frac{|T|^2}{\|T\|^2}\right\| \|R\| \, \|T\| \, f\left(s_j\left(S\right)\right) \text{ (by Lemma 2.2)} \\ & \quad = \frac{1}{2}\left\|\frac{|R|^2}{\|R\|^2} + \frac{|T|^2}{\|T\|^2}\right\| \|R\| \, \|T\| \, s_j\left(f\left(X \oplus Y\right)\right). \end{aligned} \tag{2.3}$$

Now,

$$\|R\|^2 = \left\| \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^* & 0 \\ C^* & 0 \end{bmatrix} \right\| = \left\|AA^* + CC^*\right\| = a, \tag{2.4}$$

$$\|T\|^2 = \left\| \begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B^* & 0 \\ D^* & 0 \end{bmatrix} \right\| = \left\|BB^* + DD^*\right\| = b, \tag{2.5}$$

and

$$\begin{aligned} & \left\| \frac{|R|^2}{a} + \frac{|T|^2}{b} \right\| \\ & \quad = \left\| \begin{bmatrix} \frac{|A|^2}{a} + \frac{|B|^2}{b} & \frac{A^*C}{a} + \frac{B^*D}{b} \\ \frac{C^*A}{a} + \frac{D^*B}{b} & \frac{|C|^2}{a} + \frac{|D|^2}{b} \end{bmatrix} \right\| \qquad (2.6) \\ & \leq \left\| \begin{bmatrix} \left\| \frac{|A|^2}{a} + \frac{|B|^2}{b} \right\| & \left\| \frac{A^*C}{a} + \frac{B^*D}{b} \right\| \\ \left\| \frac{A^*C}{a} + \frac{B^*D}{b} \right\| & \left\| \frac{|C|^2}{a} + \frac{|D|^2}{b} \right\| \end{bmatrix} \right\| \text{ (by Lemma 2.3)} \\ & \quad = \frac{1}{2}\left\| \frac{|A|^2}{a} + \frac{|B|^2}{b} \right\| + \frac{1}{2}\left\| \frac{|C|^2}{a} + \frac{|D|^2}{b} \right\| \\ & \quad\quad + \frac{1}{2}\sqrt{\left(\left\| \frac{|A|^2}{a} + \frac{|B|^2}{b} \right\| - \left\| \frac{|C|^2}{a} + \frac{|D|^2}{b} \right\| \right)^2 + 4 \left\| \frac{A^*C}{a} + \frac{B^*D}{b} \right\|^2}. \qquad (2.7) \end{aligned}$$

Thus, inequality (2.1) follows from (2.3), (2.4), (2.5), and (2.7). Inequality (2.2) follows from (2.1) by replacing C and D by B and A, respectively. ◻

To prove our next result, we need the following lemma. A more general form of this lemma can be found in [10].

Lemma 2.5 *If* $X, Y, Z \in \mathbb{M}_n(\mathbb{C})$ *are such that the block matrix* $\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}$ *is positive semidefinite, then*

$$\left\|\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}\right\| \leq ||X|| + ||Y||.$$

Based on equation (2.6) and Lemma 2.5, we have the following result.

Corollary 2.6 *Let* $A, B, C, D, X, Y \in \mathbb{M}_n(\mathbb{C})$ *be such that* $A, B, C,$ *and* D *are nonzero with* $AA^* + CC^* \leq I$, $BB^* + DD^* \leq I$, *and X,Y are positive semidefinite. If* f *is a nonnegative convex function on* $[0, \infty)$ *with* $f(0) = 0$, *then*

$$s_j\left(f\left(\left|AXB^* + CYD^*\right|\right)\right) \leq \frac{1}{2}\beta\sqrt{ab}\, s_j\left(f(X) \oplus f(Y)\right)$$

for $j = 1, 2, \ldots, n$, *where* $\beta = \left\|\frac{|A|^2}{a} + \frac{|B|^2}{b}\right\| + \left\|\frac{|C|^2}{a} + \frac{|D|^2}{b}\right\|$, $a = \|AA^* + CC^*\|$, *and* $b = \|BB^* + DD^*\|$. *In particular, if* $C = B$ *and* $D = A$, *we have*

$$s_j\left(f\left(\left|AXB^* + BYA^*\right|\right)\right) \leq \left\|AA^* + BB^*\right\| \; s_j\left(f(X) \oplus f(Y)\right)$$

for $j = 1, 2, \ldots, n$.

Proof By equation (2.6), we have

$$\begin{aligned} &\left\|\frac{|R|^2}{a} + \frac{|T|^2}{b}\right\| \\ &= \left\|\begin{bmatrix} \frac{|A|^2}{a} + \frac{|B|^2}{b} & \frac{A^*C}{a} + \frac{B^*D}{b} \\ \frac{C^*A}{a} + \frac{D^*B}{b} & \frac{|C|^2}{a} + \frac{|D|^2}{b} \end{bmatrix}\right\| \\ &\leq \left\|\frac{|A|^2}{a} + \frac{|B|^2}{b}\right\| + \left\|\frac{|C|^2}{a} + \frac{|D|^2}{b}\right\| \text{ (by Lemma 2.5).} \end{aligned} \tag{2.8}$$

Now, the result follows from (2.3), (2.4), (2.5), and (2.8). □

To prove our next result, we need the following two lemmas; the first one is Theorem 2.6(a) in [2], while the second one can be found in [9].

Lemma 2.7 *Let* $A, B, X \in \mathbb{M}_n(\mathbb{C})$ *be such that* X *is a positive semidefinite contraction. If* f *is a nonnegative convex function on* $[0, \infty)$ *with* $f(0) = 0$, *then*

$$s_j\left(f\left(\left|AXB^*\right|\right)\right) \leq \left\|f\left(\frac{|A|^2 + |B|^2}{2}\right)\right\| s_j(X)$$

for $j = 1, 2, \ldots, n$.

Lemma 2.8 *If* $A, B \in \mathbb{M}_n(\mathbb{C})$ *are normal, then*

$$|||\, |A + B| \,||| \leq |||\, |A| + |B| \,|||$$

for all unitarily invariant norms.

Theorem 2.9 *Let $A, B, C, D, X, Y \in \mathbb{M}_n(\mathbb{C})$ be such that X and Y are positive semidefinite contractions. If f is a nonnegative convex function on $[0,\infty)$ with $f(0)=0$, then*

$$s_j\left(f\left(\left|AXB^* + CYD^*\right|\right)\right) \le \frac{\left(\max\left(\left\| f\left(|A|^2+|B|^2\right)\right\|, \left\| f\left(|C|^2+|D|^2\right)\right\|\right) + \left\| f\left(|A^*C+B^*D|\right)\right\|\right)}{2} s_j(X \oplus Y)$$

for $j = 1, 2, \ldots, n$. In particular, letting $C = B$ and $D = A$, we have

$$s_j\left(f\left(\left|AXB^* + BYA^*\right|\right)\right) \le \frac{\left(\left\| f\left(|A|^2\right) + f\left(|B|^2\right)\right\| + \left\| f\left(|A^*B+B^*A|\right)\right\|\right)}{2} s_j(X \oplus Y) \tag{2.9}$$

for $j = 1, 2, \ldots, n$.

Proof Let $R = \begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}$, $S = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$, and $T = \begin{bmatrix} B & D \\ 0 & 0 \end{bmatrix}$. Then

$$\begin{aligned} s_j\left(f\left(\left|AXB^* + CYD^*\right|\right)\right) &= \left(s_j(f\left(\left|RST^*\right|\right)\right) \\ &\le \left\| f\left(\frac{|R|^2+|T|^2}{2}\right)\right\| s_j(S) \text{ (by Lemma 2.7)} \\ &= \left\| f\left(\frac{|R|^2+|T|^2}{2}\right)\right\| s_j(X \oplus Y). \end{aligned} \tag{2.10}$$

Moreover,

$$\begin{aligned} &\left\| f\left(\frac{|R|^2+|T|^2}{2}\right)\right\| \\ &= f\left(\left\| \frac{|R|^2+|T|^2}{2} \right\|\right) \\ &= f\left(\left\| \begin{bmatrix} \frac{|A|^2+|B|^2}{2} & \frac{A^*C+B^*D}{2} \\ \frac{C^*A+D^*B}{2} & \frac{|C|^2+|D|^2}{2} \end{bmatrix} \right\|\right) \\ &\le f\left(\left\| \begin{bmatrix} \frac{\| |A|^2+|B|^2\|}{2} & \frac{\|A^*C+B^*D\|}{2} \\ \frac{\|A^*C+B^*D\|}{2} & \frac{\| |C|^2+|D|^2\|}{2} \end{bmatrix} \right\|\right) \text{ (by Lemma 2.3)} \\ &= f\left(\left\| \begin{bmatrix} \frac{\| |A|^2+|B|^2\|}{2} & 0 \\ 0 & \frac{\| |C|^2+|D|^2\|}{2} \end{bmatrix} + \begin{bmatrix} 0 & \frac{\|A^*C+B^*D\|}{2} \\ \frac{\|A^*C+B^*D\|}{2} & 0 \end{bmatrix} \right\|\right) \\ &\le f\left(\left\| \begin{bmatrix} \frac{\| |A|^2+|B|^2\|}{2} & 0 \\ 0 & \frac{\| |C|^2+|D|^2\|}{2} \end{bmatrix} + \begin{bmatrix} \frac{\|A^*C+B^*D\|}{2} & 0 \\ 0 & \frac{\|A^*C+B^*D\|}{2} \end{bmatrix} \right\|\right) \\ &\quad \text{(by Lemma 2.8)} \\ &= f\left(\left\| \begin{bmatrix} \frac{\| |A|^2+|B|^2\|}{2} + \frac{\|A^*C+B^*D\|}{2} & 0 \\ 0 & \frac{\| |C|^2+|D|^2\|}{2} + \frac{\|A^*C+B^*D\|}{2} \end{bmatrix} \right\|\right) \\ &= \left\| f\left(\begin{bmatrix} \frac{\| |A|^2+|B|^2\|}{2} + \frac{\|A^*C+B^*D\|}{2} & 0 \\ 0 & \frac{\| |C|^2+|D|^2\|}{2} + \frac{\|A^*C+B^*D\|}{2} \end{bmatrix}\right)\right\| \end{aligned}$$

$$
\begin{aligned}
&= \left\| \begin{bmatrix} f\left(\frac{\| |A|^2+|B|^2 \|}{2} + \frac{\|A^*C+B^*D\|}{2}\right) & 0 \\ 0 & f\left(\frac{\| |C|^2+|D|^2 \|}{2} + \frac{\|A^*C+B^*D\|}{2}\right) \end{bmatrix} \right\| \\
&\leq \left\| \begin{bmatrix} \frac{f(\| |A|^2+|B|^2 \|)+f(\| |A^*C+B^*D| \|)}{2} & 0 \\ 0 & \frac{f(\| |C|^2+|D|^2 \|)+f(\| A^*C+B^*D \|)}{2} \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} \frac{\| f(|A|^2+|B|^2) \| + \| f(|A^*C+B^*D|) \|}{2} & 0 \\ 0 & \frac{\| f(|C|^2+|D|^2) \| + \| f(|A^*C+B^*D|) \|}{2} \end{bmatrix} \right\| \\
&\leq \left\| \begin{bmatrix} \frac{\| f(|A|^2+|B|^2) \|}{2} & 0 \\ 0 & \frac{\| f(|C|^2+|D|^2) \|}{2} \end{bmatrix} \right\| + \left\| \begin{bmatrix} \frac{\| f(|A^*C+B^*D|) \|}{2} & 0 \\ 0 & \frac{\| f(|A^*C+B^*D|) \|}{2} \end{bmatrix} \right\| \\
&= \frac{1}{2} \max \left(\left\| f\left(|A|^2+|B|^2\right) \right\|, \left\| f\left(|C|^2+|D|^2\right) \right\| \right) + \frac{1}{2} \left\| f\left(\left|A^*C+B^*D\right|\right) \right\|. \qquad (2.11)
\end{aligned}
$$

Thus, the result follows from (2.10) and (2.11). □

Corollary 2.10 *If $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$ are such that X and Y are positive semidefinite contractions, then*

$$
\left\| AXB^* + BYA^* \right\| \leq \left(\|A\| \, \|B\| + \frac{1}{2} \left\| A^*B + B^*A \right\| \right) |||X \oplus Y|||
$$

for all unitarily invariant norms.

Proof Since unitarily invariant norms are increasing functions of singular values, by inequality (2.9) with $f(t) = t$, we have

$$
\begin{aligned}
\left\| AXB^* + BYA^* \right\| &\leq \frac{1}{2} \left(\left\| |A|^2 + |B|^2 \right\| + \left\| A^*B + B^*A \right\| \right) |||X \oplus Y||| \\
&\leq \frac{1}{2} \left(\|A\|^2 + \|B\|^2 + \left\| A^*B + B^*A \right\| \right) |||X \oplus Y||| .
\end{aligned}
$$

Now, for $t > 0$, replacing A by $\sqrt{t}A$ and B by $\frac{1}{\sqrt{t}}B$, and taking the minimum over $t > 0$, we have

$$
\left\| AXB^* + BYA^* \right\| \leq \left(\|A\| \, \|B\| + \frac{1}{2} \left\| A^*B + B^*A \right\| \right) |||X \oplus Y||| ,
$$

as required. □

Remark 2.11 Specifying Corollary 2.10 for the spectral norm, we have

$$
\left\| AXB^* + BYA^* \right\| \leq \left(\|A\| \, \|B\| + \frac{1}{2} \left\| A^*B + B^*A \right\| \right) \max \left(\|X\|, \|Y\| \right).
$$

In particular, if $X = Y = I$, then

$$
\left\| AB^* + BA^* \right\| \leq \|A\| \, \|B\| + \frac{1}{2} \left\| A^*B + B^*A \right\|, \qquad (2.12)
$$

which is related to inequality (1.2). Replacing B by B^* in inequality (2.12), we have the equivalent inequality

$$
\left\| AB + B^*A^* \right\| \leq \|A\| \, \|B\| + \frac{1}{2} \left\| A^*B^* + BA \right\|,
$$

i.e.,

$$2\left\|\Re(AB)\right\| \leq \left\|A\right\|\left\|B\right\| + \left\|\Re(BA)\right\|, \tag{2.13}$$

where $\Re(T)$ is the real part of T, that is, $\Re(T) = \frac{T+T^*}{2}$. It follows from the triangle inequality and the submultiplicativity of the spectral norm that if $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$\left\|\Re(AB)\right\| \leq \left\|A\right\|\left\|B\right\|.$$

By symmetry, it follows from inequality (2.13) that $\left\|\Re(AB)\right\| = \left\|A\right\|\left\|B\right\|$ if and only if $\left\|\Re(BA)\right\| = \left\|A\right\|\left\|B\right\|$. It should be mentioned here that in this result the real part cannot be deleted. In fact, the inequality $2\left\|AB\right\| \leq \left\|A\right\|\left\|B\right\| + \left\|BA\right\|$ can be refuted by considering the following example: if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $\left\|AB\right\| = \left\|A\right\|\left\|B\right\| = 1$, while $BA = 0$.

3 Generalizations of Inequalities (1.1) and (1.5)–(1.10)

We start this section with the following lemmas. For the first and second lemmas, see [7] and [5], respectively, while the third lemma is a well-known fact about Hermitian matrices.

Lemma 3.1 *If $A, B \in \mathbb{M}_n(\mathbb{C})$, then*

$$s_j\left(AB^*\right) \leq s_j\left(\frac{|A|^2 + |B|^2}{2}\right)$$

for $j = 1, 2, \ldots, n$.

Lemma 3.2 *If $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, and f is a nonnegative convex function on $[0, \infty)$, then*

$$\left|\left|\left| f\left(\frac{A+B}{2}\right)\right|\right|\right| \leq \left|\left|\left|\frac{f(A) + f(B)}{2}\right|\right|\right|$$

for all unitarily invariant norms.

Lemma 3.3 *If $A \in \mathbb{M}_n(\mathbb{C})$ is Hermitian, then $\pm A \leq |A|$.*

Now, we present our main result of this section.

Theorem 3.4 *Let $A, B, X, Y, Z \in \mathbb{M}_n(\mathbb{C})$ be such that A and B are positive semidefinite and Z is a contraction. If f is a nonnegative convex function on $[0, \infty)$ with $f(0) = 0$, then*

$$\begin{aligned}
&|||f\left(|ZAX \pm YBZ|\right)||| \\
&\le \frac{\|Z\|^2}{4}|||f\left(2\left(A\oplus B\right)\right)||| + \frac{1}{4}\left|\left|\left|f\left(2A^{1/2}\left|X^*\right|^2A^{1/2}\right)\oplus f\left(2B^{1/2}\left|Y\right|^2B^{1/2}\right)\right|\right|\right| \\
&\quad + \frac{1}{2}\left|\left|\left|f\left(\left|A^{1/2}(Z^*Y \pm XZ^*)B^{1/2}\right|\right)\oplus f\left(\left|A^{1/2}(Z^*Y \pm XZ^*)B^{1/2}\right|\right)\right|\right|\right|
\end{aligned}$$

for all unitarily invariant norms.

Proof Let $K_1 = \begin{bmatrix} ZA^{1/2} & YB^{1/2} \\ 0 & 0 \end{bmatrix}$, $K_2 = \begin{bmatrix} X^*A^{1/2} & -Z^*B^{1/2} \\ 0 & 0 \end{bmatrix}$. Observe that

$$\begin{aligned}
&|K_1|^2 + |K_2|^2 \\
&= \begin{bmatrix} A^{1/2}|Z|^2A^{1/2} & A^{1/2}Z^*YB^{1/2} \\ B^{1/2}Y^*ZA^{1/2} & B^{1/2}|Y|^2B^{1/2} \end{bmatrix} + \begin{bmatrix} A^{1/2}|X^*|^2A^{1/2} & -A^{1/2}XZ^*B^{1/2} \\ -B^{1/2}ZX^*A^{1/2} & B^{1/2}|Z^*|^2B^{1/2} \end{bmatrix} \\
&= \left(A^{1/2}|Z|^2A^{1/2}\oplus B^{1/2}\left|Z^*\right|^2B^{1/2}\right) + \left(A^{1/2}\left|X^*\right|^2A^{1/2}\oplus B^{1/2}|Y|^2B^{1/2}\right) \\
&\quad + \begin{bmatrix} 0 & A^{1/2}(Z^*Y - XZ^*)B^{1/2} \\ B^{1/2}(Y^*Z - ZX^*)A^{1/2} & 0 \end{bmatrix} \\
&\le \|Z\|^2(A\oplus B) + \left(A^{1/2}\left|X^*\right|^2A^{1/2}\oplus B^{1/2}|Y|^2B^{1/2}\right) \\
&\quad + \begin{bmatrix} 0 & A^{1/2}(Z^*Y - XZ^*)B^{1/2} \\ B^{1/2}(Y^*Z - ZX^*)A^{1/2} & 0 \end{bmatrix} \\
&\le \|Z\|^2(A\oplus B) + \left(A^{1/2}\left|X^*\right|^2A^{1/2}\oplus B^{1/2}|Y|^2B^{1/2}\right) \\
&\quad + \left(\left|A^{1/2}(Z^*Y - XZ^*)B^{1/2}\right|\oplus\left|B^{1/2}(Y^*Z - ZX^*)A^{1/2}\right|\right) \text{ (by Lemma 3.3).}
\end{aligned} \tag{3.1}$$

So,

$$\begin{aligned}
&s_j\left(f\left(|ZAX - YBZ|\right)\right) \\
&\quad = s_j\left(f\left(\left|K_1K_2^*\right|\right)\right) \\
&\quad = f\left(s_j\left(K_1K_2^*\right)\right) \\
&\quad \le f\left(s_j\left(\frac{|K_1|^2 + |K_2|^2}{2}\right)\right) \text{ (by Lemma 3.1)} \\
&\quad \le f\left(\frac{1}{2}s_j\left(\begin{array}{c} \|Z\|^2(A\oplus B) + \left(A^{1/2}|X^*|^2A^{1/2}\oplus B^{1/2}|Y|^2B^{1/2}\right) \\ +\left|A^{1/2}(Z^*Y - XZ^*)B^{1/2}\right|\oplus\left|B^{1/2}(Y^*Z - ZX^*)A^{1/2}\right| \end{array}\right)\right) \\
&\qquad \text{(by inequality(3.1))} \\
&= s_j\left(f\left(\frac{1}{2}\left(\begin{array}{c} \|Z\|^2(A\oplus B) + \left(A^{1/2}|X^*|^2A^{1/2}\oplus B^{1/2}|Y|^2B^{1/2}\right) \\ +\left|A^{1/2}(Z^*Y - XZ^*)B^{1/2}\right|\oplus\left|B^{1/2}(Y^*Z - ZX^*)A^{1/2}\right| \end{array}\right)\right)\right).
\end{aligned} \tag{3.2}$$

It follows from inequality (3.2) that

$$\begin{aligned}
&|||f\left(|ZAX - YBZ|\right)||| \\
&\quad \le \left|\left|\left|f\left(\frac{1}{2}\left(\begin{array}{c} \|Z\|^2(A\oplus B) + \left(A^{1/2}|X^*|^2A^{1/2}\oplus B^{1/2}|Y|^2B^{1/2}\right) \\ +\left|A^{1/2}(Z^*Y - XZ^*)B^{1/2}\right|\oplus\left|B^{1/2}(Y^*Z - ZX^*)A^{1/2}\right| \end{array}\right)\right)\right|\right|\right|.
\end{aligned} \tag{3.3}$$

Now,

$$\begin{aligned}
&\left|\left|\left| f\left(\frac{1}{2}\left(\begin{array}{c} \|Z\|^2(A\oplus B)+\left(A^{1/2}\left|X^*\right|^2A^{1/2}\oplus B^{1/2}\left|Y\right|^2B^{1/2}\right)\\ +\left|A^{1/2}(Z^*Y-XZ^*)B^{1/2}\right|\oplus\left|B^{1/2}(Y^*Z-ZX^*)A^{1/2}\right|\end{array}\right)\right)\right|\right|\right|\\
&\leq \left|\left|\left|\begin{array}{c}\frac{1}{2}f\left(\|Z\|^2(A\oplus B)+\left(A^{1/2}\left|X^*\right|^2A^{1/2}\oplus B^{1/2}\left|Y\right|^2B^{1/2}\right)\right)\\ +\frac{1}{2}f\left(\left(\left|A^{1/2}(Z^*Y-XZ^*)B^{1/2}\right|\right)\oplus\left(\left|B^{1/2}(Y^*Z-ZX^*)A^{1/2}\right|\right)\right)\end{array}\right|\right|\right|\\
&\quad\text{(by Lemma 3.2)}\\
&\leq \frac{1}{2}\left|\left|\left| f\left(\frac{2\|Z\|^2(A\oplus B)+2\left(A^{1/2}\left|X^*\right|^2A^{1/2}\oplus B^{1/2}\left|Y\right|^2B^{1/2}\right)}{2}\right)\right|\right|\right|\\
&\quad+\frac{1}{2}\left|\left|\left| f\left(\left|A^{1/2}(Z^*Y-XZ^*)B^{1/2}\right|\right)\oplus f\left(\left|B^{1/2}(Y^*Z-ZX^*)A^{1/2}\right|\right)\right|\right|\right|\\
&\quad\text{(by the triangle inequality)}\\
&\leq \frac{1}{4}\left|\left|\left| f\left(2\|Z\|^2(A\oplus B)\right)+f\left(2\left(A^{1/2}\left|X^*\right|^2A^{1/2}\oplus B^{1/2}\left|Y\right|^2B^{1/2}\right)\right)\right|\right|\right|\\
&\quad+\frac{1}{2}\left|\left|\left| f\left(\left|A^{1/2}(Z^*Y-XZ^*)B^{1/2}\right|\right)\oplus f\left(\left|A^{1/2}(Z^*Y-XZ^*)B^{1/2}\right|\right)\right|\right|\right|\\
&\quad\text{(by Lemma 3.2)}\\
&\leq \frac{\|Z\|^2}{4}\left|\left|\left| f\left(2(A\oplus B)\right)\right|\right|\right|+\frac{1}{4}\left|\left|\left| f\left(2A^{1/2}\left|X^*\right|^2A^{1/2}\right)\oplus f\left(2B^{1/2}\left|Y\right|^2B^{1/2}\right)\right|\right|\right|\\
&\quad+\frac{1}{2}\left|\left|\left| f\left(\left|A^{1/2}(Z^*Y-XZ^*)B^{1/2}\right|\right)\oplus f\left(\left|A^{1/2}(Z^*Y-XZ^*)B^{1/2}\right|\right)\right|\right|\right|\\
&\quad\text{(by the triangle inequality and Lemma 2.2).}
\end{aligned}\tag{3.4}$$

Inequalities (3.3) and (3.4) imply

$$\begin{aligned}
&\left|\left|\left| f\left(\left|ZAX-YBZ\right|\right)\right|\right|\right|\\
&\leq \frac{\|Z\|^2}{4}\left|\left|\left| f\left(2(A\oplus B)\right)\right|\right|\right|+\frac{1}{4}\left|\left|\left| f\left(2A^{1/2}\left|X^*\right|^2A^{1/2}\right)\oplus f\left(2B^{1/2}\left|Y\right|^2B^{1/2}\right)\right|\right|\right|\\
&\quad+\frac{1}{2}\left|\left|\left| f\left(\left|A^{1/2}(Z^*Y-XZ^*)B^{1/2}\right|\right)\oplus f\left(\left|A^{1/2}(Z^*Y-XZ^*)B^{1/2}\right|\right)\right|\right|\right|.
\end{aligned}\tag{3.5}$$

On the other hand, from (3.5), replacing Y by $-Y$, we obtain

$$\begin{aligned}
&\left|\left|\left| f\left(\left|ZAX+YBZ\right|\right)\right|\right|\right|\\
&\leq \frac{\|Z\|^2}{4}\left|\left|\left| f\left(2(A\oplus B)\right)\right|\right|\right|+\frac{1}{4}\left|\left|\left| f\left(2A^{1/2}\left|X^*\right|^2A^{1/2}\right)\oplus f\left(2B^{1/2}\left|Y\right|^2B^{1/2}\right)\right|\right|\right|\\
&\quad+\frac{1}{2}\left|\left|\left| f\left(\left|A^{1/2}(Z^*Y+XZ^*)B^{1/2}\right|\right)\oplus f\left(\left|A^{1/2}(Z^*Y+XZ^*)B^{1/2}\right|\right)\right|\right|\right|.
\end{aligned}\tag{3.6}$$

Now, the result follows from (3.5) and (3.6). □

It can be easily seen that when specializing Theorem 3.4 for the function $f(t)=t$ the contractive condition that is imposed on the matrix Z can be dropped. A stronger version of this special case can be seen in the following result.

Corollary 3.5 *If $A, B, X, Y, Z \in \mathbb{M}_n(\mathbb{C})$ are such that A and B are positive semidefinite, then*

$$\begin{aligned}&|||ZAX \pm YBZ|||\\&\le \|Z\| \sqrt{|||A \oplus B|||\ |||(X^*AX) \oplus (YBY^*)|||}\\&\quad + \frac{1}{2} \left|\left|\left|(A^{1/2}\left(Z^*Y \pm XZ^*\right) B^{1/2}) \oplus (A^{1/2}\left(Z^*Y \pm XZ^*\right) B^{1/2})\right|\right|\right| \end{aligned} \tag{3.7}$$

for all unitarily invariant norms. In particular, we have

$$\begin{aligned}|||ZX \pm XZ||| \le \|Z\| &\sqrt{|||I \oplus I|||\ \ |||(X^*X) \oplus (X^*X)|||}\\&+\frac{1}{2} \left|\left|\left|\left(Z^*X \pm XZ^*\right) \oplus \left(Z^*X \pm XZ^*\right)\right|\right|\right| \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}|||AX \pm YB||| \le &\sqrt{|||A \oplus B|||\ \ |||(X^*AX) \oplus (YBY^*)|||}\\&+\frac{1}{2} \left|\left|\left|(A^{1/2}\,(Y \pm X)\, B^{1/2}) \oplus (A^{1/2}\,(Y \pm X)\, B^{1/2})\right|\right|\right|. \end{aligned} \tag{3.9}$$

Proof We only prove (3.7), the other inequalities are special cases of this. In Theorem 3.4, letting $f(t) = t$, we get

$$\begin{aligned}&|||ZAX \pm YBZ|||\\&\le \frac{\|Z\|^2}{2} |||A \oplus B||| + \frac{1}{2} \left|\left|\left|\left(A^{1/2} \left|X^*\right|^2 A^{1/2}\right) \oplus \left(B^{1/2} |Y|^2 B^{1/2}\right)\right|\right|\right|\\&\quad+\frac{1}{2} \left|\left|\left|\left(A^{1/2}(Z^*Y \pm XZ^*)B^{1/2}\right) \oplus \left(A^{1/2}(Z^*Y \pm XZ^*)B^{1/2}\right)\right|\right|\right|. \end{aligned} \tag{3.10}$$

From (3.10), replacing A, B, X, Y by $tA, tB, \frac{1}{t}X, \frac{1}{t}Y$ $(t > 0)$, respectively, we get

$$\begin{aligned}&|||ZAX \pm YBZ|||\\&\le \frac{\|Z\|^2 t}{2} |||A \oplus B\quad||| + \frac{1}{2t} \left|\left|\left|\left(A^{1/2} \left|X^*\right|^2 A^{1/2}\right) \oplus \left(B^{1/2} |Y|^2 B^{1/2}\right)\right|\right|\right|\\&\quad+\frac{1}{2} \left|\left|\left|\left(A^{1/2}(Z^*Y \pm XZ^*)B^{1/2}\right) \oplus \left(A^{1/2}(Z^*Y \pm XZ^*)B^{1/2}\right)\right|\right|\right|, \end{aligned}$$

and so

$$\begin{aligned}&|||ZAX \pm YBZ|||\\&\le \min_{t>0} \left(\frac{\|Z\|^2 t}{2} |||A \oplus B||| + \frac{1}{2t} \left|\left|\left|A^{1/2} \left|X^*\right|^2 A^{1/2} \oplus B^{1/2} |Y|^2 B^{1/2}\right|\right|\right|\right)\\&\quad+\frac{1}{2} \left|\left|\left|\left(A^{1/2}(Z^*Y \pm XZ^*)B^{1/2}\right) \oplus \left(A^{1/2}(Z^*Y \pm XZ^*)B^{1/2}\right)\right|\right|\right|\\&= \|Z\| \sqrt{|||A \oplus B|||\ \ |||(X^*AX) \oplus (YBY^*)|||}\\&\quad+\frac{1}{2} \left|\left|\left|\left(A^{1/2}(Z^*Y \pm XZ^*)B^{1/2}\right) \oplus \left(A^{1/2}(Z^*Y \pm XZ^*)B^{1/2}\right)\right|\right|\right|, \end{aligned}$$

as required. □

Remark 3.6 Corollary 3.5 presents a generalization inequalities (1.1) and (1.5)–(1.8). This can be seen as follows:

(1) Inequality (1.1) can be retained by applying (3.8) for the spectral norm, that is,

$$\|ZX - XZ\| \leq \|Z\| \sqrt{\|I \oplus I\| \, \|(XX^*) \oplus (XX^*)\|} + \frac{1}{2} \left\| \left(Z^*X - XZ^*\right) \oplus \left(Z^*X - XZ^*\right) \right\|,$$

which is equivalent to saying that

$$\|ZX - XZ\| \leq \|Z\| \, \|X\| + \frac{1}{2} \left\| Z^*X - XZ^* \right\|.$$

(2) Inequality (1.5) can be retained directly from (3.9) by taking $Y = X$. Consequently, (1.6) and (1.7) can also be retained by applying (3.9) for the spectral norm and the Schatten p-norms, for $p \geq 1$, and taking $Y = X$.

(3) Inequality (1.8) can be retained from (3.9) as follows: as a consequence of (3.9), by letting $Y = X$, we have

$$|||AX + XB||| \leq \sqrt{|||A \oplus B||| \;\; |||(X^*AX) \oplus (XBX^*)|||} + \left|\left|\left| \left(A^{1/2}XB^{1/2}\right) \oplus \left(A^{1/2}XB^{1/2}\right) \right|\right|\right|.$$

This inequality, together with the fact that

$$\left|\left|\left| \left(X^*AX\right) \oplus \left(XBX^*\right) \right|\right|\right| \leq \|X\|^2 \, |||A \oplus B|||,$$

enables us to get (1.8). Since (1.9) and (1.10) are particular cases of (1.8), they also can be retained by applying (3.9) for the spectral norm and the Schatten p-norms, for $p \geq 1$, and taking $Y = X$.

Now, we need the following lemma from [8].

Lemma 3.7 *If $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then for $r \geq 0$*

$$\left|\left|\left| A^{1/2} (A + B)^r \, B^{1/2} \right|\right|\right| \leq \frac{1}{2} \left|\left|\left| (A + B)^{r+1} \right|\right|\right|$$

for all unitarily invariant norms.

An application of inequality (3.9) can be stated as follows.

Corollary 3.8 *If $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then*

$$|||AB||| \leq \frac{1}{2} \max\left(\|A\|, \|B\|\right) |||A \oplus B||| + \frac{1}{8} \left|\left|\left| (A + B)^2 \oplus (A + B)^2 \right|\right|\right|$$

for all unitarily invariant norms.

Proof From (3.9), letting $X = B$ and $Y = A$, we obtain

$$2\,|||AB||| \leq \sqrt{|||A \oplus B||| \;\; |||(BAB) \oplus (ABA)|||} + \frac{1}{2} \left|\left|\left| \left(A^{1/2}(A + B)\, B^{1/2}\right) \oplus \left(A^{1/2}(A + B)\, B^{1/2}\right) \right|\right|\right|. \tag{3.11}$$

Observe that

$$|||(BAB) \oplus (ABA)||| \leq \left|\left|\left| (\|B\|^2 A) \oplus (\|A\|^2 B) \right|\right|\right| \leq \max\left(\|A\|^2, \|B\|^2\right) |||A \oplus B|||. \tag{3.12}$$

Also,

$$\begin{aligned}
&\left|\left|\left|(A^{1/2}(A+B)B^{1/2}) \oplus (A^{1/2}(A+B)B^{1/2})\right|\right|\right| \\
&= \left|\left|\left|\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}^{1/2} \left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}\right) \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}^{1/2}\right|\right|\right| \\
&\leq \frac{1}{2}\left|\left|\left|\left(\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}\right)^2\right|\right|\right| \text{(by Lemma 3.7)} \\
&= \frac{1}{2}\left|\left|\left|(A+B)^2 \oplus (A+B)^2\right|\right|\right|. \qquad (3.13)
\end{aligned}$$

Now, the result follows from (3.11), (3.12), and (3.13). □

Specializing Corollary 3.8 for the spectral norm and for Schatten p-norms, for $p \geq 1$, we obtain the following result.

Corollary 3.9 *If $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then*

$$\|AB\| \leq \frac{1}{2}\max\left(\|A\|^2, \|B\|^2\right) + \frac{1}{8}\|A+B\|^2$$

and

$$\|AB\|_p \leq \frac{1}{2}\max\left(\|A\|, \|B\|\right)\left(\|A\|_p^p + \|B\|_p^p\right)^{1/p} + 2^{1/p-3}\left\|(A+B)^2\right\|_p$$

for $p \geq 1$.

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