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Existence and nonexistence results for fractional mixed boundary value problems via a Lyapunov-type inequality

Barbara Łupińska[1]

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Abstract
In this article, we prove a Lyapunov-type inequality for a fractional differential equation under mixed boundary conditions. As applications, we deduce nonexistence results for some fractional boundary value problems. Moreover, we obtain numerical approximations of a lower bound for the eigenvalues of the corresponding equations.

Keywords Lyapunov inequality · Fractional differential equations · Katugampola derivative · Eigenvalue of fractional differential equation · Mixed boundary problem

1 Introduction

The well-known Lyapunov result states that if a nontrivial solution to the boundary value problem

$$\begin{aligned} &u''(t) + g(t)u(t) = 0, \qquad a < t < b,\\ &u(a) = u(b) = 0 \end{aligned}$$

exists, where $g : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |g(t)|dt > \frac{4}{b-a}.$$

By now, this result and its extensions have been found to be useful, e.g., in oscillation theory, disconjugacy, eigenvalue problems, and many other theories based on differential and difference equations (see [1–7] and the references therein).

Together with the rising popularity of fractional operators, which are interesting because of their non-local character allowing us to model non-local or time dependent processes, many modifications of the Lyapunov inequality appeared. The first work in this direction is due to Ferreira [8], who derived a Lyapunov-type inequality for differential equations depending on

✉ Barbara Łupińska
bpietruczuk@math.uwb.edu.pl

1 Institute of Computer Science, University of Bialystok, Konstantego Ciołkowskiego 1 M, 15-245 Białystok, Poland

the Riemann–Liouville fractional derivative. For other, similar works we refer the reader to [9–14].

Motivated by the above works as well as useful applications of Lyapunov's inequalities, we focus here on the Katugampola fractional differential equation with mixed boundary conditions. More precisely, we consider the fractional boundary value problem

$$D_{a+}^{\alpha,\rho} u(t) + g(t)u(t) = 0,\ a < t < b,\ 1 < \alpha \leq 2, \tag{1.1}$$

$$u(a) = u'(b) = 0, \tag{1.2}$$

where $D_{a+}^{\alpha,\rho}$ denotes the Katugampola fractional derivative of order α and $g : [a, b] \to \mathbb{R}$ is a continuous function. The Katugampola fractional derivative was chosen because it generalizes two other fractional operators: the Riemann–Liouville and the Hadamard fractional derivatives. The main goal of this work is to obtain a Lyapunov-type inequality for the above fractional boundary value problem and to present applications to demonstrate the effectiveness of this inequality.

2 Preliminaries

Before presenting the main results, let us start by recalling the concept of Katugampola fractional operators which was introduced in 2014 by Udita Katugampola. For more details, we refer to [15, 16].

Definition 2.1 Let $\alpha > 0,\ \rho > 0,\ -\infty < a < b < \infty$. The operators

$$I_{a+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau,$$

$$I_{b-}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\alpha}} f(\tau) d\tau,$$

for $t \in (a, b)$, are called the left-sided and right-sided Katugampola integrals of fractional order α, respectively. The operators $I_{a+}^{\alpha,\rho}$ and $I_{b-}^{\alpha,\rho}$ are well defined in $L^p(a, b),\ p \geq 1$.

Definition 2.2 Let $\alpha > 0,\ \rho > 0,\ n = [\alpha] + 1,\ 0 < a < t < b \leq \infty$. The operators

$$D_{a+}^{\alpha,\rho} f(t) = \left(t^{1-\rho} \frac{d}{dt}\right)^n I_{a+}^{n-\alpha,\rho} f(t),$$

$$D_{b+}^{\alpha,\rho} f(t) = \left(-t^{1-\rho} \frac{d}{dt}\right)^n I_{b-}^{n-\alpha,\rho} f(t),$$

for $t \in (a, b)$, are called the left-sided and right-sided Katugampola derivatives of fractional order α, respectively.

The Katugampola derivative generalizes two other fractional operators by introducing a new parameter $\rho > 0$ in the definition. Indeed, if we take $\rho \to 1$, we have the Riemann–Liouville fractional derivative i.e.,

$$\lim_{\rho \to 1} D_{a+}^{\alpha,\rho} f(t) = \left(\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau.$$

Moreover, if we take $\rho \to 0^+$, we get the Hadamard fractional derivative, i.e.,

$$\lim_{\rho \to 0^+} D_{a+}^{\alpha,\rho} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{\tau}\right)^{n-\alpha-1} f(\tau) \frac{d\tau}{\tau}.$$

The higher order Katugampola fractional operators satisfy the following properties, which were precisely discussed and proven in [17].

1. If $\alpha > 0$, $\rho > 0$ and $\lambda > -1$ then
$$I_{a+}^{\alpha,\rho}\left(\frac{t^\rho - a^\rho}{\rho}\right)^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\lambda}.$$
2. For $\rho > 0,\ \alpha > 0,\ \lambda > \alpha - 1$, we have
$$D_{a+}^{\alpha,\rho}\left(\frac{t^\rho - a^\rho}{\rho}\right)^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\lambda-\alpha}.$$
3. If $n-1 < \alpha < n, n \in \mathbb{N}, \rho > 0$ then
$$I_{a+}^{\alpha,\rho} D_{a+}^{\alpha,\rho} f(t) = f(t) + \sum_{i=0}^{n-1} \tilde{c}_i \left(\frac{t^\rho - a^\rho}{\rho}\right)^{i-n+\alpha},$$
where $\tilde{c}_i$ are real constants.
4. If $\alpha > 0,\ \rho > 0$ and $f \in L^p(a,b)$ then
$$D_{a+}^{\alpha,\rho} I_{a+}^{\alpha,\rho} f(t) = f(t).$$

3 Main results

In this section we prove a necessary condition for the existence of a solution to the boundary value problem with Katugampola derivative of order $1 < \alpha \le 2$.

3.1 Integral representation of the solution

We start by writing (1.1)–(1.2) in its equivalent integral form.

Theorem 3.1 *The function $u \in C[a,b]$ is a solution to the boundary value problem (1.1)–(1.2) if and only if u is a solution to the integral equation*
$$u(t) = \int_a^b G(t,s)g(s)u(s)\,ds, \tag{3.1}$$
where the Green function G is given by
$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1} \begin{cases} \left(\dfrac{b^\rho - a^\rho}{b^\rho - s^\rho}\right)^{2-\alpha} \left(t^\rho - a^\rho\right)^{\alpha-1}, & a \le t \le s \le b, \\ \left(\dfrac{b^\rho - a^\rho}{b^\rho - s^\rho}\right)^{2-\alpha} \left(t^\rho - a^\rho\right)^{\alpha-1} - \left(t^\rho - s^\rho\right)^{\alpha-1}, & a \le s < t \le b. \end{cases} \tag{3.2}$$

Proof The general solution to (1.1) is
$$u(t) = c_1\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} + c_2\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-2} - I_{a+}^{\alpha,\rho}\big(g(t)u(t)\big).$$

Taking the derivative of u, we obtain

$$u'(t) = -\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}(\alpha-1)\rho\int_a^t t^{\rho-1}\frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{2-\alpha}}g(\tau)u(\tau)\,d\tau$$
$$+c_1(\alpha-1)t^{\rho-1}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-2}+c_2t^{\rho-1}(\alpha-2)\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-3}.$$

Using the boundary conditions (1.2), we get $c_2=0$ and

$$c_1=\frac{(b^\rho-a^\rho)^{2-\alpha}}{\Gamma(\alpha)}\int_a^b\frac{\tau^{\rho-1}}{(b^\rho-\tau^\rho)^{2-\alpha}}g(\tau)u(\tau)\,d\tau.$$

Therefore,

$$u(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\Bigg[\int_a^t\Bigg[\left(\frac{b^\rho-a^\rho}{b^\rho-\tau^\rho}\right)^{2-\alpha}\left(t^\rho-a^\rho\right)^{\alpha-1}-\left(t^\rho-\tau^\rho\right)^{\alpha-1}\Bigg]\tau^{\rho-1}g(\tau)u(\tau)\,d\tau$$
$$+\int_t^b\left(\frac{b^\rho-a^\rho}{b^\rho-\tau^\rho}\right)^{2-\alpha}\left(t^\rho-a^\rho\right)^{\alpha-1}\tau^{\rho-1}g(\tau)u(\tau)\,d\tau\Bigg],$$

which concludes the proof. □

3.2 Green function estimates

Theorem 3.2 *The function G defined by (3.2) satisfies the following properties:*

1. $G(t,s)\geq 0$ *for* $t\in[a,b]$, $s\in[a,b]$;
2. $\max\limits_{t\in[a,b]}G(t,s)=G(s,s)\leq\dfrac{\rho^{1-\alpha}(b^\rho-a^\rho)}{\Gamma(\alpha)}s^{\rho-1}(b^\rho-s^\rho)^{\alpha-2}.$

Proof First we prove the positivity of G. For $t\leq s$ it is obvious, but for $s<t$ we can rewrite G in the form

$$\begin{aligned}G(t,s)&=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}s^{\rho-1}\Bigg[\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\left(\frac{b^\rho-a^\rho}{b^\rho-s^\rho}\right)\left(t^\rho-a^\rho\right)^{\alpha-1}-\left(t^\rho-s^\rho\right)^{\alpha-1}\Bigg]\\
&\geq\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}s^{\rho-1}\Bigg[\left(\frac{b^\rho-s^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\left(t^\rho-a^\rho\right)^{\alpha-1}-\left(t^\rho-s^\rho\right)^{\alpha-1}\Bigg]\\
&=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}s^{\rho-1}\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\\
&\quad\times\left[\left(b^\rho-s^\rho\right)^{\alpha-1}-\left(\frac{(t^\rho-s^\rho)(b^\rho-a^\rho)}{t^\rho-a^\rho}\right)^{\alpha-1}\right]\\
&=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}s^{\rho-1}\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\\
&\quad\times\left[\left(b^\rho-s^\rho\right)^{\alpha-1}-\left(b^\rho-\left(a^\rho+\frac{(s^\rho-a^\rho)(b^\rho-a^\rho)}{t^\rho-a^\rho}\right)\right)^{\alpha-1}\right].\end{aligned}$$

Let us note that

$$s^\rho \leq a^\rho + \frac{(s^\rho - a^\rho)(b^\rho - a^\rho)}{t^\rho - a^\rho} \leq b^\rho, \tag{3.3}$$

because the following inequalities hold:

$$\frac{(s^\rho - a^\rho)(b^\rho - t^\rho)}{t^\rho - a^\rho} \geq 0$$

and

$$\frac{(t^\rho - s^\rho)(b^\rho - a^\rho)}{t^\rho - a^\rho} \geq 0.$$

Therefore $G(t, s) \geq 0$ also for $s < t$.

Now, we show that $G(t, s) \leq G(s, s)$. First, let us take the interval $a \leq t \leq s \leq b$. Differentiating G with respect to t we get

$$\frac{\partial G}{\partial t} = \frac{\rho^{2-\alpha}(\alpha - 1)}{\Gamma(\alpha)} s^{\rho-1} t^{\rho-1} \left(\frac{(b^\rho - s^\rho)(t^\rho - a^\rho)}{b^\rho - a^\rho} \right)^{\alpha-2} > 0.$$

This means that the function G with respect to t is increasing on the given interval. We conclude

$$G(t, s) \leq G(s, s) \quad \text{for} \quad a \leq t \leq s \leq b. \tag{3.4}$$

Now, we turn our attention to the function G on the interval $a \leq s < t \leq b$. We start by fixing an arbitrary $s \in [a, b]$. Differentiating G with respect to t we have

$$\begin{aligned}
\frac{\partial G}{\partial t} &= \frac{\rho^{2-\alpha}(\alpha - 1)}{\Gamma(\alpha)} s^{\rho-1} t^{\rho-1} \left[\left(\frac{(b^\rho - s^\rho)(t^\rho - a^\rho)}{b^\rho - a^\rho} \right)^{\alpha-2} - \left(t^\rho - s^\rho\right)^{\alpha-2} \right] \\
&= \frac{\rho^{2-\alpha}(\alpha - 1)}{\Gamma(\alpha)} s^{\rho-1} t^{\rho-1} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\alpha-2} \\
&\quad \times \left[(b^\rho - s^\rho)^{\alpha-2} - \left(\frac{(t^\rho - s^\rho)(b^\rho - a^\rho)}{t^\rho - a^\rho} \right)^{\alpha-2} \right] \\
&= \frac{\rho^{2-\alpha}(\alpha - 1)}{\Gamma(\alpha)} s^{\rho-1} t^{\rho-1} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\alpha-2} \\
&\quad \times \left[(b^\rho - s^\rho)^{\alpha-2} - \left(b^\rho - \left(a^\rho + \frac{(s^\rho - a^\rho)(b^\rho - a^\rho)}{t^\rho - a^\rho} \right) \right)^{\alpha-2} \right].
\end{aligned}$$

Since the inequalities (3.3) hold, it follows that $\frac{\partial G}{\partial t} < 0$. Therefore, the function G with respect to t is decreasing on the given interval. We get

$$G(t, s) \leq G(s, s) \quad \text{for} \quad a < s < t < b. \tag{3.5}$$

From (3.4) to (3.5) we infer

$$
\begin{aligned}
G(t,s) \le G(s,s) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} s^{\rho-1} \left(\frac{b^\rho - s^\rho}{b^\rho - a^\rho}\right)^{\alpha-2} \left(s^\rho - a^\rho\right)^{\alpha-1} \\
&\le \frac{\rho^{1-\alpha}(b^\rho - a^\rho)}{\Gamma(\alpha)} s^{\rho-1}(b^\rho - s^\rho)^{\alpha-2}
\end{aligned}
$$

for all $t, s \in [a, b]$.

The proof is completed. □

3.3 A Lyapunov-type inequality

We are ready to state and prove our main results in the Banach space $C[a, b]$ with the maximum norm $\|u\| = \max\limits_{t\in[a,b]} |u(t)|$.

Theorem 3.3 *If a nontrivial continuous solution of the fractional boundary value problem (1.1)–(1.2) exists, where g is a real and continuous function, then*

$$
\int_a^b s^{\rho-1}(b^\rho - s^\rho)^{\alpha-2}|g(s)|\,ds \ge \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)}. \tag{3.6}
$$

Proof It follows from Theorem 3.1 that the solution of the fractional boundary value problem (1.1)–(1.2) satisfies the integral equation (3.1). Thus,

$$
|u(t)| \le \int\limits_a^b G(t,s)|g(s)||u(s)|\,ds, \qquad t \in [a, b].
$$

Using the estimation of the function G which was obtained in Theorem 3.2 we get

$$
\|u\| \le \frac{\rho^{1-\alpha}(b^\rho - a^\rho)}{\Gamma(\alpha)} \|u\| \int_a^b s^{\rho-1}(b^\rho - s^\rho)^{\alpha-2}|g(s)|\,ds.
$$

Thus, we have

$$
\int_a^b s^{\rho-1}(b^\rho - s^\rho)^{\alpha-2}|g(s)|\,ds \ge \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)}.
$$

The proof is completed. □

Note that taking $\rho = 1$ in Theorem 3.3 we have Lyapunov's inequality with the Riemman–Liouville fractional derivative D^α_{a+}.

Corollary 3.4 *If a nontrivial continuous solution of the fractional boundary value problem*

$$
\begin{aligned}
&D^\alpha_{a+}u(t) + g(t)u(t) = 0,\ a < t < b,\ 1 < \alpha \le 2, \\
&u(a) = u'(b) = 0,
\end{aligned}
$$

exists, where g is a real and continuous function, then

$$
\int_a^b (b-s)^{\alpha-2}|g(s)|\,ds \ge \frac{\Gamma(\alpha)}{b-a}.
$$

Moreover, taking $\rho \to 0^+$ in Theorem 3.3 we have Lyapunov's inequality with the Hadamard fractional derivative ${}^H D^\alpha_{a+}$.

Corollary 3.5 *If a nontrivial continuous solution of the fractional boundary value problem*

$$
\begin{aligned}
&{}^{H}D^{\alpha}_{a+}u(t) + g(t)u(t) = 0,\ a < t < b,\ 1 < \alpha \leq 2,\\
&u(a) = u'(b) = 0,
\end{aligned}
$$

exists, where g is a real and continuous function, then

$$
\int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-2} |g(s)| \frac{ds}{s} \geq \frac{\Gamma(\alpha)}{\ln \frac{b}{a}}.
$$

4 Applications

In this section, we apply the results on the Lyapunov-type inequality obtained previously to study the nonexistence of solutions for certain fractional boundary value problems.

Theorem 4.1 *If*

$$
\int_a^b s^{\rho-1}(b^\rho - s^\rho)^{\alpha-2}|g(s)|\,ds < \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)}, \tag{4.1}
$$

then (1.1)–(1.2) has no nontrivial solution.

Proof Assume the contrary, i.e., (1.1)–(1.2) has a nontrivial solution u. By Theorem 3.3, we obtain

$$
\int_a^b s^{\rho-1}(b^\rho - s^\rho)^{\alpha-2}|g(s)|\,ds \geq \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)}
$$

which contradicts assumption (4.1). □

The other application is that we derive an estimation related to the eigenvalue of the corresponding equation by using our Lyapunov-type inequality (3.6). For given $\lambda \in \mathbb{R}$, we consider the following boundary value problem:

$$
\begin{cases}
D^{\alpha,\rho}_{a+}u(t) + \lambda u(t) = 0,\ a < t < b,\ 1 < \alpha \leq 2,\\
u(a) = u'(b) = 0.
\end{cases} \tag{4.2}
$$

If (4.2) admits a nontrivial solution u_λ, we say that λ is an eigenvalue of (4.2).

Corollary 4.2 *If λ is an eigenvalue of (4.2), then*

$$
|\lambda| \geq (\alpha - 1)\Gamma(\alpha)\left(\frac{\rho}{b^\rho - a^\rho}\right)^{\alpha}.
$$

Proof Since λ is an eigenvalue of (4.2), it has a nontrivial solution u_λ. According to Theorem 3.3, we get

$$
|\lambda| \int_a^b s^{\rho-1}(b^\rho - s^\rho)^{\alpha-2}\,ds \geq \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)}.
$$

Therefore, calculating the integral by using the substitution $b^\rho - s^\rho = t$, we obtain

$$
|\lambda| \frac{(b^\rho - a^\rho)^{\alpha-1}}{\rho(\alpha-1)} \geq \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho - a^\rho)},
$$

which is the desired result. □

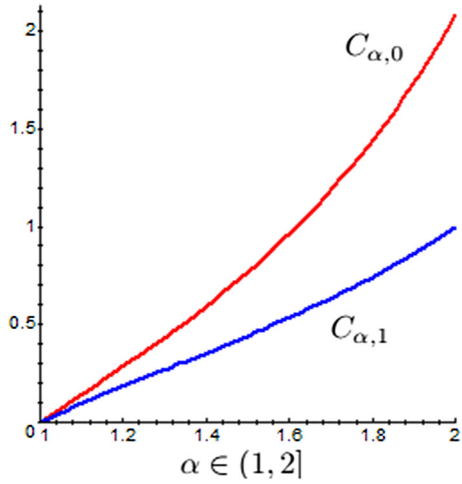

Fig. 1 Graph of the function $C_{\alpha,\rho}$ for $\alpha \in (1, 2]$ and $\rho = 1, 2$

Example 4.3 Let us consider the following problem:

$$\begin{cases} D_{1+}^{\alpha,\rho} u(t) + \lambda u(t) = 0, \;\; 1 < t < 2, \;\; 1 < \alpha \leq 2, \\ u(1) = u'(2) = 0. \end{cases} \tag{4.3}$$

By Corollary 4.2, if a continuous solution to (4.3) exists, then necessarily

$$|\lambda| > (\alpha - 1)\Gamma(\alpha) \left(\frac{\rho}{2^\rho - 1} \right)^\alpha . \tag{4.4}$$

Note that the existence of solutions depends on the parameters α, λ and ρ. Indeed, for $\rho = 1$ inequality (4.4) is satisfied if and only if

$$|\lambda| > C_{\alpha,1} := (\alpha - 1)\Gamma(\alpha),$$

and for $\rho \to 0^+$ we have

$$|\lambda| > C_{\alpha,0} := (\alpha - 1)\Gamma(\alpha)(\ln 2)^{-\alpha}.$$

The Fig. 1 shows the behavior of $C_{\alpha,\rho}$ ($\rho = 1, 2$) with respect to $\alpha \in (1, 2)$ (Fig. 1).

Moreover, let $\lambda = 0.5$. If we take $\alpha = 1.25$, then inequality (4.4) is satisfied if and only if $\rho > 0$. Hence, in this case, a solution to (4.3) may exist for all values of ρ for which the Katugampola derivative is defined. Now, let us choose $\alpha = 1.5$. In this case inequality (4.4) is satisfied if and only if $\rho > 0.7893$. Therefore, for $\rho = 1$, a solution to (4.3) may exist but for $\rho \to 0^+$ it does not. Finally, for $\alpha = 1.95$, inequality (4.4) is satisfied if and only if $\rho > 1.79$. In particular, for $\rho = 1$ and $\rho \to 0^+$ a solution to (4.3) does not exist.

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