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On the consecutive *k*-free values for certain classes of polynomials

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Abstract
In the present paper we propose an asymptotic formula for $R(H, k)$, the number of triples of positive integers $x, y, z \leq H$ such that $x^2 + y^2 + z^2 + 1, x^2 + y^2 + z^2 + 2$ are k-free with $k \geq 2$. Especially, in the case of $k = 2$ we prove that $R(H, 2) = \sigma_2 H^3 + O(H^{9/4+\varepsilon})$, where σ_2 is an absolute constant and ε is an arbitrary small positive number, which improves the error term $O(H^{7/3+\varepsilon})$ given by Chen (Indian J Pure Appl Math, 2022. https://doi.org/10.1007/s13226-022-00292-z). The key point of the new result is a refinement of Dimitrov's method.

Keywords K-free number · Asymptotic formula · Gauss sum · Salié sum

1 Introduction

For a natural number $k \geq 2$, we say an integer n is k-free if $p^k \nmid n$ for any prime p. The distribution of k-free numbers is an important theme for number theory scholars. It is well known that

$$\sum_{n \leq x} \mu_k(n) = \frac{x}{\zeta(k)} + O(x^{1/(k+1)+\varepsilon}),$$

proved by Montgomery and Vanghan [13], where $\mu_k(n)$ denotes the characteristic function of the k-free integers and $\zeta(k)$ is the usual Riemann zeta function. There also exist many articles that consider the k-free values of polynomials. In the case of $k = 2$, we mention Estermann's work [8], where it is showed that for an absolute constant a

$$\sum_{1 \leq x \leq H} \mu^2(x^2 + 1) = aH + O(H^{2/3+\varepsilon}). \tag{1.1}$$

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Heath-Brown [9] obtained (1.1) with the error term replaced by $O(H^{7/12+\varepsilon})$. In 1932, Carlitz [2] considered the polynomial $x(x+1)$ and he proved

$$\sum_{1\le x\le H}\mu^2(x)\,\mu^2(x+1)=\prod_p\left(1-\frac{2}{p^2}\right)H+O(H^{2/3+\varepsilon}). \tag{1.2}$$

Later, the exponent $2/3+\varepsilon$ was improved to $7/11+\varepsilon$ by Heath-Brown [10] using the square sieve, and to $(26+\sqrt{433})/81+\varepsilon$ by Reuss [15].

Evaluating square-free values of the polynomial in multiple variables is an essential generalization that has attracted many authors, including Dimitrov, Tolev, Zhou and Chen. In 2012, Tolev [16] studied the square-free values of the polynomial x^2+y^2+1, and he proved

$$\sum_{1\le x,y\le H}\mu^2(x^2+y^2+1)=\prod_p\left(1-\frac{\lambda(p^2)}{p^4}\right)H^2+O(H^{3/4+\varepsilon}),$$

where $\lambda(q)$ is the number of the integer solutions to the following congruence equation:

$$x^2+y^2+1\equiv 0\ (\mathrm{mod}\ q),\quad 1\le x,y\le q.$$

Recently, by using Tolev's method and some estimate for the Salié sum, Zhou and Ding [17] got an asymptotic formula for $\Sigma_{1\le x,y,z\le H}\,\mu^2(x^2+y^2+z^2+k)$. On the other hand, motivated by Carlitz's result (1.2), consecutive square-free values of polynomials in multiple variables have also been considered. Dimitrov [6, 7] found asymptotic formulas for consecutive square-free numbers of the form x^2+y^2+1, x^2+y^2+2 and respectively of the form x^2+1, x^2+2. Chen [3] considered the consecutive square-free numbers $x_1^2+\cdots+x_k^2+1$, $x_1^2+\cdots+x_k^2+2$ and showed

$$\sum_{1\le x_1,\dots,x_k\le H}\mu^2(x_1^2+\cdots+x_k^2+1)\,\mu^2(x_1^2+\cdots+x_k^2+2)=cH^k+O(H^{k-\frac{1}{2}-\frac{1}{2k}+\varepsilon}) \tag{1.3}$$

for any given integer $k\ge 3$ and an absolute constant c.

For the k-free values of polynomials, the classical problem is to study the following:

$$S_k(H)=\sum_{x\le H}\mu_k(x)\mu_k(x+1).$$

In 1932, Carlitz [2] obtained

$$S_k(H)=c_1H+O(H^{2(k+1)+\varepsilon}),$$

where the exponent $2(k+1)+\varepsilon$ may be considered as trivial. Later on, Brandes [1] derived an improvement upon the trivial exponent which is of order $1/k^2$ as $k\to\infty$. Using the approximative determinant method of Heath-Brown, Dictmann and Marmon [5] obtained the exponent $14/9k+\varepsilon$, which sharpens previous bound for $k\ge 6$. The k-free values of the polynomial $(x+a_1)(x+a_2)$ were considered by Mirsky [12]. Recently, Chen and Wang [4] studied the r-free values of $x^2+y^2+z^2+k$ and gave an asymptotic formula.

Let

$$R(H,k)=\sum_{1\le x,y,z\le H}\mu_k(x^2+y^2+z^2+1)\,\mu_k(x^2+y^2+z^2+2) \tag{1.4}$$

and

$$\lambda(q_1,q_2;m,n,l)=\sum_{\substack{1\le x,y,z\le q_1q_2\\ x^2+y^2+z^2+1\equiv 0\ (\mathrm{mod}\ q_1)\\ x^2+y^2+z^2+2\equiv 0\ (\mathrm{mod}\ q_2)}}e_{q_1q_2}(mx+ny+lz). \tag{1.5}$$

For simplicity, we also define

$$\lambda(q_1,q_2)=\lambda(q_1,q_2;0,0,0),\ \lambda(q_1,q_2;m)=\lambda(q_1,q_2;m,0,0) \tag{1.6}$$

and

$$\lambda(q_1,q_2;m,n)=\lambda(q_1,q_2;m,n,0).$$

Inspired by the above results, we shall study the k-free values of the polynomials $x^2+y^2+z^2+1$ and $x^2+y^2+z^2+2$ by following the method in Dimitrov [6] and pruning some details referring to Chen and Wang [4]. The key ingredient is still the estimation of $\lambda(q_1,q_2;m,n,l)$, which we give with the help of the elementary properties of Gauss sums and Salié sums. We prove the following.

Theorem 1.1 *For any given integer $k\geq 2$, the asymptotic formula*

$$R(H,k)=\sigma_k H^3+O(H^{\frac{3}{2}+\frac{3}{2k}+\varepsilon}+H^2)$$

holds. Here

$$\sigma_k=\prod_p\left(1-\frac{\lambda(p^k,1)+\lambda(1,p^k)}{p^{3k}}\right).$$

In the case of $k=2$, we obtain from Theorem 1.1 the following:

Theorem 1.2 *If $\varepsilon>0$ is an arbitrary positive number, then*

$$R(H,2)=\prod_p\left(1-\frac{\lambda(p^2,1)+\lambda(1,p^2)}{p^6}\right)H^3+O(H^{9/4+\varepsilon}).$$

Theorem 1.2 improves upon (1.3) with $k=3$.

2 Notations and preliminary lemmas

Throughout this paper, H is a sufficiently large positive number and m,n,l denote integers. By ε we denote an arbitrary positive number which may have different values in different places. As usual, $\mu(n)$ denotes the Möbius function; $\tau(n)$ and $\omega(n)$ represent the number of positive divisors of n and the number of distinct prime factors of n, respectively. Instead of $m\equiv n \pmod d$ we write for simplicity $m\equiv n\,(d)$. (m,n,l) denotes the greatest common divisor of m,n,l and $\|\xi\|$ denotes the distance from ξ to its nearest integer. We write $e(t)=\exp(2\pi i t)$ and $e_q(t)=e(t/q)$. For any x and q such that $(x,q)=1$, we denote by $\overline{x_q}$ the inverse of n modulo q. If we can understand the value of the modulus from the context, then we write for simplicity $\overline{x}$. For any odd q, $\left(\frac{\cdot}{q}\right)$ is the Jacobi symbol.

We define the Gauss sum and the Salié sum as follows:

$$G(q;n,m)=\sum_{1\leq x\leq q}e_q(nx^2+mx),\quad G(q;n)=\sum_{1\leq x\leq q}e_q(nx^2) \tag{2.1}$$

and, for odd integers q,

$$S(q;n,m)=\sum_{\substack{1\leq x\leq q\\(x,q)=1}}\left(\frac{x}{q}\right)e_q(nx+m\overline{x}). \tag{2.2}$$

In what follows, we present some lemmas used in the proof of the theorems. First we quote some important properties of the Gauss sum.

Lemma 2.1 *For the Gauss sum, the following hold:*
(1) If $(q_1, q_2) = 1$, *then*

$$G(q_1q_2; m_1q_2 + m_2q_1, n) = G(q_1; m_1q_2^2, n)\, G(q_2; m_2q_1^2, n).$$

(2) If $(q, m) = d$, *then*

$$G(q; m, n) = \begin{cases} d\, G(q/d; m/d, n/d), & \text{if } d \mid n, \\ 0, & \text{if } d \nmid n. \end{cases}$$

(3) If $(q, 2m) = 1$, *then*

$$G(q; m, n) = e_q\left(-\overline{(4m)}n^2\right)\left(\frac{m}{q}\right) G(q; 1).$$

Proof See Lemma 3.1 of [6]. □

In the next lemma, we present the upper bound result of the Salié sum.

Lemma 2.2 *If q is an odd integer, then*

$$\mid S(q; n, m) \mid \le 2^{\omega(q)}\sqrt{q}.$$

Proof See p. 524 in [11]. □

Lemma 2.3 *If* $(q_1'q_1'', q_2'q_2'') = (q_1', q_1'') = (q_2', q_2'') = 1$, *then*

$$\lambda\left(q_1'q_1'', q_2'q_2''; m, n, l\right) = \lambda\left(q_1', q_2'; m\overline{(q_1''q_2'')}_{q_1'q_2'}, n\overline{(q_1''q_2'')}_{q_1'q_2'}, l\overline{(q_1''q_2'')}_{q_1'q_2'}\right)$$
$$\times \lambda\left(q_1'', q_2''; m\overline{(q_1'q_2')}_{q_1''q_2''}, n\overline{(q_1'q_2')}_{q_1''q_2''}, l\overline{(q_1'q_2')}_{q_1''q_2''}\right).$$

Proof Let

$$x = x_1q_1''q_2'' + x_2q_1'q_2',\ y = y_1q_1''q_2'' + y_2q_1'q_2',\ z = z_1q_1''q_2'' + z_2q_1'q_2',$$

where $1 \le x_1, y_1, z_1 \le q_1'q_2'$ and $1 \le x_2, y_2, z_2 \le q_1''q_2''$.
From the Chinese remainder theorem, we obtain

$$\begin{aligned}
\lambda\left(q_1'q_1'', q_2'q_2''; m, n, l\right) &= \sum_{\substack{1\le x,y,z\le q_1'q_1''q_2'q_2'' \\ x^2+y^2+z^2+1\equiv 0\ (q_1'q_1'') \\ x^2+y^2+z^2+2\equiv 0\ (q_2'q_2'')}} e_{q_1'q_1''q_2'q_2''}(mx+ny+lz) \\
&= \sum_{\substack{1\le x_1,y_1,z_1\le q_1'q_2' \\ (q_1''q_2''x_1)^2+(q_1''q_2''y_1)^2+(q_1''q_2''z_1)^2+1\equiv 0\ (q_1') \\ (q_1''q_2''x_1)^2+(q_1''q_2''y_1)^2+(q_1''q_2''z_1)^2+2\equiv 0\ (q_2')}} e_{q_1'q_2'}(mx_1+ny_1+lz_1) \\
&\quad \times \sum_{\substack{1\le x_2,y_2,z_2\le q_1''q_2'' \\ (q_1'q_2'x_2)^2+(q_1'q_2'y_2)^2+(q_1'q_2'z_2)^2+1\equiv 0\ (q_1'') \\ (q_1'q_2'x_2)^2+(q_1'q_2'y_2)^2+(q_1'q_2'z_2)^2+2\equiv 0\ (q_2'')}} e_{q_1''q_2''}(mx_2+ny_2+lz_2).
\end{aligned} \tag{2.3}$$

By using the substitutions $q_1''q_2''x_1 \to x_1$, $q_1''q_2''y_1 \to y_1$, $q_1''q_2''z_1 \to z_1$, we infer

$$\sum_{\substack{1\le x_1,y_1,z_1\le q_1'q_2' \\ (q_1''q_2''x_1)^2+(q_1''q_2''y_1)^2+(q_1''q_2''z_1)^2+1\equiv 0\ (q_1') \\ (q_1''q_2''x_1)^2+(q_1''q_2''y_1)^2+(q_1''q_2''z_1)^2+2\equiv 0\ (q_2')}} e_{q_1'q_2'}(mx_1+ny_1+lz_1)$$

$$= \sum_{\substack{\overline{(q_1''q_2'')}_{q_1'q_2'}x_1,\overline{(q_1''q_2'')}_{q_1'q_2'}y_1,\overline{(q_1''q_2'')}_{q_1'q_2'}z_1(q_1'q_2') \\ x_1^2+y_1^2+z_1^2+1\equiv 0\ (q_1') \\ x_1^2+y_1^2+z_1^2+2\equiv 0\ (q_2')}} e_{q_1'q_2'}\big(m\overline{(q_1''q_2'')}_{q_1'q_2'}x_1+n\overline{(q_1''q_2'')}_{q_1'q_2'}y_1$$

$$+l\overline{(q_1''q_2'')}_{q_1'q_2'}z_1\big) = \lambda\left(q_1',q_2'; m\overline{(q_1''q_2'')}_{q_1'q_2'}, n\overline{(q_1''q_2'')}_{q_1'q_2'}, l\overline{(q_1''q_2'')}_{q_1'q_2'}\right). \tag{2.4}$$

Similarly,

$$\sum_{\substack{1\le x_2,y_2,z_2\le q_1''q_2'' \\ (q_1'q_2'x_2)^2+(q_1'q_2'y_2)^2+(q_1'q_2'z_2)^2+1\equiv 0\ (q_1'') \\ (q_1'q_2'x_2)^2+(q_1'q_2'y_2)^2+(q_1'q_2'z_2)^2+2\equiv 0\ (q_2'')}} e_{q_1''q_2''}(mx_2+ny_2+lz_2)$$

$$= \lambda\left(q_1'',q_2''; m\overline{(q_1'q_2')}_{q_1''q_2''}, n\overline{(q_1'q_2')}_{q_1''q_2''}, l\overline{(q_1'q_2')}_{q_1''q_2''}\right). \tag{2.5}$$

Thus, Lemma 2.3 follows immediately from (2.3)–(2.5). □

Now we give an upper bound estimate of $\lambda(q_1,q_2;m,n,l)$ by applying Lemmas 2.1, 2.2 and 2.3.

Lemma 2.4 *If* $8\nmid q_1q_2$ *and* $(q_1,q_2)=1$, *then*

$$\lambda(q_1,q_2;m,n,l) \ll 64\, q_1q_2\, \tau(q_1q_2)\, 2^{\omega(q_1)}2^{\omega(q_2)}(q_1q_2,m,n,l).$$

Proof Case 1. $2\nmid q_1q_2$.

In view of the orthogonality relations

$$\frac{1}{q}\sum_{1\le h\le q} e_q(ht) = \begin{cases} 1, & \text{if } t\equiv 0\,(q), \\ 0, & \text{otherwise,}\end{cases}$$

we obtain from (1.5), (2.1) and Lemma 2.1

$$\begin{aligned}
&\lambda(q_1,q_2;m,n,l) \\
&= \frac{1}{q_1q_2}\sum_{1\le x,y,z\le q_1q_2} e_{q_1q_2}(mx+ny+lz) \\
&\quad\times \sum_{1\le h_1\le q_1} e_{q_1}(h_1(x^2+y^2+z^2+1)) \sum_{1\le h_2\le q_2} e_{q_2}(h_2(x^2+y^2+z^2+2)) \\
&= \frac{1}{q_1q_2}\sum_{1\le h_1\le q_1} e_{q_1}(h_1)\sum_{1\le h_2\le q_2} e_{q_2}(2h_2)\, G(q_1q_2;h_1q_2+h_2q_1,m) \\
&\quad\times G(q_1q_2;h_1q_2+h_2q_1,n)\, G(q_1q_2;h_1q_2+h_2q_1,l) \\
&= \frac{1}{q_1q_2}\sum_{1\le h_1\le q_1} e_{q_1}(h_1)\, G(q_1;h_1q_2^2,m)\, G(q_1;h_1q_2^2,n)\, G(q_1;h_1q_2^2,l)
\end{aligned}$$

$$\times \sum_{1\le h_2\le q_2} e_{q_2}(2h_2)\, G(q_2; h_2q_1^2, m)\, G(q_2; h_2q_1^2, n)\, G(q_2; h_2q_1^2, l)$$
$$= \frac{1}{q_1q_2} \sum_{l_1|q_1} \sum_{\substack{1\le h_1\le q_1\\ (h_1,q_1)=\frac{q_1}{l_1}}} e_{q_1}(h_1)\, G(q_1; h_1q_2^2, m)\, G(q_1; h_1q_2^2, n)\, G(q_1; h_1q_2^2, l)$$
$$\times \sum_{l_2|q_2} \sum_{\substack{1\le h_2\le q_2\\ (h_2,q_2)=\frac{q_2}{l_2}}} e_{q_2}(2h_2)\, G(q_2; h_2q_1^2, m)\, G(q_2; h_2q_1^2, n)\, G(q_2; h_2q_1^2, l).$$

Since $(q_1, q_2) = 1$, $(q_i, h_i) = \frac{q_i}{l_i}$, $l_i \mid q_i$ and $2 \nmid q_1q_2$, by Lemma 2.1

$$\lambda(q_1, q_2; m, n, l)$$
$$= q_1^2q_2^2 \sum_{\substack{l_1|q_1\\ \frac{q_1}{l_1}|(m,n,l)}} \frac{1}{l_1^3} \sum_{\substack{1\le r_1\le l_1\\ (r_1,l_1)=1}} e_{l_1}(r_1) G(l_1; r_1q_2^2, ml_1q_1^{-1})$$
$$\times G(l_1; r_1q_2^2, nl_1q_1^{-1})\, G(l_1; r_1q_2^2, l\,l_1q_1^{-1}) \sum_{\substack{l_2|q_2\\ \frac{q_2}{l_2}|(m,n,l)}} \frac{1}{l_2^3} \sum_{\substack{1\le r_2\le l_2\\ (r_2,l_2)=1}} e_{l_2}(2r_2)$$
$$\times G(l_2; r_2q_1^2, ml_2q_2^{-1})\, G(l_2; r_2q_1^2, nl_2q_2^{-1})\, G(l_2; r_2q_1^2, l\,l_2q_2^{-1})$$
$$= q_1^2q_2^2 \sum_{\substack{l_1|q_1\\ \frac{q_1}{l_1}|(m,n,l)}} \frac{G^3(l_1; 1)}{l_1^3} \sum_{\substack{1\le r_1\le l_1\\ (r_1,l_1)=1}} \left(\frac{r_1}{l_1}\right)^3 e_{l_1}\left(r_1 - \overline{(4r_1q_2^2)}(m^2+n^2+l^2)l_1^2q_1^{-2}\right)$$
$$\times \sum_{\substack{l_2|q_2\\ \frac{q_2}{l_2}|(m,n,l)}} \frac{G^3(l_2; 1)}{l_2^3} \sum_{\substack{1\le r_2\le l_2\\ (r_2,l_2)=1}} \left(\frac{r_2}{l_2}\right)^3 e_{l_2}\left(2r_2 - \overline{(4r_2q_1^2)}(m^2+n^2+l^2)l_2^2q_2^{-2}\right)$$
$$= q_1^2q_2^2 \sum_{\substack{l_1|q_1\\ \frac{q_1}{l_1}|(m,n,l)}} \frac{G^3(l_1; 1)}{l_1^3} S(l_1; 1, -\overline{(4q_2^2)}(m^2+n^2+l^2)l_1^2q_1^{-2})$$
$$\times \sum_{\substack{l_2|q_2\\ \frac{q_2}{l_2}|(m,n,l)}} \frac{G^3(l_2; 1)}{l_2^3} S(l_2; 1, -\overline{(4q_1^2)}(m^2+n^2+l^2)l_2^2q_2^{-2}).$$

It is well known that $|G(l_1, 1)| = \sqrt{l_1}$, thus, by Lemma 2.2,

$$\lambda(q_1, q_2; m, n, l) \ll q_1^2q_2^2 \sum_{\substack{l_1|q_1\\ \frac{q_1}{l_1}|(m,n,l)}} l_1^{-3/2}2^{\omega(l_1)}l_1^{1/2} \sum_{\substack{l_2|q_2\\ \frac{q_2}{l_2}|(m,n,l)}} l_2^{-3/2}2^{\omega(l_2)}l_2^{1/2}$$
$$\ll q_1^2q_2^2 2^{\omega(q_1)}2^{\omega(q_2)} \sum_{\frac{q_1}{l_1}|(q_1,m,n,l)} l_1^{-1} \sum_{\frac{q_2}{l_2}|(q_2,m,n,l)} l_2^{-1}$$
$$\ll q_1^2q_2^2 2^{\omega(q_1)}2^{\omega(q_2)} \sum_{r_1|(q_1,m,n,l)} q_1^{-1}r_1 \sum_{r_2|(q_2,m,n,l)} q_2^{-1}r_2$$
$$\ll q_1q_2\tau(q_1q_2)2^{\omega(q_1)}2^{\omega(q_2)}(q_1q_2, m, n, l). \tag{2.6}$$

Case 2. $q_1 = 2^h q_1'$, where $2 \nmid q_1'$, $h \le 2$ and $2 \nmid q_2$.

By Lemma 2.3, we have

$$\lambda(2^h q_1', q_2; m, n, l) = \lambda\left(2^h, 1; m\overline{(q_1'q_2)}_{2^h}, n\overline{(q_1'q_2)}_{2^h}, l\overline{(q_1'q_2)}_{2^h}\right) \times \lambda\left(q_1', q_2; m\overline{2^h}_{q_1'q_2}, n\overline{2^h}_{q_1'q_2}, l\overline{2^h}_{q_1'q_2}\right).$$

A combination of the trivial estimate $\lambda\left(2^h, 1; m\overline{(q_1'q_2)}_{2^h}, n\overline{(q_1'q_2)}_{2^h}, l\overline{(q_1'q_2)}_{2^h}\right) \ll 8^h$ and (2.6) yields

$$\begin{aligned}\lambda(2^h q_1', q_2; m, n, l) &\ll 8^h q_1' q_2 \tau(q_1'q_2) 2^{\omega(q_1')} 2^{\omega(q_2)} \left(q_1'q_2, m\overline{2^h}_{q_1'q_2}, n\overline{2^h}_{q_1'q_2}, l\overline{2^h}_{q_1'q_2}\right)\\ &\ll 64\, q_1 q_2\, \tau(q_1q_2)\, 2^{\omega(q_1)} 2^{\omega(q_2)} (q_1q_2, m, n, l).\end{aligned}$$

Case 3. $q_2 = 2^h q_2'$, where $2 \nmid q_2'$, $h \le 2$ and $2 \nmid q_1$.

Similarly to Case 2, we obtain

$$\lambda(q_1', 2^h q_2'; m, n, l) \ll 64\, q_1 q_2\, \tau(q_1q_2)\, 2^{\omega(q_1)} 2^{\omega(q_2)} (q_1q_2, m, n, l).$$

Combining the estimates for the three cases gives the proof of Lemma 2.4. □

Lemma 2.5 *If* $8 \nmid q_1 q_2$ *and* $(q_1, q_2) = 1$, *then for the sums*

$$V_1 = \sum_{1 \le m \le H} \frac{\lambda(q_1, q_2; m)}{m}, \quad V_2 = \sum_{1 \le m, n \le H} \frac{\lambda(q_1, q_2; m, n)}{mn}$$

and

$$V_3 = \sum_{1 \le m, n, l \le H} \frac{\lambda(q_1, q_2; m, n, l)}{mnl},$$

we have the estimates

$$V_1 \ll (q_1q_2)^{1+\varepsilon} H^{\varepsilon}, \; V_2 \ll (q_1q_2)^{1+\varepsilon} H^{\varepsilon}, \; V_3 \ll (q_1q_2)^{1+\varepsilon} H^{\varepsilon}.$$

Proof By Lemma 2.4,

$$\begin{aligned}V_1 &\ll (q_1q_2)^{1+\varepsilon} \sum_{1 \le m \le H} \frac{(q_1q_2, m)}{m}\\ &\ll (q_1q_2)^{1+\varepsilon} \sum_{r \mid q_1q_2} r \sum_{\substack{m \le H\\ m \equiv 0\,(r)}} \frac{1}{m}\\ &\ll (q_1q_2)^{1+\varepsilon} \sum_{r \mid q_1q_2} 1 \sum_{t \le H/r} \frac{1}{t}\\ &\ll (q_1q_2)^{1+\varepsilon} H^{\varepsilon}.\end{aligned}$$

In a similar way, using Lemma 2.4 we obtain

$$\begin{aligned}V_2 &\ll (q_1q_2)^{1+\varepsilon} \sum_{1 \le m, n \le H} \frac{(q_1q_2, m, n)}{mn}\\ &\ll (q_1q_2)^{1+\varepsilon} \sum_{1 \le m, n \le H} \frac{(q_1q_2, m)(q_1q_2, n)}{mn}\end{aligned}$$

$$\ll (q_1q_2)^{1+\varepsilon} H^{\varepsilon}$$

and

$$\begin{aligned} V_3 &\ll (q_1q_2)^{1+\varepsilon} \sum_{1\le m,n,l\le H} \frac{(q_1q_2,m,n,l)}{mnl} \\ &\ll (q_1q_2)^{1+\varepsilon} \sum_{1\le m,n,l\le H} \frac{(q_1q_2,m)(q_1q_2,n)(q_1q_2,l)}{mnl} \\ &\ll (q_1q_2)^{1+\varepsilon} H^{\varepsilon}. \end{aligned}$$

□

Lemma 2.6 *For any real number σ and positive integers N_1, N_2 with $N_1 < N_2$, we have*

$$\sum_{N_1<n\le N_2} e(\sigma n) \ll \min\{N_2 - N_1, \|\sigma\|^{-1}\}.$$

Proof See Lemma 4.7 of [14]. □

3 Proof of Theorem 1.1

Upon using the well-known identity

$$\mu_k(n) = \sum_{d^k|n} \mu(d),$$

we find by (1.4) that

$$\begin{aligned} R(H,k) &= \sum_{\substack{d_1,d_2\\(d_1,d_2)=1}} \mu(d_1)\mu(d_2) \sum_{\substack{1\le x,y,z\le H\\ x^2+y^2+z^2+1\equiv 0\ (d_1^k)\\ x^2+y^2+z^2+2\equiv 0\ (d_2^k)}} 1 \\ &= R_1(H) + R_2(H), \end{aligned} \tag{3.1}$$

where

$$R_1(H) = \sum_{\substack{d_1d_2<\xi\\(d_1,d_2)=1}} \mu(d_1)\mu(d_2)\, S(H; d_1^k, d_2^k), \tag{3.2}$$

$$R_2(H) = \sum_{\substack{d_1d_2>\xi\\(d_1,d_2)=1}} \mu(d_1)\mu(d_2)\, S(H; d_1^k, d_2^k),$$

$$S(H; d_1^k, d_2^k) = \sum_{\substack{1\le x,y,z\le H\\ x^2+y^2+z^2+1\equiv 0\ (d_1^k)\\ x^2+y^2+z^2+2\equiv 0\ (d_2^k)}} 1, \tag{3.3}$$

ξ is a parameter to be chosen so that $H^{1/k} \le \xi \le H^{2/k}$.

3.1 Estimation of $R_1(H)$

To estimate the contribution of $R_1(H)$, we suppose that $q_1 = d_1^k$, $q_2 = d_2^k$, where d_1 and d_2 are square-free, $(q_1, q_2) = 1$ and $d_1 d_2 \le \xi$.

We first analyze $S(H; q_1, q_2)$. Define

$$\Sigma(H, q_1, q_2, x) = \sum_{\substack{h \le H \\ h \equiv x\ (q_1 q_2)}} 1.$$

By orthogonality, $\Sigma(H, q_1, q_2, x)$ may be written as

$$\begin{aligned}\Sigma(H, q_1, q_2, x) &= (q_1 q_2)^{-1} \sum_{1 \le h \le H} \sum_{1 \le t \le q_1 q_2} e_{q_1 q_2}((h - x)t) \\ &= (q_1 q_2)^{-1} \sum_{1 \le t \le q_1 q_2} e_{q_1 q_2}(-xt) \sum_{1 \le h \le H} e_{q_1 q_2}(ht) \\ &= H (q_1 q_2)^{-1} + (q_1 q_2)^{-1} \sum_{1 \le t \le q_1 q_2 - 1} e_{q_1 q_2}(-xt) \sum_{1 \le h \le H} e_{q_1 q_2}(ht). \qquad (3.4)\end{aligned}$$

From the definition of $S(H; q_1, q_2)$, we easily see that

$$S(H; q_1, q_2) = \sum_{\substack{1 \le x, y, z \le q_1 q_2 \\ x^2 + y^2 + z^2 + 1 \equiv 0\ (q_1) \\ x^2 + y^2 + z^2 + 2 \equiv 0\ (q_2)}} \Sigma(H, q_1, q_2, x) \Sigma(H, q_1, q_2, y) \Sigma(H, q_1, q_2, z), \qquad (3.5)$$

which combined with (3.4) yields

$$S(H; q_1, q_2) = \sum_{\substack{1 \le x, y, z \le q_1 q_2 \\ x^2 + y^2 + z^2 + 1 \equiv 0\ (q_1) \\ x^2 + y^2 + z^2 + 2 \equiv 0\ (q_2)}} \left(\frac{H^3}{(q_1 q_2)^3} + 3 \frac{H^2}{(q_1 q_2)^2} W_1 + 3 \frac{H}{q_1 q_2} W_2 + W_3 \right), \qquad (3.6)$$

where

$$\begin{aligned} W_1 &:= W_1(x; q_1, q_2, H) = \frac{1}{q_1 q_2} \sum_{1 \le t \le q_1 q_2 - 1} e_{q_1 q_2}(-xt) \sum_{1 \le h \le H} e_{q_1 q_2}(ht), \\ W_2 &:= W_2(x, y; q_1, q_2, H) = \frac{1}{(q_1 q_2)^2} \sum_{1 \le t_1, t_2 \le q_1 q_2 - 1} e_{q_1 q_2}(-x t_1 - y t_2) \\ &\quad \times \prod_{i=1}^{2} \left(\sum_{1 \le h_i \le H} e_{q_1 q_2}(h_i t_i) \right), \\ W_3 &:= W_3(x, y, z; q_1, q_2, H) = \frac{1}{(q_1 q_2)^3} \sum_{1 \le t_1, t_2, t_3 \le q_1 q_2 - 1} e_{q_1 q_2}(-x t_1 - y t_2 - z t_3) \\ &\quad \times \prod_{i=1}^{3} \left(\sum_{1 \le h_i \le H} e_{q_1 q_2}(h_i t_i) \right). \end{aligned}$$

By exchanging the order of summations and noting the definitions of $\lambda(q_1, q_2; m, n, l)$, we obtain

$$\sum_{\substack{1\le x,y,z\le q_1q_2\\ x^2+y^2+z^2+1\equiv 0\ (q_1)\\ x^2+y^2+z^2+2\equiv 0\ (q_2)}} W_1 = \frac{1}{q_1q_2}\sum_{1\le t\le q_1q_2-1}\lambda(q_1,q_2;-t)\sum_{1\le h\le H} e_{q_1q_2}(ht)$$
$$:= L_1(q_1, q_2; H),$$

$$\sum_{\substack{1\le x,y,z\le q_1q_2\\ x^2+y^2+z^2+1\equiv 0\ (q_1)\\ x^2+y^2+z^2+2\equiv 0\ (q_2)}} W_2 = \frac{1}{(q_1q_2)^2}\sum_{1\le t_1,t_2\le q_1q_2-1}\lambda(q_1,q_2;-t_1,-t_2)$$
$$\times\prod_{i=1}^{2}\left(\sum_{1\le h_i\le H} e_{q_1q_2}(h_it_i)\right)$$
$$:= L_2(q_1, q_2; H),$$

$$\sum_{\substack{1\le x,y,z\le q_1q_2\\ x^2+y^2+z^2+1\equiv 0\ (q_1)\\ x^2+y^2+z^2+2\equiv 0\ (q_2)}} W_3 = \frac{1}{(q_1q_2)^3}\sum_{1\le t_1,t_2,t_3\le q_1q_2-1}\lambda(q_1,q_2;-t_1,-t_2,-t_3)$$
$$\times\prod_{i=1}^{3}\left(\sum_{1\le h_i\le H} e_{q_1q_2}(h_it_i)\right)$$
$$:= L_3(q_1, q_2; H).$$

Now we treat $L_1(q_1, q_2; H)$. Employing Lemma 2.6 we get

$$\sum_{1\le h\le H} e_{q_1q_2}(ht) \ll \left\|\frac{t}{q_1q_2}\right\|^{-1}.$$

Hence by Lemma 2.5, it follows that

$$L_1(q_1, q_2; H) \ll \sum_{1\le t\le q_1q_2-1}\frac{\lambda(q_1,q_2;-t)}{t} \ll (q_1q_2)^{1+\varepsilon}.$$

The same estimates hold for $L_2(q_1, q_2; H)$ and $L_3(q_1, q_2; H)$. Gathering the estimates for $L_1(q_1, q_2; H)$, $L_2(q_1, q_2; H)$ and $L_3(q_1, q_2; H)$ and noting (3.6), we arrive at

$$S(H; q_1, q_2) = \frac{H^3}{(q_1q_2)^3}\lambda(q_1, q_2) + O(H^2(q_1q_2)^{-1+\varepsilon} + H(q_1q_2)^{\varepsilon} + (q_1q_2)^{1+\varepsilon}). \quad (3.7)$$

According to (3.2) and (3.7), $R_1(H)$ can be estimated as follows:

$$R_1(H) = \sum_{\substack{d_1d_2<\xi\\ (d_1,d_2)=1}} \mu(d_1)\mu(d_2)\frac{\lambda(d_1^k, d_2^k)}{(d_1d_2)^{3k}}H^3$$
$$+ O\left(\sum_{\substack{d_1d_2<\xi\\ (d_1,d_2)=1}}((d_1d_2)^{-k+\varepsilon}H^2 + H(d_1d_2)^{\varepsilon} + (d_1d_2)^{k+\varepsilon})\right)$$

$$= \sum_{\substack{d_1 d_2 = 1 \\ (d_1, d_2) = 1}}^{\infty} \mu(d_1)\mu(d_2) \frac{\lambda(d_1^k, d_2^k)}{(d_1 d_2)^{3k}} H^3 + O(H^3 \xi^{1-k+\varepsilon} + H^2 + \xi^{1+k+\varepsilon}). \tag{3.8}$$

In the last step above, we can check that by using Lemma 2.4

$$\sum_{\substack{d_1 d_2 > \xi \\ (d_1, d_2) = 1}} \mu(d_1)\mu(d_2) \frac{\lambda(d_1^k, d_2^k)}{(d_1 d_2)^{3k}}$$

$$\ll \sum_{\substack{d_1 d_2 > \xi \\ (d_1, d_2) = 1}} \frac{(d_1 d_2)^{2k+\varepsilon}}{(d_1 d_2)^{3k}} \ll \sum_{n > \xi} \frac{\tau(n)}{n^{k-\varepsilon}} \ll \xi^{1-k+\varepsilon}.$$

Put

$$\sigma_k = \sum_{\substack{d_1 d_2 = 1 \\ (d_1, d_2) = 1}}^{\infty} \frac{\mu(d_1)\mu(d_2)\,\lambda(d_1^k, d_2^k)}{(d_1 d_2)^{3k}}. \tag{3.9}$$

From (1.6), Lemma 2.3 and $(d_1, d_2) = 1$, we get

$$\lambda(d_1^k, d_2^k) = \lambda(d_1^k, 1)\,\lambda(1, d_2^k). \tag{3.10}$$

Combining (3.9) and (3.10) we obtain

$$\sigma_k = \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\,\lambda(d_1^k, 1)}{d_1^{3k}} \sum_{d_2=1}^{\infty} \frac{\mu(d_2)\,\lambda(1, d_2^k)}{d_2^{3k}} \delta_{d_1}(d_2), \tag{3.11}$$

where

$$\delta_{d_1}(d_2) = \begin{cases} 1, & \text{if } (d_1, d_2) = 1, \\ 0, & \text{if } (d_1, d_2) > 1. \end{cases}$$

Since the function

$$\frac{\mu(d_2)\lambda(1, d_2^k)}{d_2^{3k}} \delta_{d_1}(d_2)$$

is multiplicative with respect to d_2, we have the Euler product representation:

$$\sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1, d_2^k)}{d_2^{3k}} \delta_{d_1}(d_2) = \prod_{p \nmid d_1} \left(1 - \frac{\lambda(1, p^k)}{p^{3k}}\right)$$

$$= \prod_{p} \left(1 - \frac{\lambda(1, p^k)}{p^{3k}}\right) \prod_{p \mid d_1} \left(1 - \frac{\lambda(1, p^k)}{p^{3k}}\right)^{-1}. \tag{3.12}$$

From (3.11) and (3.12) we infer

$$\sigma_k = \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\,\lambda(d_1^k, 1)}{d_1^{3k}} \prod_{p} \left(1 - \frac{\lambda(1, p^k)}{p^{3k}}\right) \prod_{p \mid d_1} \left(1 - \frac{\lambda(1, p^k)}{p^{3k}}\right)^{-1}$$

$$= \prod_{p} \left(1 - \frac{\lambda(1, p^k)}{p^{3k}}\right) \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\,\lambda(d_1^k, 1)}{d_1^{3k}} \prod_{p \mid d_1} \left(1 - \frac{\lambda(1, p^k)}{p^{3k}}\right)^{-1}$$

$$= \prod_p \left(1 - \frac{\lambda(1, p^k)}{p^{3k}}\right) \prod_p \left(1 - \frac{\lambda(p^k, 1)}{p^{3k}} \left(1 - \frac{\lambda(1, p^k)}{p^{3k}}\right)^{-1}\right)$$
$$= \prod_p \left(1 - \frac{\lambda(p^k, 1) + \lambda(1, p^k)}{p^{3k}}\right). \tag{3.13}$$

3.2 Estimation of $R_2(H)$

From (3.3), we derive by a splitting argument

$$R_2(H) \ll (\log H)^2 \sum_{D_1 \le d_1 < 2D_1} \sum_{D_2 \le d_2 < 2D_2} \sum_{\substack{t \le (3H^2+1)d_1^{-k} \\ td_1^k + 1 \equiv 0 \ (d_2^k)}} \sum_{\substack{1 \le x, y, z \le H \\ x^2+y^2+z^2 = td_1^k - 1}} 1,$$

where

$$\frac{1}{2} \le D_1, D_2 \le (3H^2+2)^{1/k}, \, D_1, D_2 > \frac{\xi}{4}. \tag{3.14}$$

We therefore obtain

$$R_2(H) \ll H^\varepsilon \sum_{D_1 \le d_1 < 2D_1} \sum_{t \le (3H^2+1)D_1^{-k}} \sum_{D_2 \le d_2 < 2D_2} \sum_{\substack{s \le (3H^2+2)D_2^{-k} \\ td_1^k + 1 = sd_2^k}} 1$$
$$\ll H^\varepsilon \sum_{D_1 \le d_1 < 2D_1} \sum_{t \le (3H^2+1)D_1^{-k}} \tau(td_1^k + 1)$$
$$\ll H^\varepsilon \sum_{D_1 \le d_1 < 2D_1} \sum_{t \le (3H^2+1)D_1^{-k}} 1 \ll H^{2+\varepsilon} D_1^{1-k}. \tag{3.15}$$

Similarly, one can obtain

$$R_2(H) \ll H^{2+\varepsilon} D_2^{1-k}. \tag{3.16}$$

Hence from (3.14) to (3.16) we get

$$R_2(H) \ll H^{2+\varepsilon} \xi^{\frac{1}{2} - \frac{k}{2}}. \tag{3.17}$$

Combining (3.1), (3.8), (3.13) and (3.17) gives

$$R(H) = \sigma_k H^3 + O(H^3 \xi^{1-k} + H^2 + \xi^{1+k+\varepsilon}),$$

where σ_k is defined in (3.13).

Now we obtain Theorem 1.1 by choosing $\xi = H^{\frac{3}{2k}}$.

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