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Oscillation results via comparison theorems for fourth-order delay three-terms difference equations

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Abstract
In this paper we establish sufficient conditions for the oscillation of all solutions of equation

$$\Delta^4 x(n) + p(n)\Delta x(n+1) + q(n)x(n-\tau) = 0$$

via comparison with some first order delay difference equations whose oscillatory characters are known. The presented criterion is easily verifiable. Examples are also given to illustrate the main result.

Keywords Fourth-order difference equation · Delay argument · Oscillation · Nonoscillation · Quasidifferences

Mathematics Subject Classification 39A10 · 39A22

1 Introduction

In this paper we assume that p and q are sequences of positive real numbers, $\tau \in \mathbb{N} = \{0, 1, 2, \dots, \}$. We consider the linear fourth-order difference equation of the form

$$\Delta^4 x(n) + p(n)\Delta x(n+1) + q(n)x(n-\tau) = 0, \tag{E}$$

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where $n \in \mathbb{N}_{n_0} = \{n_0, n_0 + 1, \dots\}$ and $n_0 \in \mathbb{N}$, Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$. By a solution of equation (E) we understand a sequence x of real numbers, that is defined for $n \geq n_0 - \tau$ and satisfies the equation for sufficiently large n. A solution is nonoscillatory if it is eventually positive or eventually negative. Otherwise, the solution is said to be oscillatory. We call equation (E) oscillatory if all its solutions are oscillatory.

The background of difference equations and discrete oscillation theory can be found in the monographs of Agarwal [1] and Agarwal, Bohner, Grace and O'Regan [2]. In recent years, the study of oscillatory and asymptotic behavior of solutions of second order difference equations has received great attention. Compared to this, the study of higher-order difference equations receives considerably less attention in literature. Some recent results for the oscillation of third-order difference equations can be found in [3, 6, 7, 9, 12, 13, 24, 25], and for fourth-order in [4, 10, 16, 18, 23]. In particular, in [7] and [13], the authors studied oscillation and asymptotic properties of solutions of the three-term difference equation

$$\Delta^3 x(n) + p(n)\Delta x(n+1) - q(n) f(x(n-\tau)) = 0,$$

by assuming the coefficient of the damping term is nonnegative. In [12], the difference equation

$$\Delta^3 x(n) + p(n)\Delta x(n+1) + q(n)x(n-\tau) = 0$$

with positive coefficient p was investigated. In these papers the comparison theorems with suitable first-order difference equations were used. It seemed to be natural to continue the research in the case of higher-order equations especially since fourth-order differential and difference equations often occur as models in mathematical biology, economics and engineering (for example see [4, 5, 8, 26]).

In [15], the authors investigated the oscillatory behavior of solutions of the fourth order difference equations with damping

$$\Delta^4 u(n) + p(n)\Delta u(n+1) + q(n)u(s(n)) = 0,$$

under the assumption that the auxiliary third order difference equation

$$\Delta^3 z(n) + p(n)z(n+1) = 0$$

is nonoscillatory.

The purpose of this paper is to study the oscillation of equation (E). Since for three-terms equations it is difficult to introduce the classification of nonoscillatory solutions, first we prove two lemmas that allow us to rewrite equation (E) in equivalent form as a two-terms equation with quasidifferences. Then, in our main result we give sufficient conditions for the oscillation of all solutions of equation (E) in terms of the two associated first order delay difference equations. Applying the appropriate criteria to these delay difference equations allows us to obtain new, easily verifiable oscillation results for equation (E) (see Theorem 2.10). It is worth noticing, that in this criterion the explicit form of solutions of the auxiliary equations is not needed. Numerical methods are frequently used in the investigation of the properties of the solutions of differential and difference equations. However, such methods are difficult to apply when investigating the oscillation of solutions. Our results may be helpful in the interpretation of numerical solutions, see Examples 3.1 and 3.2.

Let us recall some results that will be used in the sequel.

Lemma 1.1 [14, Corollary 7.6.1] *Let a be a sequence of non-negative real numbers and let k be a positive integer such that*

$$\sum_{i=n-k}^{n-1} a(i) > 0$$

for large n. Then the difference inequality

$$\Delta x(n) + a(n)x(n-k) \le 0,$$

has an eventually positive solution if and only if the difference equation

$$\Delta x(n) + a(n)x(n-k) = 0,$$

has an eventually positive solution.

The next result follows from [19, Theorem 1].

Lemma 1.2 *Consider the equation*

$$\Delta(c(n)\Delta v(n)) + b(n)v(n+1) = 0, \tag{1.1}$$

where c, b are eventually positive sequences. If

$$\sum_{n=1}^{\infty} b(n) \sum_{j=1}^{n-1} \frac{1}{c(j)} < \infty,$$

then for any real constant β there exists a solution v of equation (1.1) *which converges to β at infinity.*

Lemma 1.3 [18] *Consider the difference equation*

$$\Delta(a(n)\Delta(b(n)\Delta(c(n)\Delta x(n)))) + f(n, x(n)) = 0, \tag{1.2}$$

where a, b, c are sequences of positive real numbers and $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$.
Assume that

$$\sum_{i=1}^{\infty} \frac{1}{a(i)} = \sum_{i=1}^{\infty} \frac{1}{b(i)} = \sum_{i=1}^{\infty} \frac{1}{c(i)} = \infty$$

and $uf(n, u) > 0$ for all $u \ne 0$, $n \in \mathbb{N}$ hold. Let x be an eventually positive solution of (1.2). *Then exactly one of the following statements holds for all sufficiently large n:*

(i) $x(n) > 0$, $\Delta x(n) > 0$, $\Delta(c(n)\Delta x(n)) > 0$ *and* $\Delta(b(n)(\Delta(c(n)\Delta x(n)))) > 0$;
(ii) $x(n) > 0$, $\Delta x(n) > 0$, $\Delta(c(n)\Delta x(n)) < 0$ *and* $\Delta(b(n)(\Delta(c(n)\Delta x(n)))) > 0$.

2 Main results

We start this section with two lemmas, which allow us to rewrite equation (E) in equivalent binomial form in terms of solutions of two auxiliary linear equations. Because the proofs of these lemmas are technically complicated, they are presented in Sect. 4.

Lemma 2.1 *If z is an eventually positive solution of equation*

$$\Delta^3 z(n) + p(n)z(n+1) = 0, \tag{2.1}$$

then the equality

$$\begin{aligned}&\Delta^4 x(n) + p(n)\Delta x(n+1)\\&= \Delta\left[\frac{1}{z(n+1)}\Delta\left(z(n+1)z(n)\Delta\left(\frac{\Delta x(n)}{z(n)}\right)\right)\right] + \Delta^2 z(n+1)\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\end{aligned} \tag{2.2}$$

holds for any sequence x and for large n.

Lemma 2.2 *If z is an eventually positive solution of equation (2.1) and equation*

$$\Delta\left(\frac{1}{z(n+1)}\Delta v(n)\right) + \frac{\Delta^2 z(n+1)}{z(n+1)z(n+2)}v(n+1) = 0, \tag{2.3}$$

has an eventually positive solution v, then the equality

$$\begin{aligned}&\Delta^4 x(n) + p(n)\Delta x(n+1)\\&= \frac{1}{v(n+1)}\Delta\left(\frac{v(n)v(n+1)}{z(n+1)}\Delta\left(\frac{z(n)z(n+1)}{v(n)}\Delta\frac{\Delta x(n)}{z(n)}\right)\right)\end{aligned} \tag{2.4}$$

holds for any sequence x and for large n .

Applying (2.4) to equation (E) we get

$$\frac{1}{v(n+1)}\Delta\left(\frac{v(n)v(n+1)}{z(n+1)}\Delta\left(\frac{z(n)z(n+1)}{v(n)}\Delta\frac{\Delta x(n)}{z(n)}\right)\right) + q(n)x(n-\tau) = 0.$$

Therefore, if z and v are eventually positive solutions of the equations (2.1) and (2.3) respectively, then, by Lemmas 2.1 and 2.2, equation (E) can be written in the form

$$\Delta\left(\frac{v(n)v(n+1)}{z(n+1)}\Delta\left(\frac{z(n)z(n+1)}{v(n)}\Delta\frac{\Delta x(n)}{z(n)}\right)\right) + v(n+1)q(n)x(n-\tau) = 0. \tag{E$'$}$$

The question arises whether equations (2.1) and (2.3) have eventually positive solutions. The lemma below presents a possible criterion.

Lemma 2.3 *Assume $\beta, \gamma \in (0, \infty)$ and*

$$\sum_{n=n_0}^{\infty} n^2 p(n) < \infty. \tag{2.5}$$

Then there exist an eventually positive solution z of (2.1) and an eventually positive solution v of (2.3) such that

$$\lim_{n\to\infty} z(n) = \gamma \quad \textit{and} \quad \lim_{n\to\infty} v(n) = \beta.$$

Proof By [22, Theorem 1] there exists a solution z of (2.1) such that $\lim_{n\to\infty} z(n) = \gamma$. That means, that there exists an index $n_1 > n_0$ such that $z(n) > 0$ and satisfies equality (2.1) for any $n \geq n_1$. Hence, by (2.1), we have $\Delta^3 z(n) < 0$ for $n \geq n_1$. The convergence of $z(n)$ implies

$$\lim_{n\to\infty} \Delta z(n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \Delta^2 z(n) = 0.$$

Since $\Delta^3 z(n) < 0$ for $n \geq n_1$, we get $\Delta^2 z(n) > 0$ for $n \geq n_1$. Analogously $\Delta z(n) < 0$ eventually.

By [20, Lemma 6] the series $\sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \Delta^2 z(n+1)$ is convergent. Using [20, Lemma 3] we get

$$\sum_{n=n_1}^{\infty} n\Delta^2 z(n+1) < \infty. \tag{2.6}$$

Let, for $n \geq n_1$,

$$b(n) = \frac{\Delta^2 z(n+1)}{z(n+1)z(n+2)}, \quad c(n) = \frac{1}{z(n+1)}.$$

Then, by (2.6), we have

$$\begin{aligned}\sum_{n=n_1}^{\infty} b(n) \sum_{j=n_1}^{n-1} \frac{1}{c(j)} &= \sum_{n=n_1}^{\infty} \frac{\Delta^2 z(n+1)}{z(n+1)z(n+2)} \sum_{j=n_1}^{n-1} z(j+1)\\ &\leq L \sum_{n=n_1}^{\infty} n\Delta^2 z(n+1) < \infty,\end{aligned}$$

where $L = \gamma^2 z(n_1+1)$. Hence, from Lemma 1.2, we get the existence of a solution v of (2.3) such that $\lim_{n\to\infty} v(n) = \beta$. □

Based on Trench [27], we say that a linear difference operator

$$L_m x(n) = r_{m-1}(n)\Delta L_{m-1}x(n), \quad L_0 x(n) = x(n)$$

is in canonical form if $\sum_{n=n_0}^{\infty} \frac{1}{r_j(n)} = \infty$ for $j = 1, \ldots, m-1$, where r_j are eventually positive real sequences. The sequences $L_i x$ are called quasidifferences of x.

The quasidifferences in equation (E′) have the form

$$\begin{aligned}&L_0 x(n) = x(n),\ \ L_1 x(n) = \frac{1}{z(n)}\Delta L_0 x(n),\ \ L_2 x(n) = \frac{z(n)z(n+1)}{v(n)}\Delta L_1 x(n),\\ &L_3 x(n) = \frac{v(n)v(n+1)}{z(n+1)}\Delta L_2 x(n),\ \ L_4 x(n) = \Delta L_3 x(n).\end{aligned}$$

Note, that if z and v are solutions of the equations (2.1) and (2.3) respectively, and both converge to positive constants, then

$$\sum_{n=n_0}^{\infty} z(n) = \infty, \quad \sum_{n=n_0}^{\infty} \frac{v(n)}{z(n)z(n+1)} = \infty, \quad \sum_{n=n_0}^{\infty} \frac{z(n+1)}{v(n)v(n+1)} = \infty,$$

and then the linear operator

$$L_4 x(n) = \Delta\left(\frac{v(n)v(n+1)}{z(n+1)}\Delta\left(\frac{z(n)z(n+1)}{v(n)}\Delta\frac{\Delta x(n)}{z(n)}\right)\right)$$

in (E′) is in canonical form, which means equation (E′) is in canonical form.

Hence, by Lemma 1.3, we get the following classification of the nonoscillatory solutions of (E′).

Lemma 2.4 *Assume that condition* (2.5) *is satisfied. Let x be an eventually positive solution of equation* (E'). *Then there exists $n_2 \geq n_0$ such that for all $n \geq n_2$*

$$L_4 x(n) < 0, \tag{2.7}$$

and either

$$L_1 x(n) > 0, \quad L_2 x(n) < 0, \quad L_3 x(n) > 0,$$

or

$$L_1 x(n) > 0, \quad L_2 x(n) > 0, \quad L_3 x(n) > 0, .$$

In summary, we get the following remark.

Remark 2.5 If condition (2.5) holds, then the three-term difference equation (E) can be rewritten as a two-terms equation of the form (E'), which is in the canonical form.

Our goal is to present an easily verifiable oscillation criterion for equation (E). In our investigation we utilize the form (E'). Using the comparison theorem we will deduce the oscillation of (E) from the oscillation of certain first-order difference equations whose properties are well-explored.

Let z be a solution to equation (2.1) which tends to a positive constant and let v be a solution to equation (2.3) which tends to a positive constant. Then there exist positive constants z^*, z^{**}, v^*, v^{**} and $n_3 \in \mathbb{N}$ such that

$$0 < z^* \leq z(n) \leq z^{**} \quad \text{and} \quad 0 < v^* \leq v(n) \leq v^{**} \tag{2.8}$$

for $n \geq n_3$.

Assuming

$$\sum_{n=1}^{\infty} \sum_{k=n}^{\infty} q(k) < \infty \tag{2.9}$$

we define sequences K_1, K_2 by

$$K_1(n) = \left(\frac{v^*}{v^{**}}\right)^2 \left(\frac{z^*}{z^{**}}\right)^2 (n - \tau - n_4) \sum_{s=n}^{\infty} \sum_{k=s}^{\infty} q(k)$$

$$K_2(n) = \left(\frac{v^*}{v^{**}}\right)^2 \left(\frac{z^*}{z^{**}}\right)^2 q(n) \sum_{k=n_4}^{n-\tau-1} \sum_{m=n_4}^{k-1} \sum_{s=n_4}^{m-1} 1$$

where $n_4 = \max\{n_2, n_3\} + \tau + 3$ and n_2 as in Lemma 2.4. It is easy to see that condition (2.9) is equivalent to

$$\sum_{n=1}^{\infty} n q(k) < \infty. \tag{2.10}$$

Theorem 2.6 *Assume that conditions* (2.5) *and* (2.10) *hold. If the following delayed equations with unknown sequence u*

$$\Delta u(n) + K_1(n) u(n - \tau) = 0 \tag{2.11}$$

and

$$\Delta u(n) + K_2(n) u(n - \tau) = 0 \tag{2.12}$$

are oscillatory, then equation (E) *is also oscillatory.*

Proof Assume to the contrary that there exists an eventually positive solution x to (E). Let x be such a solution. Notice that x is also an eventually positive solution to (E$'$). Let z be a positive decreasing solution of (2.1). According to Lemma 2.4 we consider two cases.

Case I. Assume conditions (2.7) and (2.8) hold. Summation of the both sides of (E$'$) from $s \geq n_4$ to infinity, leads to equality

$$\sum_{i=s}^{\infty} \Delta \left(\frac{v(i)v(i+1)}{z(i+1)} \Delta \left(\frac{z(i)z(i+1)}{v(i)} \Delta \frac{\Delta x(i)}{z(i)} \right) \right) = -\sum_{i=s}^{\infty} v(i+1)q(i)x(i-\tau).$$

From the above, by properties of the sequence $L_3 x$, we get

$$\frac{v(s)v(s+1)}{z(s+1)} \Delta \left(\frac{z(s)z(s+1)}{v(s)} \Delta \frac{\Delta x(s)}{z(s)} \right) \geq \sum_{i=s}^{\infty} v(i+1)q(i)x(i-\tau).$$

Since the sequence $L_1 x$ is a positive sequence, x is increasing,

$$\frac{v(s)v(s+1)}{z(s+1)} \Delta \left(\frac{z(s)z(s+1)}{v(s)} \Delta \frac{\Delta x(s)}{z(s)} \right) > x(s-\tau) \sum_{i=s}^{\infty} v(i+1)q(i),$$

and consequently

$$\Delta \left(\frac{z(s)z(s+1)}{v(s)} \Delta \frac{\Delta x(s)}{z(s)} \right) > \frac{z(s+1)}{v(s)v(s+1)} x(s-\tau) \sum_{i=s}^{\infty} v(i+1)q(i).$$

Summing the above inequality from n to infinity, we obtain

$$\sum_{s=n}^{\infty} \Delta \left(\frac{z(s)z(s+1)}{v(s)} \Delta \frac{\Delta x(s)}{z(s)} \right) > \sum_{s=n}^{\infty} \frac{z(s+1)}{v(s)v(s+1)} x(s-\tau) \sum_{i=s}^{\infty} v(i+1)q(i).$$

Again by (2.8),

$$\sum_{s=n}^{\infty} \Delta \left(\frac{z(s)z(s+1)}{v(s)} \Delta \frac{\Delta x(s)}{z(s)} \right) > x(n-\tau) \sum_{s=n}^{\infty} \frac{z(s+1)}{v(s)v(s+1)} \sum_{i=s}^{\infty} v(i+1)q(i).$$

By properties of the sequence $L_2 x$, we have

$$\sum_{s=n}^{\infty} \Delta \left(\frac{z(s)z(s+1)}{v(s)} \Delta \frac{\Delta x(s)}{z(s)} \right) \leq -\frac{z(n)z(n+1)}{v(n)} \Delta \frac{\Delta x(n)}{z(n)}.$$

Combining the above two inequalities, we obtain

$$-\frac{z(n)z(n+1)}{v(n)} \Delta \frac{\Delta x(n)}{z(n)} \geq x(n-\tau) \sum_{s=n}^{\infty} \frac{z(s+1)}{v(s)v(s+1)} \sum_{i=s}^{\infty} v(i+1)q(i),$$

that is

$$-\Delta \frac{\Delta x(n)}{z(n)} \geq x(n-\tau) \frac{v(n)}{z(n)z(n+1)} \sum_{s=n}^{\infty} \frac{z(s+1)}{v(s)v(s+1)} \sum_{i=s}^{\infty} v(i+1)q(i).$$

Hence, using (2.8) we get

$$-\Delta \frac{\Delta x(n)}{z(n)} \geq x(n-\tau) \left(\frac{v^*}{v^{**}} \right)^2 \frac{z^*}{(z^{**})^2} \sum_{s=n}^{\infty} \sum_{i=s}^{\infty} q(i). \tag{2.13}$$

Based on the sequence L_1x properties, we have

$$x(n) > \sum_{i=n_4}^{n-1} \Delta x(i) = \sum_{i=n_4}^{n-1} z(i)\frac{\Delta x(i)}{z(i)} > \frac{\Delta x(n-1)}{z(n-1)}\sum_{i=n_4}^{n-1} z(i) > \frac{\Delta x(n)}{z(n)}\sum_{i=n_4}^{n-1} z(i)$$

for $n > n_4$. Therefore

$$x(n-\tau) > \frac{\Delta x(n-\tau)}{z(n-\tau)}\sum_{i=n_4}^{n-\tau-1} z(i) \text{ for } n \geq n_4+\tau. \tag{2.14}$$

Combining (2.13) with (2.14), we get

$$-\Delta\frac{\Delta x(n)}{z(n)} > \frac{\Delta x(n-\tau)}{z(n-\tau)}\left(\frac{v^*}{v^{**}}\right)^2\left(\frac{z^*}{z^{**}}\right)^2(n-\tau-n_4)\sum_{s=n}^{\infty}\sum_{k=s}^{\infty} q(k),$$

that is

$$0 > \Delta\frac{\Delta x(n)}{z(n)} + \left(\frac{v^*}{v^{**}}\right)^2\left(\frac{z^*}{z^{**}}\right)^2(n-\tau-n_4)\sum_{s=n}^{\infty}\sum_{k=s}^{\infty} q(k)\frac{\Delta x(n-\tau)}{z(n-\tau)}$$

for $n \geq n_4+\tau$. Using the definition of K_1, this leads to inequality

$$0 > \Delta\frac{\Delta x(n)}{z(n)} + K_1(n)\frac{\Delta x(n-\tau)}{z(n-\tau)} \text{ for } n \geq n_4+\tau. \tag{2.15}$$

By assumption, the sequence x is an eventually positive solution of inequality (2.15). By virtue of Lemma 1.1, it is also an eventually positive solution of equation (2.11). This contradicts that all solutions of (2.11) are oscillatory.

Case II. Set $u(n) := L_3x(n)$. Let conditions (2.7) and (2.8) hold. Thus the the sequence L_2x is positive for $n \geq n_4$. Hence, the following estimation holds

$$\sum_{s=n_4}^{m-1}\Delta\left(\frac{z(s)z(s+1)}{v(s)}\Delta\frac{\Delta x(s)}{z(s)}\right) \leq \frac{z(m)z(m+1)}{v(m)}\Delta\frac{\Delta x(m)}{z(m)} \text{ for } n \geq n_4.$$

From the properties of the sequence L_3x, we have

$$\sum_{s=n_4}^{m-1}\Delta\left(\frac{z(s)z(s+1)}{v(s)}\Delta\frac{\Delta x(s)}{z(s)}\right) \geq u(m-1)\sum_{s=n_4}^{m-1}\frac{z(s+1)}{v(s)v(s+1)}.$$

Combining the above two inequalities, we get

$$\frac{v(m)}{z(m)z(m+1)}u(m-1)\sum_{s=n_4}^{m-1}\frac{z(s+1)}{v(s)v(s+1)} \leq \Delta\frac{\Delta x(m)}{z(m)}.$$

Summing both sides of the above inequality from n_4 to $k-1$, we obtain

$$\sum_{m=n_4}^{k-1}\frac{v(m)u(m-1)}{z(m)z(m+1)}\sum_{s=n_4}^{m-1}\frac{z(s+1)}{v(s)v(s+1)} \leq \sum_{m=n_4}^{k-1}\Delta\frac{\Delta x(m)}{z(m)}. \tag{2.16}$$

Since the sequence L_1x is positive for $n \geq n_4$, we have

$$\sum_{m=n_4}^{k-1}\Delta\left(\frac{\Delta x(m)}{z(m)}\right) = \frac{\Delta x(k)}{z(k)} - \frac{\Delta x(n_4)}{z(n_4)} \leq \frac{\Delta x(k)}{z(k)}. \tag{2.17}$$

On the other hand, again by (2.8), we get

$$\sum_{m=n_4}^{k-1} \frac{v(m)u(m-1)}{z(m)z(m+1)} \sum_{s=n_4}^{m-1} \frac{z(s+1)}{v(s)v(s+1)} \geq u(k-2) \sum_{m=n_4}^{k-1} \frac{v(m)}{z(m)z(m+1)} \sum_{s=n_4}^{m-1} \frac{z(s+1)}{v(s)v(s+1)}.$$

The above and inequalities (2.16), (2.17) imply

$$\Delta x(k) \geq z(k)u(k-2) \sum_{m=n_4}^{k-1} \frac{v(m)}{z(m)z(m+1)} \sum_{s=n_4}^{m-1} \frac{z(s+1)}{v(s)v(s+1)}.$$

By summation of both sides of the above inequality from n_4 to $n-1$, we obtain

$$\sum_{k=n_4}^{n-1} \Delta x(k) \geq \sum_{k=n_4}^{n-1} z(k)u(k-2) \sum_{m=n_4}^{k-1} \frac{v(m)}{z(m)z(m+1)} \sum_{s=n_4}^{m-1} \frac{z(s+1)}{v(s)v(s+1)}.$$

Sequence x is positive for $n \geq n_4$ hence $\sum_{k=n_4}^{n-1} \Delta x(k) = x(n) - x(n_4) \leq x(n)$. Consequently

$$x(n) \geq \sum_{k=n_4}^{n-1} z(k)u(k-2) \sum_{m=n_4}^{k-1} \frac{v(m)}{z(m)z(m+1)} \sum_{s=n_4}^{m-1} \frac{z(s+1)}{v(s)v(s+1)}.$$

Since the sequence $L_3 x$ is decreasing,

$$x(n) \geq u(n-2) \sum_{k=n_4}^{n-1} z(k) \sum_{m=n_4}^{k-1} \frac{v(m)}{z(m)z(m+1)} \sum_{s=n_4}^{m-1} \frac{z(s+1)}{v(s)v(s+1)}.$$

Hence,

$$v(n+1)q(n)x(n-\tau) \geq v(n+1)q(n)u(n-\tau-2) \sum_{k=n_4}^{n-\tau-1} z(k) \sum_{m=n_4}^{k-1} \frac{v(m)}{z(m)z(m+1)} \sum_{s=n_4}^{m-1} \frac{z(s+1)}{v(s)v(s+1)}.$$

Using the definition of u, equation (E′) takes the form

$$-\Delta u(n) = v(n+1)q(n)x(n-\tau).$$

We get

$$-\Delta u(n) \geq v(n+1)q(n)u(n-\tau-2) \sum_{k=n_4}^{n-\tau-1} z(k) \sum_{m=n_4}^{k-1} \frac{v(m)}{z(m)z(m+1)} \sum_{s=n_4}^{m-1} \frac{z(s+1)}{v(s)v(s+1)}.$$

Hence, using (2.8) we get

$$-\Delta u(n) \geq u(n-\tau-2) \left(\frac{v^*}{v^{**}}\right)^2 \left(\frac{v^*}{v^{**}}\right)^2 q(n) \sum_{k=n_4}^{n-\tau-1} \sum_{m=n_4}^{k-1} \sum_{s=n_4}^{m-1} 1.$$

Finally, using the definition of K_2, we obtain

$$\Delta u(n) + u(n-\tau-2)K_2(n) \leq 0 \text{ for } n \geq n_4.$$

Since u is an eventually positive solution of the above inequality, by Lemma 1.1 it is also an eventually positive solution of equation (2.12). In that way there is contradiction with the assumption that all solutions of equation (2.12) are oscillatory. □

Applying well-known oscillation criteria for first-order delay difference equations to equations (2.11) and (2.12) (see [11] and [17]), we obtain the following oscillation criteria for equation (E).

Corollary 2.7 *Assume* (2.5) *and* (2.10) *are satisfied. If for* $j = 1, 2$

$$\limsup_{n\to\infty} \sum_{i=n-\tau}^{n} K_j(i) > 1,$$

then equation (E) *is oscillatory.*

If $\tau > 0$, then we can also use the following criterion.

Corollary 2.8 *Assume* (2.5) *and* (2.9) *are satisfied. If for* $j = 1, 2$

$$\liminf_{n\to\infty} \left(\frac{1}{\tau} \sum_{i=n-\tau}^{n-1} K_j(i) \right) > \frac{\tau^\tau}{(\tau+1)^{\tau+1}},$$

then equation (E) *is oscillatory.*

Note, that in the above results the explicit form of the eventually positive solutions of (2.1) and (2.3) are needed. But, it is well known, that it is difficult to find the explicit form of solutions of second and third order linear difference equations with variable coefficients. Therefore, we present a criterion in which the assumptions depend only on the coefficients p and q of the equation (E). First we prove a simple lemma.

Lemma 2.9 *Assume* (2.5), (2.10), *and*

$$\lim_{n\to\infty} n^3 q(n) = \infty. \tag{2.18}$$

Then $\lim_{n\to\infty} K_j(i) = \infty$ *for* $j = 1, 2$.

Proof Let $j = 1$. Define a sequence Q by

$$Q(n) = \sum_{s=n}^{\infty} \sum_{k=s}^{\infty} q(k).$$

Using discrete L'Hospital's rule we obtain

$$\begin{aligned}\lim_{n\to\infty} nQ(n) &= \lim_{n\to\infty} \frac{Q(n)}{n^{-1}} = \lim_{n\to\infty} \frac{\Delta Q(n)}{\Delta(n^{-1})} = \lim_{n\to\infty} \frac{-\sum_{k=n}^{\infty} q(k)}{-(n(n+1))^{-1}} \\ &= \lim_{n\to\infty} \frac{-\Delta\left(\sum_{k=n}^{\infty} q(k)\right)}{-\Delta\left((n(n+1))^{-1}\right)} = \lim_{n\to\infty} \frac{q(n)}{2(n(n+1)(n+2))^{-1}}.\end{aligned}$$

Since $\lim_{n\to\infty} n^3 q(n) = \infty$ we get $\lim_{n\to\infty} nQ(n) = \infty$. Hence

$$\lim_{n\to\infty} K_1(n) = \infty.$$

Let $j = 2$. It is easy to verify that

$$\sum_{k=n_4}^{n-\tau-1}\sum_{m=n_4}^{k-1}\sum_{s=n_4}^{m-1} 1 = \sum_{k=n_4}^{n-\tau-1}\sum_{m=n_4}^{k-1}(m-n_4) = \sum_{k=n_4}^{n-\tau-1}\sum_{j=0}^{k-1-n_4} j$$
$$= \sum_{k=n_4}^{n-\tau-1}\frac{(k-n_4)(k-n_4-1)}{2} = \frac{1}{2}\sum_{i=0}^{n-\tau-n_4-1} i(i-1)$$
$$= \frac{1}{6}(n-\tau-n_4)(n-\tau-n_4-1)(n-\tau-n_4-2)$$
$$\geq \frac{1}{6}(n-\tau-n_4-2)^3.$$

Since $\lim_{n\to\infty} n^3 q(n) = \infty$ we get $\lim\limits_{n\to\infty} K_2(n) = \infty$. □

Combining Lemma 2.9 and Corollary 2.7 we get the following oscillation criterion.

Theorem 2.10 *Assume* (2.5), (2.10), *and* (2.18). *Then equation* (E) *is oscillatory.*

To verify conditions (2.5) and (2.10) the tests presented in [21] may be useful. For example, applying [21, Lemma 4.4] or [21, Lemma 4.5] we get the following results

Corollary 2.11 *Assume* (2.18) *and*

$$\limsup_{n\to\infty}\frac{\log q(n)}{\log n} < -2, \quad \limsup_{n\to\infty}\frac{\log p(n)}{\log n} < -3.$$

Then equation (E) *is oscillatory.*

Corollary 2.12 *Assume* (2.18) *and*

$$\liminf_{n\to\infty} n\left(\frac{q_n}{q_{n+1}} - 1\right) > 2, \quad \liminf_{n\to\infty} n\left(\frac{p_n}{p_{n+1}} - 1\right) > 3.$$

Then equation (E) *is oscillatory.*

3 Examples

By using computer algebra systems there is always possible to find recursively an approximate solution of the considered equation, but sometimes it is very difficult to determine whether the approximate solution is oscillatory or not. In many cases, it is rather easy to use our last criterion to verify that the considered equation is oscillatory. Such cases are shown in the following examples.

Example 3.1 Consider the following equation of type (E):

$$\Delta^4 x(n) + \frac{12}{n^2(n+1)\ln^2(n+1)}\Delta x(n+1) + \frac{(0.01+\sin^2 n)\sqrt[10]{n}}{n^3}x(n-1) = 0, \tag{3.1}$$

where $n \geq 2$,

$$p(n) = \frac{12}{n^2(n+1)\ln^2(n+1)}, \quad q(n) = \frac{(0.01+\sin^2 n)\sqrt[10]{n}}{n^3}, \quad \tau = 1.$$

Table 1 Numerical result of Example 3.1 for initial conditions $x(0) = 1.0$, $x(1) = 0.2$, $x(2) = 1.0$, $x(3) = 0.3$, $x(4) = 4.0$

n	$x(n)$	n	$x(n)$	n	$x(n)$
0	1	344	−986118	1750	−71573972460
1	0.2	345	−449098	1895	−913282896
10	433	346	100032	1896	−137139238
20	4894	347	661396	1897	643565756
50	59463	600	753284063	1898	1428844567
65	83149	750	1356273762	2800	3277178712650
87	11738	898	31203849	3200	4025222148557
88	3203	899	8838057	3504	29162711322
89	−5930	900	−13762530	3505	4686183595
90	−15678	901	−36599045	3506	−19880176643
200	−7636989	1300	−36883721193	3507	−44536524503
280	−15600305	1650	−83700172358	4000	−26091472292638

Using the iterative scheme

$$x(n+4) = 4x(n+3) - \big(p(n)+6\big)x(n+2) + \big(p(n)+4\big)x(n+1) - x(n) - q(n)x(n-1),$$

we solve (3.1) with the following initial conditions

$$x(0) = 1.0,\ x(1) = 0.2,\ x(2) = 1.0,\ x(3) = 0.3,\ x(4) = 4.0.$$

Except for the initial conditions all computed values of x are rounded to the nearest integer number. As we see in Table 1, the obtained terms of this solution are positive until the 88th term, then negative. The next sign change of terms is on 346th term, next on 900th term and so on. Since the length of intervals with one sign are increasing, the oscillations of the solutions are not easily visible.

On the other hand, it is easy to verified that

$$\sum_{n=1}^{\infty} n^2 \frac{12}{n^2(n+1)\ln^2(n+1)} < \infty,$$

$$\sum_{n=1}^{\infty} n \frac{\sin^2 n}{n^3} < \infty,$$

$$\lim_{n\to\infty} n^3 \frac{(0.01+\sin^2 n)\sqrt[10]{n}}{n^3} = \infty.$$

Hence, by Theorem 2.10 all solutions of equation (3.1) are oscillatory.

Example 3.2 Consider fourth-order trinomial difference equations of the form

$$\Delta^4 x(n) + \frac{a}{n^\alpha}\Delta x(n+1) + \frac{b}{n^\beta}x(n-\tau) = 0, \quad n \geq 2, \tag{3.2}$$

where $a, b, \alpha, \beta > 0$, $\tau \in \mathbb{N}$. It is easy to see that if $\alpha > 3$ and $2 < \beta < 3$, then conditions (2.5), (2.9) and (2.18) are satisfied. Hence, by Theorem 2.10 all solutions of equation (3.2) are oscillatory.

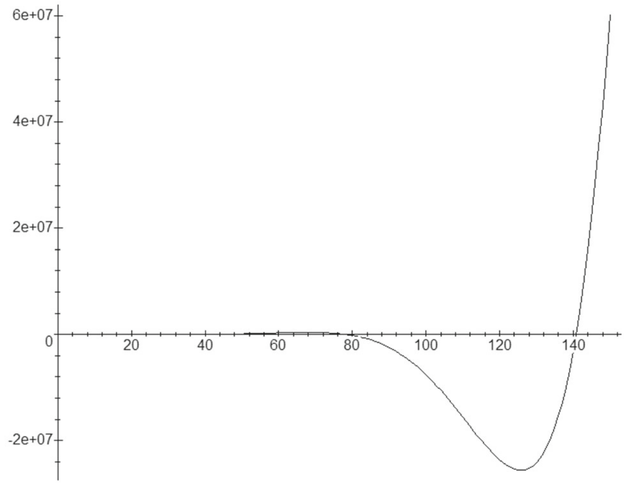
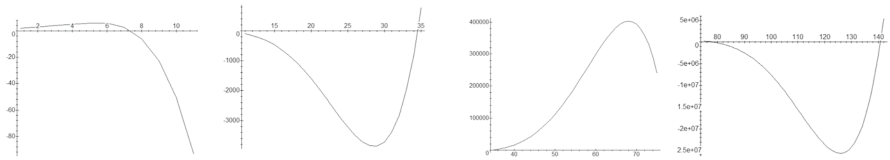


Fig. 1 Solution plot of Eq. (3.2) from Ex. 2



(a) n=1 to n=11 (b) n=11 to n=35 (c) n=35 to n=75 (d) n=75 to n=142

Fig. 2 The graph from Fig. 1 divided into four parts

For a numerical solution of (3.2), we set $a = 10,\ \ b = 5, \tau = 2, \alpha = 5, \beta = 2, 5$. The recurrence formula for the equation takes the form

$$x(n+4) = 4x(n+3) - 6x(n+2) + 4x(n+1) - x(n) - \frac{10}{n^5}\big(x(n+2) - x(n+1)\big) - \frac{5}{n^{2.5}}x(n-2).$$

Taking the initial values

$$x(0) = 1, x(1) = 2, x(2) = 3, x(3) = 4, x(4) = 5, x(5) = 6$$

we get a solution x whose trajectory from $n = 1$ to $n = 150$ is shown in Fig. 1. To show the oscillatory nature of this solution, we present its graph divided into four parts with increasingly larger scales.

4 Proofs of Lemmas

Proof of Lemma 2.1 Let z be an eventually positive solution of equation (2.1). From the left hand side of (2.2), using (2.1), we obtain

$$\Delta^4 x(n) + p(n)\Delta x(n+1) = \Delta^4 x(n) - \frac{\Delta^3 z(n)}{z(n+1)}\Delta x(n+1)$$

$$
\begin{aligned}
&= \left(\Delta^4 x(n) - \Delta^2 z(n+1)\Delta\frac{\Delta x(n+1)}{z(n+1)} - \Delta^3 z(n)\frac{\Delta x(n+1)}{z(n+1)}\right) + \Delta^2 z(n+1)\Delta\frac{\Delta x(n+1)}{z(n+1)}\\
&= \Delta\left(\Delta^3 x(n) - \Delta^2 z(n)\frac{\Delta x(n+1)}{z(n+1)}\right) + \Delta^2 z(n+1)\Delta\frac{\Delta x(n+1)}{z(n+1)})\\
&= \Delta\left(\frac{1}{z(n+1)}\big(z(n+1)\Delta^3 x(n) - \Delta x(n+1)\Delta^2 z(n)\big)\right) + \Delta^2 z(n+1)\Delta\frac{\Delta x(n+1)}{z(n+1)}\\
&= \Delta\left(\frac{1}{z(n+1)}\big(z(n+1)\Delta^2 x(n+1) - z(n+1)\Delta^2 x(n)\big)\right)\\
&\quad + \Delta\left(\frac{1}{z(n+1)}\big(-\Delta x(n+1)\Delta z(n+1) + \Delta x(n+1)\Delta z(n)\big)\right)\\
&\quad + \Delta^2 z(n+1)\Delta\frac{\Delta x(n+1)}{z(n+1)}\\
&= \Delta\left(\frac{1}{z(n+1)}\Delta\big(z(n)\Delta x(n+1) - z(n+1)\Delta x(n)\big)\right) + \Delta^2 z(n+1)\Delta\frac{\Delta x(n+1)}{z(n+1)}\\
&= \Delta\left(\frac{1}{z(n+1)}\Delta\left(z(n+1)z(n)\left(\frac{\Delta x(n+1)}{z(n+1)} - \frac{\Delta x(n)}{z(n)}\right)\right)\right) + \Delta^2 z(n+1)\Delta\frac{\Delta x(n+1)}{z(n+1)}\\
&= \Delta\left(\frac{1}{z(n+1)}\Delta\left(z(n+1)z(n)\Delta\frac{\Delta x(n)}{z(n)}\right)\right) + \Delta^2 z(n+1)\Delta\frac{\Delta x(n+1)}{z(n+1)}.
\end{aligned}
$$

Proof of Lemma 2.2 We start from the right hand side of (2.4):

$$
\begin{aligned}
&\frac{1}{v(n+1)}\Delta\left(\frac{v(n)v(n+1)}{z(n+1)}\Delta\left(\frac{1}{v(n)}z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\right)\\
&\quad = \frac{1}{v(n+1)}\Delta\left(\frac{v(n)v(n+1)}{z(n+1)}\left(\frac{1}{v(n+1)}\Delta\left(z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\right.\right.\\
&\quad + \left.\left.\left(\Delta\frac{1}{v(n)}\right)z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\right) = \frac{1}{v(n+1)}\Delta\left(\frac{v(n)}{z(n+1)}\Delta\left(z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\right.\\
&\quad + \left.\frac{v(n)v(n+1)}{z(n+1)}\left(\Delta\frac{1}{v(n)}\right)z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\\
&\quad = \frac{1}{v(n+1)}\Delta\left(\frac{v(n)}{z(n+1)}\Delta\left(z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right) - \frac{\Delta v(n)}{z(n+1)}z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\\
&\quad = \frac{1}{v(n+1)}\left(v(n+1)\Delta\left(\frac{1}{z(n+1)}\Delta\left(z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\right)\right.\\
&\quad + \frac{\Delta v(n)}{z(n+1)}\Delta\left(z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right) - \frac{\Delta v(n+1)}{z(n+2)}\Delta\left(z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\\
&\quad \left. -\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\\
&\quad = \Delta\left(\frac{1}{z(n+1)}\Delta\left(z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\right)\\
&\quad + \frac{\Delta v(n)}{z(n+1)v(n+1)}\Delta\left(z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\\
&\quad - \frac{\Delta v(n+1)}{v(n+1)z(n+2)}\Delta\left(z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right) - \Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\frac{z(n)z(n+1)}{v(n+1)}\Delta\frac{\Delta x(n)}{z(n)}.
\end{aligned}
$$

Applying (2.2) to the first term of the last equality and combining together the second and third term we obtain

$$\begin{aligned}
&\frac{1}{v(n+1)}\Delta\left(\frac{v(n)v(n+1)}{z(n+1)}\Delta\left(\frac{1}{v(n)}z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\right)\\
&\quad= \Delta^4 x(n) + p(n)\Delta x(n+1) - \Delta^2 z(n+1)\Delta\frac{\Delta x(n+1)}{z(n+1)}\\
&\qquad - \frac{1}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta\left(z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\\
&\qquad - \Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\frac{z(n)z(n+1)}{v(n+1)}\Delta\frac{\Delta x(n)}{z(n)}.
\end{aligned}$$

Now we transform separately the third term of the above expression

$$\begin{aligned}
&-\frac{1}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta\left(z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right) = -\frac{1}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\\
&\times\left(\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\Delta(z(n)z(n+1)) + z(n)z(n+1)\Delta^2\frac{\Delta x(n)}{z(n)}\right)\\
&= -\frac{1}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Bigg((z(n+1)z(n+2) - z(n)z(n+1))\Delta\frac{\Delta x(n+1)}{z(n+1)}\\
&\quad + z(n)z(n+1)\Delta^2\frac{\Delta x(n)}{z(n)}\Bigg)\\
&= -\frac{z(n+1)z(n+2)}{v(n+1)}\Delta\frac{\Delta v(n)}{z(n+1)}\Delta\frac{\Delta x(n+1)}{z(n+1)}\\
&\quad + \frac{z(n)z(n+1)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\\
&\quad - \frac{z(n)z(n+1)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta^2\left(\frac{\Delta x(n)}{z(n)}\right).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\Delta^4 x(n) + p(n)\Delta x(n+1) - \Delta^2 z(n+1)\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\\
&\quad - \frac{z(n+1)z(n+2)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\\
&\quad - \frac{z(n)z(n+1)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta^2\left(\frac{\Delta x(n)}{z(n)}\right)\\
&\quad + \left(\frac{z(n)z(n+1)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\right.\\
&\quad \left. - \Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\frac{z(n)z(n+1)}{v(n+1)}\Delta\left(\frac{\Delta x(n)}{z(n)}\right)\right)\\
&\quad = \Delta^4 x(n) + p(n)\Delta x(n+1) - \Delta^2 z(n+1)\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\\
&\quad - \frac{z(n+1)z(n+2)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\\
&\quad - \frac{z(n)z(n+1)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta^2\left(\frac{\Delta x(n)}{z(n)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{z(n)z(n+1)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\left[\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\right.\\
&\left.-\Delta\left(\frac{\Delta x(n)}{z(n)}\right)\right]\\
&=\Delta^4 x(n)+p(n)\Delta x(n+1)-\Delta^2 z(n+1)\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\\
&-\frac{z(n+1)z(n+2)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\\
&-\frac{z(n)z(n+1)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta^2\left(\frac{\Delta x(n)}{z(n)}\right)\\
&+\frac{z(n)z(n+1)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta^2\left(\frac{\Delta x(n)}{z(n)}\right)\\
&=\Delta^4 x(n)+p(n)\Delta x(n+1)-\Delta^2 z(n+1)\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\\
&-\frac{z(n+1)z(n+2)}{v(n+1)}\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\\
&=\Delta^4 x(n)+p(n)\Delta x(n+1)\\
&-\frac{z(n+1)z(n+2)}{v(n+1)}\Delta\left(\frac{\Delta x(n+1)}{z(n+1)}\right)\left[\Delta^2 z(n+1)\frac{v(n+1)}{z(n+1)z(n+2)}\right.\\
&\left.+\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)\right].
\end{aligned}
$$

From (2.3) we conclude that $\Delta^2 z(n+1)\frac{v(n+1)}{z(n+1)z(n+2)}+\Delta\left(\frac{\Delta v(n)}{z(n+1)}\right)=0$ and finally we get

$$
\begin{aligned}
&\frac{1}{v(n+1)}\Delta\left(\frac{v(n)v(n+1)}{z(n+1)}\Delta\left(\frac{1}{v(n)}z(n)z(n+1)\Delta\frac{\Delta x(n)}{z(n)}\right)\right)\\
&\quad=\Delta^4 x(n)+p(n)\Delta x(n+1).
\end{aligned}
$$

□

5 Conclusions

In this paper, we have studied the oscillation of a fourth-order delay three-terms equation (E). Our comparison method is based on the canonical representation (E′) of equation (E) and the existence of positive solutions satisfying the auxiliary equations (2.1) and (2.3). We have deduced the oscillation of equation (E) from the oscillation of certain first-order difference equations. Note, that even when we do not know the exact solutions of the auxiliary equation (2.1) or (2.3), we can easily verify the conditions presented in our criterion.

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