Periodica Mathematica Hungarica (2023) 86:43–57
https://doi.org/10.1007/s10998-022-00459-5

Lie triple derivations of standard operator algebras

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Accepted: 25 October 2021 / Published online: 8 April 2022

Abstract

Let X be a Banach space over the field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Denote by $B(X)$ the set of all bounded linear operators on X and by $F(X)$ the set of all finite rank operators on X. A subalgebra $\mathcal{A} \subseteq B(X)$ is called a standard operator algebra if $F(X) \subseteq \mathcal{A}$. Suppose that δ is a mapping from $\mathcal{A}$ into $B(X)$. First, we prove that if δ is a Lie triple derivation, then δ is standard. Next, we show that if δ is a local Lie triple derivation and $\dim(X) \geq 3$, then δ is a Lie triple derivation. Finally, we prove that if δ is a 2-local Lie triple derivation, then $\delta = d + \tau$, where d is a derivation, and τ is a homogeneous mapping from $\mathcal{A}$ into $\mathbb{F}I$ such that $\tau(A + B) = \tau(A)$ for each A, B in $\mathcal{A}$ where B is a sum of double commutators.

Keywords Lie triple derivation · Local Lie triple derivation · 2-Local Lie triple derivation · Standard operator algebra

Mathematics Subject Classification 46L57 · 47B47 · 47C15 · 47L35

1 Introduction

Let $\mathcal{A}$ be an associative algebra over the field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *derivation* if $\delta(AB) = \delta(A)B + A\delta(B)$ for each A, B in $\mathcal{A}$, and δ is called an *inner derivation* if there exists an element M in $\mathcal{M}$ such that $\delta(A) = AM - MA$ for every A in $\mathcal{A}$. Clearly, every inner derivation is a derivation. In [13, 24], Kadison and Sakai independently proved that every derivation on a von Neumann algebra is inner. In [6], Chernoff proved that every derivation from a standard operator algebra $\mathcal{A}$ into $B(X)$ is inner for a Banach space X. In [8], Christensen showed that every derivation on nest algebras is inner.

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In 1990, Kadison [14], Larson and Sourour [15] independently introduced the concept of local derivations. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *local derivation* if for every A in $\mathcal{A}$ there exists a derivation δ_A (depending on A) from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A) = \delta_A(A)$. In [14], Kadison proved that every continuous local derivation from a von Neumann algebra into its dual Banach module is a derivation. In [15], Larson and Sourour proved that every local derivation on $B(X)$ is a derivation for a Banach space X. In [12], Johnson proved that every local derivation from a C^*-algebra into its Banach bimodule is a derivation. In [29], Zhu and Xiong proved that every local derivation from a unital standard operator algebra $\mathcal{A}$ into $B(X)$ is a derivation.

In 1997, Šemrl [25] introduced the concept of 2-local derivations. A mapping (not necessarily linear) δ from $\mathcal{A}$ into $\mathcal{M}$ is called a 2-*local derivation* if for each A, B in $\mathcal{A}$, there exists a derivation $\delta_{A,B}$ (depending on A, B) from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A) = \delta_{A,B}(A)$ and $\delta(B) = \delta_{A,B}(B)$. In [25], Šemrl proved that every 2-local derivation on $B(\mathcal{H})$ is a derivation for a separable Hilbert space H. In [2], Ayupov and Kudaybergenov proved that every 2-local derivation on a von Neumann algebra is a derivation. In [10], we showed that every 2-local derivation from a standard operator algebra $\mathcal{A}$ into $B(X)$ is a derivation.

A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *Lie derivation* if $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for each A, B in $\mathcal{A}$, where $[A, B] = AB - BA$ is called a *commutator* on $\mathcal{A}$. A Lie derivation δ is said to be *standard* if it can be decomposed as $\delta = d + \tau$, where d is a derivation from $\mathcal{A}$ into $\mathcal{M}$ and τ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{M}, \mathcal{A})$ with $\tau([A, B]) = 0$ for each A, B in $\mathcal{A}$, where $\mathcal{Z}(\mathcal{M}, \mathcal{A}) = \{M \in \mathcal{M} : MA = AM \text{ for every } A \text{ in } \mathcal{A}\}$.

An interesting problem is to identify those algebras on which every Lie derivation is standard. In [22], Mathieu and Villena proved that every Lie derivation on a C^*-algebra is standard. In [7], Cheung characterized Lie derivations on triangular algebras. In [20, 21], Lu studied Lie derivations on CDCSL algebras and reflexive algebras, respectively. In [3], Benkovič proved that every Lie derivation on a matrix algebra $M_n(\mathcal{A})$ is standard, where $n \geq 2$ and $\mathcal{A}$ is a unital algebra.

Similarly to local derivations and 2-local derivations, in [4], Chen et al. introduced the concepts of local Lie derivations and 2-local Lie derivations. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *local Lie derivation* if for every A in $\mathcal{A}$ there exists a Lie derivation δ_A (depending on A) from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A) = \delta_A(A)$. A mapping (not necessarily linear) δ from $\mathcal{A}$ into $\mathcal{M}$ is called a 2-*local Lie derivation* if for every A, B in $\mathcal{A}$ there exists a Lie derivation $\delta_{A,B}$ (depending on A, B) from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A) = \delta_{A,B}(A)$ and $\delta(B) = \delta_{A,B}(B)$.

In [4], Chen et al. study local Lie derivations and 2-local Lie derivations on $B(X)$. In [5], Chen and Lu proved that every local Lie derivation on nest algebras is a Lie derivation. In [18, 19], Liu and Zhang proved that under certain conditions every local Lie derivation on triangular algebras is a Lie derivation, and every local Lie derivation on factor von Neumann algebras with dimension exceeding 1 is a Lie derivation. In [9], He et al. proved that every local Lie derivation on some algebras such as finite von Neumann algebras, nest algebras, Jiang–Su algebras and UHF algebras is a Lie derivation, and every 2-local Lie derivation on on some algebras such as factor von Neumann algebras, Jiang–Su algebra and UHF algebras is also a Lie derivation. In [16, 17], Liu proved that under certain conditions every local Lie derivation on generalized matrix algebras is a Lie derivation, and he showed that every 2-local Lie derivation of nest subalgebras of factors is a Lie derivation.

A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is a *Lie triple derivation* if $\delta([[A, B], C]) = [[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$ for each A, B and C in $\mathcal{A}$. We call $[[A, B], C]$ a *double commutator* on $\mathcal{A}$. It is clear that every Lie derivation is a Lie triple derivation. A Lie triple derivation δ from $\mathcal{A}$ into $\mathcal{M}$ is said to be *standard* if it can be decomposed as

$\delta = d + \tau$, where d is a derivation from $\mathcal{A}$ into $\mathcal{M}$ and τ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{M}, \mathcal{A})$ with $\tau([[A, B], C]) = 0$ for each A, B and C in $\mathcal{A}$.

Similarly to Lie derivations, the authors always consider the problem of identifying those algebras on which every Lie triple derivation is standard. In [23], Miers proved that if $\mathcal{A}$ is a von Neumann algebra with no central abelian summands, then every Lie triple derivation on $\mathcal{A}$ is standard. In [11], Ji and Wang proved that every continuous Lie triple derivation on TUHF algebras is standard. In [28], Zhang et al. proved that if $\mathcal{N}$ is a nest on a complex separable Hilbert space $\mathcal{H}$, then every Lie triple derivation on the nest algebra Alg $\mathcal{N}$ is standard. In [27], Yu and Zhang studied the Lie triple derivations on commutative subspace lattice algebras. In [3], Benkovič showed that if $\mathcal{A}$ is a unital algebra with a nontrivial idempotent, then under suitable assumptions every Lie triple derivation d on $\mathcal{A}$ is of the form $d = \Delta + \delta + \tau$, where Δ is a derivation on $\mathcal{A}$, δ is a Jordan derivation on $\mathcal{A}$ and τ is a linear mapping from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ that vanishes on $[[\mathcal{A}, \mathcal{A}], \mathcal{A}]$. In [1], Ashraf and Akhtar proved that every Lie triple derivation on a generalized matrix algebra is standard. In [26], Wani proved that every Lie triple derivation from standard operator algebra into itself is standard.

Now we give the concepts of local Lie triple derivations and 2-local Lie triple derivations. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *local Lie triple derivation* if for every A in $\mathcal{A}$ there exists a Lie triple derivation δ_A (depending on A) from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A) = \delta_A(A)$. A mapping (not necessarily linear) δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *2-local Lie triple derivation* if for every A, B in $\mathcal{A}$ there exists a Lie triple derivation $\delta_{A,B}$ (depending on A, B) from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(A) = \delta_{A,B}(A)$ and $\delta(B) = \delta_{A,B}(B)$.

In this paper, we always suppose that X is a Banach space over the field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Denote by $B(X)$ the set of all linear mappings on X and by $F(X)$ the set of all finite rank operators on X. A subalgebra $\mathcal{A} \subseteq B(X)$ is called a *standard operator algebra* if $F(X) \subseteq \mathcal{A}$. Suppose that δ is a mapping from $\mathcal{A}$ into $B(X)$. In Sect. 2, we prove that if δ is a Lie triple derivation, then δ is standard. In Sect. 3, we prove that if δ is a local Lie triple derivation and $\dim(X) \geq 3$, then δ is a Lie triple derivation. In Sect. 4, we prove that if δ is a 2-local Lie triple derivation, then $\delta = d + \tau$, where d is a derivation and τ is a homogeneous mapping from $\mathcal{A}$ into $\mathbb{F}I$ such that $\tau(A + B) = \tau(A)$ for each A, B in $\mathcal{A}$ where B is a sum of double commutators.

We shall review some simple properties of rank one operators and finite rank operators. Denote by X^* the set of all bounded linear functionals on X. For each x in X and f in X^*, one can define an operator $x \otimes f$ by $(x \otimes f)y = f(y)x$ for every y in X. Obviously, $x \otimes f \in B(X)$. If both x and f are nonzero, then $x \otimes f$ is an operator of rank one. The following properties are evident and will be used frequently in this paper.

Proposition 1.1 *Suppose that X is a Banach space and $\mathcal{A} \subseteq B(X)$ is a standard operator algebra. For each x, y in X, f, g in X^* and A, B in $B(X)$, the following statements hold:*

(1) $(x \otimes f)A = x \otimes (fA)$ *and* $A(x \otimes f) = (Ax) \otimes f$*;*
(2) $(x \otimes f)(y \otimes g) = f(y)(x \otimes g)$*;*
(3) $\mathcal{Z}(B(X), \mathcal{A}) = \mathbb{F}I$.

2 Lie triple derivations

In this section, we choose $x_0 \in X$ and $f_0 \in X^*$ such that $f_0(x_0) = 1$, and denote by I the unit operator in $B(X)$. For the convenience of expression, we give some symbols firstly. Let $P_1 = x_0 \otimes f_0$ and $P_2 = I - P_1$. It is easy to see that P_1 and P_2 are two idempotents in $B(X)$.

Denote $P_i\mathcal{A}P_j$ and $P_iB(X)P_j$ by $\mathcal{A}_{ij}$ and $B(X)_{ij}$, respectively, denote P_iAP_j by A_{ij} for every A in $\mathcal{A}$, where $1 \leq i, j \leq 2$.

Lemma 2.1 *$P_1AP_1 = f_0(Ax_0)P_1 = f_0(P_1AP_1x_0)P_1$ for every A in $B(X)$. Moreover, $B(X)_{11}$ is commutative.*

Proof For every A in $B(X)$, by Proposition 1.1 (1) and (2), we have

$$P_1AP_1 = x_0 \otimes f_0Ax_0 \otimes f_0 = f_0(Ax_0)x_0 \otimes f_0 = f_0(Ax_0)P_1. \tag{2.1}$$

Replacing A by P_1AP_1 in (2.1), we get

$$P_1AP_1 = P_1P_1AP_1P_1 = f_0(P_1AP_1x_0)P_1.$$

It follows that $B(X)_{11}$ is commutative. □

Lemma 2.2 (1) *If $BA_{21} = 0$ for every A_{21} in $\mathcal{A}_{21}$, then $BP_2 = 0$.*
(2) *If $A_{12}B = 0$ for every A_{12} in $\mathcal{A}_{12}$, then $P_2B = 0$.*

Proof (1) Let $A_{21} = P_2x \otimes f_0P_1$, where x is an arbitrary element in X. We obtain

$$0 = BP_2x \otimes f_0P_1x_0 = f_0(P_1x_0)BP_2x = BP_2x.$$

It follows that $BP_2 = 0$.

(2) Let $A_{12} = P_1x_0 \otimes fP_2$, where f is an arbitrary element in X^*. We obtain

$$0 = P_1x_0 \otimes fP_2Bx = f(P_2Bx)P_1x_0 = f(P_2Bx)x_0$$

for every x in X. It follows that $f(P_2Bx) = 0$ for each $f \in X^*$ and x in X. Thus, $P_2B = 0$. □

Next we consider Lie triple derivations from a unital standard operator algebra $\mathcal{A}$ into $B(X)$. The following theorem is the main result in this section.

Theorem 2.3 *Let X be a Banach space and $\mathcal{A} \subseteq B(X)$ be a unital standard operator algebra. If δ is a Lie triple derivation δ from $\mathcal{A}$ into $B(X)$, then δ is standard.*

Before we prove Theorem 2.3, we present some lemmas.

Lemma 2.4 $\delta(I) \in \mathbb{F}I$.

Proof Let P be an idempotent in $\mathcal{A}$. We have

$$0 = \delta([[I, P], P]) = [[\delta(I), P], P] = [\delta(I)P - P\delta(I), P] = \delta(I)P + P\delta(I) - 2P\delta(I)P.$$

Multiplying the above equation by P from the right, we obtain $P\delta(I)P = \delta(I)P$. It means that $(I - P)\delta(I)P = 0$. Thus, $P_1\delta(I)P_2 = P_2\delta(I)P_1 = 0$; it follows that $\delta(I) \in B(X)_{11} + B(X)_{22}$. By Lemma 2.1, we know that $\mathcal{A}_{11}$ is commutative, so $[\delta(I), A_{11}] = 0$ for every A_{11} in $\mathcal{A}_{11}$. In the following, we show

$$[\delta(I), A_{22}] = [\delta(I), A_{12}] = [\delta(I), A_{21}] = 0$$

for every A_{22} in $\mathcal{A}_{22}$, A_{12} in $\mathcal{A}_{12}$ and A_{21} in $\mathcal{A}_{21}$.

For each A, B in $\mathcal{A}$, we have

$$[[A, B], \delta(I)] = \delta([[A, B], I]) - [[A, \delta(B)], I] - [[\delta(A), B], I] = 0.$$

By $A_{12} = [P_1, A_{12}]$ and $A_{21} = [A_{21}, P_1]$, we have

$$[\delta(I), A_{12}] = [\delta(I), A_{21}] = 0. \tag{2.2}$$

By (2.2), it follows that

$$0 = [\delta(I), A_{22}B_{21}] = [\delta(I), A_{22}]B_{21} + A_{22}[\delta(I), B_{21}] = [\delta(I), A_{22}]B_{21}$$

for every A_{22} in $\mathcal{A}_{22}$ and B_{21} in $\mathcal{A}_{21}$. By Lemma 2.2, we have $[\delta(I), A_{22}]P_2 = 0$. By $\delta(I) \in B(X)_{11} + B(X)_{22}$, we obtain $[\delta(I), A_{22}] \in B(X)_{22}$, it follows that $[\delta(I), A_{22}] = 0$. Hence by Proposition 1.1 (3), we have $\delta(I) \in \mathcal{Z}(B(X), \mathcal{A}) = \mathbb{F}I$. □

Lemma 2.5 $P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 \in \mathbb{F}I$.

Proof By Lemma 2.1, we know that $P_1\delta(P_1)P_1 = \lambda P_1$, where $\lambda = f_0(P_1\delta(P_1)P_1x_0) \in \mathbb{F}$. Let x be in X and let $P_2x \otimes f_0P_1 = A_{21}$. It follows that

$$\begin{aligned}
-\delta(A_{21}) = \delta([[P_2, A_{21}], P_2])&\\
= [[\delta(P_2), A_{21}], P_2] + [[P_2, \delta(A_{21})], P_2] + [[P_2, A_{21}], \delta(P_2)]&\\
= -A_{21}\delta(P_2)P_2 - P_2\delta(P_2)A_{21} + A_{21}\delta(P_2) + 2P_2\delta(A_{21})P_2&\\
- \delta(A_{21})P_2 - P_2\delta(A_{21}) + A_{21}\delta(P_2) - \delta(P_2)A_{21}.&
\end{aligned} \tag{2.3}$$

Multiplying (2.3) by P_2 from the left and by P_1 from the right, we obtain

$$P_2\delta(P_2)A_{21} = A_{21}\delta(P_2)P_1.$$

That is,

$$P_2\delta(P_2)P_2x \otimes f_0P_1 = P_2x \otimes f_0P_1\delta(P_2)P_1. \tag{2.4}$$

By letting both sides of (2.3) act on x_0 in X, we have

$$f_0(P_1x_0)P_2\delta(P_2)P_2x = f_0(P_1\delta(P_2)P_1x_0)P_2x.$$

Since $f_0(P_1x_0) = f_0(x_0) = 1$, it follows that

$$P_2\delta(P_2)P_2 = f_0(P_1\delta(P_2)P_1x_0)P_2. \tag{2.5}$$

By Lemma 2.4, we know that $\delta(I) \in \mathbb{F}I$. It follows that

$$P_2\delta(I)P_2 = \delta(I)P_2 = \delta(I)f_0(x_0)P_2 = f_0(\delta(I)x_0)P_2 = f_0(P_1\delta(I)P_1x_0)P_2.$$

Now replacing $\delta(P_2)$ by $\delta(I) - \delta(P_1)$ in (2.5), we obtain

$$P_2\delta(P_1)P_2 = f_0(P_1\delta(P_1)P_1x_0)P_2 = \lambda P_2.$$

This implies $P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 = \lambda(P_1 + P_2) = \lambda I$. □

Let $G = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$ and define a mapping Δ from $\mathcal{A}$ into $B(X)$ by

$$\Delta(A) = \delta(A) - [A, G]$$

for every A in $\mathcal{A}$. Obviously, Δ is also a Lie triple derivation from $\mathcal{A}$ into $B(X)$. Moreover,

$$\Delta(P_1) = \delta(P_1) - [P_1, G] = P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2$$

and, by Lemma 2.5, we know that $\Delta(P_1) \in \mathbb{F}I$. In Lemmas 2.6, 2.7 and 2.8, we show some properties of Δ.

Lemma 2.6 *$\Delta(\mathcal{A}_{ij}) \subseteq B(X)_{ij}$, where $1 \le i, j \le 2$ and $i \ne j$.*

Proof Since $\Delta(P_1) \in \mathbb{F}I$, for each A_{12} in $\mathcal{A}_{12}$, we have

$$\begin{aligned}\Delta(A_{12}) &= \Delta([[A_{12}, P_1], P_1])\\ &= [[\Delta(A_{12}), P_1], P_1] + [[A_{12}, \Delta(P_1)], P_1] + [[A_{12}, P_1], \Delta(P_1)]\\ &= [[\Delta(A_{12}), P_1], P_1]\\ &= P_1\Delta(A_{12})P_2 + P_2\Delta(A_{12})P_1. \end{aligned} \tag{2.6}$$

In the following, we show that $P_2\Delta(A_{12})P_1 = 0$.

Let B_{12} be in $\mathcal{A}_{12}$, then $[A_{12}, B_{12}] = 0$. Thus,

$$\begin{aligned}0 = \Delta(0) = \Delta([[A_{12}, B_{12}], C]) &= [[\Delta(A_{12}), B_{12}], C] + [[A_{12}, \Delta(B_{12})], C]\\ &= [[\Delta(A_{12}), B_{12}] + [A_{12}, \Delta(B_{12})], C]\end{aligned}$$

for every C in $\mathcal{A}$. It means that $J = [\Delta(A_{12}), B_{12}] + [A_{12}, \Delta(B_{12})] \in \mathbb{F}I$. Since $A_{12} = [P_1, A_{12}]$, we have

$$\begin{aligned}[\Delta(A_{12}), B_{12}] &= J - [A_{12}, \Delta(B_{12})] = J - [[P_1, A_{12}], \Delta(B_{12})]\\ &= J - (\Delta([[P_1, A_{12}], B_{12}]) - [[\Delta(P_1), A_{12}], B_{12}] - [[P_1, \Delta(A_{12})], B_{12}])\\ &= J + [[P_1, \Delta(A_{12})], B_{12}].\end{aligned}$$

By (2.6), we have

$$\begin{aligned}[P_1\Delta(A_{12})P_2 + P_2\Delta(A_{12})P_1, B_{12}] &= J + [[P_1, P_1\Delta(A_{12})P_2 + P_2\Delta(A_{12})P_1], B_{12}]\\ &= J + [P_1\Delta(A_{12})P_2 - P_2\Delta(A_{12})P_1, B_{12}].\end{aligned}$$

Hence

$$[P_2\Delta(A_{12})P_1, B_{12}] = \frac{1}{2}J \in \mathbb{F}I.$$

It is well known that $[P_2\Delta(A_{12})P_1, B_{12}] = 0$. Thus, $P_2\Delta(A_{12})B_{12} = B_{12}\Delta(A_{12})P_1 = 0$ for every B_{12} in $\mathcal{A}_{12}$. By Lemma 2.2, we know that $P_2\Delta(A_{12})P_1 = 0$. Similarly, we have $\Delta(\mathcal{A}_{21}) \subseteq B(X)_{21}$. □

Lemma 2.7 *$\Delta(\mathcal{A}_{11}) \subseteq \mathbb{F}I$.*

Proof For every A_{11} in $\mathcal{A}_{11}$, by Lemma 2.1, we have

$$\Delta(A_{11}) = \Delta(P_1A_{11}P_1) = \Delta(f_0(A_{11}x_0)P_1) = f_0(A_{11}x_0)\Delta(P_1).$$

Since $\Delta(P_1) \in \mathbb{F}I$, it follows that $\Delta(A_{11}) \in \mathbb{F}I$. □

Lemma 2.8 *$\Delta(A_{22}) - f_0(\Delta(A_{22})x_0)I \in B(X)_{22}$ for every A_{22} in $\mathcal{A}_{22}$. In particular, $\Delta(P_2) = f_0(\Delta(P_2)x_0)I$.*

Proof Through simple calculation, we get

$$0 = \Delta([[P_1, A_{22}], P_1]) = [[P_1, \Delta(A_{22})], P_1] = -P_1\Delta(A_{22})P_2 - P_2\Delta(A_{22})P_1.$$

It follows that $\Delta(A_{22}) \in B(X)_{11} + B(X)_{22}$. By Lemma 2.1, we obtain

$$\Delta(A_{22}) = P_1\Delta(A_{22})P_1 + P_2\Delta(A_{22})P_2 = f_0(\Delta(A_{22})x_0)P_1 + P_2\Delta(A_{22})P_2,$$

that is,

$$\Delta(A_{22}) - f_0(\Delta(A_{22})x_0)I = -f_0(\Delta(A_{22})x_0)P_2 + P_2\Delta(A_{22})P_2 \in B(X)_{22}.$$

Since $\Delta(P_2) = \Delta(I) - \Delta(P_1) \in \mathbb{F}I$, we have

$$\Delta(P_2) - f_0(\Delta(P_2)x_0)I \in \mathbb{F}I \cap B(X)_{22} = \{0\}.$$

Thus, $\Delta(P_2) = f_0(\Delta(P_2)x_0)I$. □

In the following, we prove Theorem 2.3.

Proof Define two mappings τ and D on from $\mathcal{A}$ into $B(X)$ by

$$\tau(A) = f_0(P_1AP_1x_0)\Delta(P_1) + f_0(\Delta(P_2AP_2)x_0)I$$

and

$$D(A) = \Delta(A) - \tau(A)$$

for every A in $\mathcal{A}$. It is clear that τ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(B(X), \mathcal{A})$ and D is a linear mapping from $\mathcal{A}$ into $B(X)$. Moreover, according to the previous lemmas and the definitions of τ and D, we have

(1) $D(A_{ij}) = \Delta(A_{ij}) \in B(X)_{ij}$ for every A_{ij} in $\mathcal{A}_{ij}$, where $1 \leq i, j \leq 2$ and $i \neq j$;
(2) $D(P_1) = D(P_2) = D(I) = 0$;
(3) $D(A_{11}) = 0$ for every A_{11} in $\mathcal{A}_{11}$;
(4) $D(A_{22}) \in B(X)_{22}$ for every A_{22} in $\mathcal{A}_{22}$.

To prove that Δ is standard, it is sufficient to show that D is a derivation and $\tau([[A, B], C]) = 0$ for each A, B and C in $\mathcal{A}$.

In the following we show

$$D(A_{ij}B_{sk}) = D(A_{ij})B_{sk} + A_{ij}D(B_{sk})$$

for every A_{ij} in $\mathcal{A}_{ij}$ and B_{sk} in $\mathcal{A}_{sk}$, where $1 \leq i, j, s, k \leq 2$.

Since $D(\mathcal{A}_{ij}) \in B(X)_{ij}$, we have

$$D(A_{ij}B_{sk}) = D(A_{ij})B_{sk} + A_{ij}D(B_{sk})$$

for $j \neq s$. Thus, we only need to prove the following 8 cases:

(1) $D(A_{11}B_{11}) = D(A_{11})B_{11} + A_{11}D(B_{11})$;
(2) $D(A_{11}B_{12}) = D(A_{11})B_{12} + A_{11}D(B_{12})$;
(3) $D(A_{12}B_{22}) = D(A_{12})B_{22} + A_{12}D(B_{22})$;
(4) $D(A_{21}B_{11}) = D(A_{21})B_{11} + A_{21}D(B_{11})$;
(5) $D(A_{22}B_{21}) = D(A_{22})B_{21} + A_{22}D(B_{21})$;
(6) $D(A_{22}B_{22}) = D(A_{22})B_{22} + A_{22}D(B_{22})$;
(7) $D(A_{12}B_{21}) = D(A_{12})B_{21} + A_{12}D(B_{21})$;
(8) $D(A_{21}B_{12}) = D(A_{21})B_{12} + A_{21}D(B_{12})$.

Since $D(A_{11}) = 0$ for every A_{11} in $\mathcal{A}_{11}$, case (1) is trivial.

For each A, B in $\mathcal{A}$, by $\Delta(A) - D(A) = \tau(A) \in \mathcal{Z}(B(X), \mathcal{A})$, we have $[\Delta(A), B] = [D(A), B]$. Therefore

$$\begin{aligned} D(A_{11}B_{12}) &= \Delta(A_{11}B_{12}) = -\Delta([[P_1, B_{12}], A_{11}]) \\ &= -[[P_1, \Delta(B_{12})], A_{11}] - [[P_1, B_{12}], \Delta(A_{11})] \\ &= -[\Delta(B_{12}), A_{11}] - [B_{12}, \Delta(A_{11})] \\ &= [A_{11}, D(B_{12})] + [D(A_{11}), B_{12}] \\ &= A_{11}D(B_{12}) + D(A_{11})B_{12} \end{aligned}$$

for each A_{11} in $\mathcal{A}_{11}$ and B_{12} in $\mathcal{A}_{12}$. Thus, case (2) holds. The cases (3), (4) and (5) are similar to case (2), so we omit the proofs.

For every C_{21} in $\mathcal{A}_{21}$, according to case (5), we have the following two equations:

$$D(A_{22}B_{22}C_{21}) = D(A_{22}B_{22})C_{21} + A_{22}B_{22}D(C_{21}) \tag{2.7}$$

and

$$\begin{aligned} D(A_{22}B_{22}C_{21}) &= D(A_{22})B_{22}C_{21} + A_{22}D(B_{22}C_{21}) \\ &= D(A_{22})B_{22}C_{21} + A_{22}D(B_{22})C_{21} + A_{22}B_{22}D(C_{21}) \end{aligned} \tag{2.8}$$

for each A_{22}, B_{22} in $\mathcal{A}_{22}$. Comparing (2.7) and (2.8), we have

$$D(A_{22}B_{22})C_{21} = D(A_{22})B_{22}C_{21} + A_{22}D(B_{22})C_{21}.$$

It follows that $(D(A_{22}B_{22}) - D(A_{22})B_{22} - A_{22}D(B_{22}))C_{21} = 0$ for every C_{21} in $\mathcal{A}_{21}$. By Lemma 2.2 and $D(A_{22}) \in \mathcal{A}_{22}$, we know that

$$D(A_{22}B_{22}) - D(A_{22})B_{22} - A_{22}D(B_{22}) = 0.$$

Finally, we show cases (7) and (8). Let A_{12} be in $\mathcal{A}_{12}$ and B_{21} be in $\mathcal{A}_{21}$. Through simple calculation, we obtain

$$\begin{aligned} &\Delta([[A_{12}, P_2], B_{21}]) - D([[A_{12}, P_2,], B_{21}]) \\ &\quad = [[\Delta(A_{12}), P_2], B_{21}] + [[A_{12}, P_2], \Delta(B_{21})] - D([[A_{12}, P_2,], B_{21}]) \\ &\quad = [\Delta(A_{12}), B_{21}] + [A_{12}, \Delta(B_{21})] - D[A_{12}, B_{21}] \\ &\quad = [D(A_{12}), B_{21}] + [A_{12}, D(B_{21})] - D(A_{12}B_{21} - B_{21}A_{12}) \\ &\quad = D(A_{12})B_{21} - B_{21}D(A_{12}) + A_{12}D(B_{21}) - D(B_{21})A_{12} - D(A_{12}B_{21}) + D(B_{21}A_{12}) \\ &\quad = (D(A_{12})B_{21} + A_{12}D(B_{21}) - D(A_{12}B_{21})) + (D(B_{21}A_{12}) - B_{21}D(A_{12}) - D(B_{21})A_{12}). \end{aligned}$$

Since $\Delta([A_{12}, B_{21}]) - D([A_{12}, B_{21}])$ belongs to $\mathbb{F}I$, we may assume that

$$\Delta([A_{12}, B_{21}]) - D([A_{12}, B_{21}]) = \lambda I$$

holds for some λ in $\mathbb{F}$. That is,

$$\begin{aligned} \lambda I &= (D(A_{12})B_{21} + A_{12}D(B_{21}) - D(A_{12}B_{21})) \\ &\quad + (D(B_{21}A_{12}) - B_{21}D(A_{12}) - D(B_{21})A_{12}). \end{aligned} \tag{2.9}$$

Since $D(\mathcal{A}_{ij}) \in B(X)_{ij}$, we get

$$D(A_{12})B_{21} + A_{12}D(B_{21}) - D(A_{12}B_{21}) \in B(X)_{11}$$

and

$$D(B_{21}A_{12}) - B_{21}D(A_{12}) - D(B_{21})A_{12} \in B(X)_{22}.$$

Multiplying (2.9) by P_1 and P_2 respectively from the right, we obtain the following two equations:

$$D(A_{12}B_{21}) = D(A_{12})B_{21} + A_{12}D(B_{21}) - \lambda P_1 \tag{2.10}$$

and

$$D(B_{21}A_{12}) = B_{21}D(A_{12}) + D(B_{21})A_{12} + \lambda P_2. \tag{2.11}$$

By case (2) and equation (2.10), we obtain

$$\begin{aligned} &D(A_{12}B_{21}A_{12}) \\ &\quad = D(A_{12}B_{21})A_{12} + A_{12}B_{21}D(A_{12}) \\ &\quad = D(A_{12})B_{21}A_{12} + A_{12}D(B_{21})A_{12} - \lambda A_{12} + A_{12}B_{21}D(A_{12}). \end{aligned} \tag{2.12}$$

By case (3) and equation (2.9), we obtain

$$\begin{aligned} &D(A_{12}B_{21}A_{12}) \\ &\quad = D(A_{12})B_{21}A_{12} + A_{12}D(B_{21}A_{12}) \\ &\quad = D(A_{12})B_{21}A_{12} + A_{12}D(B_{21})A_{12} + A_{12}B_{21}D(A_{12}) + \lambda A_{12}. \end{aligned} \tag{2.13}$$

Comparing (2.12) and (2.13), we have $\lambda A_{12} = 0$. Thus, $\lambda = 0$. By (2.10) and (2.9), cases (7) and (8) hold.

By cases (1)–(8), this implies immediately that D is a derivation. Now we show that $\tau([[A, B], C]) = 0$ for each A, B and C in $\mathcal{A}$. Indeed,

$$\begin{aligned} \tau([[A, B], C]) &= \Delta([[A, B], C]) - D([[A, B], C]) \\ &= [[\Delta(A), B], C] + [[A, \Delta(B)], C] + [[A, B], \Delta(C)] - D([[A, B], C]) \\ &= [[D(A), B], C] + [[A, D(B)], C] + [[A, B], D(C)] - D([[A, B], C]) \\ &= 0. \end{aligned}$$

It follows that $\Delta(A) = D(A) + \tau(A)$ is a standard Lie triple derivation from $\mathcal{A}$ into $B(X)$. Define a linear mapping from $\mathcal{A}$ into $B(X)$ by

$$d(A) = D(A) + [A, G]$$

for every A in $\mathcal{A}$. Thus, we have

$$\delta(A) = \Delta(A) + [A, G] = D(A) + \tau(A) + [A, G] = d(A) + \tau(A),$$

where d is a derivation from $\mathcal{A}$ into $B(X)$ and τ is a linear mapping from $\mathcal{A}$ into $\mathcal{Z}(B(X), \mathcal{A})$ such that $\tau([[A, B], C]) = 0$ for each A, B and C in $\mathcal{A}$. □

For a non-unital standard operator algebra, the following result holds.

Corollary 2.9 *Let X be a Banach space and $\mathcal{A} \subseteq B(X)$ be a non-unital standard operator algebra. If δ is a Lie triple derivation δ from $\mathcal{A}$ into $B(X)$, then δ is standard.*

Proof Denote the unital algebra $\mathcal{A} \oplus \mathbb{F}I$ by $\widetilde{\mathcal{A}}$. Thus, $\widetilde{\mathcal{A}}$ is a unital standard operator algebra. Define a linear mapping $\widetilde{\delta}$ from $\widetilde{\mathcal{A}}$ into $B(X)$ by

$$\widetilde{\delta}(A + \lambda I) = \delta(A)$$

for every A in $\mathcal{A}$ and λ in $\mathbb{F}$. Through a simple calculation, it is easy to show that $\widetilde{\delta}$ is also a Lie triple derivation. By Theorem 2.3, we know that $\widetilde{\delta}$ is standard, and so is δ. □

3 Local Lie triple derivations

In this section, we study local Lie triple derivations and the following theorem is the main result.

Theorem 3.1 *Let X be a Banach space of dimension at least 3 and $\mathcal{A} \subseteq B(X)$ be a unital standard operator algebra. If δ is a local Lie triple derivation δ from $\mathcal{A}$ into $B(X)$, then δ is a Lie triple derivation.*

Proof For every A in $B(X)$, there is a Lie triple derivation δ_A from $\mathcal{A}$ into $B(X)$ such that $\delta(A) = \delta_A(A)$. By Theorem 2.3, we know $\delta_A(A)$ is standard, then there exist a derivation d_A from $\mathcal{A}$ into $B(X)$ and a scalar operator $\tau_A(A)$ in $\mathbb{F}I$ such that $\delta(A) = d_A(A) + \tau_A(A)$. By [6, Corollary 3.4], we know that d_A is an inner derivation, then there exists an element T_A in $B(X)$ such that $d_A(A) = [A, T_A]$. Thus, we have

$$\delta(A) = d_A(A) = [A, T_A] + \tau_A(A).$$

We claim that $\tau_A(A)$ is unique. In fact, if

$$\delta(A) = [A, S_A] + \tau'_A(A)$$

for some S_A in $B(X)$ and $\tau'_A(A)$ in $\mathbb{F}I$, then

$$[A, S_A - T_A] = \tau_A(A) - \tau'_A(A) = \lambda I$$

for some λ in $\mathbb{F}$. It is well known that $\tau_A(A) = \tau'_A(A)$. Hence we can define a mapping from $\mathcal{A}$ into $\mathbb{F}I$ by

$$\tau(A) = \tau_A(A)$$

for every A in $\mathcal{A}$. Moreover, by the definition of τ and Theorem 2.3, we know that $\tau(A) = \tau_A(A) = 0$ if A is a sum of double commutators.

For each x in X and f in X^*, define $\psi(x, f) = \tau(x \otimes f)$. Then we have

$$\delta(x \otimes f) = [x \otimes f, T_{x\otimes f}] + \psi(x, f) \tag{3.1}$$

for some $T_{x\otimes f}$ in $B(X)$. In the following we show that $\psi(x, f)$ is a bilinear mapping.

Firstly, we show the homogeneity of ψ. For each x in X, f in X^* and λ in $\mathbb{F}$, by (3.1), we have

$$\delta(x \otimes f) = [x \otimes f, T_{x\otimes f}] + \psi(x, f) \text{ and } \delta(\lambda x \otimes f) = [\lambda x \otimes f, T_{\lambda x\otimes f}] + \psi(\lambda x, f).$$

By $\delta(\lambda x \otimes f) = \lambda\delta(x \otimes f)$, we infer

$$[\lambda x \otimes f, T_{x\otimes f} - T_{\lambda x\otimes f}] = \psi(\lambda x, f) - \lambda\psi(x, f) \in \mathbb{F}I.$$

Thus, $\lambda\psi(x, f) = \psi(\lambda x, f)$. This proved that ψ is homogenous in the first variable. In the same way, we can show that ψ is homogenous in the second variable.

Secondly, we show that $\psi(x, f)$ is biadditive. We note that $\psi(x, f) = 0$ for x in X and f in X^* with $f(x) = 0$. Indeed, we may choose an element z in X such that $f(z) = 1$, then $x \otimes f = [[x \otimes f, z \otimes f], z \otimes f]$ is a double commutator and hence $\psi(x, f) = \tau(x \otimes f) = 0$.

Let x_1, x_2 be in X and f be in X^*. If both x_1 and x_2 belong to $\ker f$, then

$$\psi(x_1, f) = \psi(x_2, f) = \psi(x_1 + x_2, f) = 0$$

and so

$$\psi(x_1 + x_2, f) = \psi(x_1, f) + \psi(x_2, f).$$

If one of x_1 and x_2 is not in $\ker f$, then $\dim(\mathrm{span}\{x_1, x_2\} \cap \ker f) \leq 1$. Since $\dim(X) \geq 3$, we know that $\dim(\ker f) \geq 2$. Thus, we can take $y \in \ker f$ such that $y \notin \mathrm{span}\{x_1, x_2\}$. By (3.1), we have the following equations:

$$\delta(x_1 \otimes f)y = \psi(x_1, f)y + \mu_1 x_1 \quad \delta(x_2 \otimes f)y = \psi(x_2, f)y + \mu_2 x_2$$

and

$$\delta((x_1 + x_2) \otimes f)y = \psi(x_1 + x_2, f)y + \mu(x_1 + x_2)$$

for some $\mu, \mu_1, \mu_2 \in \mathbb{F}$. Since δ is an additive mapping, we know that

$$(\psi(x_1 + x_2, f) - \psi(x_1, f) - \psi(x_2, f))y = \mu_1 x_1 + \mu_2 x_2 - \mu(x_1 + x_2).$$

Since $y \notin$ span $\{x_1, x_2\}$, it follows that

$$\psi(x_1 + x_2, f) = \psi(x_1, f) + \psi(x_2, f).$$

It means that ψ is additive in the first variable.

Let f_1, f_2 be in X^* and x be in X. If $x \in \ker f_1 \cap \ker f_2$, then

$$\psi(x, f_1 + f_2) = \psi(x, f_1) = \psi(x, f_2) = 0$$

and so

$$\psi(x, f_1 + f_2) = \psi(x, f_1) + \psi(x, f_2).$$

If $x \notin \ker f_1 \cap \ker f_2$, then we can take $z \in \ker f_1 \cap \ker f_2$ which is linearly independent of x, By (3.1), we have

$$\delta(x \otimes f_1)z = \psi(x, f_1)z + \lambda_1 x \quad \delta(x \otimes f_2)z = \psi(x, f_2)z + \lambda_2 x$$

and

$$\delta(x \otimes (f_1 + f_2))z = \psi(x, f_1 + f_2)z + \lambda x$$

for some $\lambda, \lambda_1, \lambda_2 \in \mathbb{F}$. Since δ is an additive mapping, we know that

$$(\psi(x, f_1 + f_2) - \psi(x, f_1) - \psi(x, f_2))z = (\lambda_1 + \lambda_2 - \lambda)x.$$

Since z and x are linearly independent, it follows that

$$\psi(x, f_1 + f_2) = \psi(x, f_1) + \psi(x, f_2).$$

The next goal is to show that there is an element J in $B(X)$ such that

$$\delta(x \otimes f) = [x \otimes f, J] + \psi(x, f)$$

for every rank one operator $x \otimes f$ in $B(X)$.

For each x in X and f in X^*, define

$$\phi(x, f) = [x \otimes f, T_{x \otimes f}] = \delta(x \otimes f) - \psi(x, f). \tag{3.2}$$

It is easy to see that $\phi(x, f)$ is a bilinear mapping and $\phi(x, f)\ker f \subseteq \mathbb{F}x$. Hence by [21, Proposition 1.1], there are two linear mappings $T : X \to X$ and $S^* : X^* \to X^*$ such that

$$\phi(x, f) = [x \otimes f, T_{x\otimes f}] = Tx \otimes f + x \otimes S^* f \tag{3.3}$$

for each x in X and f in X^*. It follows that

$$(T + T_{x\otimes f})x \otimes f = x \otimes (T^*_{x\otimes f} - S^*)f \tag{3.4}$$

for each x in X and f in X^*.

We claim that $S^* = -T^*$. We only have to show that $S^* f(x) = -f(Tx)$ for each x in X and f in X^*. It is trivial if one of x and f is zero. Suppose that neither of x and f is zero. If both sides of (3.4) are zeros, then

$$(T + T_{x\otimes f})x = (T^*_{x\otimes f} - S^*)f = 0.$$

It follows that

$$S^* f(x) = T^*_{x\otimes f} f(x) = f(T_{x\otimes f}x) = -f(Tx).$$

If both sides of (3.4) are not zeros, then we have

$$[(T + T_{x\otimes f})x \otimes f]^2 = [x \otimes (T^*_{x\otimes f} - S^*)f]^2,$$

that is,

$$f((T + T_{x\otimes f})x)((T + T_{x\otimes f})x \otimes f) = ((T^*_{x\otimes f} - S^*)f)(x)(x \otimes (T^*_{x\otimes f} - S^*)f).$$

It follows that

$$f((T + T_{x\otimes f})x) = ((T^*_{x\otimes f} - S^*)f)(x)$$

and then $S^* f(x) = -f(Tx)$. Consequently, we always have $S^* = -T^*$. By (3.2) and (3.3), we have

$$\delta(x \otimes f) = Tx \otimes f + x \otimes S^* f + \psi(x, f) = [x \otimes f, -T] + \psi(x, f)$$

for every $x \otimes f$ in $\mathcal{A}$. Let $J = -T$ and by $\psi(x, f) = \tau(x \otimes f) \in \mathbb{F}I$, we obtain

$$\delta(A) = [A, J] + \lambda_A I \tag{3.5}$$

for every $A = x \otimes f$ in $\mathcal{A}$ and some $\lambda_A \in \mathbb{F}$. Finally, we show that

$$\delta(A) = [A, J] + \lambda_A I$$

holds for every A in $\mathcal{A}$. Suppose that P, Q are two idempotents of rank one and let $P^\perp = I - P$, $Q^\perp = I - Q$. By Proposition 1.1(1) and (3.5), it follows that

$$\begin{aligned}
\delta(A) &= \delta(PA + P^\perp AQ + P^\perp AQ^\perp) \\
&= \delta(PA) + \delta(P^\perp AQ) + \delta(P^\perp AQ^\perp) \\
&= [PA, J] + \lambda_{PA} I + [P^\perp AQ, J] + \lambda_{P^\perp AQ} I + [P^\perp AQ^\perp, T_{P^\perp AQ^\perp}] + \lambda_{P^\perp AQ^\perp} I \\
&= PAJ - JPA + P^\perp AQJ - JP^\perp AQ + [P^\perp AQ^\perp, T_{P^\perp AQ^\perp}] + \lambda_A I \\
&= PAJ - JAQ + P^\perp AQJ - JPAQ^\perp + [P^\perp AQ^\perp, T_{P^\perp AQ^\perp}] + \lambda_A I,
\end{aligned} \tag{3.6}$$

where $\lambda_A = \lambda_{PA} + \lambda_{P^\perp AQ} + \lambda_{P^\perp AQ^\perp}$. Multiplying (3.6) by P on the left and by Q on the right, we have

$$P\delta(A)Q = P[A, J]Q + \lambda PQ,$$

that is,

$$P(\delta(A) - [A, J] - \lambda_A I)Q = 0.$$

By the arbitrariness of P and Q, it follows that $\delta(A) = [A, J] + \lambda_A I$, where J is a fixed element and λ_A is depends on A. By the uniqueness of τ, we know that $\tau(A) = \lambda_A I$ and τ is a linear mapping from $\mathcal{A}$ into $\mathbb{F}I$ vanishing on every double commutator, which means that δ is a Lie triple derivation. □

Corollary 3.2 *Let X be a Banach space of dimension at least 3 and $\mathcal{A} \subseteq B(X)$ be a non-unital standard operator algebra. If δ is a local Lie triple derivation δ from $\mathcal{A}$ into $B(X)$, then δ is a Lie triple derivation.*

Proof Denote the unital algebra $\mathcal{A} \oplus \mathbb{F}I$ by $\widetilde{\mathcal{A}}$. Thus, $\widetilde{\mathcal{A}}$ is a unital standard operator algebra. Define a linear mapping $\widetilde{\delta}$ from $\widetilde{\mathcal{A}}$ into $B(X)$ by

$$\widetilde{\delta}(A + \lambda I) = \delta(A)$$

for every A in $\mathcal{A}$ and λ in $\mathbb{F}$.

Since δ is a local Lie triple derivation from $\mathcal{A}$ into $B(X)$, for each $A \in \mathcal{A}$ and $\lambda \in \mathbb{F}$, there exists a Lie triple derivation δ_A such that $\delta(A) = \delta_A(A)$. Define a linear mapping $\widetilde{\delta_A}$ from $\widetilde{\mathcal{A}}$ into $B(X)$ by

$$\widetilde{\delta_A}(B + \lambda I) = \delta_A(B)$$

for every B in $\mathcal{A}$ and λ in $\mathbb{F}$. It is easy to show that $\widetilde{\delta_A}$ is also a Lie triple derivation. Moreover, we have

$$\widetilde{\delta}(A + \lambda I) = \delta(A) = \delta_A(A) = \widetilde{\delta_A}(A + \lambda I).$$

It means that $\widetilde{\delta}$ is a local Lie triple derivation from $\widetilde{\mathcal{A}}$ into $B(X)$. By the result of the case that $\mathcal{A}$ contains the unit, $\widetilde{\delta}$ is a Lie triple derivation. Hence δ is also a Lie triple derivation. □

4 2-Local Lie triple derivations

In this section, we study the 2-local Lie triple derivations and the following theorem is the main result.

Theorem 4.1 *Let X be a Banach space and $\mathcal{A} \subseteq B(X)$ be a unital standard operator algebra. If δ is a 2-local Lie triple derivation from $\mathcal{A}$ into $B(X)$, then $\delta = d + \tau$, where d is a derivation and τ is a homogeneous mapping from $\mathcal{A}$ into $\mathbb{F}I$ such that $\tau(A + B) = \tau(A)$ for each A, B in $\mathcal{A}$ where B is a sum of double commutators.*

Proof Similarly to the proof of Theorem 3.1, we can show that δ has a unique decomposition at each point A in $\mathcal{A}$, i.e.

$$\delta(A) = \delta_A(A) = d_A(A) + \tau_A(A),$$

where δ_A is a Lie triple derivation, d_A is a derivation and τ_A is a linear mapping from $\mathcal{A}$ into $\mathbb{F}I$ such that $\tau_A[[X, Y], Z] = 0$ each X, Y and Z in $\mathcal{A}$.

Thus, we can define

$$d(A) = d_A(A) \text{ and } \tau(A) = \tau_A(A)$$

for every A in $\mathcal{A}$.

In the following we show that d is a derivation and τ is a homogeneous mapping. Given A and B in $\mathcal{A}$, there exists a Lie triple derivation $\delta_{A,B}$ from $\mathcal{A}$ into $B(X)$ such that

$$\delta(A) = \delta_{A,B}(A) = d_{A,B}(A) + \tau_{A,B}(A),$$

and

$$\delta(B) = \delta_{A,B}(B) = d_{A,B}(B) + \tau_{A,B}(B),$$

where $d_{A,B} + \tau_{A,B}$ is the standard decomposition of $\delta_{A,B}$. By the uniqueness of the decomposition, $d(A) = d_{A,B}(A)$ and $d(B) = d_{A,B}(B)$. Hence d is a 2-local derivation and by [10, Theorem 3.1], we know d is a derivation from $\mathcal{A}$ into $B(X)$.

For every A in $\mathcal{A}$ and λ in $\mathbb{F}$, there exists a Lie triple derivation $\delta_{A,\lambda A}$ from $\mathcal{A}$ into $B(X)$ such that

$$\delta(A) = \delta_{A,\lambda A}(A) \text{ and } \delta(\lambda A) = \delta_{A,\lambda A}(\lambda A).$$

It follows that δ is homogeneous, and so is τ.

Moreover, for each A, B in $\mathcal{A}$ where B is a sum of double commutators, there is a linear mapping $\tau_{A,A+B}$ from $\mathcal{A}$ into $\mathbb{F}I$ vanishing on every double commutator such that

$$\tau(A + B) = \tau_{A,A+B}(A + B) = \tau_{A,A+B}(A) = \tau(A).$$

□

Acknowledgements The authors thank the referee for his or her suggestions. This research was partly supported by the National Natural Science Foundation of China (Grant Nos. 11801342, 11801005); Natural Science Foundation of Shaanxi Province (Grant No. 2020JQ-693); Scientific research plan projects of Shanxi Education Department (Grant No. 19JK0130).

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