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Corner's theorem on modules with anti-isomorphic endomorphism algebras

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Abstract
We present a version of an unpublished result of A.L.S. Corner on p-adic modules with anti-isomorphic endomorphism algebras. The result gives a complete description of necessary conditions for two such modules to have anti-isomorphic endomorphism algebras and a sufficient condition is also given. A main difference in the current version is that extensive use is made of our ability to describe certain homomorphism groups.

Keywords P-adic modules · Abelian groups · Endomorphism algebras · Anti-isomorphisms

Mathematics Subject Classification Primary: 20K30, 20K10 · Secondary: 13C05

1 Introduction

Throughout let R and Q denote the ring of p-adic integers and the field of p-adic numbers respectively and for rings T, S we write $T \eqsim S$ if there is an anti-isomorphism from T onto S. As usual $\mathrm{End}(X)$ denotes the R-endomorphism algebra of X and we use the terminology 'G is a p-adic module' as a shorthand for saying 'G is a module over the ring of p-adic integers'.

Corner proved the result stated below sometime around 1961/62 but never published it. The only version [1] that exists is in Corner's handwriting and corresponds to the reference [U4] in the listing of Corner's unpublished works in [8]. This present work is based on the original but many of the arguments used there have been modified to reflect the advances that have been made in our understanding of the structure of homomorphism groups in more recent years.

In Memoriam Peter Vámos.

Noel White passed away on 1st September 2021 after a long illness.

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Theorem 1.1 (A.L.S. Corner) *Let G, H be p-adic modules with anti-isomorphic endomorphism algebras,* $\mathrm{End}(G) \approx \mathrm{End}(H)$. *Then either*

(I) $G \cong H \cong \bigoplus_r R$ *or* $G \cong H \cong \bigoplus_r \mathbb{Z}(p^\infty)$ *for some integer r,or*

(II)

$$\begin{cases} G \cong \bigoplus_r Q \oplus \bigoplus_s R \oplus \bigoplus_t \mathbb{Z}(p^\infty) \oplus G', \\ H \cong \bigoplus_r Q \oplus \bigoplus_s \mathbb{Z}(p^\infty) \oplus \bigoplus_t R \oplus H', \end{cases}$$

for integers r, s, t and either

(a) *G', H' are isomorphic finite p-groups or*

(b) *there is a sequence of finite cyclic groups $\langle a_j \rangle$ with exponents tending to infinity such that $\prod_{j=1}^{\infty} \langle a_j \rangle = P \geq G', H' \geq T$, the torsion submodule of P and $G'/T, H'/T$ are divisible. Thus, G', H' are submodules of D, where D/T is the maximal divisible submodule of P/T.*

If $r \geq 1$ then G', H' are p-groups; if $s \geq 1$ or $t \geq 1$, then G', H' are finite p-groups.

Note that an immediate consequence of this result is that if G, H are p-groups with anti-isomorphic endomorphism rings then G is actually isomorphic to H. In addition, the rank of G is either finite or the continuum $2^{\aleph_0}$; moreover either G is divisible or it has no elements of infinite height, i.e., G is a separable p-group.

Corner also provided a sufficient condition for anti-isomorphism of endomorphism algebras but we will defer discussion of this until later.

There have been several other approaches to this question of determining necessary conditions for p-adic modules to have anti-isomorphic endomorphism algebras. Most notably, Gabriella d'Este [3] proved in 1978 that for torsion p-adic modules (i.e., p-groups) $\mathrm{End}(G) \approx \mathrm{End}(H)$ if, and only if, G is torsion-compact. She also established a number of interesting results in both the non-local and non-torsion situations. However, to date no result seems to be known which has the generality of Corner's original theorem.

Corner also introduced the following terminology which we shall continue to use: if G, H are R-modules, then we say that the module G is *opposed under* ω to H if there exists an R-algebra anti-isomorphism $\omega : \mathrm{End}(G) \to \mathrm{End}(H)$; in this situation we say that G (and of course H) is opposable. The relationship is clearly symmetrical and, as above, we write $\mathrm{End}(G) \approx \mathrm{End}(H)$.

Otherwise our teminology is standard and follows that of Fuchs [5–7]; standard notions in Abelian group and module theory may be found in these works and also in [11]. In particular, if G is a torsion R-module, then G is an Abelian p-group and the rank of G is the cardinality of a maximal independent subset consisting only of elements of order a power of p; equivalently the rank of G is then the vector-space dimension of its socle $G[p]$—see, for example, [5, Section 16]. We have used the terminology 'semi-standard' to denote a torsion R-module G having the property that each Ulm invariant $f_G(n)$, with $n < \omega$, is finite; equivalently a basic submodule B of G is of the form $B = \bigoplus_{n<\omega} B_n$ where each B_n is a homocyclic p-group of exponent n and finite (possibly zero) rank.

2 Basic facts about p-adic modules

In this section we highlight some basic facts relating to modules over the ring R of p-adic integers. These results are well known and we mostly omit proofs or just give a reference to standard material.

A key tool in our approach to proving Corner's result is, inevitably, based on an idea going back to Kaplansky's work on torsion modules over a complete discrete valuation ring [11, §19]: if e is an idempotent in the endomorphism ring of a module M, then the rings $e\mathrm{End}(M)e$ and $\mathrm{End}(e(M))$ are isomorphic.

Proposition 2.1 *Suppose that A, A' are R-modules and $\omega : \mathrm{End}(A) \to \mathrm{End}(A')$ is an anti-isomorphism. If A has a decomposition $A = B \oplus C$, then A' has a decomposition $A' = B' \oplus C'$, and ω induces an anti-isomorphism $\mathrm{End}(B) \approxeq \mathrm{End}(B')$. In particular, a direct summand of an opposable module is opposable.*

Proof If π is the projection of A onto B along C, then $\pi' = \omega(\pi)$ is an idempotent in $\mathrm{End}(A')$ and gives rise to a decomposition $A' = B' \oplus C'$ where $B' = \omega(\pi)(B)$. However, $\mathrm{End}(B') = \mathrm{End}(\pi'(A')) \cong \pi'\mathrm{End}(A')\pi' = \omega(\pi\mathrm{End}(A)\pi) = \omega(\mathrm{End}(\pi(A)))$. Thus, ω induces an anti-isomorphism $\mathrm{End}(B) \approxeq \mathrm{End}(B')$. The final statement is immediate. □

Our next result, which is contained in Corner's original manuscript, is simple but extremely useful.

Lemma 2.2 *Let G, H be opposed under ω and let α_1, α_2 be two projections of G. Set $A_i = \alpha_i(G)$, $\beta_i = \omega(\alpha_i)$ and $B_i = \beta_i(H)$. Then if $\mathrm{Hom}(A_1, A_2) = 0$, we have $\mathrm{Hom}(B_2, B_1) = 0$.*

Proof Let $\beta : B_2 \to B_1$ be an arbitrary homomorphism. Extend β to the whole of H by defining $\beta(1 - \beta_2)(H) = 0$. Then an easy check gives that $\beta = \beta_1\beta\beta_2$. Now taking images under ω^{-1} and setting $\alpha = \omega^{-1}(\beta)$, we get $\alpha = \omega^{-1}(\beta_1\beta\beta_2) = \omega^{-1}(\beta_2)\omega^{-1}(\beta)\omega^{-1}(\beta_1) = \alpha_2\alpha\alpha_1$ and so α is a homomorphism : $A_1 \to A_2$. Hence α must be the zero homomorphism and thus, $\beta = 0$ since ω^{-1} is an anti-isomorphism. □

- The nonzero indecomposable p-adic modules are $R, Q, \mathbb{Z}(p^\infty)$ and $\mathbb{Z}(p^n)$ for each $n \geq 1$. An indecomposable p-adic module has a commutative endomorphism ring which has rank 1 as a p-adic module; the possible rings are just $R, Q, R, \mathbb{Z}(p^n)$ respectively. If X is an indecomposable R-module, then it follows from Proposition 2.1 and the structure of the endomorphism rings of $R, Q, \mathbb{Z}(p^\infty), \mathbb{Z}(p^n)$ as given above that:
- If X is opposed to Y, then Y is indecomposable.
- If $X \in \{Q, \mathbb{Z}(p^n)\ n \geq 1\}$ and X is opposed to Y, then $X \cong Y$.
- If R is opposed to Y then either $Y \cong R$ or $Y \cong \mathbb{Z}(p^\infty)$.
- If $\mathbb{Z}(p^\infty)$ is opposed to Y then either $Y \cong \mathbb{Z}(p^\infty)$ or $Y \cong R$.

We remark that the main source of problems in trying to establish Corner's theorem derives from the fact that R may be opposed to either R itself or to $\mathbb{Z}(p^\infty)$.

Our next basic results focus on the structure of homomorphism groups and proofs of these may be found in [5, Chapter VIII] or [7, Chapter 7].

($\bullet$ X) If either A or C is a torsion-free and divisible R-module, then $\mathrm{Hom}(A, C)$ is always torsion-free and divisible.
($\bullet$ Y) If $\mathrm{End}(A)$ is torsion-free divisible, then A is torsion-free divisible.
($\bullet$ Z) If A is divisible then $\mathrm{Hom}(A, C)$ is torsion-free.
($\bullet$ W) $\mathrm{Hom}(\mathbb{Z}(p^\infty), \bigoplus_\lambda \mathbb{Z}(p^\infty)) \cong \widehat{\bigoplus_\lambda R}$.

We require one further elementary result and include the proof for the convenience of the reader.

Lemma 2.3 *If G is a reduced R-module having a summand isomorphic to R, then $\mathrm{Hom}(G, R) \cong \prod_\lambda R$ for some cardinal $\lambda \neq 0$.*

Proof Observe firstly that $\mathrm{Hom}(G, R) \cong \mathrm{Hom}(G/tG, R)$ always holds so there is no loss in assuming that G is torsion-free. Thus, G has the form $G = D \oplus X$ where D is divisible and X is reduced; note that $X \neq 0$ since G has a summand isomorphic to R. Clearly $\mathrm{Hom}(G, R) = \mathrm{Hom}(X, R)$. However, since X is torsion-free, it has a free basic submodule B of rank λ say and $0 \to B \to X \to D \to 0$ is an exact sequence where D is torsion-free divisible. Since R is complete, $\mathrm{Ext}(D, R) = 0$ and it follows that $\mathrm{Hom}(X, R) = \mathrm{Hom}(B, R) \cong \prod_\lambda R$. □

We finish this section with a simple property of R-modules having torsion basic submodules; this result will be useful in the final stages of our arguments.

Lemma 2.4 *If the torsion module B is basic in the R-module G and $G/B \cong \bigoplus_\lambda \mathbb{Z}(p^\infty) \oplus \bigoplus_\mu Q$, then* $\mathrm{Hom}(G, \mathbb{Z}(p^\infty))$ *has a summand of the form $\prod_\lambda R \oplus \prod_\mu Q$.*

Proof Consider the pure exact sequence $0 \to B \to G \to G/B \to 0$ and apply the functor $\mathrm{Hom}(-, \mathbb{Z}(p^\infty))$ to obtain the pure exact sequence

$$0 \to \mathrm{Hom}(G/B, \mathbb{Z}(p^\infty)) \to \mathrm{Hom}(G, \mathbb{Z}(p^\infty)) \to \mathrm{Hom}(B, \mathbb{Z}(p^\infty)).$$

Now

$$\mathrm{Hom}(G/B, \mathbb{Z}(p^\infty)) \cong \prod_\lambda \mathrm{Hom}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) \oplus \prod_\mu \mathrm{Hom}(Q, \mathbb{Z}(p^\infty)) \cong \prod_\lambda R \oplus \prod_\mu Q,$$

since it is straightforward to show, using the completeness of R, that $\mathrm{Ext}^1(Q, R) = 0$ and hence $\mathrm{Hom}(Q, Q/R) \cong Q$. Now the latter is algebraically compact and pure in $\mathrm{Hom}(G, \mathbb{Z}(p^\infty))$ and so it is a summand. □

2.1 Some cardinality relationships

In this subsection we isolate some arguments that will appear a number of times in our later discussions. The main results that we shall need are all well known but it is not easy to give specific references to them. Since they are reasonably easy to demonstrate, we give the short proofs.

Proposition 2.5 *Suppose λ, μ are cardinals.*

(i) *If $\bigoplus_\lambda Q \cong \prod_\mu Q$ and $\prod_\lambda Q \cong \bigoplus_\mu Q$, then $\lambda = \mu$ is finite.*
(ii) *If $\prod_\lambda R \cong \widehat{\bigoplus_\mu R}$, then $\mu = 2^\lambda$ if μ is infinite; if μ is finite then $\mu = \lambda$.*

Proof (i) The module $\prod_\mu Q$ is the vector space dual of the Q-space $\bigoplus_\mu Q$. If μ is infinite then so too is λ and the first isomorphism in (i) gives $\lambda = \dim \bigoplus_\lambda Q = \dim \prod_\mu Q = |Q|^\mu = 2^\mu$, while the second isomorphism gives $\mu = \dim \bigoplus_\mu Q = \dim \prod_\lambda Q = |Q|^\lambda = 2^\lambda$—here we are using the classical result on the dimension of the dual space of an infinite-dimensional vector space, see, for example, [10, Chapter IX, Theorem 2]. So $\lambda = 2^\mu$ and $\mu = 2^\lambda$ which yields $\lambda = 2^{2^\lambda}$—contradiction. Hence λ, μ are both finite and their equality is then immediate.

(ii) The submodule $\bigoplus_\mu R$ is pure and dense in its completion $\widehat{\bigoplus_\mu R}$ and so

$$\widehat{\bigoplus_\mu R}/p(\widehat{\bigoplus_\mu R}) \cong \bigoplus_\mu R/p(\bigoplus_\mu R) \cong \bigoplus_\mu \mathbb{Z}(p).$$

However, $\prod_\lambda R/p(\prod_\lambda R) \cong \prod_\lambda \mathbb{Z}(p)$ and so it follows from the isomorphism in (ii) that $\prod_\lambda \mathbb{Z}(p) \cong \widehat{\bigoplus_\mu} \mathbb{Z}(p)$. If λ is infinite then so too is μ and we have, on equating cardinalities, that $\mu = 2^\lambda$, as required. If μ is finite, then $\widehat{\bigoplus_\mu R} = \bigoplus_\mu R$ and it follows immediately that $\mu = \lambda$.

□

We shall also need the following result which can be derived in greater generality from [9, Proposition 1.2]; we give a simple version adequate for our purposes.

Lemma 2.6 $\mathrm{Hom}(\prod_\lambda R, R) \cong \prod_{2^\lambda} R$ *if* λ *is infinite; if* λ *is finite then* $\mathrm{Hom}(\prod_\lambda R, R) \cong \prod_\lambda R$.

Proof Since R is complete and λ is infinite, $\prod_\lambda R$ is of the form $\widehat{\bigoplus_\kappa R}$ for some infinite cardinal κ. Then, reducing modulo p, we get $\prod_\lambda \mathbb{Z}(p) \cong \widehat{\bigoplus_\kappa R}/p\widehat{\bigoplus_\kappa R} \cong \bigoplus_\kappa \mathbb{Z}(p)$. Since λ is infinite, $\kappa = 2^\lambda$. Thus, $\mathrm{Hom}(\prod_\lambda R, R) = \mathrm{Hom}(\hat{F}, R)$ where F is free of rank $\kappa = 2^\lambda$. However, as the quotient $\hat{F}/F$ is torsion-free and R is complete, $\mathrm{Ext}(\hat{F}/F, R) = 0$ and so $\mathrm{Hom}(\hat{F}, R) = \mathrm{Hom}(F, R) \cong \prod_{2^\lambda} R$. The situation when λ is finite is straightforward since, in that case, $\prod_\lambda R$ is just the free R-module of rank λ. □

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2.2 The fundamental relations

Suppose that G, H are R-modules which are opposed under the anti-isomorphism ω. We shall use the following notation: if $\phi \in \mathrm{End}(G)$, then $\phi' = \omega(\phi)$ is the corresponding element of $\mathrm{End}(H)$. For simplicity of notation and when there is no danger of confusion, we shall often write $E = \mathrm{End}(G)$, $E' = \mathrm{End}(H) = \omega(E)$.

If we assume that G is indecomposable then, as we have shown in the previous section, Corner's result holds when the module G is assumed to be indecomposable. So from here on we assume that G, and hence H, is decomposable.

Assume π is an indecomposable idempotent of $E = \mathrm{End}(G)$, so that $\pi(G)$ is an indecomposable R-module, then as we have seen in Proposition 2.1, there is an indecomposable idempotent $\pi' \in E' = \mathrm{End}(H)$. Since ω is an anti-isomorphism, we get isomorphisms $E\pi \cong \pi' E'$ and $\pi E \cong E'\pi'$. Since the relationships $\mathrm{Hom}(\pi(G), G) \cong \mathrm{End}(G)\pi$, $\mathrm{Hom}(G, \pi(G)) \cong \pi\mathrm{End}(G)$ always hold for an idempotent $\pi \in \mathrm{End}(G)$, we have the following *Fundamental Relations* for an indecomposable idempotent π and its image π' under the anti-isomorphism ω:

$$\mathrm{Hom}(\pi(G), G) \cong \mathrm{Hom}(H, \pi'(H)) \quad \text{and} \quad \mathrm{Hom}(G, \pi(G)) \cong \mathrm{Hom}(\pi'(H), H).$$

The main thrust of the rest of our approach to proving Corner's result will be in applying these Fundamental Relations in a systematic way.

Before proceeding to the next section where we introduce a series of reductions, we give four useful consequences of the Fundamental Relations relating to direct sums of a fixed indecomposable module.

Proposition 2.7 *If* $G = \bigoplus_\lambda Q$ *and* G *is opposed to* H*, then* $G \cong H$ *and* λ *is finite.*

Proof By $(\bullet X)$, $\mathrm{End}(G)$ is divisible and torsion-free and so it follows from $(\bullet Y)$ that H must be torsion-free and divisible, so that $H = \bigoplus_\mu Q$ for some cardinal μ. It follows from the Fundamental Relations that $\prod_\lambda Q \cong \mathrm{Hom}(G, Q) \cong \mathrm{Hom}(Q, H) \cong \bigoplus_\mu Q$ and also $\bigoplus_\lambda Q \cong \mathrm{Hom}(Q, G) \cong \mathrm{Hom}(H, Q) \cong \prod_\mu Q$. It follows immediately from Proposition 2.5 (i) that $\lambda = \mu$ is finite and $G \cong H$. □

Proposition 2.8 *Suppose* $G = \bigoplus_\lambda \mathbb{Z}(p^\infty)$ *and* $\mathrm{End}(G) \approx \mathrm{End}(H)$ *for some* R*-module* H*. Then either* $H = \bigoplus_\lambda \mathbb{Z}(p^\infty)$ *or* $H = \bigoplus_\lambda R$ *and in both cases* λ *must be finite.*

Proof G has a projection π onto a summand isomorphic to $\mathbb{Z}(p^\infty)$ and this gives rise to a summand $\pi'(H)$ of H which may be isomorphic to either $\mathbb{Z}(p^\infty)$ or to R. From the Fundamental Relations we have two scenarios to consider:

Case (a): $\mathrm{Hom}(\mathbb{Z}(p^\infty), G) \cong \mathrm{Hom}(H, \mathbb{Z}(p^\infty))$ and $\mathrm{Hom}(G, \mathbb{Z}(p^\infty)) \cong \mathrm{Hom}(\mathbb{Z}(p^\infty), H)$.
Case (b): $\mathrm{Hom}(\mathbb{Z}(p^\infty), G) \cong \mathrm{Hom}(H, R)$ and $\mathrm{Hom}(G, \mathbb{Z}(p^\infty)) \cong \mathrm{Hom}(R, H)$.

In Case (a) we have by ($\bullet$ W) that $\mathrm{Hom}(H, \mathbb{Z}(p^\infty)) \cong \mathrm{Hom}(\mathbb{Z}(p^\infty), \bigoplus_\lambda \mathbb{Z}(p^\infty)) \cong \widehat{\bigoplus_\lambda R}$; in particular $\mathrm{Hom}(H, \mathbb{Z}(p^\infty))$ is torsion-free and reduced. Now if H is not torsion divisible, then it has a summand isomorphic to one of $\mathbb{Z}(p^n)$ (for some n), R or Q. The first of these possibilities would yield a torsion direct summand of $\mathrm{Hom}(H, \mathbb{Z}(p^\infty))$ and the third would yield a summand isomorphic to Q since, as noted in the Proof of Lemma 2.4, $\mathrm{Hom}(Q, \mathbb{Z}(p^\infty)) \cong Q$, hence neither can occur. If H has a summand isomorphic to R, then $\mathrm{Hom}(H, \mathbb{Z}(p^\infty))$ would have a summand isomorphic to $\mathbb{Z}(p^\infty)$ which is also impossible. Hence H is torsion divisible and of the form $H = \bigoplus_\mu \mathbb{Z}(p^\infty)$ for some cardinal μ. The Fundamental Relations in this case then reduce to $\prod_\lambda R \cong \widehat{\bigoplus_\mu R}$ and $\widehat{\bigoplus_\lambda R} \cong \prod_\mu R$. It follows immediately from Proposition 2.5 that $\lambda = \mu$ is finite.

In Case (b) $H \cong \mathrm{Hom}(G, \mathbb{Z}(p^\infty)) = \prod_\lambda R$ and $\mathrm{Hom}(\mathbb{Z}(p^\infty), G) = \widehat{\bigoplus_\lambda R} \cong \mathrm{Hom}(H, R)$. Thus, we have by Lemma 2.6 above that $\widehat{\bigoplus_\lambda R} \cong \prod_{2^\lambda} R$ if λ is infinite; if λ is finite, the latter term is just the product of λ copies of R. However, λ infinite is impossible since it would then follow from Proposition 2.5 (ii) that $\lambda = 2^{2^\lambda}$. Thus, we conclude that λ is finite and $H = \bigoplus_\lambda R$. □

Proposition 2.9 *If* G *is a reduced torsion-free* R*-module and* $\mathrm{End}(G) \approx \mathrm{End}(H)$ *for some* R*-module* H*, then* G *is free of finite rank* r *and either* $H \cong G$ *or* $H \cong \bigoplus_r \mathbb{Z}(p^\infty)$.

Proof Since G is reduced torsion-free, it has a summand R and this gives rise to a summand of H which can be either R or $\mathbb{Z}(p^\infty)$. The Fundamental Relations then give either (i) $\mathrm{Hom}(R, G) \cong \mathrm{Hom}(H, R)$ and $\mathrm{Hom}(G, R) \cong \mathrm{Hom}(R, H)$ or (ii) $\mathrm{Hom}(R, G) \cong \mathrm{Hom}(H, \mathbb{Z}(p^\infty))$ and $\mathrm{Hom}(G, R) \cong \mathrm{Hom}(\mathbb{Z}(p^\infty), H)$.

In the first case this yields $G \cong \mathrm{Hom}(H, R)$ and $H \cong \mathrm{Hom}(G, R)$. However, since G is torsion-free reduced, it has a free basic submodule B, of rank r say and it follows from Lemma 2.3 that $H \cong \mathrm{Hom}(G, R) = \mathrm{Hom}(B, R) \cong \prod_r R$. If r is infinite, substituting for H will give that $G \cong \mathrm{Hom}(H, R) \cong \prod_{2^r} R$, the last equality coming from Proposition 2.6. Substituting now for G we get $H \cong \mathrm{Hom}(G, R) \cong \prod_{2^{2^r}} R$ and a simple calculation of cardinalities shows this cannot hold. Hence r is finite and $B = G \cong H$.

In the second case we see that $G \cong \mathrm{Hom}(H, \mathbb{Z}(p^\infty))$ is torsion-free and reduced and so the argument in the proof of Case (b) of Proposition 2.8 gives us that H is torsion divisible, say $H = \bigoplus_\mu \mathbb{Z}(p^\infty)$ for some μ. As $\mathrm{End}(G) \approx \mathrm{End}(H)$ and $H = \bigoplus_\mu \mathbb{Z}(p^\infty)$, it now follows from Proposition 2.8 that G is either free of rank μ or torsion divisible of rank μ; in both cases μ is finite.

This completes the proof of the proposition. □

Proposition 2.10 *Suppose that G is opposed to H and $G = \bigoplus_\lambda \mathbb{Z}(p^n)$ for some $\lambda \geq 1$ and n a fixed positive integer. Then $G \cong H$ and λ is finite.*

Proof Since $p^n G = 0$, $p^n\mathrm{End}(G) = 0$ also and the anti-isomorphism between $\mathrm{End}(G)$ and $\mathrm{End}(H)$ implies that $p^n\mathrm{End}(H) = 0$; in particular, $p^n 1_H = 0$ and thus, $p^n H = 0$. Moreover, H cannot have a summand $\mathbb{Z}(p^k)$ for any $k < n$ since it would follow from Proposition 2.1 that G has then a summand opposed to $\mathbb{Z}(p^k)$—impossible since the only R-module oppose to $\mathbb{Z}(p^k)$ is $\mathbb{Z}(p^k)$ itself. So $H = \bigoplus_\mu \mathbb{Z}(p^n)$ for some $\mu \geq 1$. By the Fundamental Relations, $\mathrm{Hom}(G, \mathbb{Z}(p^n)) \cong \mathrm{Hom}(\mathbb{Z}(p^n), H)$ and $\mathrm{Hom}(\mathbb{Z}(p^n), G) \cong \mathrm{Hom}(H, \mathbb{Z}(p^n))$. Thus, we have $G = G[p^n] \cong \mathrm{Hom}(G, \mathbb{Z}(p^n)) \cong \mathrm{Hom}(\mathbb{Z}(p^n), H)$ and $H = H[p^n] \cong \mathrm{Hom}(\mathbb{Z}(p^n), H) \cong \mathrm{Hom}(G, \mathbb{Z}(p^n)) \cong \prod_\lambda \mathbb{Z}(p^n)$. So

$$H \cong \begin{cases} \bigoplus_{\lambda} \mathbb{Z}(p^n) & : \quad \lambda \text{ finite}, \\ \bigoplus_{2^\lambda} \mathbb{Z}(p^n) & : \quad \lambda \text{ infinite}. \end{cases}$$

Thus, $\mu = \lambda$ if λ is finite; $\mu = 2^\lambda$ if λ is infinite. However, $G \cong \mathrm{Hom}(\mathbb{Z}(p^n), H)$ is then homocyclic of rank μ or 2^μ depending on whether μ is finite or infinite. It then follows that $\lambda = \mu$ if μ is finite. However, if μ is infinite, we are forced to conclude that λ is also infinite leading to the absurdity that $\mu = 2^\lambda = 2^{2^\mu}$. Thus, we must have $\lambda = \mu$ and then $G \cong H$, as required. □

3 Some reductions

For our first reduction we consider the possibility that G has a summand D which is of the form $D = \bigoplus_\lambda Q$ and $G = D \oplus G_1$, where G_1 has no summand isomorphic to Q. Then H has a summand K with $\mathrm{End}(D) \approx \mathrm{End}(K)$. Now it follows from Proposition 2.7 that $D \cong K$ and λ is finite. Furthermore, if $H = K \oplus H_1$ and H_1 has a summand isomorphic to Q, then G_1 would also have a summand isomorphic to Q since $\mathrm{End}(G_1) \approx \mathrm{End}(H_1)$—contradiction. So H_1 has no summands isomorphic to Q.

Thus, we have our first reduction:

(I) If $G = D \oplus G_1$ where $0 \neq D$ is torsion-free divisible and G_1 has no summands isomorphic to Q, then if $\mathrm{End}(G) \approx \mathrm{End}(H)$, we have $H = D' \oplus H_1$ where $D' \cong D$ is a finite-dimensional Q-space, $\mathrm{End}(H_1) \approx \mathrm{End}(G_1)$ and H_1 has no summands isomorphic to Q.

For our second reduction we focus on the situation where G_1 has no summands isomorphic to Q, $G_1 = C \oplus G_2$, $0 \neq C = \bigoplus_\lambda \mathbb{Z}(p^\infty)$ and G_2 is reduced. Now if π is an idempotent corresponding to the summand C and $\mathrm{End}(G_1) \approx \mathrm{End}(H_1)$, the corresponding idempotent π' gives rise to a summand $0 \neq D$ of H_1, say $H_1 = D \oplus H_2$, and the endomorphism rings of corresponding summands are anti-isomorphic: $\mathrm{End}(C) \approx \mathrm{End}(D)$, $\mathrm{End}(G_2) \approx \mathrm{End}(H_2)$. Moreover, as $\mathrm{Hom}(C, G_2) = 0$ it follows from Lemma 2.2 that $\mathrm{Hom}(H_2, D) = 0$. Since $0 \neq C = \bigoplus_\lambda \mathbb{Z}(p^\infty)$, it follows from Proposition 2.8 that either $D = \bigoplus_\lambda \mathbb{Z}(p^\infty)$ or $D = \bigoplus_\lambda R$; in both cases λ must be finite. The first option is impossible since $\mathrm{Hom}(X, \mathbb{Z}(p^\infty)) \neq 0$ for all nonzero R-modules X. Hence $D = \bigoplus_\lambda R$ where λ is finite and $\mathrm{End}(H_2) \approx \mathrm{End}(G_2)$. Note that H_2 cannot have a summand isomorphic to R: if it did, then $0 = \mathrm{Hom}(H_2, D)$ would have a summand $\mathrm{Hom}(R, D) \cong D$, contrary to $0 \neq D$. Thus, we have our second reduction:

(II) If $G = \bigoplus_r Q \oplus \bigoplus_t \mathbb{Z}(p^\infty) \oplus G_2$, where $0 \neq r, t$, G_2 is reduced and $\mathrm{End}(G) \approx \mathrm{End}(H)$, then $H = \bigoplus_r Q \oplus \bigoplus_t R \oplus H_2$, r, t are finite, H_2 has no summand isomorphic to R or Q and H_2 is opposed to G_2.

We want to make one further reduction to enable us to reduce the problem to its core case. In the situation above, the module G_2 may have a summand isomorphic to R. In order to handle this situation we need to develop a further basic result.

Proposition 3.1 *Suppose that X, Y are R-modules with* $\operatorname{End}(X) \approxeq \operatorname{End}(Y)$ *and X is reduced and has a summand isomorphic to R. Then either (i) $Y \cong X$ and both are free of finite rank or (ii) $X = \bigoplus_\lambda R \oplus X_1$, $Y = \bigoplus_\lambda \mathbb{Z}(p^\infty) \oplus Y_1$ where λ is finite and neither X_1 nor Y_1 have a summand isomorphic to R.*

Proof Since X is opposed to Y and X has a summand isomorphic to R, Y has a summand isomorphic to either R or $\mathbb{Z}(p^\infty)$. The Fundamental Relations then present two possibilities:

(i) $\operatorname{Hom}(R, X) \cong \operatorname{Hom}(Y, R)$ and $\operatorname{Hom}(X, R) \cong \operatorname{Hom}(R, Y)$,
(ii) $\operatorname{Hom}(R, X) \cong \operatorname{Hom}(Y, \mathbb{Z}(p^\infty))$ and $\operatorname{Hom}(X, R) \cong \operatorname{Hom}(\mathbb{Z}(p^\infty), Y)$.

Case (i). By Lemma 2.3 we get $\prod_\lambda R \cong \operatorname{Hom}(X, R) \cong \operatorname{Hom}(R, Y) \cong Y$ for some cardinal $\lambda \neq 0$. So Y is torsion-free reduced and it follows from Proposition 2.9 that $X \cong Y \cong \bigoplus_\lambda R$ for some finite λ; the possibility arising in Proposition 2.9 that $X \cong \bigoplus_r \mathbb{Z}(p^\infty)$ cannot occur since X is reduced.

Case (ii). Since X is reduced and X, Y are opposed, Y cannot have a summand isomorphic to Q. Suppose then that $Y = D \oplus Y_1$, where Y_1 is reduced and $D = \bigoplus_\mu \mathbb{Z}(p^\infty)$. In this case the Fundamental Relations and Lemma 2.3 yield

$$\operatorname{Hom}(\mathbb{Z}(p^\infty), Y) = \operatorname{Hom}(\mathbb{Z}(p^\infty), \bigoplus_\mu \mathbb{Z}(p^\infty)) \cong \operatorname{Hom}(X, R) \cong \prod_\lambda R \quad \text{for some } \lambda \neq 0.$$

By $(\bullet W)$ we then have $\widehat{\bigoplus_\mu R} \cong \prod_\lambda R$ and it follows by Proposition 2.5 (ii) that $\mu = \lambda$ if μ is finite; but $\mu = 2^\lambda$ if μ is infinite.

Note also that the Fundamental Relations also give

$$X \cong \operatorname{Hom}(R, X) \cong \operatorname{Hom}(Y, \mathbb{Z}(p^\infty)) = \operatorname{Hom}(D, \mathbb{Z}(p^\infty)) \oplus \operatorname{Hom}(Y_1, \mathbb{Z}(p^\infty))$$

so that $X \cong \prod_\mu R \oplus X_1$, where $X_1 \cong \operatorname{Hom}(Y_1, \mathbb{Z}(p^\infty))$.

Consider firstly the situation where μ is finite. In this case we have $X = \bigoplus_\mu R \oplus X_1$ and $Y = \bigoplus_\mu \mathbb{Z}(p^\infty) \oplus Y_1$, where Y_1 is reduced. However, Y_1 cannot have a summand isomorphic to R, for if it did, then $X_1 \cong \operatorname{Hom}(Y, \mathbb{Z}(p^\infty))$ would have a summand isomorphic to $\mathbb{Z}(p^\infty)$ contradicting the fact that X is reduced. It also follows that X_1 cannot have a summand isomorphic to R: if it did then $\operatorname{Hom}(X, R) \cong \operatorname{Hom}(\mathbb{Z}(p^\infty), Y)$ would have a free summand of rank greater than μ which is impossible since $\operatorname{Hom}(\mathbb{Z}(p^\infty), Y_1) = 0$ as Y_1 is reduced.

To establish the proposition it remains to handle the case in which μ is infinite. We claim that this case cannot occur.

Now $X \cong \prod_\mu R \oplus X_1$ and so it follows by using Lemma 2.6 that $|\operatorname{Hom}(X, R)| \geq |\operatorname{Hom}(\prod_\mu R, R)| = |\prod_{2^\mu} R|$. But $\operatorname{Hom}(X, R) = \prod_\lambda R$, so $|\operatorname{Hom}(X, R)| = (2^{\aleph_0})^\lambda = 2^\lambda = \mu$. However, $|\prod_{2^\mu} R| = (2^{\aleph_0})^{2^\mu} = 2^{2^\mu}$ leading to the contradiction that $\mu \geq 2^{2^\mu}$. So the case with μ infinite cannot occur. □

For our next reduction we consider the situation where G_2 is reduced and has a summand isomorphic to R, G_2 is opposed to H_2 but H_2 does not have a summand isomorphic to either R or Q. It follows from Proposition 3.1 above that either G_2 is free of finite rank and isomorphic to H_2—impossible in the present situation as H_2 has no free summands—or $G_2 = \bigoplus_\lambda R \oplus G_3$ for some finite λ and G_3 does not have a summand isomorphic to R. Now this decomposition of G_2 gives a corresponding decomposition of H_2 as $H_2 = A \oplus H_3$, where A is opposed to $\bigoplus_\lambda R$ and H_3 is opposed to G_3. It follows from Proposition 2.9 that either A is free of finite rank λ or $A = \bigoplus_\lambda \mathbb{Z}(p^\infty)$. As G_3 does not have a summand isomorphic to R, $\mathrm{Hom}(G_3, R) = 0$ and so $\mathrm{Hom}(G_3, \bigoplus_\lambda R) = 0$. It follows now from Lemma 2.2 that $\mathrm{Hom}(A, H_3) = 0$ so that A is clearly not free. Hence we must have $A = \bigoplus_\lambda \mathbb{Z}(p^\infty)$ and $H_2 = \bigoplus_\lambda \mathbb{Z}(p^\infty) \oplus H_3$, where H_3 is opposed to G_3. Note also that H_3 cannot have a summand isomorphic to $\mathbb{Z}(p^\infty)$ since $\mathrm{Hom}(A, H_3) = 0$; as H_2 does not have a summand isomorphic to Q, the same is true of H_3. Thus, we have that H_3 is reduced.

Thus, we have our third reduction:

(III) Assume $G = \bigoplus_r Q \oplus \bigoplus_t \mathbb{Z}(p^\infty) \oplus \bigoplus_s R \oplus G_3$, where $0 \neq r, s, t$, G_3 is reduced and has no summand isomorphic to R, and $\mathrm{End}(G) \approxeq \mathrm{End}(H)$. Then $H = \bigoplus_r Q \oplus \bigoplus_t R \oplus \bigoplus_s \mathbb{Z}(p^\infty) \oplus H_3$, r, s, t are finite, H_3 is reduced and has no summand isomorphic to R, and H_3 is opposed to G_3.

We have now reached the core of the proof of Corner's theorem. To simplify notation in the next subsection we will write G, H for G_3, H_3, respectively.

3.1 Reduced groups with no summand R

Suppose now that G is a reduced group which does not have a summand isomorphic to R.

Recall that basic submodules of R-modules exist and have the form $B = B_0 \oplus \bigoplus_{n \geq 1} B_n$, where B_0 is a free R-module and each B_n is a direct sum (possibly zero) of cyclic groups $\mathbb{Z}(p^n)$.

Proposition 3.2 *If G is a reduced R-module which does not have a summand isomorphic to R and B is a basic submodule of G, then (i) G/tG is divisible and (ii) $G/B \cong (tG/B) \oplus D$, where tG/B is torsion divisible and D is torsion-free divisible isomorphic to G/tG.*

Proof (i) Let $G/tG = D \oplus F$, where D is divisible and F is torsion-free reduced. Since G has no summand isomorphic to R, $\mathrm{Hom}(G, R) = 0$ and hence it follows that $\mathrm{Hom}(G/tG, R) = 0$. Thus, $\mathrm{Hom}(F, R) = 0$ which forces $F = 0$ since F is torsion-free and reduced.

(ii) If B is basic in G then B_0 must be zero for otherwise G would have a summand isomorphic to R. Thus, B is torsion and G/B is divisible. Furthermore, tG/B is pure in G/B since tG is pure in G and so tG/B is divisible. Thus, $G/B = tG/B \oplus D$ and since $D \cong G/tG$, it follows from part (i) that D is divisible.

□

Notice that Proposition 3.2 tells us that if G is a reduced R-module which does not have a summand isomorphic to R and B is a basic submodule of G, then B is, in fact, precisely a basic submodule of tG. Before we can exploit this situation further, we need a result on p-groups which we establish below.

Proposition 3.3 *If G is opposed to H and G, H are reduced R-modules with torsion basic submodules B, C respectively, then $B \cong C$ and both are semi-standard.*

Proof Let $B = \bigoplus_{n \geq 1} B_n$, $C = \bigoplus_{n \geq 1} C_n$, where B_n is homocyclic of exponent n and rank λ_n, C_n is homocyclic of exponent n and rank μ_n; of course, some of the λ_n, μ_n may be zero. If $\lambda_n \neq 0$, then $G = B_n \oplus G'$ and since $\mathrm{End}(G) \approxeq \mathrm{End}(H)$, we have $H = A_n \oplus H'$ for some A_n with $\mathrm{End}(B_n) \approxeq \mathrm{End}(A_n)$. It follows from Proposition 2.10 that $B_n \cong A_n$ and λ_n is finite. Hence $0 < \lambda_n = f_G(n-1) \leq f_H(n-1) = \mu_n$. Now reverse roles noting that

$\mu_n \neq 0$ and begin with a decomposition of H as $H = C_n \oplus H''$. By a similar argument we get $0 < \mu_n = f_H(n-1) \leq f_G(n-1)$ and so $\mu_n \leq \lambda_n \leq \mu_n$ for all nonzero λ_n, μ_n. Furthermore, these cardinals are finite, so the nonzero Ulm invariants of B, C are equal and all are finite, hence $B \cong C$ and both are semi-standard. □

Note that if a direct sum of cyclic groups B is semi-standard, then it can be expressed either in the form $B = \bigoplus_{i=1}^{N} \langle b_{n_i} \rangle$ where N is finite or as $B = \bigoplus_{i=1}^{\infty} \langle b_i \rangle$ and each $\langle b_i \rangle$ is cyclic of order p^{n_i} with $n_1 \leq n_2 \leq \cdots$.

Let us return now to the situation where G, H are reduced R-modules with torsion basic submodules B, C respectively. If G, H are opposed then by Proposition 3.3 $B \cong C$ and both are semi-standard. First we dispose of the situation where B is bounded. Here it follows that $B = G$ and $C = H$ since bounded pure submodules are summands. Thus, $G \cong H$ and since G is semi-standard and bounded, it is a finite p-group. This corresponds to Case (a) of (II) in the statement of Corner's theorem.

For the remaining case, B and hence, of course, C are both unbounded. Then there is a sequence of cyclic direct summands of B of increasing orders, say $\langle b_i \rangle$ is such a summand where b_i is of order n_i and $n_1 \leq n_2 \leq \cdots$. Thus, $B = \bigoplus_{i=1}^{\infty} \langle b_i \rangle$ is basic in G and so the p-adic completions of B, G coincide and G may be regarded as a pure submodule of the product $P = \prod_{i=1}^{\infty} \langle b_i \rangle$. Thus, $B \leq tG \leq tP$. Our objective is to show that the equality $tG = tP$ holds.

If $\mathbf{x} = (x_1, x_2, \ldots)$ is an arbitrary element of tP, then there is an integer k, dependent on $\mathbf{x}$, such that $o(\mathbf{x}) \leq p^{n_k}$; for a given $\mathbf{x}$ fix such a k. Then the components x_i of $\mathbf{x}$ satisfy, for some suitable integers $r_i < p^{n_k}$,

$$x_i = \begin{cases} r_i b_i & : \ 1 \leq i \leq k, \\ r_i p^{n_i - n_k} b_i & : \ k \leq i < \infty. \end{cases}$$

Now define endomorphisms α_{ij} of B by

$$\alpha_{ij}(b_j) = \begin{cases} b_i & : \ i \leq j, \\ p^{n_i - n_j} b_i & : \ j \leq i, \end{cases}$$

and extend these to endomorphisms of G by setting $\alpha_{ij}(1 - \alpha_j)(G) = 0$.

Suppose we have shown that for all i, $\mathrm{End}(G)$ *contains an endomorphism* α *such that* $\alpha_i \alpha = r_i \alpha_{ik}$, where α_i denotes the projection mapping taking an element $g \in G$ onto its i^{th} component regarding g as an element of $P = \prod_{i=1}^{\infty} \langle b_i \rangle$. Then if $g = \alpha(b_k)$, then $g \in tG$ and, furthermore,

$$\alpha_i(g) = \alpha_i \alpha(b_k) = r_i \alpha_{ik}(b_k) = \begin{cases} r_i b_i & : \ i \leq k, \\ r_i p^{n_i - n_k} b_k & : \ k \leq i. \end{cases}$$

Since the last display is precisely the value x_i, we conclude that $\mathbf{x} = g \in tG$ and so $tP = tG = \bar{B}$, the torsion-completion of B.

If $tG = tP$ then $G/tG = G/tP \leq P/tP$ and as G/tG is divisible, as noted in Proposition 3.2 above, we have $G/tP \leq D/tP$, where the latter is the maximal divisible submodule of P/tP. Thus, the requirement in Case (b) of (II) in the statement of Corner's theorem will hold.

The remainder of this section is devoted to showing that G has the appropriate endomorphism α. Not surprisingly, the key to establishing this lies in the fact that G is opposed to H.

Note that the endomorphisms α_i, α_{ij} of G satisfy the relations

$$\begin{cases} \alpha_{ii} = \alpha_i & : \ all\ i, \\ \alpha_{ij}\alpha_{jk} = \alpha_{ik} & : \ i \leq j \leq k \ or\ i \geq j \geq k, \\ \alpha_{ij}\alpha_{ji} = p^{|n_i - n_j|}\alpha_i & : \ all\ i, j, \\ \alpha_i\alpha_{ij}\alpha_j = \alpha_{ij} & : \ all\ i, j. \end{cases}$$

Let Φ denote the anti-isomorphism $\mathrm{End}(G) \overline{\approx} \mathrm{End}(H)$ and denote by β_i, β_{ji} the images $\Phi(\alpha_i), \Phi(\alpha_{ij})$; then if $\beta_i = \Phi(\alpha_i)$, the image $\beta_i(H)$ is a direct summand, $\langle c_i \rangle$ say, of H with $o(c_i) = p^{n_i}$. Furthermore, the uniqueness of opposition of modules of the form $\bigoplus_\nu \langle b_\nu \rangle$, where each b_ν has order p^{n_i} (Proposition 2.10) and ν indexes all such summands in a direct decomposition of B, means that the corresponding summand $\bigoplus_\nu \langle c_\nu \rangle$ contains all such summands of order p^{n_i} from a decomposition of H. It follows from a result of Szele [5, Theorem 33.2] that the submodule $C = \bigoplus_{i=1}^\infty \langle c_i \rangle$ is a basic submodule of H. Also if $\gamma = \Phi(\alpha_{ij})$, we have $\beta_j\gamma\beta_i = \gamma$ and it follows exactly as in Kaplansky's proof of Theorem 28 in [11] that, absorbing units if necessary, $\gamma = \beta_{ji}$. Hence the analogues of the equations displayed above hold with the α_i replaced by β_i and the α_{ij} replaced by β_{ji}.

Now the map $\beta' : C \to \langle c_k \rangle$ given by $\beta' = \bigoplus_{i=1}^\infty r_i\beta_{ki}$ extends to a map $\beta : H \to \langle c_k \rangle$ with $\beta \restriction C = \beta'$ since $\langle c_k \rangle$ is a complete R-module and C is basic in H. Direct calculation gives that $\beta\beta_i = r_i\beta_{ki}$ for each $i = 1, 2, \ldots$. Hence if we set $\alpha = \Phi^{-1}(\beta)$, we get for each i that $\alpha_i\alpha = r_i\alpha_{ik}$ so that the required mapping α exists. Thus, we have established that $tG = tP = \bar{B}$, the torsion-completion of B. A similar result holds of course for H by interchanging the roles of G and H.

Summarising, we have established Case II (b) of Corner's result Theorem 1.1.

We can obtain some additional information on the submodules we labelled G_3, H_3 when the integers r, s, t appearing in the decompositions

$$\begin{cases} G \cong \bigoplus_r Q \oplus \bigoplus_s R \oplus \bigoplus_t \mathbb{Z}(p^\infty) \oplus G_3, \\ H \cong \bigoplus_r Q \oplus \bigoplus_s \mathbb{Z}(p^\infty) \oplus \bigoplus_t R \oplus H_3, \end{cases}$$

are nonzero.

Suppose firstly that $r \neq 0$; we claim that G_3 (and hence of course H_3) is torsion. The module $X = Q \oplus G_3$ being a summand of an opposable module, is itself opposable by some module Y. Then Y decomposes as $Y = Q \oplus Z$ for some R-module Z, by the uniqueness of opposition of torsion-free divisible modules Proposition 2.7 and Z is opposed to G_3; in particular, Z is reduced, $\mathrm{Hom}(Q, Z) = 0$ and thus, $\mathrm{Hom}(G_3, Q) = 0$. However, if G_3 is not torsion, we know that G_3/tG_3 is nonzero divisible and hence $\mathrm{Hom}(G_3, Q) = \mathrm{Hom}(G_3/tG_3, Q) \neq 0$—contradiction. So in this case G_3 is necessarily torsion.

Suppose now that $s \geq 1$, then $A = \mathbb{Z}(p^\infty) \oplus G_3$ is opposed by some module W and we know that W is then of the form $\mathbb{Z}(p^\infty) \oplus Z$ or $R \oplus Z$, for some R-module Z. The first possibility cannot happen since $\mathrm{Hom}(\mathbb{Z}(p^\infty), G_3) = 0$ would imply that $\mathrm{Hom}(Z, \mathbb{Z}(p^\infty)) = 0$ and this can never occur. Thus, $W = R \oplus Z$ for some Z which is opposed to G_3. It follows from the Fundamental Relations that $\mathrm{Hom}(A, \mathbb{Z}(p^\infty)) \cong \mathrm{Hom}(R, W) = R \oplus Z$. So $R \oplus Z \cong R \oplus \mathrm{Hom}(G_3, \mathbb{Z}(p^\infty))$ and since free summands of finite rank have the cancellation property, we conclude that $Z \cong \mathrm{Hom}(G_3, \mathbb{Z}(p^\infty))$. Furthermore, since Z is opposed to G_3 it has no summand isomorphic to R or Q. Now if G_3 has an unbounded basic submodule B, then G_3/B has a summand of the form $tG_3/B = \bar{B}/B \cong \bigoplus_\lambda \mathbb{Z}(p^\infty)$ for some $\lambda \neq 0$. It follows from Lemma 2.4 that G_3 must have a summand isomorphic to R—contradiction. Hence G_3 is itself bounded and, as noted in Proposition 3.3, it is also semi-standard so that it is in fact finite. It follows then that H_3 is also finite.

The case in which $t \geq 1$ is essentially identical: one can simply interchange the roles played by G_3 and H_3 in the previous argument.

3.2 A sufficient condition

In Corner's handwritten manuscript a partial converse is stated at the end of his statement of Theorem 1.1: "*Conversely, if G and H are related as above, then* $\mathrm{End}(G) \approx \mathrm{End}(H)$ *provided in case (b) of (II) that G' and H' are fully invariant submodules of D*."No proof of the claim is provided but the key idea he intended to use is probably that which is contained in the publication [2]; we remark that the result in [2] shows that there is no possibility of strengthening Theorem 1.1 to a statement saying that G' and H' are isomorphic. Indeed, as explained in [2, Section 3], the examples contained in that work show that Kaplansky's hope of using some sort of duality to clarify the situation was unfounded.

We shall merely look at the situation that occurs in relation to condition (II) Case (b) in terms of sufficiency; the other situations are standard and derive easily from the fact that $R, \mathbb{Z}(p^\infty)$ and finite p-groups possess an anti-isomorphism. Suppose then that X is a fully invariant submodule of D obeying the conditions of (II) Case (b). It is easy to see that D corresponds to the p-adic completion of T and so any endomorphism of T extends uniquely to an endomorphism of D, which in turn restricts to an endomorphism of X since X is, by assumption, fully invariant in D. Thus, if $\theta \in \mathrm{End}(T)$, there is a unique $\phi \in \mathrm{End}(X)$ with $\phi \restriction T = \theta$. Also if $\phi \in \mathrm{End}(X)$, then ϕ restricts to a unique endomorphism of T since if $\phi \restriction T = 0$ then ϕ induces a map $X/T \to X$ and this later must be the zero map since X/T is divisible while X is reduced. It follows that in this situation we have a ring isomorphism $\mathrm{End}(X) \cong \mathrm{End}(T)$; in particular if G, H are any pair of submodules obeying the conditions of (II) Case (b), then there is a ring isomorphism $\Psi : \mathrm{End}(G) \cong \mathrm{End}(H)$.

Assume for the moment that we have an anti-isomorphism $\Phi : \mathrm{End}(T) \to \mathrm{End}(T)$, then the composition $\Psi^{-1}\Phi\Psi$ is easily seen to be an anti-isomorphism $\mathrm{End}(G) \to \mathrm{End}(H)$. Thus, to establish the sufficiency of the conditions it suffices to show T has an anti-automorphism. Since T is the maximal torsion subgroup of a direct product of cyclic p-groups of increasing order, T is a torsion-complete group and may be viewed as $T = \bar{B}$, where B is the corresponding direct sum of the cyclic groups. The existence of such an anti-automorphism is well known with a proof being given in [2, Theorem 3.2]; alternative proofs may be found in [3] or [4].

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