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Characterizing linear mappings through zero products or zero Jordan products

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Abstract

Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}$ be a $*$-$\mathcal{A}$-bimodule. We study the local properties of $*$-derivations and $*$-Jordan derivations from $\mathcal{A}$ into $\mathcal{M}$ under the following orthogonality conditions on elements in $\mathcal{A}$: $ab^* = 0$, $ab^* + b^*a = 0$ and $ab^* = b^*a = 0$. We characterize the mappings on zero product determined algebras and zero Jordan product determined algebras. Moreover, we give some applications on C^*-algebras, group algebras, matrix algebras, algebras of locally measurable operators and von Neumann algebras.

Keywords $*$-(Jordan) derivation · $*$-(Jordan) left derivation · Zero (Jordan) product determined algebra · C^*-algebra · von Neumann algebra.

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1 Introduction

Throughout this paper, let $\mathcal{A}$ be an associative algebra over the complex field $\mathbb{C}$ and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. For each a, b in $\mathcal{A}$, we define the *Jordan product* by $a \circ b = ab + ba$. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *derivation* if $\delta(ab) = a\delta(b) + \delta(a)b$ for each a, b in $\mathcal{A}$; and δ is called a *Jordan derivation* if $\delta(a \circ b) = a \circ \delta(b) + \delta(a) \circ b$ for each a, b in $\mathcal{A}$. It follows from the results in [9,20,21] that every Jordan derivation from a C^*-algebra into its Banach bimodule is a derivation.

By an *involution* on an algebra $\mathcal{A}$ we mean a mapping $*$ from $\mathcal{A}$ into itself such that

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$$(\lambda a+\mu b)^*=\bar{\lambda}a^*+\bar{\mu}b^*,\ (ab)^*=b^*a^* \text{ and } (a^*)^*=a,$$

whenever $a, b \in \mathcal{A}$, $\lambda, \mu \in \mathbb{C}$ and $\bar{\lambda}, \bar{\mu}$ denote the conjugate complex numbers. An algebra $\mathcal{A}$ equipped with an involution is called a $*$-algebra. Moreover, if $\mathcal{A}$ is a $*$-algebra, then an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a $*$-$\mathcal{A}$-bimodule if $\mathcal{M}$ is equipped with a $*$-mapping from $\mathcal{M}$ into itself such that

$$(\lambda m+\mu n)^*=\bar{\lambda}m^*+\bar{\mu}n^*,\ (am)^*=m^*a^*,\ (ma)^*=a^*m^* \text{ and } (m^*)^*=m,$$

whenever $a \in \mathcal{A}$, $m, n \in \mathcal{M}$ and $\lambda, \mu \in \mathbb{C}$. An element a in $\mathcal{A}$ is called *self-adjoint* if $a^*=a$; an element p in $\mathcal{A}$ is called an *idempotent* if $p^2=p$; and p is called a *projection* if p is both a self-adjoint element and an idempotent.

In [24], A. Kishimoto studied the $*$-derivations on a C^*-algebra, and proved that the closure of a normal $*$-derivation of a UHF algebra satisfying a special condition is a generator of a one-parameter group of $*$-automorphisms. Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}$ be a $*$-$\mathcal{A}$-bimodule. A derivation δ from $\mathcal{A}$ into $\mathcal{M}$ is called a $*$-*derivation* if $\delta(a^*)=\delta(a)^*$ for every a in $\mathcal{A}$. Obviously, every derivation δ is a linear combination of two $*$-derivations. In fact, we can define a linear mapping $\hat{\delta}$ from $\mathcal{A}$ into $\mathcal{M}$ by $\hat{\delta}(a)=\delta(a^*)^*$ for every a in $\mathcal{A}$, therefore $\delta=\delta_1+i\delta_2$, where $\delta_1=\frac{1}{2}(\delta+\hat{\delta})$ and $\delta_2=\frac{1}{2i}(\delta-\hat{\delta})$. It is easy to show that δ_1 and δ_2 are both $*$-derivations. We can define $*$-Jordan derivations similarly.

For $*$-derivations and $*$-Jordan derivations, in [3,13,17,18], the authors characterized the following two conditions on a linear mapping δ from a $*$-algebra $\mathcal{A}$ into its $*$-bimodule $\mathcal{M}$:

($\mathbb{D}_1$) $a, b \in \mathcal{A},\ ab^*=0$ implies $a\delta(b)^*+\delta(a)b^*=0$;

($\mathbb{D}_2$) $a, b \in \mathcal{A},\ ab^*=b^*a=0$ implies $a\delta(b)^*+\delta(a)b^*=\delta(b)^*a+b^*\delta(a)=0$,

where $\mathcal{A}$ is a C^*-algebra, a zero product determined algebra or a group algebra $L^1(G)$.

Let $\mathcal{J}$ be an ideal of $\mathcal{A}$. We say that $\mathcal{J}$ is a *right separating set* or *left separating set* of $\mathcal{M}$ if for every m in $\mathcal{M}$, $\mathcal{J}m=\{0\}$ implies $m=0$ or $m\mathcal{J}=\{0\}$ implies $m=0$, respectively. We denote by $\mathfrak{J}(\mathcal{A})$ the subalgebra of $\mathcal{A}$ generated algebraically by all idempotents in $\mathcal{A}$.

In Sect. 2, we suppose that $\mathcal{A}$ is a $*$-algebra and $\mathcal{M}$ is a $*$-$\mathcal{A}$-bimodule that satisfy one of the following conditions:

(1) $\mathcal{A}$ is a zero product determined Banach $*$-algebra with a bounded approximate identity and $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule;
(2) $\mathcal{A}$ is a von Neumann algebra and $\mathcal{M}=\mathcal{A}$;
(3) $\mathcal{A}$ is a unital $*$-algebra and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule with a left or right separating set $\mathcal{J}\subseteq\mathfrak{J}(\mathcal{A})$;

and we investigate whether the linear mappings from $\mathcal{A}$ into $\mathcal{M}$ satisfying condition $\mathbb{D}_1$ characterize $*$-derivations. In particular, we generalize some results from [13,17,18].

An $\mathcal{A}$-bimodule $\mathcal{M}$ is said to have *property* $\mathbb{M}$, if there is an ideal $\mathcal{J}\subseteq\mathfrak{J}(\mathcal{A})$ of $\mathcal{A}$ such that

$$\{m\in\mathcal{M} : xmx=0 \text{ for every } x\in\mathcal{J}\}=\{0\}.$$

It is clear that if $\mathcal{A}=\mathfrak{J}(\mathcal{A})$, then $\mathcal{M}$ has property $\mathbb{M}$.

For $*$-Jordan derivations, we can study the following conditions on a linear mapping δ from a $*$-algebra $\mathcal{A}$ into its $*$-$\mathcal{A}$-bimodule $\mathcal{M}$:

($\mathbb{D}_3$) $a, b \in \mathcal{A},\ a\circ b^*=0$ implies $a\circ\delta(b)^*+\delta(a)\circ b^*=0$,

($\mathbb{D}_4$) $a, b \in \mathcal{A},\ ab^*=b^*a=0$ implies $a\circ\delta(b)^*+\delta(a)\circ b^*=0$.

It is obvious that condition $\mathbb{D}_2$ or $\mathbb{D}_3$ implies condition $\mathbb{D}_4$.

In Sect. 3, we suppose that $\mathcal{A}$ is a $*$-algebra and $\mathcal{M}$ is a $*$-$\mathcal{A}$-bimodule that satisfy one of the following conditions:

(1) $\mathcal{A}$ is a unital zero Jordan product determined $*$-algebra and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule;
(2) $\mathcal{A}$ is a unital $*$-algebra and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule such that the property $\mathbb{M}$;
(3) $\mathcal{A}$ is a C^*-algebra (not necessary unital) and $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule;

and we investigate whether the linear mappings from $\mathcal{A}$ into $\mathcal{M}$ satisfying condition $\mathbb{D}_3$ or $\mathbb{D}_4$ characterize $*$-Jordan derivations. In particular, we improve some results from [13,17,18].

2 $*$-derivations on some algebras

A (Banach) algebra $\mathcal{A}$ is said to be *zero product determined* if every (continuous) bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into any (Banach) linear space $\mathcal{X}$ satisfying

$$\phi(a, b) = 0 \text{ whenever } ab = 0$$

can be written as $\phi(a, b) = T(ab)$, for some (continuous) linear mapping T from $\mathcal{A}$ into $\mathcal{X}$. In [7], M. Brešar showed that if $\mathcal{A} = \mathfrak{J}(\mathcal{A})$, then $\mathcal{A}$ is zero product determined, and in [1], the authors proved that every C^*-algebra $\mathcal{A}$ is zero product determined.

Let $\mathcal{A}$ be a Banach $*$-algebra and $\mathcal{M}$ be a Banach $*$-$\mathcal{A}$-bimodule. Denote by $\mathcal{M}^{\sharp\sharp}$ the second dual space of $\mathcal{M}$. Next, we show that $\mathcal{M}^{\sharp\sharp}$ is also a Banach $*$-$\mathcal{A}$-bimodule.

Since $\mathcal{M}$ is a Banach $*$-$\mathcal{A}$-bimodule, $\mathcal{M}^{\sharp\sharp}$ turns into a dual Banach $\mathcal{A}$-bimodule with the operation defined by

$$a \cdot m^{\sharp\sharp} = \lim_{\mu} am_{\mu} \text{ and } m^{\sharp\sharp} \cdot a = \lim_{\mu} m_{\mu} a$$

for every a in $\mathcal{A}$ and every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$, where (m_μ) is a net in $\mathcal{M}$ with $\|m_\mu\| \leq \|m^{\sharp\sharp}\|$ and $(m_\mu) \to m^{\sharp\sharp}$ in the weak*-topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$.

We define an involution $*$ in $\mathcal{M}^{\sharp\sharp}$ by

$$(m^{\sharp\sharp})^*(\rho) = \overline{m^{\sharp\sharp}(\rho^*)}, \quad \rho^*(m) = \overline{\rho(m^*)},$$

where $m^{\sharp\sharp} \in \mathcal{M}^{\sharp\sharp}$, $\rho \in \mathcal{M}^{\sharp}$ and $m \in \mathcal{M}$. Moreover, if (m_μ) is a net in $\mathcal{M}$ and $m^{\sharp\sharp}$ is an element in $\mathcal{M}^{\sharp\sharp}$ such that $m_\mu \to m^{\sharp\sharp}$ in $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$, then for every ρ in $\mathcal{M}^{\sharp}$, we have that

$$\rho(m_\mu) = m_\mu(\rho) \ \to \ m^{\sharp\sharp}(\rho).$$

It follows that

$$(m_\mu^*)(\rho) = \rho(m_\mu^*) = \overline{\rho^*(m_\mu)} \ \to \ \overline{m^{\sharp\sharp}(\rho^*)} = (m^{\sharp\sharp})^*(\rho)$$

for every ρ in $\mathcal{M}^{\sharp}$. This means that the involution $*$ in $\mathcal{M}^{\sharp\sharp}$ is continuous in $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$. Thus, we obtain

$$(a \cdot m^{\sharp\sharp})^* = (\lim_{\mu} am_\mu)^* = \lim_{\mu} m_\mu^* a^* = (m^{\sharp\sharp})^* \cdot a^*.$$

Similarly, we can show $(m^{\sharp\sharp} \cdot a)^* = a^* \cdot (m^{\sharp\sharp})^*$. This implies that $\mathcal{M}^{\sharp\sharp}$ is a Banach $*$-$\mathcal{A}$-bimodule.

If $\mathcal{A}$ is a Banach $*$-algebra, then a *bounded approximate identity* for $\mathcal{A}$ is a net $(e_i)_{i\in\Gamma}$ of self-adjoint elements in $\mathcal{A}$ such that $\lim_{i} \|ae_i - a\| = \lim_{i} \|e_i a - a\| = 0$ for every a in $\mathcal{A}$ and $\sup_{i\in\Gamma} \|e_i\| \leq k$ for some $k > 0$.

In [18], H. Ghahramani and Z. Pan proved that if $\mathcal{A}$ is a unital zero product determined $*$-algebra and a linear mapping δ from $\mathcal{A}$ into itself satisfies the condition

$$(\mathbb{D}_1)\ a, b \in \mathcal{A},\ ab^* = 0 \text{ implies } a\delta(b)^* + \delta(a)b^* = 0,$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $\mathcal{A}$, where Δ is a $*$-derivation.

For general zero product determined Banach $*$-algebras with a bounded approximate identity, the following result holds.

Theorem 2.1 *Suppose that $\mathcal{A}$ is a zero product determined Banach $*$-algebra with a bounded approximate identity, and $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule. If δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$a, b \in \mathcal{A},\ ab^* = 0 \text{ implies } a\delta(b)^* + \delta(a)b^* = 0,$$

then there exists a $*$-derivation Δ from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ and an element ξ in $\mathcal{M}^{\sharp\sharp}$ such that $\delta(a) = \Delta(a) + \xi \cdot a$ for every a in $\mathcal{A}$. Furthermore, ξ can be chosen in $\mathcal{M}$ in each of the following cases:

(1) *$\mathcal{A}$ is a unital $*$-algebra,*
(2) *$\mathcal{M}$ is a dual $*$-$\mathcal{A}$-bimodule.*

Proof Let $(e_i)_{i\in\Gamma}$ be a bounded approximate identity of $\mathcal{A}$. Since δ is continuous, the net $(\delta(e_i))_{i\in\Gamma}$ is bounded and we can assume that it converges to ξ in $\mathcal{M}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$.

Since $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule, we know that the nets $(e_i m)_{i\in\Gamma}$ and $(me_i)_{i\in\Gamma}$ converge to m with the norm topology for every m in $\mathcal{M}$. Thus, we have

$$\mathrm{Ann}_{\mathcal{M}}(\mathcal{A}) = \{m \in \mathcal{M} : amb = 0 \text{ for each } a, b \in \mathcal{A}\} = \{0\}.$$

By the hypothesis, we obtain that

$$a, b, c \in \mathcal{A},\ ab^* = b^*c = 0 \text{ implies } a\delta(b)^*c = 0.$$

It follows that

$$a, b, c \in \mathcal{A},\ ab = bc = 0 \Rightarrow c^*b^* = b^*a^* = 0 \Rightarrow c^*\delta(b)^*a^* = 0 \Rightarrow a\delta(b)c = 0. \tag{2.1}$$

By (2.1) and [1, Theorem 4.5], we know that

$$\delta(ab) = \delta(a)b + a\delta(b) - a \cdot \xi \cdot b$$

for each a, b in $\mathcal{A}$, and ξ can be chosen in $\mathcal{M}$ if $\mathcal{A}$ is a unital $*$-algebra or $\mathcal{M}$ is a dual $*$-$\mathcal{A}$-bimodule.

Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by

$$\Delta(a) = \delta(a) - \xi \cdot a$$

for every a in $\mathcal{A}$. It is easy to show that Δ is a norm-continuous derivation from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ and we only need to show that $\Delta(b^*) = \Delta(b)^*$ for every b in $\mathcal{A}$.

First we claim that $\Delta(e_i) = \delta(e_i) - \xi \cdot e_i$ converges to zero in $\mathcal{M}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$. In fact, since $(e_i)_{i\in\Gamma}$ is bounded in $\mathcal{A}$, we assume $(e_i)_{i\in\Gamma}$ converges to ζ in $\mathcal{A}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^{\sharp})$. For every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$, define

$$m^{\sharp\sharp} \cdot \zeta = \lim_i m^{\sharp\sharp} \cdot e_i.$$

Thus, $m \cdot \zeta = m$ for every m in $\mathcal{M}$. By [10, Proposition A.3.52], we know that the mapping $m^{\sharp\sharp} \mapsto m^{\sharp\sharp} \cdot \zeta$ from $\mathcal{M}^{\sharp\sharp}$ into itself is $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$-continuous, and by the $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$-denseness of $\mathcal{M}$ in $\mathcal{M}^{\sharp\sharp}$, we have

$$m^{\sharp\sharp} \cdot \zeta = m^{\sharp\sharp} \tag{2.2}$$

for every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$. Hence $\Delta(e_i) = \delta(e_i) - \xi \cdot e_i$ converges to zero in $\mathcal{M}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$.

Next we prove $\Delta(b^*) = \Delta(b)^*$ for every b in $\mathcal{A}$. By the definition of Δ, we know that $a\Delta(b)^* + \Delta(a)b^* = 0$ for each a, b in $\mathcal{A}$ with $ab^* = 0$. Define a bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ by

$$\phi(a, b) = a\Delta(b^*)^* + \Delta(a)b.$$

Thus, $ab = 0$ implies $\phi(a, b) = 0$. Since $\mathcal{A}$ is a zero product determined algebra, there exists a norm-continuous linear mapping T from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ such that

$$T(ab) = \phi(a, b) = a\Delta(b^*)^* + \Delta(a)b \tag{2.3}$$

for each a, b in $\mathcal{A}$. If $b = e_i$ in (2.3), then we obtain

$$T(ae_i) = a\Delta(e_i)^* + \Delta(a)e_i.$$

By the continuity of T and (2.2), it follows that $T(a) = \Delta(a)$ for every a in $\mathcal{A}$. Thus,

$$T(ab) = \Delta(ab) = a\Delta(b^*)^* + \Delta(a)b.$$

Since Δ is a derivation, we have $a\Delta(b^*)^* = a\Delta(b)$ and $\Delta(b^*)a^* = \Delta(b)^*a^*$. If $a = e_i$, then taking $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$-limits, by (2.2) it follows that $\Delta(b^*) = \Delta(b)^*$ for every b in $\mathcal{A}$. □

Let G be a locally compact group. The group algebra and the measure convolution algebra of G are denoted by $L^1(G)$ and $M(G)$, respectively. The convolution product is denoted by $\cdot$ and the involution is denoted by $*$. It is well known that $M(G)$ is a unital Banach $*$-algebra, and $L^1(G)$ is a closed ideal in $M(G)$ with a bounded approximate identity. By [3, Lemma 1.1], we know that $L^1(G)$ is zero product determined. By [10, Theorem 3.3.15(ii)], it follows that $M(G)$ with respect to convolution product is the dual of $C_0(G)$ as a Banach $M(G)$-bimodule.

By [27, Corollary 1.2], we know that every continuous derivation Δ from $L^1(G)$ into $M(G)$ is an inner derivation, that is, there exists μ in $M(G)$ such that $\Delta(f) = f \cdot \mu - \mu \cdot f$ for every f in $L^1(G)$. Thus, by Theorem 2.1, we can prove [17, Theorem 3.1(ii)] as follows.

Corollary 2.2 *Let G be a locally compact group. If δ is a continuous linear mapping from $L^1(G)$ into $M(G)$ such that*

$$f, g \in L^1(G),\ \ f \cdot g^* = 0 \text{ implies } f \cdot \delta(g)^* + \delta(f) \cdot g^* = 0,$$

then there are μ, ν in $M(G)$ such that

$$\delta(f) = f \cdot \mu - \nu \cdot f$$

for every f in $L^1(G)$ and $\mathrm{Re}\mu \in \mathcal{Z}(M(G))$.

Proof By Theorem 2.1, we know that there exists a $*$-derivation Δ from $L^1(G)$ into $M(G)$ and an element ξ in $M(G)$ such that $\delta(f) = \Delta(f) + \xi \cdot f$ for every f in $L^1(G)$. By [27, Corollary 1.2], it follows that there exists μ in $M(G)$ such that $\Delta(f) = f \cdot \mu - \mu \cdot f$. Since $\Delta(f^*) = \Delta(f)^*$, we have that

$$f^* \cdot \mu - \mu \cdot f^* = \mu^* \cdot f^* - f^* \cdot \mu^*$$

for every f in $L^1(G)$. By [3, Lemma 1.3(ii)], we know $\mathrm{Re}\mu = \frac{1}{2}(\mu + \mu^*) \in \mathcal{Z}(M(G))$. If $\nu = \mu - \xi$, then from the definition of Δ we have $\delta(f) = f \cdot \mu - \nu \cdot f$ for every f in $L^1(G)$. □

For a general C^*-algebra $\mathcal{A}$, in [13], B. Fadaee and H. Ghahramani proved that if δ is a continuous linear mapping from $\mathcal{A}$ into its second dual space $\mathcal{A}^{\sharp\sharp}$ such that condition $\mathbb{D}_1$ holds, then there exists a $*$-derivation Δ from $\mathcal{A}$ into $\mathcal{A}^{\sharp\sharp}$ and an element ξ in $\mathcal{A}^{\sharp\sharp}$ such that $\delta(a) = \Delta(a) + \xi a$ for every a in $\mathcal{A}$.

In [1], the authors proved that every C^*-algebra $\mathcal{A}$ is zero product determined, and it is well known that $\mathcal{A}$ has a bounded approximate identity. Thus, by Theorem 2.1, we can improve the result in [13] for any essential Banach $*$-bimodule.

Corollary 2.3 *Suppose that $\mathcal{A}$ is a C^*-algebra and $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule. If δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$a, b \in \mathcal{A},\ ab^* = 0 \textit{ implies } a\delta(b)^* + \delta(a)b^* = 0,$$

then there exists a $*$-derivation Δ from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ and an element ξ in $\mathcal{M}^{\sharp\sharp}$ such that $\delta(a) = \Delta(a) + \xi \cdot a$ for every a in $\mathcal{A}$. Furthermore, ξ can be chosen in $\mathcal{M}$ in each of the following cases:

(1) *$\mathcal{A}$ has an identity,*
(2) *$\mathcal{M}$ is a dual $*$-$\mathcal{A}$-bimodule.*

For von Neumann algebras, we have the following result.

Theorem 2.4 *Suppose that $\mathcal{A}$ is a von Neumann algebra. If δ is a linear mapping from $\mathcal{A}$ into itself such that*

$$a, b \in \mathcal{A},\ ab^* = 0 \textit{ implies } a\delta(b)^* + \delta(a)b^* = 0,$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $\mathcal{A}$, where Δ is a $*$-derivation. In particular, δ is a $*$-derivation when $\delta(1) = 0$.

Proof Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by

$$\Delta(a) = \delta(a) - \delta(1)a$$

for every a in $\mathcal{A}$. In the following we show that Δ is a $*$-derivation. It is clear that $\Delta(1) = 0$ and $ab^* = 0$ imply $a\Delta(b)^* + \Delta(a)b^* = 0$.

Case 1 Suppose that $\mathcal{A}$ is an abelian von Neumann algebra. First we show that for Δ the following holds:

$$a, b \in \mathcal{A},\ ab = 0 \text{ implies } a\Delta(b) = 0.$$

It is well known that $\mathcal{A} \cong C(X)$, where X is a compact Hausdorff space and $C(X)$ denotes the C^*-algebra of all continuous complex-valued functions on X. Thus, we have $ab = 0$ if and only if $ab^* = 0$ for each a, b in $\mathcal{A}$. Indeed, if f and g are functions in $C(X)$ corresponding to a and b, respectively, then

$$ab^* = 0 \Leftrightarrow f \cdot \bar{g} = 0 \Leftrightarrow f \cdot g = 0 \Leftrightarrow ab = 0.$$

If a and b are in $\mathcal{A}$ with $ab^* = ab = 0$, then $a\Delta(b)^* + \Delta(a)b^* = 0$. Multiplying by a on the left side of the above equation, we obtain $a^2\Delta(b)^* = 0$. If f and h are functions in $C(X)$ corresponding to a and $\Delta(b)$, respectively, then

$$0 = f^2\bar{g} = f^2 g = fg.$$

This implies that $a\Delta(b) = 0$. By [23, Theorem 3], the function Δ is continuous. By [19, Lemma 2.5] and $\Delta(1) = 0$, we obtain $\Delta(a) = \Delta(1)a = 0$ for every a in $\mathcal{A}$.

Case 2 Suppose $\mathcal{A} \cong M_n(\mathcal{B})$, where $\mathcal{B}$ is also a von Neumann algebra and $n \geq 2$. By [6,7] we know that $\mathcal{A}$ is a zero product determined algebra. Thus, by [18, Theorem 3.1] it follows that Δ is a $*$-derivation.

Case 3 Suppose that $\mathcal{A}$ is a von Neumann algebra without abelian direct summands. By the type decomposition theorem, we have

$$\mathcal{A} = \left(\sum_{n\in E} \bigoplus \mathcal{A}_n\right) \oplus \mathcal{A}_{\mathrm{I}_\infty} \oplus \mathcal{A}_{\mathrm{II}} \oplus \mathcal{A}_{\mathrm{III}},$$

where E is some set of different finite cardinal numbers and $\mathcal{A}_n$ is type $\mathrm{I_n}$ ($n \geq 2$).

By [22, Theorem 6.6.5], we know that $\mathcal{A}_n$ is $*$-isomorphic to $M_n(\mathcal{Z})$, where $\mathcal{Z}$ is the center of $\mathcal{A}_n$. Since $\mathcal{A}_{\mathrm{I}_\infty}$ is a properly infinite von Neumann algebra and $(\mathcal{A}_{\mathrm{II}} \oplus \mathcal{A}_{\mathrm{III}})$ is a continuous von Neumann algebra, by [22, Lemma 6.3.3] and [26, Theorem 6.8.3], we know that there are two equivalent projections in $(\mathcal{A}_{\mathrm{I}_\infty} \oplus \mathcal{A}_{\mathrm{II}} \oplus \mathcal{A}_{\mathrm{III}})$ with sum the unit element of $(\mathcal{A}_{\mathrm{I}_\infty} \oplus \mathcal{A}_{\mathrm{II}} \oplus \mathcal{A}_{\mathrm{III}})$. By [22, Lemmas 6.6.3 and 6.6.4], it follows that $(\mathcal{A}_{\mathrm{I}_\infty} \oplus \mathcal{A}_{\mathrm{II}} \oplus \mathcal{A}_{\mathrm{III}})$ is $*$-isomorphic to $M_2(\mathcal{B})$ for some von Neumann algebra $\mathcal{B}$.

Hence, for a general von Neumann algebra $\mathcal{A}$, we have $\mathcal{A} \cong \sum_{i=1}^{n} \bigoplus \mathcal{A}_i$ (n is a finite integer or infinite), where each $\mathcal{A}_i$ coincides with either Case 1 or Case 2. Denote the unit element of $\mathcal{A}_i$ by 1_i and the restriction of Δ in $\mathcal{A}_i$ by Δ_i. Since $1_i(1-1_i) = 0$ and $\Delta(1) = 0$, we have

$$1_i\Delta(1-1_i)^* + \Delta(1_i)(1-1_i) = 0,$$

therefore

$$-1_i\Delta(1_i)^* + \Delta(1_i) - \Delta(1_i)1_i = 0. \tag{2.4}$$

Multiplying by 1_i on the left side of (2.4) and using $1_i\Delta(1_i) = \Delta(1_i)1_i$, we obtain $1_i\Delta(1_i)^* = 0$. This implies $\Delta(1_i) = 0$. For every a in $\mathcal{A}$, we write $a = \sum_{i=1}^{n} a_i$ with a_i in $\mathcal{A}_i$. Since $a_i(1-1_i) = 0$, we have $\Delta(a_i)(1-1_i) = 0$, which means $\Delta(a_i) \in \mathcal{A}_i$. If a_i, b_i are in $\mathcal{A}_i$ with $a_ib_i^* = 0$, then

$$\Delta(a_i)b_i^* + a_i\Delta(b_i)^* = \Delta_i(a_i)b_i^* + a_i\Delta_i(b_i)^* = 0.$$

By Cases 1 and 2, we know that every Δ_i is a $*$-derivation. Thus, Δ is a $*$-derivation. □

In what follows, we characterize the linear mappings δ that satisfy condition $\mathbb{D}_1$ from a unital $*$-algebra into a unital $*$-$\mathcal{A}$-bimodule with a right or left separating set $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$.

Lemma 2.5 *([7, Theorem 4.1]) Suppose that $\mathcal{A}$ is a unital algebra and $\mathcal{X}$ is a linear space. If ϕ is a bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{X}$ such that*

$$a, b \in \mathcal{A},\ ab = 0 \text{ implies } \phi(a, b) = 0,$$

then

$$\phi(a, x) = \phi(ax, 1) \text{ and } \phi(x, a) = \phi(1, xa)$$

for every a in $\mathcal{A}$ and every x in $\mathfrak{J}(\mathcal{A})$.

Theorem 2.6 *Suppose that $\mathcal{A}$ is a unital $*$-algebra and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule with a right or left separating set $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$a, b \in \mathcal{A},\ ab^* = 0 \text{ implies } a\delta(b)^* + \delta(a)b^* = 0,$$

then $\delta(a) = \Delta(a) + \delta(1)a$ *for every* a *in* $\mathcal{A}$*, where* Δ *is a* $*$*-derivation. In particular,* δ *is a* $*$*-derivation when* $\delta(1) = 0$.

Proof Since $\mathcal{A}$ is a unital $*$-algebra and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule, we know that $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$ is a right separating set of $\mathcal{M}$ if and only if $\mathcal{J}^* = \{x^* : x \in \mathcal{J}\} \subseteq \mathfrak{J}(\mathcal{A})$ is a left separating set of $\mathcal{M}$. Thus, without loss of generality, we can assume that $\mathcal{J}$ is a left separating set of $\mathcal{A}$, otherwise, we replace $\mathcal{J}$ by $\mathcal{J}^*$.

Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by

$$\Delta(a) = \delta(a) - \delta(1)a$$

for every a in $\mathcal{A}$. In what follows, we show that Δ is a $*$-derivation.

It is clear that $\Delta(1) = 0$ and $ab^* = 0$ imply $a\Delta(b)^* + \Delta(a)b^* = 0$. Define a bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by

$$\phi(a, b) = a\Delta(b^*)^* + \Delta(a)b$$

for each a and b in $\mathcal{A}$. By the assumption, $ab = 0$ implies $\phi(a, b) = 0$.

Let a, b be in $\mathcal{A}$ and x be in $\mathcal{J}$. By Lemma 2.5, we obtain

$$\phi(x, 1) = \phi(1, x) \text{ and } \phi(a, x) = \phi(ax, 1).$$

Hence, the following two identities hold:

$$x\Delta(1)^* + \Delta(x) = \Delta(x^*)^* + \Delta(1)x \tag{2.5}$$

and

$$a\Delta(x^*)^* + \Delta(a)x = ax\Delta(1)^* + \Delta(ax). \tag{2.6}$$

By (2.5) and $\Delta(1) = 0$, we obtain $\Delta(x)^* = \Delta(x^*)$. Thus, by (2.6), this implies

$$\Delta(ax) = a\Delta(x) + \Delta(a)x.$$

Similarly to the proof of [4, Theorem 2.3], we obtain $\Delta(ab) = a\Delta(b) + \Delta(a)b$ for each a and b in $\mathcal{A}$.

It remains to show that $\Delta(a)^* = \Delta(a^*)$ holds for every a in $\mathcal{A}$. Indeed, for every a in $\mathcal{A}$ and every x in $\mathcal{J}$, we have $\Delta(ax)^* = \Delta((ax)^*)$. This implies

$$(\Delta(a)x + a\Delta(x))^* = \Delta(x^*)a^* + x^*\Delta(a^*).$$

Thus, we obtain $x^*(\Delta(a)^* - \Delta(a^*)) = 0$, hence $(\Delta(a) - \Delta(a^*)^*)x = 0$. Therefore $\Delta(a)^* = \Delta(a^*)$ for every a in $\mathcal{A}$. □

Remark 1 Let $\mathcal{A}$ be a $*$-algebra, $\mathcal{M}$ a $*$-$\mathcal{A}$-bimodule, and δ a linear mapping from $\mathcal{A}$ into $\mathcal{M}$. Similarly to condition $\mathbb{D}_1$ which we have characterized in Sect. 2 as follows:

$$(\mathbb{D}_1)\ a, b \in \mathcal{A},\ \ ab^* = 0 \text{ implies } a\delta(b)^* + \delta(a)b^* = 0,$$

we can consider condition $\mathbb{D}_1'$:

$$(\mathbb{D}_1')\ a, b \in \mathcal{A},\ \ a^*b = 0 \text{ implies } a^*\delta(b) + \delta(a)^*b = 0.$$

Through minor modifications, we can obtain the corresponding results.

Remark 2 A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *local derivation* if, for every a in $\mathcal{A}$, there exists a derivation δ_a (depending on a) from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(a) = \delta_a(a)$. It is clear that every local derivation satisfies the following condition:

$$(\mathbb{H})\ a, b, c \in \mathcal{A},\ ab = bc = 0 \text{ implies } a\delta(b)c = 0.$$

In [1], the authors proved that every continuous linear mapping from a unital C^*-algebra into its unital Banach bimodule such that condition $\mathbb{H}$ holds and $\delta(1) = 0$ is a derivation.

Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}$ a $*$-$\mathcal{A}$-bimodule. The natural way to translate condition $\mathbb{H}$ to the context of $*$-derivations is to consider the following condition:

$$(\mathbb{H}')\ a, b, c \in \mathcal{A},\ ab^* = b^*c = 0 \text{ implies } a\delta(b)^*c = 0.$$

However, conditions $\mathbb{H}'$ and $\mathbb{H}$ are equivalent. Indeed, if condition $\mathbb{H}'$ holds, then

$$a, b, c \in \mathcal{A},\ ab = bc = 0 \ \Rightarrow\ c^*b^* = b^*a^* = 0 \ \Rightarrow\ c^*\delta(b)^*a^* = 0 \ \Rightarrow\ a\delta(b)c = 0,$$

and if condition $\mathbb{H}$ holds, then

$$a, b, c \in \mathcal{A},\ ab^* = b^*c = 0 \ \Rightarrow\ c^*b = ba^* = 0 \ \Rightarrow\ c^*\delta(b)a^* = 0 \ \Rightarrow\ a\delta(b)^*c = 0.$$

This means that condition $\mathbb{H}'$ and $\delta(1) = 0$ do not imply that δ is a $*$-derivation.

3 $*$-Jordan derivations on some algebras

A (Banach) algebra $\mathcal{A}$ is said to be *zero Jordan product determined* if every (continuous) bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into any (Banach) linear space $\mathcal{X}$ satisfying

$$\phi(a, b) = 0 \text{ whenever } a \circ b = 0$$

can be written as $\phi(a, b) = T(a \circ b)$ for some (continuous) linear mapping T from $\mathcal{A}$ into $\mathcal{X}$. In [5], we showed that if $\mathcal{A}$ is a unital algebra with $\mathcal{A} = \mathfrak{J}(\mathcal{A})$, then $\mathcal{A}$ is a zero Jordan product determined algebra.

Theorem 3.1 *Suppose that $\mathcal{A}$ is a unital zero Jordan product determined $*$-algebra, and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$a, b \in \mathcal{A},\ a \circ b^* = 0 \textit{ implies } a \circ \delta(b)^* + \delta(a) \circ b^* = 0 \text{ and } \delta(1)a = a\delta(1),$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $\mathcal{A}$, where Δ is a $*$-Jordan derivation. In particular, δ is a $*$-Jordan derivation when $\delta(1) = 0$.

Proof Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by $\Delta(a) = \delta(a) - \delta(1)a$ for every a in $\mathcal{A}$. It is sufficient to show that Δ is a $*$-Jordan derivation.

It is clear that $\Delta(1) = 0$, and by $\delta(1)a = a\delta(1)$ we have

$$a, b \in \mathcal{A},\ a \circ b^* = 0 \text{ implies } a \circ \Delta(b)^* + \Delta(a) \circ b^* = 0.$$

Define a bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by

$$\phi(a, b) = a \circ \Delta(b^*)^* + \Delta(a) \circ b.$$

Thus, $a \circ b = 0$ implies $\phi(a, b) = 0$. Since $\mathcal{A}$ is a zero Jordan product determined algebra, there exists a linear mapping T from $\mathcal{A}$ into $\mathcal{M}$ such that

$$T(a \circ b) = \phi(a, b) = a \circ \Delta(b^*)^* + \Delta(a) \circ b \tag{3.1}$$

for each a, b in $\mathcal{A}$. Let $a = 1$ and $b = 1$ in (3.1). By $\Delta(1) = 0$, we obtain

$$T(a) = \Delta(a) \text{ and } T(b) = \Delta(b^*)^*.$$

It follows that $\Delta(a^*) = \Delta(a)^*$ holds for every a in $\mathcal{A}$. By (3.1),

$$T(a \circ b) = \Delta(a \circ b) = \phi(a, b) = a \circ \Delta(b) + \Delta(a) \circ b.$$

This means that Δ is a $*$-Jordan derivation. □

In [5], we proved that the matrix algebra $M_n(\mathcal{B})$ for $n \geq 2$ is zero Jordan product determined, where $\mathcal{B}$ is a unital algebra. In [16], H. Ghahramani showed that every Jordan derivation from $M_n(\mathcal{B})$ with $n \geq 2$ into its unital bimodule $\mathcal{M}$ is a derivation. Hence we have the following result.

Corollary 3.2 *Suppose that $\mathcal{B}$ is a unital $*$-algebra, $M_n(\mathcal{B})$ is a matrix algebra with $n \geq 2$, and $\mathcal{M}$ is a unital $*$-$M_n(\mathcal{B})$-bimodule. If δ is a linear mapping from $M_n(\mathcal{B})$ into $\mathcal{M}$ such that*

$$a, b \in M_n(\mathcal{B}),\ a \circ b^* = 0 \textit{ implies } a \circ \delta(b)^* + \delta(a) \circ b^* = 0 \text{ and } \delta(1)a = a\delta(1),$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $M_n(\mathcal{B})$, where Δ is a $*$-derivation. In particular, δ is a $*$-derivation when $\delta(1) = 0$.

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Suppose that $\mathcal{A}$ is a von Neumann algebra on $\mathcal{H}$ and $LS(\mathcal{A})$ is the set of all locally measurable operators affiliated with the von Neumann algebra $\mathcal{A}$.

In [28], M. Muratov and V. Chilin proved that $LS(\mathcal{A})$ is a unital $*$-algebra and $\mathcal{A} \subset LS(\mathcal{A})$. By [25, Proposition 21.20, Exercise 21.18], we know that if $\mathcal{A}$ is a von Neumann algebra without abelian direct summands, and $\mathcal{B}$ is a $*$-algebra with $\mathcal{A} \subseteq \mathcal{B} \subseteq LS(\mathcal{A})$, then $\mathcal{B} \cong \sum_{i=1}^{k} \bigoplus M_{n_i}(\mathcal{B}_i)$ (k is a finite integer or infinite), where $\mathcal{B}_i$ is a unital algebra. By Theorem 3.1, we have the following result.

Corollary 3.3 *Suppose that $\mathcal{A}$ is a von Neumann algebra without abelian direct summands, and $\mathcal{B}$ is a $*$-algebra with $\mathcal{A} \subseteq \mathcal{B} \subseteq LS(\mathcal{A})$. If δ is a linear mapping from $\mathcal{B}$ into $LS(\mathcal{A})$ such that*

$$a, b \in \mathcal{B},\ a \circ b^* = 0 \textit{ implies } a \circ \delta(b)^* + \delta(a) \circ b^* = 0 \text{ and } \delta(1)a = a\delta(1),$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $\mathcal{B}$, where Δ is a $*$-Jordan derivation. In particular, δ is a $*$-Jordan derivation when $\delta(1) = 0$.

For von Neumann algebras, by Corollary 3.2 and similarly to the proof of Theorem 2.4, we can easily obtain the following result and we omit the proof.

Corollary 3.4 *Suppose that $\mathcal{A}$ is a von Neumann algebra. If δ is a linear mapping from $\mathcal{A}$ into itself such that*

$$a, b \in \mathcal{A},\ a \circ b^* = 0 \textit{ implies } a \circ \delta(b)^* + \delta(a) \circ b^* = 0 \text{ and } \delta(1)a = a\delta(1),$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $\mathcal{A}$, where Δ is a $*$-derivation. In particular, δ is a $*$-derivation when $\delta(1) = 0$.

Lemma 3.5 *([5, Theorem 2.1]) Suppose that $\mathcal{A}$ is a unital algebra and $\mathcal{X}$ is a linear space. If ϕ is a bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{X}$ such that*

$$a, b \in \mathcal{A},\ a \circ b = 0 \textit{ implies } \phi(a, b) = 0,$$

then

$$\phi(a, x) = \frac{1}{2}\phi(ax, 1) + \frac{1}{2}\phi(xa, 1)$$

for every a in $\mathcal{A}$ and every x in $\mathfrak{J}(\mathcal{A})$.

Suppose that $\mathcal{A}$ is a unital algebra and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule satisfying

$$\{m \in \mathcal{M} : xmx = 0 \text{ for every } x \in \mathcal{J}\} = \{0\},$$

where $\mathcal{J}$ is an ideal of $\mathcal{A}$ linear generated by idempotents in $\mathcal{A}$. In [15, Theorem 4.3], H. Ghahramani studied the linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ that satisfies

$$a, b \in \mathcal{A},\ a \circ b = 0 \text{ implies } a \circ \delta(b) + \delta(a) \circ b = 0,$$

and showed that δ is a generalized Jordan derivation. In what follows, we suppose that $\mathcal{J}$ is an ideal of $\mathcal{A}$ generated algebraically by all idempotents in $\mathcal{A}$, and have the following result.

Theorem 3.6 *Suppose that $\mathcal{A}$ is a unital $*$-algebra, $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule, and $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$ is an ideal of $\mathcal{A}$ such that*

$$\{m \in \mathcal{M} : xmx = 0 \text{ for every } x \in \mathcal{J}\} = \{0\}.$$

If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that

$$a, b \in \mathcal{A},\ a \circ b^* = 0 \text{ implies } a \circ \delta(b)^* + \delta(a) \circ b^* = 0 \text{ and } \delta(1)a = a\delta(1),$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $\mathcal{A}$, where Δ is a $*$-Jordan derivation. In particular, δ is a $*$-Jordan derivation when $\delta(1) = 0$.

Proof Let $\widehat{\mathcal{J}}$ be an algebra generated algebraically by $\mathcal{J}$ and $\mathcal{J}^*$. Since $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$ is an ideal of $\mathcal{A}$, it is easy to show that $\widehat{\mathcal{J}} \subseteq \mathfrak{J}(\mathcal{A})$ is also an ideal of $\mathcal{A}$, and also

$$\{m \in \mathcal{M} : xmx = 0 \text{ for every } x \in \widehat{\mathcal{J}}\} = \{0\}.$$

Thus, without loss of generality, we can assume that $\mathcal{J}$ is a self-adjoint ideal of $\mathcal{A}$, otherwise we may replace $\mathcal{J}$ by $\widehat{\mathcal{J}}$.

Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by

$$\Delta(a) = \delta(a) - \delta(1)a$$

for every a in $\mathcal{A}$. Next, we show that Δ is a $*$-derivation.

It is clear that $\Delta(1) = 0$ holds and, by $\delta(1)a = a\delta(1)$, the equation $a \circ b^* = 0$ implies $a \circ \Delta(b)^* + \Delta(a) \circ b^* = 0$.

Define a bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by

$$\phi(a, b) = a \circ \Delta(b^*)^* + \Delta(a) \circ b$$

for each a and b in $\mathcal{A}$. By the assumption, $a \circ b = 0$ implies $\phi(a, b) = 0$.

Let a, b be in $\mathcal{A}$ and x be in $\mathcal{J}$. By Lemma 3.5, we obtain

$$\phi(x, 1) = \phi(1, x),$$

hence

$$x \circ \Delta(1)^* + \Delta(x) \circ 1 = 1 \circ \Delta(x^*)^* + \Delta(1) \circ x. \tag{3.2}$$

By (3.2) and $\Delta(1)=0$, we know that $\Delta(x)^*=\Delta(x^*)$. Again by Lemma 3.5, it follows that

$$a\circ\Delta(x^*)^*+\Delta(a)\circ x=\frac{1}{2}[\Delta(ax)\circ 1+\Delta(xa)\circ 1]. \tag{3.3}$$

By (3.3) and $\Delta(x)^*=\Delta(x^*)$, it is easy to show that

$$\Delta(a\circ x)=a\circ\Delta(x)+\Delta(a)\circ x. \tag{3.4}$$

Next, we prove that Δ is a Jordan derivation.

Define $\{a,m,b\}=amb+bma$ and $\{a,b,m\}=\{m,b,a\}=abm+mba$ for each a,b in $\mathcal{A}$ and every m in $\mathcal{M}$. Let a be in $\mathcal{A}$ and x,y be in $\mathcal{M}$.

By the technique of the proof of [15, Theorem 4.3] and (3.4), we obtain the following two identities:

$$\Delta\{x,a,y\}=\{\Delta(x),a,y\}+\{x,\Delta(a),y\}+\{x,a,\Delta(y)\}, \tag{3.5}$$

and

$$\Delta\{x,a^2,y\}=\{\Delta(x),a^2,y\}+\{x,a\circ\Delta(a),y\}+\{x,a^2,\Delta(y)\}. \tag{3.6}$$

On the other hand, by (3.5),

$$\Delta\{x,a^2,x\}=\{\Delta(x),a^2,x\}+\{x,\Delta(a^2),x\}+\{x,a^2,\Delta(x)\}. \tag{3.7}$$

By comparing (3.6) and (3.7), it follows that $\{x,\Delta(a^2),x\}=\{x,a\circ\Delta(a),x\}$ holds. That is, $x(\Delta(a^2)-a\circ\Delta(a))x=0$. By the assumption, this implies that $\Delta(a^2)-a\circ\Delta(a)=0$ is true for every a in $\mathcal{A}$.

It remains to show that $\Delta(a)^*=\Delta(a^*)$ holds for every a in $\mathcal{A}$. Indeed, for every a in $\mathcal{A}$ and every x in $\mathcal{J}$, we have $\Delta(xax)^*=\Delta((xax)^*)$. Since Δ is a Jordan derivation, this implies

$$(\Delta(x)ax+x\Delta(a)x+xa\Delta(x))^*=\Delta(x^*)a^*x^*+x^*\Delta(a^*)x^*+x^*a^*\Delta(x^*).$$

Thus, we can obtain $x^*(\Delta(a)^*-\Delta(a^*))x^*=0$. Since $\mathcal{J}$ is a self-adjoint ideal of $\mathcal{A}$, the equation $\Delta(a)^*=\Delta(a^*)$ follows. □

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ a Banach $*$-$\mathcal{A}$-bimodule. Denote by $\mathcal{A}^{\sharp\sharp}$ and $\mathcal{M}^{\sharp\sharp}$ the second dual space of $\mathcal{A}$ and $\mathcal{M}$, respectively. By [11, p. 26], we can define a product $\diamond$ in $\mathcal{A}^{\sharp\sharp}$ by

$$a^{\sharp\sharp}\diamond b^{\sharp\sharp}=\lim_{\lambda}\lim_{\mu}\alpha_\lambda\beta_\mu$$

for each $a^{\sharp\sharp},b^{\sharp\sharp}\in\mathcal{A}^{\sharp\sharp}$, where (α_λ) and (β_μ) are two nets in $\mathcal{A}$ with $\|\alpha_\lambda\|\le\|a^{\sharp\sharp}\|$ and $\|\beta_\mu\|\le\|b^{\sharp\sharp}\|$, such that $\alpha_\lambda\to a^{\sharp\sharp}$ and $\beta_\mu\to b^{\sharp\sharp}$ in the weak*-topology $\sigma(\mathcal{A}^{\sharp\sharp},\mathcal{A}^{\sharp})$. Moreover, we can define an involution $*$ in $\mathcal{A}^{\sharp\sharp}$ by

$$(a^{\sharp\sharp})^*(\rho)=\overline{a^{\sharp\sharp}(\rho^*)},\qquad \rho^*(a)=\overline{\rho(a^*)},$$

where $a^{\sharp\sharp}\in\mathcal{A}^{\sharp\sharp}$, $\rho\in A^{\sharp}$ and $a\in\mathcal{A}$. By [22, p. 726], we deduce that $\mathcal{A}^{\sharp\sharp}$ is a von Neumann algebra with the product $\diamond$ and the involution $*$.

Since $\mathcal{M}$ is a Banach $\mathcal{A}$-bimodule, $\mathcal{M}^{\sharp\sharp}$ turns into a dual Banach $(\mathcal{A}^{\sharp\sharp},\diamond)$-bimodule with the operation defined by

$$a^{\sharp\sharp}\cdot m^{\sharp\sharp}=\lim_{\lambda}\lim_{\mu}a_\lambda m_\mu \text{ and } m^{\sharp\sharp}\cdot a^{\sharp\sharp}=\lim_{\mu}\lim_{\lambda}m_\mu a_\lambda$$

for every $a^{\sharp\sharp}$ in $\mathcal{A}^{\sharp\sharp}$ and every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$, where (a_λ) is a net in $\mathcal{A}$ with $\|a_\lambda\| \leq \|a^{\sharp\sharp}\|$ and $(a_\lambda) \to a^{\sharp\sharp}$ in $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^\sharp)$, (m_μ) is a net in $\mathcal{M}$ with $\|m_\mu\| \leq \|m^{\sharp\sharp}\|$ and $(m_\mu) \to m^{\sharp\sharp}$ in $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^\sharp)$.

We remarked in the discussion preceding Theorem 2.1 that $\mathcal{M}^{\sharp\sharp}$ has an involution $*$ and it is continuous in $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^\sharp)$. By [1, p. 553], we know that every continuous bilinear map φ from $\mathcal{A} \times \mathcal{M}$ into $\mathcal{M}$ is Arens regular, which means that

$$\lim_\lambda \lim_\mu \varphi(a_\lambda, m_\mu) = \lim_\mu \lim_\lambda \varphi(a_\lambda, m_\mu)$$

holds for every $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^\sharp)$-convergent net (a_λ) in $\mathcal{A}$ and every $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^\sharp)$-convergent net (m_μ) in $\mathcal{M}$. Thus, we obtain

$$(a^{\sharp\sharp} \cdot m^{\sharp\sharp})^* = (\lim_\lambda \lim_\mu a_\lambda m_\mu)^* = \lim_\lambda \lim_\mu m_\mu^* a_\lambda^* = \lim_\mu \lim_\lambda m_\mu^* a_\lambda^* = (m^{\sharp\sharp})^* \cdot (a^{\sharp\sharp})^*,$$

where (a_λ) is a net in $\mathcal{A}$ with $(a_\lambda) \to a^{\sharp\sharp}$ in $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^\sharp)$ and (m_μ) is a net in $\mathcal{M}$ with $(m_\mu) \to m^{\sharp\sharp}$ in $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^\sharp)$. Similarly, we can show $(m^{\sharp\sharp} \cdot a^{\sharp\sharp})^* = (a^{\sharp\sharp})^* \cdot (m^{\sharp\sharp})^*$. This implies that $\mathcal{M}^{\sharp\sharp}$ is a Banach $*$-$\mathcal{A}^{\sharp\sharp}$-bimodule.

A projection p in $\mathcal{A}^{\sharp\sharp}$ is called *open* if there exists an increasing net (a_α) of positive elements in $\mathcal{A}$ such that $p = \lim_\alpha a_\alpha$ in the weak*-topology of $\mathcal{A}^{\sharp\sharp}$. If p is open, then we say that the projection $1 - p$ is *closed*.

For a unital C^*-algebra, the following result holds.

Theorem 3.7 *Suppose that $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{M}$ is a unital Banach $*$-$\mathcal{A}$-bimodule. If δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(1)a = a\delta(1)$ holds for every a in $\mathcal{A}$, then the following three statements are equivalent:*

(1) $a, b \in \mathcal{A},\ a \circ b^* = 0$ implies $a \circ \delta(b)^* + \delta(a) \circ b^* = 0$;
(2) $a, b \in \mathcal{A},\ ab^* = b^*a = 0$ implies $a \circ \delta(b)^* + \delta(a) \circ b^* = 0$;
(3) $\delta(a) = \Delta(a) + \delta(1)a$ holds for every a in $\mathcal{A}$, where Δ is a $*$-derivation from $\mathcal{A}$ into $\mathcal{M}$.

Proof It is clear that (1) implies (2) and (3) implies (1). It is sufficient the prove that (2) implies (3).

Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by $\Delta(a) = \delta(a) - \delta(1)a$ for every a in $\mathcal{A}$. It is sufficient to show that Δ is a $*$-derivation. First we prove $\Delta(a^*) = \Delta(a)^*$ for every a in $\mathcal{A}$.

By assumption, we can easily to show that

$$a, b \in \mathcal{A},\ ab^* = b^*a = 0 \text{ implies } a \circ \Delta(b)^* + \Delta(a) \circ b^* = 0 \text{ and } \Delta(1) = 0.$$

Next, we verify $\Delta(b) = \Delta(b)^*$ for every self-adjoint element b in $\mathcal{A}$.

Since Δ is a norm-continuous linear mapping form $\mathcal{A}$ into $\mathcal{M}$, we know that $\Delta^{\sharp\sharp} : (\mathcal{A}^{\sharp\sharp}, \diamond) \to \mathcal{M}^{\sharp\sharp}$ is the weak*-continuous extension of Δ to the double duals of $\mathcal{A}$ and $\mathcal{M}$.

Let b be a nonzero self-adjoint element in $\mathcal{A}$, $\sigma(b) \subseteq [-\|b\|, \|b\|]$ the spectrum of b and $r(b) \in \mathcal{A}^{\sharp\sharp}$ the range projection of b.

Denote by $\mathcal{A}_b$ the C^*-subalgebra of $\mathcal{A}$ generated by b, and by $C(\sigma(b))$ the C^*-algebra of all continuous complex-valued functions on $\sigma(b)$. By Gelfand theory we know that there is an isometric $*$ isomorphism between $\mathcal{A}_b$ and $C(\sigma(b))$.

For every n in $\mathbb{N}$, let p_n be the projection in $\mathcal{A}_b^{\sharp\sharp} \subseteq \mathcal{A}^{\sharp\sharp}$ corresponding to the characteristic function $\chi_{([-\|b\|, -\frac{1}{n}] \cup [\frac{1}{n}, \|b\|]) \cap \sigma(b)}$ in $C(\sigma(b))$, and let b_n be in $\mathcal{A}_b$ such that

$$b_n p_n = p_n b_n = b_n = b_n^* \text{ and } \|b_n - b\| < \frac{1}{n}.$$

By [29, Section 1.8], we know that (p_n) converges to $r(b)$ in the strong*-topology of $\mathcal{A}^{\sharp\sharp}$, and hence in the weak*-topology.

It is well known that p_n is a closed projection in $\mathcal{A}_b^{\sharp\sharp} \subseteq \mathcal{A}^{\sharp\sharp}$ and $1 - p_n$ is an open projection in $\mathcal{A}_b^{\sharp\sharp}$. Thus, there exists an increasing net (z_λ) of positive elements in $((1-p_n)\mathcal{A}^{\sharp\sharp}(1-p_n)) \cap \mathcal{A}$ such that

$$0 \leq z_\lambda \leq 1 - p_n$$

and (z_λ) converges to $1 - p_n$ in the weak*-topology of $\mathcal{A}^{\sharp\sharp}$. Since

$$0 \leq ((1-p_n) - z_\lambda)^2 \leq (1-p_n) - z_\lambda \leq (1-p_n),$$

the net (z_λ) also converges to $1 - p_n$ in the strong*-topology of $\mathcal{A}^{\sharp\sharp}$.

By $b_n = b_n^*$ and $z_\lambda b_n = b_n z_\lambda = 0$, it follows that

$$z_\lambda \circ \Delta^{\sharp\sharp}(b_n)^* + \Delta^{\sharp\sharp}(z_\lambda) \circ b_n = 0. \tag{3.8}$$

Taking weak*-limits in (3.8) and since $\Delta^{\sharp\sharp}$ is weak*-continuous, we deduce

$$(1-p_n) \circ \Delta^{\sharp\sharp}(b_n)^* + \Delta^{\sharp\sharp}((1-p_n)) \circ b_n = 0. \tag{3.9}$$

Since (p_n) converges to $r(b)$ in the weak*-topology of $\mathcal{A}^{\sharp\sharp}$ and (b_n) converges to b in the norm-topology of $\mathcal{A}$, by (3.9), we have that

$$(1-r(b)) \circ \Delta^{\sharp\sharp}(b)^* + \Delta^{\sharp\sharp}(1-r(b)) \circ b = 0. \tag{3.10}$$

Now the range projection of every power b^m with $m \in \mathbb{N}$ coincides with the $r(b)$, and by (3.10), hence

$$(1-r(b)) \circ \Delta^{\sharp\sharp}(b^m)^* + \Delta^{\sharp\sharp}(1-r(b)) \circ b^m = 0$$

holds for every $m \in \mathbb{N}$, and by the linearity and norm continuity of the product we obtain

$$(1-r(b)) \circ \Delta^{\sharp\sharp}(z)^* + \Delta^{\sharp\sharp}(1-r(b)) \circ z = 0$$

for every $z = z^*$ in $\mathcal{A}_b$. A standard argument involving the weak*-continuity of $\Delta^{\sharp\sharp}$ gives

$$(1-r(b)) \circ \Delta^{\sharp\sharp}(r(b))^* + \Delta^{\sharp\sharp}(1-r(b)) \circ r(b) = 0. \tag{3.11}$$

By (3.11), we obtain

$$(\Delta^{\sharp\sharp}(r(b))^* + \Delta^{\sharp\sharp}(r(b)) - \Delta^{\sharp\sharp}(1)) \circ r(b) = 2\Delta^{\sharp\sharp}(r(b))^*.$$

By $\Delta(1) = 0$, the equality $\Delta^{\sharp\sharp}(1) = 0$ holds, hence

$$\Delta^{\sharp\sharp}(r(b))^* = \Delta^{\sharp\sharp}(r(b)). \tag{3.12}$$

It is clear that every characteristic function

$$p = \chi_{([-\|b\|,-\alpha]\cup[\alpha,\|b\|])\cap\sigma(b)} \tag{3.13}$$

in $C_0(\sigma(b))^{\sharp\sharp}$ with $0 < \alpha < \|b\|$ is the range projection of a function in $C(\sigma(b))$. Moreover, every projection of the form

$$q = \chi_{([-\beta,-\alpha]\cup[\alpha,\beta])\cap\sigma(b)} \tag{3.14}$$

in $C_0(\sigma(b))^{\sharp\sharp}$ with $0 < \alpha < \beta < \|b\|$ can be written as the difference of two projections of the type in (3.13).

Since $\mathcal{A}_b$ and $C(\sigma(b))$ are isometric $*$-isomorph and $\Delta^{\sharp\sharp}(r(b))^* = \Delta^{\sharp\sharp}(r(b))$ holds for the range projection of b in $\mathcal{A}^{\sharp\sharp}$, we infer $\Delta^{\sharp\sharp}(p)^* = \Delta^{\sharp\sharp}(p)$ for every projection p of the type in (3.13). It follows that $\Delta^{\sharp\sharp}(q)^* = \Delta^{\sharp\sharp}(q)$ holds for every projection q of the type in (3.14).

It is well known that b can be approximated in norm by finite linear combinations of mutually orthogonal projections q_j of the type in (3.14). Therefore, using the continuity of Δ, we obtain $\Delta(b)^* = \Delta(b)$. Thus, $\Delta(a)^* = \Delta(a)$ for every a in $\mathcal{A}$.

By the assumption, it follows that

$$a, b \in \mathcal{A},\ ab = ba = 0 \text{ implies } a \circ \Delta(b) + \Delta(a) \circ b = 0.$$

By [2, Theorem 4.1], we infer that Δ is a $*$-derivation. $\square$

Next, we consider general C^*-algebras $\mathcal{A}$. If $(e_i)_{i\in\Gamma}$ is a bounded approximate identity of $\mathcal{A}$, $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule, and δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$, then $(\delta(e_i))_{i\in\Gamma}$ is bounded and we can assume that it converges to ξ in $\mathcal{M}^{\sharp\sharp}$ in the topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$.

Theorem 3.8 *Suppose that $\mathcal{A}$ is a C^*-algebra (not necessary unital) and $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule. If δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that $\xi \cdot a = a \cdot \xi$ for every a in $\mathcal{A}$, then the following three statements are equivalent:*

(1) $a, b \in \mathcal{A},\ a \circ b^* = 0$ *implies* $a \circ \delta(b)^* + \delta(a) \circ b^* = 0$;
(2) $a, b \in \mathcal{A},\ ab^* = b^*a = 0$ *implies* $a \circ \delta(b)^* + \delta(a) \circ b^* = 0$;
(3) $\delta(a) = \Delta(a) + \xi \cdot a$ *for every a in $\mathcal{A}$, where Δ is a $*$-derivation from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$.*

Proof It is clear that (1) implies (2) and (3) implies (1). We only need to prove that (2) implies (3).

Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ by

$$\Delta(a) = \delta(a) - \xi \cdot a$$

for every a in $\mathcal{A}$. It is sufficient to show that Δ is a $*$-derivation.

By the definition of Δ and $\xi \cdot a = a \cdot \xi$ for every a in $\mathcal{A}$, we can easily to show that

$$a, b \in \mathcal{A},\ ab^* = b^*a = 0 \text{ implies } a \circ \Delta(b)^* + \Delta(a) \circ b^* = 0.$$

By [10, Proposition 2.9.16], we know that $(e_i)_{i\in\Gamma}$ converges to the identity 1 in $\mathcal{A}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^{\sharp})$. By the proof of Theorem 2.1, we infer that $\Delta(e_i) = \delta(e_i) - e_i \cdot \xi$ converges to zero in $\mathcal{M}^{\sharp\sharp}$ in the topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$, and we obtain

$$m^{\sharp\sharp} \cdot 1 = m^{\sharp\sharp}$$

for every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$. Since $\mathcal{M}^{\sharp\sharp}$ is a Banach $*$-$\mathcal{A}^{\sharp\sharp}$-bimodule,

$$1 \cdot m^{\sharp\sharp} = m^{\sharp\sharp}$$

holds for every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$. Since Δ is a norm-continuous linear mapping form $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$, the mapping $\Delta^{\sharp\sharp} : (\mathcal{A}^{\sharp\sharp}, \diamond) \to \mathcal{M}^{\sharp\sharp\sharp\sharp}$ is the weak*-continuous extension of Δ to the double duals of $\mathcal{A}$ and $\mathcal{M}^{\sharp\sharp}$ such that $\Delta^{\sharp\sharp}(1) = 0$.

By [10, Proposition A.3.52], we know that the mapping $m^{\sharp\sharp\sharp\sharp} \mapsto m^{\sharp\sharp\sharp\sharp} \cdot 1$ from $\mathcal{M}^{\sharp\sharp\sharp\sharp}$ into itself is $\sigma(\mathcal{M}^{\sharp\sharp\sharp\sharp}, \mathcal{M}^{\sharp\sharp\sharp})$-continuous and, by the $\sigma(\mathcal{M}^{\sharp\sharp\sharp\sharp}, \mathcal{M}^{\sharp\sharp\sharp})$-denseness of $\mathcal{M}^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp\sharp\sharp}$, the equality

$$m^{\sharp\sharp\sharp\sharp} \cdot 1 = m^{\sharp\sharp\sharp\sharp}$$

holds for every $m^{\sharp\sharp\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp\sharp\sharp}$. Since $\mathcal{M}^{\sharp\sharp\sharp\sharp}$ is a Banach $*$-$\mathcal{A}^{\sharp\sharp}$-bimodule,

$$1 \cdot m^{\sharp\sharp\sharp\sharp} = m^{\sharp\sharp\sharp\sharp}$$

holds for every $m^{\sharp\sharp\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp\sharp\sharp}$.

Finally, we use the proof of Theorem 3.7 to show that Δ is a $*$-derivation from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$. □

Remark 3 In [12], A. Essaleh and A. Peralta investigated the concept of triple derivation on C^*-algebras. Suppose that $\mathcal{A}$ is a C^*-algebra. If a, b and c be in $\mathcal{A}$, define the *ternary product* by $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$. A linear mapping δ from $\mathcal{A}$ into itself is called a *triple derivation* if

$$\delta\{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}$$

holds for each a, b and c in $\mathcal{A}$. If z is an element in $\mathcal{A}$, then δ is called a *triple derivation at* z if

$$a, b, c \in \mathcal{A},\ \{a, b, c\} = z \text{ implies } \delta(z) = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}.$$

In [12], A. Essaleh and A. Peralta proved that every continuous linear mapping δ which is a triple derivation at zero from a unital C^*-algebra into itself with $\delta(1) = 0$ is a $*$-derivation.

On the other hand, it is easy to show that if δ is a triple derivation at zero, then

$$a, b \in \mathcal{A},\ ab^* = b^*a = 0 \text{ implies } a \circ \delta(b)^* + \delta(a) \circ b^* = 0.$$

Thus, Theorem 3.7 generalizes [12, Corollary 2.10].

Remark 4 In [8], M. Brešar and J. Vukman introduced left derivations and Jordan left derivations. A linear mapping δ from an algebra $\mathcal{A}$ into its bimodule $\mathcal{M}$ is called a *left derivation* if $\delta(ab) = a\delta(b) + b\delta(a)$ holds for each a, b in $\mathcal{A}$; and δ is called a *Jordan left derivation* if $\delta(a \circ b) = 2a\delta(b) + 2b\delta(a)$ holds for each a, b in $\mathcal{A}$.

Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}$ a $*$-$\mathcal{A}$-bimodule. A left derivation (Jordan left derivation) δ from $\mathcal{A}$ into $\mathcal{M}$ is called a $*$*-left derivation* ($*$*-Jordan left derivation*) if $\delta(a^*) = \delta(a)^*$ for every a in $\mathcal{A}$.

We also can investigate the following conditions on a linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$:

$$(\mathbb{J}_1)\ a, b \in \mathcal{A},\ ab^* = 0 \text{ implies } a\delta(b)^* + b^*\delta(a) = 0;$$
$$(\mathbb{J}_2)\ a, b \in \mathcal{A},\ a \circ b^* = 0 \text{ implies } a\delta(b)^* + b^*\delta(a) = 0;$$
$$(\mathbb{J}_3)\ a, b \in \mathcal{A},\ ab^* = b^*a = 0 \text{ implies } a\delta(b)^* + b^*\delta(a) = 0.$$

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