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The Schatten–von Neumann class associated with the Gabor–Riemann–Liouville operator

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Abstract
In this paper, we define the localization operator associated with the Riemann–Liouville operator, and show that it is not only bounded, but it is also in the Schatten–von Neumann class. We also give a trace formula when the symbol function is nonnegative.

Keywords Localization operator · Fourier transform · Trace formula · The continuous Gabor transform · Schatten class operators

Mathematics Subject Classification 42A38 · 44A35

1 Introduction

The localization operators were introduced by Daubechies in [6–8]. She highlighted the role of these operators in localizing a signal simultaneously in time and frequency; this can be seen as an uncertainty principle.

Nowadays, localization operators have found many applications in timefrequency analysis, the theory of differential equationsm and quantum mechanics. Arguing from these points of view, many works deal with them; we refer in particular to the papers of Balazs et al. [3,4] (see also [14,20]).

In [1], the authors have defined the Riemann–Liouville operator $\mathscr{R}_\alpha$, $\alpha \geq 0$, by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \frac{\alpha}{\pi}\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}\,dt\;\,ds & \text{if } \alpha > 0, \\ \frac{1}{\pi}\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)\frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha = 0, \end{cases}$$

where f is any continuous function on $\mathbb{R}^2$, even with respect to the first variable. The dual operator ${}^t\mathscr{R}_\alpha$ is defined by

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$$
{}^t\mathscr{R}_\alpha(g)(r,x) = \begin{cases} \frac{1}{2^{\alpha-\frac{1}{2}}\sqrt{\pi}\,\Gamma(\alpha+1)} \int_r^{+\infty}\int_{-\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} g(u,x+v)(u^2-v^2-r^2)^{\alpha-1} u\,du\,dv, & \text{if } \alpha>0,\\ \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g\big(\sqrt{r^2+(x-y)^2},y\big)dy, & \text{if } \alpha=0, \end{cases}
$$

where g is any continuous function on $\mathbb{R}^2$, even with respect to the first variable and with compact support. In particular, for $\alpha = 0$ and by a change of variables, we get

$$
\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f(r\cos\theta, x + r\sin\theta)\,d\theta.
$$

This means that $\mathscr{R}_0(f)(r,x)$ is the mean value of f on the circle centered at $(0,x)$ and with radius r. The mean operator $\mathscr{R}_0$ and its dual ${}^t\mathscr{R}_0$ play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [13,16] or in the linearized inverse scattering problem in acoustics [9]. The operators $\mathscr{R}_\alpha$ and its dual ${}^t\mathscr{R}_\alpha$ have the same properties as the Radon transform [15]; for this reason, $\mathscr{R}_\alpha$ is called sometimes the generalized Radon transform.

Motivated by their impact in real-life signals, we define in this paper the localization operators by means of the most used time-frequency representation that is the continuous Gabor transform connected with the Riemann–Liouville operator, which was introduced in [2,5,10–12]. Other names of the continuous Gabor transform frequently used in the literature are Weyl-Heisenberg transform, short time Fourier transform and windowed Fourier transform.

In this paper, building signs on the idea of [21], we will define one kind of localization operator associated to the Riemann–Liouville operator, and will show that this kind of operator is not only bounded, but it is also in the Schatten–von Neumann class. We also give a trace formula when the symbol function is nonnegative.

The rest of this paper is arranged as follows. In Sect. 2, we recall some harmonic analysis results related to the Fourier and the continuous Gabor transforms associated with the Riemann–Liouville operator. In Sect. 3, we define the localization operators for the continuous Gabor transform associated with the Riemann–Liouville operator and we study the boundedness and compactness properties of the localization operators for the continuous Gabor transform; we show that they are in the Schatten–von Neumann class. We also give a trace formula.

2 Preliminaries

2.1 Harmonic analysis results related to the Fourier transform associated with the Riemann–Liouville operator

In this part, we recall some harmonic analysis results related to the Riemann–Liouville operator (see [1]).

The Lebesgue space with respect to the measure $d\nu_\alpha$ defined on $[0,+\infty[\times\mathbb{R}$ by

$$
d\nu_\alpha(r,x) = \frac{r^{2\alpha+1}\,dr\,dx}{2^{\alpha+\frac{1}{2}}\,\Gamma(\alpha+1)\,\sqrt{\pi}},
$$

equipped with the L^p-norm $\|.\|_{p,\nu_\alpha}$ is denoted by $L^p(d\nu_\alpha)$.

For every $f \in L^1(d\nu_\alpha)$, the Fourier transform of f is defined by

$$\mathscr{F}_\alpha(f)(\lambda_0,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x) j_\alpha\big(r\sqrt{\lambda_0^2+\lambda^2}\big)\, e^{-i\lambda x} d\nu_\alpha(r,x), \quad \forall(\lambda_0,\lambda)\in\Upsilon,$$

where j_α is the modified Bessel function defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\Big(\frac{z}{2}\Big)^{2k}, \quad \forall z\in\mathbb{C},$$

and Υ is the set given by

$$\Upsilon = \mathbb{R}^2 \cup \big\{(i\lambda_0,\lambda);\ (\lambda_0,\lambda)\in\mathbb{R}^2;\ |\lambda_0|\le|\lambda|\big\}. \tag{2.1}$$

In the following, we give some properties of this transform (see [18,19]):

- For every $f \in L^1(d\nu_\alpha)$, the function $\mathscr{F}_\alpha(f)$ is bounded on the set Υ and for every $(\lambda_0,\lambda)\in\Upsilon$, $\big|\mathscr{F}_\alpha(f)(\lambda_0,\lambda)\big| \le \|f\|_{1,\nu_\alpha}$.
- For every $f \in L^1(d\nu_\alpha)$ and $(r,x)\in[0,+\infty[\times\mathbb{R}$, the function $\tau_{(r,x)}(f)$ belongs to $L^1(d\nu_\alpha)$ and we have

$$\mathscr{F}_\alpha\big(\tau_{(r,x)}(f)\big)(\lambda_0,\lambda) = \overline{j_\alpha\big(r\sqrt{\lambda_0^2+\lambda^2}\big)\, e^{-i\lambda x}}\mathscr{F}_\alpha(f)(\lambda_0,\lambda), \quad \forall(\lambda_0,\lambda)\in\Upsilon. \tag{2.2}$$

- For all $f, g \in L^1(d\nu_\alpha)$, the function $f * g$ belongs to $L^1(d\nu_\alpha)$ and

$$\mathscr{F}_\alpha(f*g)(\lambda_0,\lambda) = \mathscr{F}_\alpha(f)(\lambda_0,\lambda)\mathscr{F}_\alpha(g)(\lambda_0,\lambda), \quad \forall(\lambda_0,\lambda)\in\Upsilon. \tag{2.3}$$

- For every $f \in L^1(d\nu_\alpha)$ we have $\mathscr{F}_\alpha(f)(\lambda_0,\lambda) = \widetilde{\mathscr{F}}_\alpha(f)\big(\sqrt{\lambda_0^2+\lambda^2},\lambda\big)$, where $\widetilde{\mathscr{F}}_\alpha$ is the so-called Fourier-Bessel transform defined on $L^1(d\nu_\alpha)$ by

$$\widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda) = \int_0^{\infty}\int_{\mathbb{R}} f(r,x)\ j_\alpha(r\mu)\ e^{-i\lambda x} d\nu_\alpha(r,x), \quad \forall(\mu,\lambda)\in[0,+\infty[\times\mathbb{R}. \tag{2.4}$$

- (Inversion formula) For every $f \in L^1(d\nu_\alpha)$, such that $\widetilde{\mathscr{F}}_\alpha(f)$ belongs to $L^1(d\nu_\alpha)$ and for almost every $(r,x)\in[0,+\infty[\times\mathbb{R}$, we have

$$f(r,x) = \int_0^{\infty}\int_{\mathbb{R}} \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)\ j_\alpha(r\mu)\ e^{i\lambda x}\ d\nu_\alpha(\mu,\lambda) = \widetilde{\mathscr{F}}_\alpha\big(\widetilde{\mathscr{F}}_\alpha(f)\big)(r,-x). \tag{2.5}$$

- (Plancherel theorem) The transform $\widetilde{\mathscr{F}}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself and for every $f \in L^2(d\nu_\alpha)$,

$$\widetilde{\mathscr{F}}_\alpha^{-1}(f) = \widetilde{\mathscr{F}}_\alpha(\check{f}) = \widetilde{\mathscr{F}}_\alpha\check{\ }(f). \tag{2.6}$$

- For every $f \in L^1(d\nu_\alpha)$, $g \in L^p(d\nu_\alpha)$, $p\in\{1,2\}$, the function $f*g$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\widetilde{\mathscr{F}}_\alpha(f*g) = \widetilde{\mathscr{F}}_\alpha(f)\ \widetilde{\mathscr{F}}_\alpha(g). \tag{2.7}$$

2.2 The continuous Gabor transform associated with the Riemann–Liouville operator

Following [12], for every $g \in L^2(d\nu_\alpha)$, the modulation of g by $(\xi_1, \xi_2) \in [0, +\infty[\times\mathbb{R}$ is defined by

$$M_{(\xi_1,\xi_2)}(g) = \widetilde{\mathscr{F}}_\alpha\Big(\sqrt{\tau_{(\xi_1,\xi_2)}\big(|\widetilde{\mathscr{F}}_\alpha(g)|^2\big)}\Big) = g_{(\xi_1,\xi_2)}. \tag{2.8}$$

Then

$$\|M_{(\xi_1,\xi_2)}(g)\|_{2,\nu_\alpha} = \|g\|_{2,\nu_\alpha}. \tag{2.9}$$

For a non-zero window function g in $L^2(d\nu_\alpha)$ and $(r,x),\ (\xi_1,\xi_2) \in [0,+\infty[\times\mathbb{R}$, we consider the function $g_{(r,x),(\xi_1,\xi_2)}$ defined by

$$g_{(r,x),(\xi_1,\xi_2)} = \tau_{(r,x)}\big(M_{(\xi_1,\xi_2)}(g)\big). \tag{2.10}$$

Therefore, for any function $f \in L^2(d\nu_\alpha)$, we define the continuous Gabor transform associated with the Riemann–Liouville operator with respect to window g by

$$\mathscr{V}_g(f)\big((r,x),(\xi_1,\xi_2)\big) = \int_0^\infty \int_{\mathbb{R}} f(s,y)\, \overline{g_{(r,x),(\xi_1,\xi_2)}(s,y)}\, d\nu_\alpha(s,y), \tag{2.11}$$

which can be also written in the form

$$\mathscr{V}_g(f)\big((r,x),(\xi_1,\xi_2)\big) = \langle f|g_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha} = f * g_{(\xi_1,\xi_2)}(r,-x). \tag{2.12}$$

Moreover, from Cauchy–Schwarz's inequality and relation (2.9), we get

$$\|\mathscr{V}_g(f)\|_{\infty,\nu_\alpha\otimes\nu_\alpha} \le \|f\|_{2,\nu_\alpha}\ \|g\|_{2,\nu_\alpha}, \tag{2.13}$$

where $\nu_\alpha \otimes \nu_\alpha$ is the product measure on $([0,+\infty[\times\mathbb{R})^2$ defined by

$$d(\nu_\alpha \otimes \nu_\alpha)\big((r,x),(s,y)\big) = d\nu_\alpha(r,x) \otimes d\nu_\alpha(s,y),$$

then $L^2(d\nu_\alpha \otimes d\nu_\alpha)$ is the Hilbert space of square integrable functions on $([0,+\infty[\times\mathbb{R})^2$ with respect to the measure $\nu_\alpha \otimes \nu_\alpha$ equipped with the inner product

$$\langle f|g\rangle_{\nu_\alpha\otimes\nu_\alpha} = \iint_{\big([0,+\infty[\times\mathbb{R}\big)^2} f\big((r,x),(s,y)\big)\, \overline{g\big((r,x),(s,y)\big)}\, d\nu_\alpha(r,x)\, d\nu_\alpha(s,y)$$

and the norm $\|f\|_{2,\nu_\alpha\otimes\nu_\alpha} = \sqrt{\langle f|f\rangle_{\nu_\alpha\otimes\nu_\alpha}}$.

The continuous Gabor transform associated with the Riemann–Liouville operator $\mathscr{R}_\alpha$ possesses the following properties (see [12]).

Let $g \in L^2(d\nu_\alpha)$ be a non-zero window function. Then the following hold.

- (Plancherel's formula for $\mathscr{V}_g$) For every $f \in L^2(d\nu_\alpha)$, we have

$$\|\mathscr{V}_g(f)\|_{2,\nu_\alpha\otimes\nu_\alpha} = \|f\|_{2,\nu_\alpha}\ \|g\|_{2,\nu_\alpha}. \tag{2.14}$$

- (Parseval's formula for $\mathscr{V}_g$) For all $f, h \in L^2(d\nu_\alpha)$, we have

$$\langle \mathscr{V}_g(f)|\mathscr{V}_g(h)\rangle_{\nu_\alpha\otimes\nu_\alpha} = \|g\|^2_{2,\nu_\alpha}\ \langle f|h\rangle_{\nu_\alpha}. \tag{2.15}$$

- For every $f \in L^2(d\nu_\alpha)$, the function $\mathscr{V}_g(f)$ belongs to $L^q(d\nu_\alpha \otimes d\nu_\alpha)$, $2 \leq q \leq \infty$ ($L^p(d\nu_\alpha \otimes d\nu_\alpha)$ $p \in [1, +\infty]$, the Lebesgue space on $([0, +\infty[\times\mathbb{R})^2$ with respect to the measure $\nu_\alpha \otimes \nu_\alpha$ equipped with the L^p-norm denoted by $\|.\|_{p,\nu_\alpha\otimes\nu_\alpha}$), with

$$\|\mathscr{V}_g(f)\|_{q,\nu_\alpha\otimes\nu_\alpha} \leq \|f\|_{2,\nu_\alpha} \, \|g\|_{2,\nu_\alpha}. \tag{2.16}$$

- (Inversion formula for $\mathscr{V}_g$) For every $f \in L^2(d\nu_\alpha)$ such that $\mathscr{V}_g(f)$ belongs to $L^1(d\nu_\alpha \otimes d\nu_\alpha)$, we have

$$f(s, y) = \frac{1}{\|g\|^2_{2,\nu_\alpha}} \int\int_{([0,+\infty[\times\mathbb{R})^2} \mathscr{V}_g(f)((r, x), (\xi_1, \xi_2)) g_{(r,x),(\xi_1,\xi_2)}(s, y) d\nu_\alpha(r, x)\, d\nu_\alpha(\xi_1, \xi_2),$$

weakly in $L^2(d\nu_\alpha)$.

- (Reproducing kernel Hilbert space) The space $\mathscr{V}_g(L^2(d\nu_\alpha))$ is a reproducing kernel Hilbert space in $L^2(d\nu_\alpha \otimes d\nu_\alpha)$ with kernel function $\mathscr{K}_g$ defined by

$$\mathscr{K}_g\big((r, x), (\xi_1, \xi_2), (s, y), (\mu, \lambda)\big) = \frac{1}{\|g\|^2_{2,\nu_\alpha}} \mathscr{V}_g\big(\tau_{(s,y)}(g_{(\mu,\lambda)})\big)\big((r, x), (\xi_1, \xi_2)\big).$$

Furthermore, the kernel $\mathscr{K}_g$ is pointwise bounded, that is

$$|\mathscr{K}_g\big((r, x), (\xi_1, \xi_2), (s, y), (\mu, \lambda)\big)| \leq 1, \quad \forall (r, x), (\xi_1, \xi_2), (s, y),\ (\mu, \lambda) \in [0, +\infty[\times\mathbb{R}.$$

Remark 2.1 For every non-zero window $g \in L^2(d\nu_\alpha)$, we denote by P_g the orthogonal projection of $L^2(d\nu_\alpha \otimes d\nu_\alpha)$ into $\mathscr{V}_g(L^2(d\nu_\alpha))$. The reproducing kernel $\mathscr{K}_g$ gives explicitly the orthogonal projection P_g, more precisely, for every $F \in L^2(d\nu_\alpha \otimes d\nu_\alpha)$,

$$P_g(F)\big((s, y), (\mu, \lambda)\big) = \langle F|\mathscr{K}_g\big((., .), (., .), (s, y), (\mu, \lambda)\big)\rangle_{\nu_\alpha\otimes\nu_\alpha}. \tag{2.17}$$

3 Localization operators for the continuous Gabor transform associated with the Riemann–Liouville operator

In this section, we will study the boundedness and the compactness of the localization operators for the continuous Gabor transform associated with the Riemann–Liouville operator. To do so, let g_1 and g_2 be two window functions in $L^2(d\nu_\alpha)$ such that $\|g_1\|_{2,\nu_\alpha} = \|g_2\|_{2,\nu_\alpha} = 1$.

Let $\mathcal{S}$ be a symbol in $L^1(d\nu_\alpha \otimes d\nu_\alpha) \cup L^\infty(d\nu_\alpha \otimes d\nu_\alpha)$. The localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ for the continuous Gabor transform associated with the Riemann–Liouville operator is defined on $L^2(d\nu_\alpha)$ by

$$\begin{aligned} &\mathcal{L}_{\mathcal{S}}^{g_1,g_2} f(s, y) \\ &= \iint_{([0,+\infty[\times\mathbb{R})^2} \mathcal{S}((r, x), (\xi_1, \xi_2)) \mathscr{V}_{g_1}(f)((r, x), \\ &(\xi_1, \xi_2))(g_2)_{(r,x),(\xi_1,\xi_2)}(s, y)\, d\nu_\alpha(r, x)\, d\nu_\alpha(\xi_1, \xi_2), \end{aligned} \tag{3.1}$$

for all $(s, y) \in [0, +\infty[\times\mathbb{R}$. Often it is more convenient to interpret the definition of $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ in a weak sense, that is, for functions $f, h \in L^2(d\nu_\alpha)$

$$\langle \mathcal{L}_{\mathcal{S}}^{g_1,g_2} f|h\rangle_{\nu_\alpha} = \langle \mathcal{S}\mathscr{V}_{g_1}(f)|\mathscr{V}_{g_2}(h)\rangle_{\nu_\alpha\otimes\nu_\alpha} = \langle \mathcal{S}|\overline{\mathscr{V}_{g_1}(f)}\mathscr{V}_{g_2}(h)\rangle_{\nu_\alpha\otimes\nu_\alpha}. \tag{3.2}$$

Let us recall the notation of the Schatten–von Neumann class S_p. The singular values $(s_k(A))_{k\geq 1}$ of a compact operator $A \in \mathfrak{B}(L^2(d\nu_\alpha))$ (the space of bounded operators A from $L^2(d\nu_\alpha)$ into $L^2(d\nu_\alpha)$) are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$. We say that the compact operator $A\colon L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ is in the Schatten–von Neumann class S_p, $1 \leq p < \infty$, if

$$\sum_k s_k(A)^p < \infty.$$

Hence S_p is equipped with the norm

$$\|A\|_{S_p} = \Big(\sum_{k=1}^{\infty} s_k(A)^p\Big)^{1/p}. \tag{3.3}$$

In particular, S_1 is the space of trace class operators. It is well known that the trace of an operator A in S_1 is defined by (see [21, Theorem 2.6])

$$\mathrm{Tr}(A) = \sum_{n=1}^{\infty} \langle A\psi_n|\psi_n\rangle_{\nu_\alpha}, \tag{3.4}$$

where $(\psi_n)_n$ is an orthonormal basis of $L^2(d\nu_\alpha)$. $\mathrm{Tr}(A)$ is independent of the choice of the orthonormal basis. In addition, if A is non-negative, then

$$\mathrm{Tr}(A) = \|A\|_{S_1}. \tag{3.5}$$

S_1 is called the trace class.

For consistency, we define $S_\infty := \mathfrak{B}\big(L^2(d\nu_\alpha)\big)$, equipped with the norm

$$\|A\|_{S_\infty} = \sup_{\|f\|_{2,\nu_\alpha}\leq 1} \|A(f)\|_{2,\nu_\alpha}. \tag{3.6}$$

3.1 Boundedness

In this section we prove that the linear operators $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\colon L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ are bounded for all symbol $\mathcal{S} \in L^p(d\nu_\alpha \otimes d\nu_\alpha)$, $1 \leq p \leq \infty$. We first tackle this problem for $\mathcal{S} \in L^1(d\nu_\alpha \otimes d\nu_\alpha)$ or $S \in L^\infty(d\nu_\alpha \otimes d\nu_\alpha)$ and then we conclude using interpolation theory.

Proposition 3.1 *Let $\mathcal{S}$ be a symbol in $L^1(d\nu_\alpha \otimes d\nu_\alpha)$. Then the localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is in S_∞ and we have*

$$\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_\infty} \leq \|\mathcal{S}\|_{1,\nu_\alpha\otimes\nu_\alpha} \tag{3.7}$$

Proof Let f, h be two functions in $L^2(d\nu_\alpha)$. Then, by the relations (2.13) and (3.2),

$$\begin{aligned}|\langle \mathcal{L}_{\mathcal{S}}^{g_1,g_2} f|h\rangle_{\nu_\alpha}| &\leq \iint_{([0,+\infty[\times\mathbb{R})^2} |\mathcal{S}((r,x),(\xi_1,\xi_2))|\,|\mathscr{V}_{g_1}(f)((r,x),(\xi_1,\xi_2))| \\ &\quad \times |\mathscr{V}_{g_2}(h)((r,x),(\xi_1,\xi_2))|\,d\nu_\alpha(r,x)\,d\nu_\alpha(\xi_1,\xi_2) \\ &\leq \|\mathscr{V}_{g_1}(f)\|_{\infty,\nu_\alpha\otimes\nu_\alpha}\|\mathscr{V}_{g_2}(h)\|_{\infty,\nu_\alpha\otimes\nu_\alpha}\|S\|_{1,\nu_\alpha\otimes\nu_\alpha} \\ &\leq \|f\|_{2,\nu_\alpha\otimes\nu_\alpha}\|h\|_{2,\nu_\alpha\otimes\nu_\alpha}\|\mathcal{S}\|_{1,\nu_\alpha\otimes\nu_\alpha},\end{aligned}$$

and the proof of the proposition is complete. □

We also have the following proposition.

Proposition 3.2 *Let S be a symbol in $L^\infty(d\nu_\alpha \otimes d\nu_\alpha)$. Then the localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is in $\mathcal{S}_\infty$ and we have*

$$\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_\infty} \leq \|\mathcal{S}\|_{\infty,\nu_\alpha\otimes\nu_\alpha}. \tag{3.8}$$

Proof Let f, h be two functions in $L^2(d\nu_\alpha)$. Then, by the relations (3.2), (2.14) and the Cauchy–Schwartz inequality,

$$\begin{aligned}
|\langle \mathcal{L}_{\mathcal{S}}^{g_1,g_2} f|h\rangle_{\nu_\alpha}| \leq & \iint_{([0,+\infty[\times\mathbb{R})^2} |\mathcal{S}((r,x),(\xi_1,\xi_2))|\,|\mathscr{V}_{g_1}(f)((r,x),(\xi_1,\xi_2))| \\
& \times |\mathscr{V}_{g_2}(h)((r,x),(\xi_1,\xi_2))|\,d\nu_\alpha(r,x)\,d\nu_\alpha(\xi_1,\xi_2) \\
& \leq \|\mathscr{V}_{g_1}(f)\|_{2,\nu_\alpha\otimes\nu_\alpha}\|\mathscr{V}_{g_2}(h)\|_{2,\nu_\alpha\otimes\nu_\alpha}\|\mathcal{S}\|_{\infty,\nu_\alpha\otimes\nu_\alpha} \\
& = \|f\|_{2,\nu_\alpha}\|h\|_{2,\nu_\alpha}\|\mathcal{S}\|_{\infty,\nu_\alpha\otimes\nu_\alpha}.
\end{aligned}$$

Thus, the proof of the proposition is complete. □

Corollary 3.3 *If $1 \leq p \leq 2$, then for any symbol S in $L^p(d\nu_\alpha \otimes d\nu_\alpha)$, there exists a unique bounded linear operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}: L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ satisfying the relation (3.2).*

Proof Let $\mathcal{S}$ be a symbol in $L^p(d\nu_\alpha\otimes d\nu_\alpha)$, $1 \leq p < \infty$. Then there exists a sequence $(\mathcal{S}_n)_{n\geq1}$ of functions in $L^1(d\nu_\alpha \otimes d\nu_\alpha) \cap L^\infty(d\nu_\alpha \otimes d\nu_\alpha)$ such that $\mathcal{S}_n \longrightarrow \mathcal{S}$ in $L^p(d\nu_\alpha \otimes d\nu_\alpha)$ as $n \longrightarrow \infty$. By Theorem 3.4

$$\|\mathcal{L}_{\mathcal{S}_m}^{g_1,g_2} - \mathcal{L}_{\mathcal{S}_n}^{g_1,g_2}\|_{S_\infty} \leq \|\mathcal{S}_m - \mathcal{S}_n\|_{p,\nu_\alpha\otimes\nu_\alpha}, \tag{3.9}$$

therefore $(\mathcal{S}_n)_{n\geq1}$ is a Cauchy sequence in S_∞. Let it converge to $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}: L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$. This limit $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is independent of the choice of $(\mathcal{S}_n)_{n\geq1}$ and we have

$$\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_\infty} = \lim_{n\longrightarrow\infty} \|\mathcal{L}_{\mathcal{S}_n}^{g_1,g_2}\|_{S_\infty} \leq \lim_{n\longrightarrow\infty} \|S_n\|_{p,\nu_\alpha\otimes\nu_\alpha} = \|S\|_{p,\nu_\alpha\otimes\nu_\alpha}. \tag{3.10}$$

Therefore, for $1 \leq p \leq 2$ and for any functions $f, h \in L^2(d\nu_\alpha)$,

$$\begin{aligned}
\langle \mathcal{L}_{\mathcal{S}}^{g_1,g_2} f|h\rangle_{\nu_\alpha} &= \lim_{n\to\infty} \langle \mathcal{L}_{\mathcal{S}_n}^{g_1,g_2} f|h\rangle_{\nu_\alpha} \\
&= \lim_{n\to\infty} \langle \mathcal{S}_n \mathscr{V}_{g_1}(f)|\mathscr{V}_{g_2}(h)\rangle_{\nu_\alpha\otimes\nu_\alpha} \\
&= \lim_{n\to\infty} \langle \mathcal{S}_n|\overline{\mathscr{V}_{g_1}(f)}\mathscr{V}_{g_2}(h)\rangle_{\nu_\alpha\otimes\nu_\alpha} \\
&= \langle \mathcal{S}|\overline{\mathscr{V}_{g_1}(f)}\mathscr{V}_{g_2}(h)\rangle_{\nu_\alpha\otimes\nu_\alpha},
\end{aligned} \tag{3.11}$$

and the proof is completed. □

We can now associate a localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}: L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ to every function $\mathcal{S}$ in $L^p(d\nu_\alpha \otimes d\nu_\alpha)$, $1 < p < \infty$ and prove that $\mathcal{L}_{S}^{g_1,g_2}$ is in S_∞. The precise result is the following theorem.

Theorem 3.4 *Let $\mathcal{S}$ be a symbol in $L^p(d\nu_\alpha \otimes d\nu_\alpha)$, $1 \leq p \leq \infty$. Then there exists a unique bounded linear operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}: L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ such that*

$$\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_\infty} \leq \|S\|_{p,\nu_\alpha\otimes\nu_\alpha}. \tag{3.12}$$

Proof Let $f \in L^2(d\nu_\alpha)$. We can consider the linear operators $\mathcal{A}\colon L^1(d\nu_\alpha \otimes d\nu_\alpha) \to L^2(d\nu_\alpha)$ and $\mathcal{A}\colon L^\infty(d\nu_\alpha \otimes d\nu_\alpha) \to L^2(d\nu_\alpha)$ given by

$$\mathcal{A}(\mathcal{S}) = \mathcal{L}_{\mathcal{S}}^{g_1,g_2} f, \quad \mathcal{S} \in L^1(d\nu_\alpha \otimes d\nu_\alpha) \cup L^\infty(d\nu_\alpha \otimes d\nu_\alpha).$$

Then, by Propositions 3.1 and 3.2,

$$\|\mathcal{A}(\mathcal{S})\|_{2,\nu_\alpha} \leq \|f\|_{2,\nu_\alpha} \|\mathcal{S}\|_{1,\nu_\alpha \otimes \nu_\alpha} \tag{3.13}$$

and

$$\|\mathcal{A}(\mathcal{S})\|_{2,\nu_\alpha} \leq \|f\|_{2,\nu_\alpha} \|\mathcal{S}\|_{\infty,\nu_\alpha \otimes \nu_\alpha}. \tag{3.14}$$

Therefore, by (3.13), (3.14) and the Riesz–Thorin interpolation theorem (see [17, Theorem 2] and [21, Theorem 2.11]), $\mathcal{A}$ may be uniquely extended to a linear transformation on $L^p(d\nu_\alpha \otimes d\nu_\alpha)$, $1 \leq p \leq \infty$, and we obtain

$$\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2} f\|_{2,\nu_\alpha} = \|\mathcal{A}(S)\|_{2,\nu_\alpha} \leq \|f\|_{2,\nu_\alpha} \|S\|_{p,\nu_\alpha \otimes \nu_\alpha}. \tag{3.15}$$

Since (3.15) is true for arbitrary functions $f \in L^2(d\nu_\alpha)$, then we obtain the desired result. □

3.2 Compactness

In this section we will prove that the localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\colon L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ is in the Schatten class S_p.

The first result on the Schatten property of localization operators is given in the following proposition.

Proposition 3.5 *Let $\mathcal{S}$ be a symbol in $L^1(d\nu_\alpha \otimes d\nu_\alpha)$. Then the localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is in S_1 and we have*

$$\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_1} \leq 4\|\mathcal{S}\|_{1,\nu_\alpha \otimes \nu_\alpha}. \tag{3.16}$$

Moreover, the following trace-formula holds:

$$\begin{aligned}\mathrm{Tr}(\mathcal{L}_{\mathcal{S}}^{g_1,g_2}) = &\iint_{([0,+\infty[\times\mathbb{R})^2} \mathcal{S}((r,x),(\xi_1,\xi_2)) \\ &\langle (g_1)_{(r,x),(\xi_1,\xi_2)} | (g_2)_{(r,x),(\xi_1,\xi_2)} \rangle_{\nu_\alpha} \, d\nu_\alpha(r,x)\, d\nu_\alpha(\xi_1,\xi_2).\end{aligned} \tag{3.17}$$

Proof Let $\mathcal{S} \in L^1(d\nu_\alpha \otimes d\nu_\alpha)$ and let $(\varphi_n)_n$ be an orthonormal basis for $L^2(d\nu_\alpha)$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \langle \mathcal{L}_{\mathcal{S}}^{g_1,g_2} \varphi_n | \varphi_n \rangle_{\nu_\alpha} = \sum_{n=1}^{\infty} &\iint_{([0,+\infty[\times\mathbb{R})^2} \mathcal{S}((r,x),(\xi_1,\xi_2)) \langle \varphi_n | (g_1)_{(r,x),(\xi_1,\xi_2)} \rangle_{\nu_\alpha} \\ &\times \overline{\langle \varphi_n | (g_2)_{(r,x),(\xi_1,\xi_2)} \rangle_{\nu_\alpha}} \, d\nu_\alpha(r,x)\, d\nu_\alpha(\xi_1,\xi_2).\end{aligned}$$

To prove that $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is in S_1, we first assume that $\mathcal{S}$ is real-valued and nonnegative. Therefore, by Parseval's identity and (2.9),

$$\begin{aligned}\sum_{n=1}^{\infty}\langle\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\varphi_n|\varphi_n\rangle_{\nu_\alpha} &\leq \frac{1}{2}\sum_{n=1}^{\infty}\iint_{([0,+\infty[\times\mathbb{R})^2}\mathcal{S}((r,x),(\xi_1,\xi_2))\\ &\quad\Big(\sum_{n=1}^{\infty}|\langle\varphi_n|(g_1)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}|^2\\ &\quad+\sum_{n=1}^{\infty}|\langle\varphi_n|(g_2)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}|^2\Big)\,d\nu_\alpha(r,x)\,d\nu_\alpha(\xi_1,\xi_2)\\ &\leq \|S\|_{1,\nu_\alpha\otimes\nu_\alpha}. \end{aligned}\tag{3.18}$$

Then, by [21, Proposition 2.4], the localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is in S_1.

Moreover, since in this case $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is positive, then by (3.4), (3.5) and (3.18)

$$\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_1}=\mathrm{Tr}(\mathcal{L}_{\mathcal{S}}^{g_1,g_2})\leq\|S\|_{1,\nu_\alpha\otimes\nu_\alpha}.\tag{3.19}$$

Now, if $\mathcal{S}$ is a real-valued function, then we write $\mathcal{S}=\mathcal{S}_+-\mathcal{S}_-$, where $\mathcal{S}_+=\max(\mathcal{S},0)$ and $\mathcal{S}_-=-\min(\mathcal{S},0)$. Then, by (3.19), we obtain

$$\begin{aligned}\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_1}&=\|\mathcal{L}_{\mathcal{S}_+}^{g_1,g_2}-\mathcal{L}_{\mathcal{S}_-}^{g_1,g_2}\|_{S_1}\\ &\leq\|\mathcal{L}_{\mathcal{S}_+}^{g_1,g_2}\|_{S_1}+\|\mathcal{L}_{\mathcal{S}_-}^{g_1,g_2}\|_{S_1}\\ &\leq\|\mathcal{S}_+\|_{1,\nu_\alpha\otimes\nu_\alpha}+\|\mathcal{S}_-\|_{1,\nu_\alpha\otimes\nu_\alpha}\\ &\leq 2\|\mathcal{S}\|_{1,\nu_\alpha\otimes\nu_\alpha}.\end{aligned}\tag{3.20}$$

Finally, if $\mathcal{S}$ is a complex-valued function, then we write $\mathcal{S}=\mathcal{S}_R+i\mathcal{S}_I$ where $\mathcal{S}_R$ and $\mathcal{S}_I$ are the real and imaginary parts of $\mathcal{S}$ respectively. Then, by (3.20),

$$\begin{aligned}\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_1}&=\|\mathcal{L}_{\mathcal{S}_R}^{g_1,g_2}+i\,\mathcal{L}_{\mathcal{S}_I}^{g_1,g_2}\|_{S_1}\\ &\leq\|\mathcal{L}_{\mathcal{S}_R}^{g_1,g_2}\|_{S_1}+\|\mathcal{L}_{\mathcal{S}_I}^{g_1,g_2}\|_{S_1}\\ &\leq 2\|\mathcal{S}_R\|_{1,\nu_\alpha\otimes\nu_\alpha}+2\|\mathcal{S}_I\|_{1,\nu_\alpha\otimes\nu_\alpha}\\ &\leq 4\|\mathcal{S}\|_{1,\nu_\alpha\otimes\nu_\alpha}.\end{aligned}\tag{3.21}$$

Thus for the symbol $\mathcal{S}\in L^1(d\nu_\alpha\otimes d\nu_\alpha)$, the localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is in S_1 that satisfies (3.21).

On the other hand, let $(\psi_n)_n$ be an orthonormal basis for $L^2(d\nu_\alpha)$, then, by using Fubini's theorem, the Parseval identity and the relation (2.12), we get

$$\begin{aligned}\mathrm{Tr}\left(\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\right)&=\sum_{n=1}^{\infty}\langle\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\psi_n|\psi_n\rangle_{\nu_\alpha}=\sum_{n=1}^{\infty}\iint_{([0,+\infty[\times\mathbb{R})^2}\mathcal{S}((r,x),(\xi_1,\xi_2))\\ &\quad\times\langle\psi_n|(g_1)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}\overline{\langle\psi_n|(g_2)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}}\,d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2)\\ &=\iint_{([0,+\infty[\times\mathbb{R})^2}\mathcal{S}((r,x),(\xi_1,\xi_2))\\ &\quad\langle(g_1)_{(r,x),(\xi_1,\xi_2)}|(g_2)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2).\end{aligned}$$

□

Consequently we have the following result.

Proposition 3.6 *Let $\mathcal{S}$ be a symbol in $L^p(d\nu_\alpha \otimes d\nu_\alpha)$, $1 \leq p < \infty$. Then the localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2} : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ is compact.*

Proof Let $\mathcal{S} \in L^p(d\nu_\alpha \otimes d\nu_\alpha)$ and let $(\mathcal{S}_n)_{n\geq 1}$ be a sequence of functions in $L^1(d\nu_\alpha \otimes d\nu_\alpha) \cap L^\infty(d\nu_\alpha \otimes d\nu_\alpha)$ such that $\mathcal{S}_n \longrightarrow \mathcal{S}$ in $L^p(d\nu_\alpha \otimes d\nu_\alpha)$ as $n \longrightarrow \infty$. Then by Theorem 3.4

$$\|\mathcal{L}_{\mathcal{S}_m}^{g_1,g_2} - \mathcal{L}_{\mathcal{S}_n}^{g_1,g_2}\|_{S_\infty} \leq \|\mathcal{S}_m - \mathcal{S}_n\|_{p,\nu_\alpha \otimes \nu_\alpha}. \tag{3.22}$$

Therefore $\mathcal{L}_{\mathcal{S}_n}^{g_1,g_2} \longrightarrow \mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ in S_∞ as $n \longrightarrow \infty$. Now, since, by Proposition 4.5, the operators $\mathcal{L}_{\mathcal{S}_n}^{g_1,g_2}$ are in S_1 and hence are compact, then the operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is also compact. □

More precisely we have the following theorem.

Theorem 3.7 *Let S be a symbol in $L^p(d\nu_\alpha \otimes d\nu_\alpha)$, $1 \leq p \leq \infty$. Then the localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is in $\mathcal{S}_p$. Moreover,*

$$\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_p} \leq 4^{1/p}\|\mathcal{S}\|_{p,\nu_\alpha \otimes \nu_\alpha}. \tag{3.23}$$

Proof The result follows immediately from Propositions 3.2, 3.5, 3.6 and by the interpolation theorem [21, Theorems 2.10 and 2.11]. □

Based on an idea of Wong [21] we can prove that the constant in Proposition 3.5 and then the constant in Theorem 3.7 can be improved. The next theorem improves Proposition 4.5 and gives a lower bound for the norm $\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_1}$ of the localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ in S_1 in terms of the norm of the function $\tilde{\mathcal{S}}$ defined by

$$\tilde{\mathcal{S}}((r,x),(\xi_1,\xi_2)) = \langle \mathcal{L}_{\mathcal{S}}^{g_1,g_2}(g_1)_{(r,x),(\xi_1,\xi_2)} | (g_2)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}. \tag{3.24}$$

Theorem 3.8 *Let $\mathcal{S} \in L^1(d\nu_\alpha \otimes d\nu_\alpha)$. Then*

$$\|\tilde{\mathcal{S}}\|_{1,\nu_\alpha \otimes \nu_\alpha} \leq \|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_1} \leq \|\mathcal{S}\|_{1,\nu_\alpha \otimes \nu_\alpha}. \tag{3.25}$$

Proof First, by Proposition 3.5, the localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is in S_1. Now by using the canonical form for compact operators given in [21, Theorem 2.2], we obtain for $f \in L^2(d\nu_\alpha)$

$$\mathcal{L}_{\mathcal{S}}^{g_1,g_2} f = \sum_{n=1}^{\infty} s_n \langle f | \varphi_n\rangle_{\nu_\alpha} \phi_n, \tag{3.26}$$

where $s_n := s_n(\mathcal{L}_{\mathcal{S}}^{g_1,g_2})$, $n = 1, 2, \ldots$, are the positive singular values of $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$, $(\phi_n)_{n\geq 1}$ is an orthonormal set in $L^2(d\nu_\alpha)$ and $(\varphi_n)_{n\geq 1}$ is an orthonormal basis for the orthogonal complement of the null space of $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$, consisting of eigenvectors of the positive and compact operator $|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}| : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$. Then

$$\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2} f\|_{S_1} = \sum_{n=1}^{\infty} s_n = \sum_{n=1}^{\infty} \langle \mathcal{L}_{\mathcal{S}}^{g_1,g_2} \varphi_n | \phi_n\rangle_{\nu_\alpha}. \tag{3.27}$$

Thus, by Schwartz' inequality, Bessel's inequality and (2.9) we obtain

$$\begin{aligned}\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2} f\|_{S_1} &= \sum_{n=1}^{\infty}\langle \mathcal{L}_{\mathcal{S}}^{g_1,g_2}\varphi_n|\phi_n\rangle_{\nu_\alpha} = \sum_{n=1}^{\infty}\iint_{([0,+\infty[\times\mathbb{R})^2} \mathcal{S}((r,x),(\xi_1,\xi_2)) \\ &\quad \times \mathscr{V}_{g_1}(\varphi_n)((r,x),(\xi_1,\xi_2))\overline{\mathscr{V}_{g_2}(\phi_n)((r,x),(\xi_1,\xi_2))}d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2) \\ &\leq \iint_{([0,+\infty[\times\mathbb{R})^2} |\mathcal{S}((r,x),(\xi_1,\xi_2))|\Big(\sum_{n=1}^{\infty}|\langle\varphi_n|(g_2)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}|^2\Big)^{1/2} \\ &\quad \times \Big(\sum_{n=1}^{\infty}|\langle\phi_n|(g_2)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}|^2\Big)^{1/2} d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2) \\ &\leq \|\mathcal{S}\|_{1,\nu_\alpha\otimes\nu_\alpha}.\end{aligned}$$

On the other hand, by the relation (3.26), we have

$$\begin{aligned}\tilde{\mathcal{S}}((r,x),(\xi_1,\xi_2)) &= \langle \mathcal{L}_{\mathcal{S}}^{g_1,g_2}(g_1)_{(r,x),(\xi_1,\xi_2)}|(g_2)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha} \\ &= \sum_{n=1}^{\infty} s_n\langle (g_1)_{(r,x),(\xi_1,\xi_2)}|\varphi_n\rangle_{\nu_\alpha}\langle\phi_n|(g_2)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}.\end{aligned}$$

Then

$$\begin{aligned}|\tilde{\mathcal{S}}((r,x),(\xi_1,\xi_2))| &\leq \frac{1}{2}\sum_{n=1}^{\infty} s_n\big(|\langle\varphi_n|(g_1)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}|^2 + |\langle\phi_n|(g_2)_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}|^2\big) \\ &= \frac{1}{2}\sum_{n=1}^{\infty} s_n\big(|\mathscr{V}_{g_1}(\varphi_n)((r,x),(\xi_1,\xi_2))|^2 + |\mathscr{V}_{g_2}(\phi_n)((r,x),(\xi_1,\xi_2))|^2\big).\end{aligned}$$

Therefore

$$\|\tilde{\mathcal{S}}\|_{1,\nu_\alpha\otimes\nu_\alpha} \leq \frac{1}{2}\sum_{n=1}^{\infty} s_n\big(\|\mathscr{V}_{g_1}(\varphi_n)\|^2_{2,\nu_\alpha\otimes\nu_\alpha} + \|\mathscr{V}_{g_2}(\phi_n)\|^2_{2,\nu_\alpha\otimes\nu_\alpha}\big). \tag{3.28}$$

Thus, by Plancherel's formula (2.14), we have $\tilde{\mathcal{S}} \in L^1(d\nu_\alpha \otimes d\nu_\alpha)$, and

$$\|\tilde{\mathcal{S}}\|_{1,\nu_\alpha\otimes\nu_\alpha} \leq \sum_{n=1}^{\infty} s_n = \|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_1}. \tag{3.29}$$

This completes the proof of the theorem. □

An immediate consequence of Theorem 3.8 and interpolation theory is the following improvement of Theorem 3.7.

Corollary 3.9 *Let $\mathcal{S}$ be a symbol in $L^p(d\nu_\alpha \otimes d\nu_\alpha)$, $1 \leq p \leq \infty$. Then the localization operator $\mathcal{L}_{\mathcal{S}}^{g_1,g_2}$ is in $\mathcal{S}_p$. Moreover,*

$$\|\mathcal{L}_{\mathcal{S}}^{g_1,g_2}\|_{S_p} \leq \|\mathcal{S}\|_{p,\nu_\alpha\otimes\nu_\alpha}. \tag{3.30}$$

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