Periodica Mathematica Hungarica (2021) 82:125–135
https://doi.org/10.1007/s10998-020-00345-y

On $\mathcal{I}_{<q}$- and $\mathcal{I}_{\le q}$-convergence of arithmetic functions

János T. Tóth[1] · Ferdinánd Filip[1] · József Bukor[1] · László Zsilinszky[2]

Published online: 28 May 2020

Abstract
Let $\mathbb{N}$ be the set of positive integers, and denote by

$$\lambda(A) = \inf\{t > 0 : \sum_{a\in A} a^{-t} < \infty\}$$

the convergence exponent of $A \subset \mathbb{N}$. For $0 < q \le 1$, $0 \le q \le 1$, respectively, the admissible ideals $\mathcal{I}_{<q}$, $\mathcal{I}_{\le q}$ of all subsets $A \subset \mathbb{N}$ with $\lambda(A) < q$, $\lambda(A) \le q$, respectively, satisfy $\mathcal{I}_{<q} \subsetneq \mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_{\le q}$, where

$$\mathcal{I}_c^{(q)} = \{A \subset \mathbb{N} : \sum_{a\in A} a^{-q} < \infty\}.$$

In this note we sharpen the results of Baláž et al. from (J Number Theory 183:74–83, 2018) and other papers, concerning characterizations of $\mathcal{I}_c^{(q)}$-convergence of various arithmetic functions in terms of q. This is achieved by utilizing $\mathcal{I}_{<q}$- and $\mathcal{I}_{\le q}$-convergence, for which new methods and criteria are developed.

Keywords $\mathcal{I}$-convergence · Arithmetic functions · Convergence exponent

Mathematics Subject Classification 40A35 · 11A25

✉ Ferdinánd Filip
filipf@ujs.sk

János T. Tóth
bukorj@ujs.sk

József Bukor
tothj@ujs.sk

László Zsilinszky
laszlo@uncp.edu

1 Department of Mathematics and Informatics, J. Selye University, 945 01 Komárno, Slovakia

2 Department of Mathematics and Computer Science, University of North Carolina at Pembroke, Pembroke, NC 28304, USA

1 Introduction

Denote by $\mathbb{N}$ the set of positive integers, and let λ be the convergence exponent function on the power set $2^{\mathbb{N}}$ of $\mathbb{N}$, i.e. for $A \subset \mathbb{N}$ put

$$\lambda(A) = \inf\Big\{t > 0 : \sum_{a\in A} \frac{1}{a^t} < \infty\Big\}.$$

If $q > \lambda(A)$ then $\sum_{a\in A} \frac{1}{a^q} < \infty$, and $\sum_{a\in A} \frac{1}{a^q} = \infty$ when $q < \lambda(A)$; if $q = \lambda(A)$, the convergence of $\sum_{a\in A} \frac{1}{a^q}$ is inconclusive. It follows from [11, p.26, Exercises 113, 114] that the range of λ is the interval [0, 1], moreover, for $A = \{a_1 < a_2 < \cdots < a_n < \ldots\} \subset \mathbb{N}$,

$$\lambda(A) = \limsup_{n\to\infty} \frac{\log n}{\log a_n}.$$

It is easy to see that λ is monotonic, i.e. $\lambda(A) \le \lambda(B)$ whenever $A \subset B \subset \mathbb{N}$, furthermore, $\lambda(A \cup B) = \max\{\lambda(A), \lambda(B)\}$ for all $A, B \subset \mathbb{N}$. Define the following sets:

$$\begin{aligned}\mathcal{I}_{<q} &= \{A \subset \mathbb{N} : \lambda(A) < q\}, \text{ if } 0 < q \le 1,\\ \mathcal{I}_{\le q} &= \{A \subset \mathbb{N} : \lambda(A) \le q\}, \text{ if } 0 \le q \le 1, \text{ and}\\ \mathcal{I}_0 &= \{A \subset \mathbb{N} : \lambda(A) = 0\}.\end{aligned}$$

Clearly, $\mathcal{I}_{\le 0} = \mathcal{I}_0$, and $\mathcal{I}_{\le 1} = 2^{\mathbb{N}}$. Since $\lambda(A) = 0$ when $A \subset \mathbb{N}$ is finite, then $\mathcal{I}_f = \{A \subset \mathbb{N} : A \text{ is finite}\} \subset \mathcal{I}_0$, moreover, also considering the well-known set

$$\mathcal{I}_c^{(q)} = \Big\{A \subset \mathbb{N} : \sum_{a\in A} \frac{1}{a^q} < \infty\Big\}$$

we get that whenever $0 < q < q' < 1$,

$$\mathcal{I}_f \subset \mathcal{I}_0 \subset \mathcal{I}_{<q} \subset \mathcal{I}_c^{(q)} \subset \mathcal{I}_{\le q} \subset \mathcal{I}_{<q'}. \tag{1}$$

In what follows, we will use the following definitions.

The set $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is a so-called admissible ideal, provided $\mathcal{I}$ is additive (i.e. $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$), hereditary (i.e. $A \in \mathcal{I},\ B \subset A$ implies $B \in \mathcal{I}$), it contains the singletons, and $\mathbb{N} \notin \mathcal{I}$.

Given an ideal $\mathcal{I} \subset 2^{\mathbb{N}}$, we say that a sequence $x = (x_n)_{n=1}^{\infty}$ $\mathcal{I}$-converges to a number L, and write $\mathcal{I}\text{-}\lim x_n = L$, if for each $\varepsilon > 0$ the set

$$A_\varepsilon = \{n : |x_n - L| \ge \varepsilon\} \tag{2}$$

belongs to the ideal $\mathcal{I}$. One can see, e.g., [6], [7] for a general treatment of $\mathcal{I}$-convergence. A useful property is as follows:

Lemma 1.1 [7] *If $\mathcal{I}_1 \subset \mathcal{I}_2$, then $\mathcal{I}_1$-lim $x_n = L$ implies $\mathcal{I}_2$-lim $x_n = L$.*

We will study $\mathcal{I}$-convergence in the case when $\mathcal{I}$ stands for $\mathcal{I}_{<q}, \mathcal{I}_c^{(q)}, \mathcal{I}_{\le q}$, respectively. We will establish necessary and sufficient conditions for a set $A \subset \mathbb{N}$ to belong to $\mathcal{I}_{<q}, \mathcal{I}_{\le q}$, respectively; as well as for the set $A_\varepsilon = \{n : |x_n - L| \ge \varepsilon\}$ so that $\mathcal{I}_{<q}$-lim $x_n = L$, resp. $\mathcal{I}_{\le q}$-lim $x_n = L$ hold. Note that analogous criteria were not known for $\mathcal{I}_c^{(q)}$.

In this paper, we embed the ideals $\mathcal{I}_{<q}$ and $\mathcal{I}_{\le q}$ into the structure of ideals $\mathcal{I}_c^{(q)}$. We show that these ideals are essentially distinct. Then we refine a known statement concerning the $\mathcal{I}_c^{(q)}$-convergence of some arithmetic functions. A new method is introduced and can be applied widely for consideration of $\mathcal{I}_{<q}$ and $\mathcal{I}_{\le q}$-convergence of sequences.

2 On ideals enveloping the ideal $\mathcal{I}_c^{(q)}$

Theorem 2.1 *Let* $0 < q < q' < 1$. *Then*

$$\mathcal{I}_0 \subsetneq \mathcal{I}_{<q} \subsetneq \mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_{\leq q} \subsetneq \mathcal{I}_{<q'} \subsetneq \mathcal{I}_c^{(q')} \subsetneq \mathcal{I}_{\leq q'} \subsetneq \mathcal{I}_{<1} \subsetneq \mathcal{I}_c^{(1)} \subsetneq \mathcal{I}_{\leq 1} = 2^{\mathbb{N}}. \tag{3}$$

Proof The inclusions follow from the definitions of the sets. We can show that the difference of successive sets in (3) is infinite, so equality does not hold in any of the inclusions, by considering the following four cases (as usual, $\lfloor x \rfloor$ is the integer part of the real x):

Case 1. $\mathcal{I}_0 \neq \mathcal{I}_{<q}$: let $0 < s < q < 1$, and take the set $A = \{a_1 < a_2 < \cdots\} \subset \mathbb{N}$, where for all $n \in \mathbb{N}$,

$$a_n = \lfloor n^{\frac{1}{s}} \rfloor.$$

Then $a_n = n^{\frac{1}{s}} - \varepsilon(n)$ for some $0 \leq \varepsilon(n) < 1$, and by Lagrange's Mean Value Theorem for $f(x) = x^{\frac{1}{s}}$ on $[n, n+1]$ we get that $a_n < a_{n+1}$ for all n. Since

$$\frac{\log n}{\log a_n} = \frac{\log n}{\frac{1}{s} . \log n + \log\left(1 - \frac{\varepsilon(n)}{n^{\frac{1}{s}}}\right)} \to s, \quad \text{if } n \to \infty,$$

then $0 < \lambda(A) = s < q$; thus, $A \in \mathcal{I}_{<q} \setminus \mathcal{I}_0$. It is also clear that $\mathcal{I}_{<q} \setminus \mathcal{I}_0$ is infinite, since for any $k \in \mathbb{N}$ the sets $A_k = \{ka_n : n \in \mathbb{N}\}$ satisfy

$$\lambda(A_k) = \limsup_{n\to\infty} \frac{\log n}{\log ka_n} = \lambda(A).$$

Case 2. $\mathcal{I}_{<q} \neq \mathcal{I}_c^{(q)}$: let $0 < q < 1$, and take the set $A = \{a_1 < a_2 < \cdots\} \subset \mathbb{N}$, where for all $n \in \mathbb{N}$,

$$a_n = \lfloor n^{\frac{1}{q}} \log^{\frac{2}{q}}(n+1) \rfloor + 1.$$

One can easily show that (a_n) is increasing sequence, and,

$$\sum_{n=1}^{\infty} \frac{1}{a_n^q} < \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} < \infty, \text{ thus, } A \in \mathcal{I}_c^{(q)}.$$

On the other hand

$$\lim_{n\to\infty} \frac{\log n}{\log a_n} = \lim_{n\to\infty} \frac{\log n}{\log(n^{\frac{1}{q}} \log^{\frac{2}{q}}(n+1))} = \lim_{n\to\infty} \frac{\log n}{\frac{1}{q}\log n + \frac{2}{q}\log\log(n+1)} = q,$$

hence, $\lambda(A) = q$. Similarly to Case 1 we can see that $\mathcal{I}_c^{(q)} \setminus \mathcal{I}_{<q}$ is actually infinite.

Case 3. $\mathcal{I}_c^{(q)} \neq \mathcal{I}_{\leq q}$: let $0 < q < 1$, define $A = \{a_1 < a_2 < \cdots\} \subset \mathbb{N}$, where $a_n = \lfloor n^{\frac{1}{q}} \rfloor$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} \frac{1}{a_n^q} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

so $A \notin \mathcal{I}_c^{(q)}$, but $A \in \mathcal{I}_{\leq q}$, since $\lambda(A) = q$. Analogously to Case 1, one can show that $\mathcal{I}_{\leq q} \setminus \mathcal{I}_c^{(q)}$ is infinite.

Case 4. $\mathcal{I}_{\leq q} \neq \mathcal{I}_{<q'}$: it suffices to choose the set $A = \{a_1 < a_2 < \cdots\} \subset \mathbb{N}$ such that $a_n = \lfloor n^{\frac{1}{s}} \rfloor$ for all n, where $0 < q < s < q'$. Then $\lambda(A) = s$, so $A \in \mathcal{I}_{<q'}$, however, $A \notin \mathcal{I}_{\leq q}$. Moreover, again, $\mathcal{I}_{<q'} \setminus \mathcal{I}_{\leq q}$ is infinite. □

By (3), it is worth noting that in order to decide if a given $A \subset \mathbb{N}$ belongs to $\mathcal{I}_c^{(q)}$, it may be easier, or more advantageous to first determine the convergence exponent of A. Indeed, if $\lambda(A) < q$, then $A \in \mathcal{I}_{<q} \subset \mathcal{I}_c^{(q)}$, or, if $\lambda(A) = q$, then $A \in \mathcal{I}_{\leq q} \subset \mathcal{I}_c^{(q')}$ for every $q' > q$. This view is important, since in what follows, we will establish criteria for $\mathcal{I}_{<q}, \mathcal{I}_{\leq q}$ membership, respectively.

Theorem 2.2 *Let $0 < q \leq 1$. Then each of the sets $\mathcal{I}_0$, $\mathcal{I}_{<q}$, $\mathcal{I}_{\leq q}$ forms an admissible ideal, except for $\mathcal{I}_{\leq 1}$.*

Proof Follows from properties of λ listed in the Introduction, along with (3). □

Theorem 2.3 *We have*

$$\mathcal{I}_0 = \bigcap_{0<q\leq 1} \mathcal{I}_{<q} = \bigcap_{0<q\leq 1} \mathcal{I}_{\leq q},$$

hence,

$$\mathcal{I}_0 = \bigcap_{0<q\leq 1} \mathcal{I}_c^{(q)}.$$

Proof Follows from the definitions of $\mathcal{I}_0, \mathcal{I}_{<q}, \mathcal{I}_{\leq q}$, and (3). □

3 Conditions for a set *A* to belong to $\mathcal{I}_{<q}, \mathcal{I}_{\leq q}$

Given $x \geq 1$, define the counting function of $A \subset \mathbb{N}$ by

$$A(x) = \#\{a \leq x : a \in A\}.$$

Theorem 3.1 *Let $0 \leq q < 1$ be a real number and $A \subset \mathbb{N}$. Then $A \in \mathcal{I}_{\leq q}$ if and only if for every $\delta > 0$*

$$\lim_{x\to\infty} \frac{A(x)}{x^{q+\delta}} = 0. \tag{4}$$

Proof Let $A = \{a_1 < a_2 < \ldots\}$, and $A \in \mathcal{I}_{\leq q}$. Then

$$\lambda(A) = \limsup_{n\to\infty} \frac{\log n}{\log a_n} \leq q,$$

so for any $\delta > 0$ there is an $n_0 \in \mathbb{N}$ so that, for all $n \geq n_0$,

$$\frac{\log n}{\log a_n} \leq q + \frac{\delta}{2}, \text{ thus } A(a_n) = n \leq a_n^{q+\frac{\delta}{2}}.$$

If x is sufficiently large, we can find $n \geq n_0$ with $a_n \leq x < a_{n+1}$, hence, $A(x) = n \leq x^{q+\frac{\delta}{2}}$. Consequently,

$$0 \leq \frac{A(x)}{x^{q+\delta}} \leq \frac{x^{q+\frac{\delta}{2}}}{x^{q+\delta}} = \frac{1}{x^{\frac{\delta}{2}}} \to 0, \quad \text{as } x \to \infty,$$

which implies (4) for every $\delta > 0$.

Conversely, let $\delta > 0$, and (4) be true for some $A = \{a_1 < a_2 < \ldots\}$. Then

$$\frac{A(a_n)}{a_n^{q+\delta}} \to 0 \quad \text{as } n \to \infty,$$

so there is an $n_1 \in \mathbb{N}$ such that for all $n \ge n_1$, $n \le a_n^{q+\delta}$, thus,

$$\frac{\log n}{\log a_n} \le \frac{(q+\delta)\log a_n}{\log a_n} = q + \delta.$$

Then for all $\delta > 0$, $\lambda(A) \le q + \delta$, hence, letting $\delta \to 0$, we get $\lambda(A) \le q$, so, $A \in \mathcal{I}_{\le q}$. □

The definition of $\mathcal{I}_{\le q}$-convergence immediately yields

Corollary 3.2 *Let $0 \le q < 1$, $\varepsilon > 0$, L and x_n be real numbers for all $n \in \mathbb{N}$, and $A_\varepsilon = \{n : |x_n - L| \ge \varepsilon\}$. Then $\mathcal{I}_{\le q}$-lim $x_n = L$ if and only if for every $\varepsilon > 0$ and $\delta > 0$*

$$\lim_{x\to\infty} \frac{A_\varepsilon(x)}{x^{q+\delta}} = 0.$$

Theorem 3.3 *Let $0 < q \le 1$ be a real number and $A \subset \mathbb{N}$. Then $A \in \mathcal{I}_{<q}$ if and only if there exists a $\delta > 0$ such that*

$$\lim_{x\to\infty} \frac{A(x)}{x^{q-\delta}} = 0. \tag{5}$$

Proof Let $A \in \mathcal{I}_{<q}$. Then

$$\lambda(A) = \limsup_{n\to\infty} \frac{\log n}{\log a_n} < q, \quad \text{where } A = \{a_1 < a_2 < \dots\}.$$

For each $\delta > 0$ with $0 < \delta < \frac{1}{2}(q - \lambda(A))$ there is an $n_0 \in \mathbb{N}$ so that for all $n \ge n_0$,

$$\frac{\log n}{\log a_n} \le q - 2\delta, \quad \text{thus, } n \le a_n^{q-2\delta},$$

hence, for all $n \ge n_0$,

$$A(a_n) = n \le a_n^{q-2\delta}.$$

If x is large enough, there exists some $n \ge n_0$ with $a_n \le x < a_{n+1}$, so $A(x) = n \le x^{q-2\delta}$. This implies

$$0 \le \frac{A(x)}{x^{q-\delta}} \le \frac{x^{q-2\delta}}{x^{q-\delta}} = \frac{1}{x^\delta} \to 0 \quad \text{as } x \to \infty,$$

and (5) follows.

Conversely, let $\delta > 0$ be such that (5) is true. Then by Theorems 2.1 and 3.1 we have

$$A \in \mathcal{I}_{\le q-\delta} \subset I_{<q}.$$

□

The definition of the $\mathcal{I}_{<q}$-convergence immediately yields

Corollary 3.4 *Let $0 < q \le 1$, $\varepsilon > 0$, L and x_n be real numbers for all $n \in \mathbb{N}$, and $A_\varepsilon = \{n : |x_n - L| \ge \varepsilon\}$. Then $\mathcal{I}_{<q}$-lim $x_n = L$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\lim_{x\to\infty} \frac{A_\varepsilon(x)}{x^{q-\delta}} = 0.$$

As an application of the above results, we will show that an important number-theoretic set belongs to the smallest element of (3), namely $\mathcal{I}_0$:

Lemma 3.5 *Given $k \in \mathbb{N}$, and arbitrary primes $p_1 < p_2 < \cdots < p_k$, denote*

$$D(p_1, p_2 \ldots, p_k) = \{n \in \mathbb{N} : n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \alpha_i \geq 0, i = 1, 2, \ldots, k\}.$$

Then

$$D(p_1, p_2 \ldots, p_k) \in \mathcal{I}_0 .$$

Proof For a number $x \geq 2$ denote

$$D(p_1, p_2 \ldots, p_k)(x) = \#\{n \leq x : n \in D(p_1, p_2 \ldots, p_k)\}.$$

Then by [9, p.37, Exercise 15] we have

$$D(p_1, p_2 \ldots, p_k)(x) \leq \prod_{i=1}^{k} \Big(\frac{\log x}{\log p_i} + 1\Big) \leq \Big(\frac{2}{\log 2} \log x\Big)^k .$$

From this, by Theorem 3.1 for $q = 0$ we get

$$D(p_1, p_2 \ldots, p_k) \in \mathcal{I}_0.$$

□

4 On $\mathcal{I}_{<q}$- and $\mathcal{I}_{\leq q}$-convergence of arithmetic functions

First we recall some arithmetic functions, which we will investigate with respect to $\mathcal{I}_{<q}$- and $\mathcal{I}_{\leq q}$-convergence. We refer to the papers [2,5,8,10,12,14–16] for definitions and properties of these functions.

Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the canonical representation of $n \in \mathbb{N}$. Define the following functions:

- $\omega(n)$ is the number of distinct prime factors of n (i.e. $\omega(n) = k$);
- $\Omega(n)$ is the number of prime factors of n counted with multiplicities (i.e. $\Omega(n) = \alpha_1 + \cdots + \alpha_k$);
- for $n > 1$,

  $$h(n) = \min_{1 \leq j \leq k} \alpha_j, \quad H(n) = \max_{1 \leq j \leq k} \alpha_j$$

  and $h(1) = 1$, $H(1) = 1$;
- $f(n) = \prod_{d|n} d$ and $f^*(n) = \frac{1}{n} f(n)$;
- $a_p(n)$ as follows: $a_p(1) = 0$ and $a_p(n)$ is the unique integer $j \geq 0$ satisfying $p^j \mid n$, but $p^{j+1} \nmid n$, i.e. $p^{a_p(n)} \parallel n$, for $n > 1$;
- $\gamma(n)$ is the number of all representations of a natural number n in the form $n = a^b$, where a, b are positive integers (see [8]). Let

  $$n = a_1^{b_1} = a_2^{b_2} = \cdots = a_{\gamma(n)}^{b_{\gamma(n)}}$$

  be all such representations of a given n, where $a_i, b_i \in \mathbb{N}$;
- for $n > 1$,

  $$\tau(n) = b_1 + b_2 + \cdots + b_{\gamma(n)};$$

- $N(n)$ is the number of times the positive integer n occurs in Pascal's triangle (see [1] and [15]).

Recall that $\mathcal{I}_c^{(q)}$-convergence of the following sequences has been established in [2–4]:

I. For $0 < q \le 1$ we have $\mathcal{I}_c^{(q)}$-lim $\frac{h(n)}{\log n} = 0$ (see [2], [Th.8]).
II. Only for $q = 1$ we have $\mathcal{I}_c^{(q)}$-lim $\frac{H(n)}{\log n} = 0$ (see [2], [Th.10, Th.11]).
III. For a prime number p the sequence $\big((\log p)\frac{a_p(n)}{\log n}\big)_{n=2}^{\infty}$ is $\mathcal{I}_c^{(q)}$-convergent to 0 only for $q = 1$ (see [3],[Th.2.3]),
IV. For $q > \frac{1}{2}$ we have $\mathcal{I}_c^{(q)}$-lim $\gamma(n) = 1$, and for $0 < q \le \frac{1}{2}$ the sequence $\gamma(n)$ is not $\mathcal{I}_c^{(q)}$-convergent (see [3], [Cor.3.5]),
V. For $q > \frac{1}{2}$ we have $\mathcal{I}_c^{(q)}$-lim $\tau(n) = 1$, and for $0 < q \le \frac{1}{2}$ the sequence $\tau(n)$ is not $\mathcal{I}_c^{(q)}$-convergent (see [3], [Cor.3.8]),
VI. For $q > \frac{1}{2}$ we have $\mathcal{I}_c^{(q)}$-lim $N(n) = 2$, and for $0 < q \le \frac{1}{2}$ the sequence $\big(N(n)\big)_{t=1}^{\infty}$ is not $\mathcal{I}_c^{(q)}$-convergent (see [4], [Th.2.2]),
VII. The sequences $\big(\frac{\omega(n)}{\log\log n}\big)_{n=2}^{\infty}$ and $\big(\frac{\Omega(n)}{\log\log n}\big)_{n=2}^{\infty}$ are not $\mathcal{I}_c^{(q)}$-convergent for all $0 < q \le 1$ (see [2], [Th.12]),
VIII. The sequences $\big(\frac{\log\log f(n)}{\log\log n}\big)$ and $\big(\frac{\log\log f^*(n)}{\log\log n}\big)$ are not $\mathcal{I}_c^{(q)}$-convergent for all $0 < q \le 1$ (see [2], [Th.13, Th.14]).

In what follows, we will improve and sharpen all statements I–VIII via the best convergences one can obtain from the ideals in (3) that are within $\mathcal{I}_{<q}, \mathcal{I}_{\le q}$.

The next theorem, which is readily implied by Theorem 2.3 and [2], [Th.8], gives Statement I using Theorem 2.1 and Lemma 1.1. We will, however, provide another simpler proof based on Lemma 3.5:

Theorem 4.1 *We have*

$$\mathcal{I}_0\text{-}lim\ \frac{h(n)}{\log n} = 0\,.$$

Proof Take a small $\varepsilon > 0$, and the largest prime p_0 for which $\frac{1}{\log p_0} \ge \varepsilon$. Then $\frac{1}{\log p} < \varepsilon$ whenever $p > p_0$, so if $n \in \mathbb{N}$ is such that $p|n$ for some prime $p > p_0$, then $n \ge p^{h(n)}$. It follows that

$$\frac{h(n)}{\log n} \le \frac{h(n)}{\log p^{h(n)}} = \frac{1}{\log p} < \varepsilon\,,$$

thus,

$$n \notin \left\{k \in \mathbb{N} : \frac{h(k)}{\log k} \ge \varepsilon\right\} = \left\{k \in \mathbb{N} : \left|\frac{h(k)}{\log k} - 0\right| \ge \varepsilon\right\} = A_\varepsilon.$$

This implies $A_\varepsilon \subset D(2, 3, 5, \ldots, p_0)$, so, by Lemma 3.5 and the hereditary property, $A_\varepsilon \in \mathcal{I}_0$. □

Statement II has the following strengthening:

Theorem 4.2 *We have*

$$\mathcal{I}_{<1}\text{-}lim\ \frac{H(n)}{\log n} = 0\,.$$

Proof Let $0 < \varepsilon < \frac{1}{\log 2}$. Then, according to (2), we have

$$A_\varepsilon = \left\{ n \in \mathbb{N} : \frac{H(n)}{\log n} \geq \varepsilon \right\}.$$

We will show that $A_\varepsilon \in \mathcal{I}_{<1}$: every positive integer n can be uniquely represented as $n = ab^2$, where a is a square-free number. Hence $H(a) = 1$ and $H(n) \in \{H(b^2), H(b^2)+1\}$. For any $n \in \mathbb{N}$ we have $n = p_1^{a_1} \cdots p_k^{a_k} \geq 2^{H(n)}$ and from this

$$H(n) \leq \frac{\log n}{\log 2}.$$

If $n \in A_\varepsilon$ then for $n = ab^2$ we get

$$\log n = \log(ab^2) \leq \frac{H(ab^2)}{\varepsilon} \leq \frac{H(b^2)+1}{\varepsilon} \leq \frac{\log b^2}{\varepsilon \log 2} + \frac{1}{\varepsilon},$$

thus,

$$A_\varepsilon \subseteq B = \left\{ n \in \mathbb{N} : n = ab^2,\ \log ab^2 \leq \frac{\log b^2}{\varepsilon \log 2} + \frac{1}{\varepsilon},\ \ a, b \in \mathbb{N} \right\}.$$

Furthermore, if $n \in B$, then

$$\log a \leq \frac{1 - \varepsilon \log 2}{\varepsilon \log 2} \log b^2 + \frac{1}{\varepsilon},$$

which is equivalent to

$$a^{\frac{\varepsilon \log 2}{1 - \varepsilon \log 2}} \leq b^2 e^{\frac{\log 2}{1-\varepsilon \log 2}}, \quad \text{and so} \quad a^{\frac{1}{1-\varepsilon \log 2}} \leq ab^2 e^{\frac{\log 2}{1 - \varepsilon \log 2}},$$

therefore,

$$B = \left\{ n \in \mathbb{N} : n = ab^2 \text{ and } a \leq 2n^{1-\varepsilon \log 2} \right\}.$$

If $n \in B$, and $n = ab^2 \leq x$ for $x \geq 2$, then $a \leq 2x^{1-\varepsilon \log 2}$ and $b \leq \sqrt{\frac{x}{a}}$. Consequently,

$$B(x) \leq \sum_{a < 2x^{1-\varepsilon \log 2}} \sqrt{\frac{x}{a}} = \sqrt{x} \sum_{a < 2x^{1-\varepsilon \log 2}} \frac{1}{\sqrt{a}} \leq \sqrt{x} \left(1 + \int_1^{2x^{1-\varepsilon \log 2}} \frac{1}{\sqrt{t}} \mathrm{d}t \right)$$
$$\leq \sqrt{x} \left(1 + 2\left(\sqrt{2x^{1-\varepsilon \log 2}} - 1 \right) \right) \leq 2\sqrt{2} x^{1 - \varepsilon \frac{\log 2}{2}},$$

hence, for $x \geq 2$, we have

$$A_\varepsilon(x) \leq 2\sqrt{2} x^{1 - \varepsilon \frac{\log 2}{2}}.$$

Using $q = 1$ and arbitrary $\delta \in (0, \varepsilon \frac{\log 2}{2})$ in Theorem 3.3, the above estimate gives $A_\varepsilon \in \mathcal{I}_{<1}$. □

The next result strengthens statement III.

Theorem 4.3 *For any prime number p, we have*

$$\mathcal{I}_{<1}\text{-}\mathit{lim} (\log p) \frac{a_p(n)}{\log n} = 0.$$

Proof Let $0 < \varepsilon < 1$. Then, according to (2), we have

$$A_\varepsilon = \{n > 1 : (\log p)\frac{a_p(n)}{\log n} \geq \varepsilon\}.$$

We have

$$A_\varepsilon = \bigcup_{i=0}^{\infty} A_\varepsilon^i,$$

where

$$A_\varepsilon^i = \{n \in A_\varepsilon : n = p^i u \quad \text{where } p \nmid u\} \ (i = 0, 1, 2 \dots).$$

Clearly, $A_\varepsilon^i \cap A_\varepsilon^j = \emptyset$ for $i \neq j$, and if $n \in A_\varepsilon^i$, then

$$(\log p)\frac{a_p(n)}{\log n} = (\log p)\frac{i}{i \log p + \log u} \geq \varepsilon, \quad \text{thus, } u \leq p^{i(\frac{1-\varepsilon}{\varepsilon})}.$$

In case $x \geq 2$, this implies that

$$A_\varepsilon^i(x) \leq \# \left\{u : u^{\frac{\varepsilon}{1-\varepsilon}} u \leq x\right\} = \# \left\{u : u^{\frac{1}{1-\varepsilon}} \leq x\right\} \leq x^{1-\varepsilon},$$

hence

$$A_\varepsilon(x) = \sum_{i : p^i \leq x} A_\varepsilon^i(x) \leq \frac{\log x}{\log p} x^{1-\varepsilon}.$$

Using $q = 1 - \varepsilon$ in Theorem 3.1 and using Theorem 2.1, the above estimate gives

$$A_\varepsilon \in \mathcal{I}_{\leq 1-\varepsilon} \subset \mathcal{I}_{<1}.$$

□

The statements IV, V, VI are consequences of the following result.

Theorem 4.4 *We have*

(i) $\mathcal{I}_{\leq \frac{1}{2}}$*-lim* $\gamma(n) = 1$.
(ii) $\mathcal{I}_{\leq \frac{1}{2}}$*-lim* $\tau(n) = 1$.
(iii) $\mathcal{I}_{\leq \frac{1}{2}}$*-lim* $N(n) = 2$.

Proof (i) Let $0 < \varepsilon < 1$. Then, according to (2), we have $A_\varepsilon = \{n \in \mathbb{N} : |\gamma(n) - 1| \geq \varepsilon\}$. Clearly,

$$A_\varepsilon \subseteq H = \left\{a^b : a, b \in \mathbb{N} \setminus \{1\}\right\} = \bigcup_{k=2}^{\infty} \left\{n^k : n = 2, 3, \dots\right\}.$$

Given some $x \in \mathbb{N}$, $x \geq 2^2$, there is a $k \in \mathbb{N} \setminus \{1\}$ with $2^k \leq x < 2^{k+1}$. Then $k \leq \frac{\log x}{\log 2}$, and

$$H(x) \leq \sum_{n=2}^{k} \sqrt[n]{x} \leq \sqrt{x}\frac{\log x}{\log 2},$$

thus, for all $x \geq 4$,

$$A_\varepsilon(x) \leq \frac{\log x}{\log 2} x^{\frac{1}{2}}.$$

For $q = \frac{1}{2}$ in Theorem 3.1, we get $A_\varepsilon \in \mathcal{I}_{\leq \frac{1}{2}}$.

(ii) Similar to i).

(iii) Let $0 < \varepsilon < 1$. Then, according to (2), we have $A_\varepsilon = \{n \in \mathbb{N} : |N(n) - 2| \geq \varepsilon\}$. If we take $H = \{1, 2\} \cup M$, where $M = \{n \in \mathbb{N} : N(n) > 2\}$, then $A_\varepsilon \subset H$. It was proved in [1] that $M(x) = O(\sqrt{x})$, thus, there is a $c > 0$ so that for all $x \geq 2$,

$$A_\varepsilon(x) \leq H(x) \leq cx^{\frac{1}{2}}.$$

By Theorem 3.1, $A_\varepsilon \in \mathcal{I}_{\leq \frac{1}{2}}$ follows. □

Remark 4.5 We note that the set $\mathcal{I}_d$ containing all subsets of $\mathbb{N}$ with zero asymptotic density forms an admissible ideal. The corresponding $\mathcal{I}_d$-convergence is the wellknown statistical convergence. The following results were proved in [14] and [13]:

$$\mathcal{I}_d\text{-lim}\,\frac{\omega(n)}{\log\log n} = \mathcal{I}_d\text{-lim}\,\frac{\Omega(n)}{\log\log n} = 1,$$

$$\mathcal{I}_d\text{-lim}\,\frac{\log\log f(n)}{\log\log n} = \mathcal{I}_d\text{-lim}\,\frac{\log\log f^*(n)}{\log\log n} = 1 + \log 2.$$

We note that $\mathcal{I}_c^{(1)} \subsetneq \mathcal{I}_d$.

If $\mathcal{I}_c^{(q)}$-$\lim x_n = L$ is false for every $0 < q \leq 1$, then (x_n) does not $\mathcal{I}_{<q}$-converge for any q, so $A_\varepsilon = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \notin \mathcal{I}_{<q}$ whenever $0 < q \leq 1$; thus, $\lambda(A_\varepsilon) = 1$ is the only option. Then by Statements VII and VIII it follows that for all $\varepsilon > 0$ and for every n, $a_n \in \{\omega(n), \Omega(n)\}$, and $b_n \in \{f(n), f^*(n)\}$ we have

(i) $\lambda\Big(\Big\{n \in \mathbb{N} : \Big|\frac{a_n}{\log\log n} - 1\Big| \geq \varepsilon\Big\}\Big) = 1,$

(ii) $\lambda\Big(\Big\{n \in \mathbb{N} : \Big|\frac{\log\log b_n}{\log\log n} - (1 + \log 2)\Big| \geq \varepsilon\Big\}\Big) = 1.$

As a consequence, say of i) for $a_n = \omega(n)$, we have that if

$$\Big\{n \in \mathbb{N} : \Big|\frac{\omega(n)}{\log\log n} - 1\Big| \geq \varepsilon\Big\} = \{n_1 < n_2 < \cdots < n_k < \dots\},$$

then

$$\limsup_{k\to\infty} \frac{\log k}{\log n_k} = 1.$$

References

1. H.L. Abbot, P. Erdős, D. Hanson, On the number of times an integer occurs as a binomial coefficient. Am. Math. Month. **81**, 256–260 (1974)
2. V. Baláž, J. Gogola, T. Visnyai, $\mathcal{I}_c^{(q)}$-convergence of arithmetical functions. J. Number Theory **183**, 74–83 (2018)
3. Z. Fehér, B. László, M. Mačaj, T. Šalát, Remarks on arithmetical functions $a_p(n)$, $\gamma(n)$, $\tau(n)$. Ann. Math. Inform. **33**, 35–43 (2006)
4. Š. Gubo, M. Mačaj, T. Šalát, J. Tomanová, On binomial coefficients. Acta Math. (Nitra) **6**, 33–42 (2003)
5. G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, 6th edn. (Oxford Univ. Press, Oxford, 2008)
6. P. Kostyrko, M. Mačaj, T. Šalát, M. Sleziak, $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit poits. Math. Slovaca **55**, 443–464 (2005)

7. P. Kostyrko, T. Šalát, W. Wilczyński, $\mathcal{I}$-convergence. Real Anal. Exchange **26**, 669–686 (2000)
8. J. Mycielski, Sur les représentations des nombres natural par des puissances a bàse et exposant naturales. Colloq. Math. **II**, 245–260 (1951)
9. M.B. Nathanson, *Elementary Methods in Number Theory Graduate Texts of Mathematics*, vol. 195 (Springer, New York, 2000)
10. I. Niven, Averages of exponents in factoring integers. Proc. Am. Math. Soc. **22**, 356–360 (1969)
11. G. Pólya, G. Szegő, *Problems and Theorems in Analysis I* (Springer, Berlin, Heidelberg, New York, 1978)
12. T. Šalát, On the function a_p, $p^{a_p(n)} \parallel n(n > 1)$. Math. Slovaca **44**, 143–151 (1994)
13. T. Šalát, J. Tomanová, On the product of divisors of a positive integer. Math. Slovaca **52**, 271–287 (2002)
14. A. Schinzel, T. Šalát, Remarks on maximum and minimum exponents in factoring. Math. Slovaca **44**, 505–514 (1994)
15. D. Singmaster, How often does an integer occur as a binomial coefficient? Am. Math. Month. **88**, 385–386 (1971)
16. O. Strauch, Š. Porubský, *Distribution of Sequences: A Sampler* (Peter Lang, Frankfurt a. M., 2005)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.