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Optimality of barrier dividend strategy in a jump-diffusion risk model with debit interest

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Abstract
This paper investigates the optimal dividend problem in a jump-diffusion risk model with debit interest. In this model, the insurer could borrow money at a debit interest when the surplus turns negative. However, when the negative surplus attains a certain critical level, the business stops and absolute ruin happens at this moment. A sufficient condition under which the optimal dividend strategy is of barrier type is given in such a risk model. The main result relies on the smoothness of certain function arising from the dividend problem and we prove that it is twice continuously differentiable by the probability argument. Finally, numerical examples are given to illustrate the effects of the debit interest.

Keywords Absolute ruin · Barrier strategy · Debit interest · Hamilton–Jacobi–Bellman equation · Weak infinitesimal generator

Mathematics Subject Classification 60J99 · 91B30

1 Introduction

Risk models under a dividend strategy have been discussed by many authors in the actuarial literature. The Gerber–Shiu discounted penalty function, the expectation of the discounted dividend paid until ruin, the distribution of the dividend payments, the asymptotic distribution of the time of ruin and so on are the main concerned quantities. For the references on these topics and results, see for example [1–12]. For example, Cai et al. [1] assumed that the surplus earns investment income at a constant rate of credit interest. When the surplus is negative, a higher rate of debit interest is applied. They showed how the expected discounted value of the dividends and the optimal dividend barrier can be calculated. Dickson and

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Waters [2] showed how a discrete time risk model can be used to provide approximations when analytic results are unavailable. Frostig [3] and Irbäck [4] studied a risk model with constant high dividend barrier and obtained some asymptotic distributions. Gerber and Shiu [5] studied the joint distribution of the time of ruin, the surplus before the time of ruin, and the deficit at the time of ruin by considering an expected discounted penalty involving these three random variables. Gerber and Shiu [6] considered the optimal dividend problem in a Brownian setting, while Kyprianou and Palmowski [7] and Renaud and Zhou [9] concerned constant dividend barrier problems under Lévy risk models. Li and Garrido [8] considered a Sparre Andersen risk process in the presence of a constant dividend barrier in which the claim waiting times are generalized Erlang(n) distributed. Yuen, Wang and Li [10] studied the classical surplus process with interest and a constant dividend barrier and derived an integro-differential equation for the Gerber–Shiu expected discounted penalty function. Zhou [11] first pointed out interesting connections between some previous results for this model and those for spectrally negative Lévy processes for a risk model with a constant dividend barrier. Zhu [12] investigated dividend optimization of an insurance corporation under a more realistic model, which takes into consideration refinancing or capital injections.

Another actuarial quantity frequently concerned is the optimal dividend problem for the insurance company. The classical optimal dividend problem is looking for the optimal dividend strategy to maximize the expectation of the discounted dividends until the time of ruin. De Finetti [13] firstly proposed this problem in a discrete-time model. He found that the optimal dividend strategy must be a barrier strategy and the optimal level of the barrier could be determined. In the classical risk model, Gerber [14] found that the optimal dividend strategy is the so-called band strategy and simplified to a barrier strategy for exponentially claim sizes. Using the viscosity solution approach from the stochastic control theory, Albrecher and Thonhauser [15] proved that Gerber's result is also correct in the classical risk model with a constant interest. When the risk model is modeled by a Brownian motion with a positive drift, Asmussen and Taksar [16] found that the optimal dividend strategy is a barrier strategy when the dividend rate is unbounded. A recent survey of some classical contributions and recent progress on this topic can be found in Albrecher and Thonhauser [17].

Barrier strategy is a candidate for the optimal dividend strategy in many risk models, but it is not always optimal in general. For example, Azcue and Muler [18] gave an example where the optimal strategy is not a barrier strategy. A natural question is when a barrier strategy is optimal among all admissible dividend strategies. Avram et al. [19] investigated this problem when the risk process is modeled by a spectrally negative Lévy process. They proved the optimality of the barrier strategy if the valve function under the optimal barrier is smooth enough and satisfies the variational inequality. Basing on [19], Loeffen [20] proved that barrier strategy is optimal among all admissible strategies if the scale function is convex or the Lévy measure has a complete monotone density. The risk process is still modeled by the spectrally negative Lévy process in [20].

The model considered in this paper is a jump-diffusion risk model with debit interest, that is, the insurer is allowed to borrow money at a debit interest when the surplus is negative. Meanwhile, the insurer will repay the debts continuously from his premium income, and the negative surplus may recover to a positive level. In this paper, we will discuss the optimality of the barrier dividend strategy for this risk model. The risk process considered in this paper is not a Lévy process and many powerful techniques from Lévy processes theory can not be applied any more. We have to draw on other methods from Markov process theory and conquer corresponding difficulties. The main result obtained in our paper relies on the smoothness of certain function arising from the dividend problem. This function is analogue for the so called scale function in the Lévy insurance process. Mainly exploiting the weak infinitesimal generator theory of Markov processes, we could prove that it is twice continuously differentiable.

Our paper is organized as follows. In Sect. 2, we introduce our risk model and formulate the optimal dividend problem. In Sect. 3, we aim to find the optimal barrier when dividends are paid according to the barrier strategy. Certain function which plays an important role in optimal dividend problem is proved to be twice continuously differentiable by the probability argument. A sufficient condition under which the barrier strategy is optimal among all admissible dividend strategies is given in Sect. 4. Numerical examples are presented to compare with the numerical results of the classical risk model perturbed by diffusion in order to illustrate the effects of the debit interest in Sect. 5.

2 Risk model and problem formulation

Let $\{\Omega, \mathscr{F}, \mathbb{P}\}$ be a probability space with filtration $\{\mathscr{F}_t\}$ containing all objects defined in the following. The surplus process of an insurance company before paying dividends is given by

$$R(t) = u + \int_0^t C(R(s))\,\mathrm{d}s - \sum_{k=1}^{N(t)} Z_k + \sigma W(t), \quad t \ge 0, \tag{2.1}$$

where u is the initial surplus, $C(u) = c$ for $u \ge 0$ and $C(u) = c + \delta u$ for $u < 0$, $c > 0$ is the constant premium rate, $\delta > 0$ represents the debit interest, $\{N(t), t \ge 0\}$ is a Poisson precess with parameter $\lambda > 0$, denoting the total number of claims of the insurance company. $X_1, X_2, \ldots$ independent of $\{N(t), t \ge 0\}$, are positive i.i.d. random variables with common distribution function $P(x) = \mathbb{P}(X \le x)$ with $P(0) = 0$ and density function $p(x)$. $\{W(t), t \ge 0\}$ is a standard Brownian motion independent of $\{N(t), t \ge 0\}$ and $\{X_k, k = 1, 2\cdots\}$. $\sigma > 0$ is the dispersion parameter.

The risk model (2.1) is a jump diffusion risk model with debit interest. Gerber and Yang [21] considered the absolute ruin problems for this model. When $\sigma = 0$, the surplus can never become positive once it is below $-c/\delta$ and the first time the surplus jumps below $-c/\delta$ is called the absolute ruin time. When $\sigma > 0$, as [21] have pointed out, the surplus could bounced back from below $-c/\delta$ because of oscillation. Nevertheless, they still use the term absolute ruin when the surplus attains or falls down the level $-c/\delta$. Let

$$\tau = \inf\{t \ge 0, \quad R(t) \le -c/\delta\}.$$

We call τ is the time of absolute ruin for risk model (2.1).

Now we turn to the optimal dividend problem for the insurance company. Let a dividend strategy $\pi = \{L_t^\pi, t \ge 0\}$ be a non-decreasing, left continuous $\mathscr{F}_t$-adapted process with $L_{0-}^\pi = 0$. L_t^π represents the total dividends the company has paid until time t under the dividend strategy π. Then the surplus process becomes

$$R_t^\pi = u + \int_0^t C(R_s^\pi)\ \mathrm{d}s - \sum_{k=1}^{N(t)} Z_k + \sigma W(t) - L_t^\pi, \quad t \ge 0. \tag{2.2}$$

Denote τ^π the corresponding absolute ruin time.

Our ultimate goal is the maximization of the expectation of the discounted dividends until the time of absolute ruin. We define the value function under the strategy π by

$$V_\pi(u) = \mathbb{E}[\int_0^{\tau^\pi} e^{-\beta t} dL_t^\pi | R_0^\pi = u] = \mathbb{E}^u[\int_0^{\tau^\pi} e^{-\beta t} dL_t^\pi]$$

where $\beta > 0$ is the discounted rate. As usual we assume that $0 < \beta < \delta$ in this paper. By definition we see that $V_\pi(u) = 0$ for $u \leq -c/\delta$.

We call a strategy π admissible if the absolute ruin does not occur by a dividend pay-out, that is, $L_{t+}^\pi - L_t^\pi \leq R_t^\pi$ for $t < \tau^\pi$. Let Π be the set of all admissible dividend strategies. We aim to find an admissible policy $\pi^* \in \Pi$ to maximize $V_\pi(u)$, i.e. we are looking for the value function

$$V(u) = \sup_{\pi \in \Pi} V_\pi(u), \tag{2.3}$$

and to find an optimal policy π^* from Π that satisfies $V(u) = V_{\pi^*}(u)$ for all $u > -c/\delta$.

3 Optimal barrier under the barrier strategies

It is assumed in this section that dividends are paid according to barrier strategies and we aim to find the optimal barrier. Now let us consider the barrier strategy $\pi_b = \{L_t^b, t \geq 0\}$ under the barrier b, where $L_t^b = (\sup_{0 \leq s \leq t} R(s) - b) \vee 0$ for $t \geq 0$. It can be seen that $\pi_b \in \Pi$. Denote $R_b(t)$ the surplus at time t and $V_b(u)$ the value function under the barrier dividend strategy π_b.

For a given barrier $b \geq 0$, let $T_b = \inf\{t \geq 0, R(t) = b\}$ denote the first hitting time of the surplus process $\{R(t), t \geq 0\}$, $T_b = \infty$ if the set is empty. We denote $L(u; b)$ to be the expected present value of a payment of 1 due at the time when the surplus process $\{R(t), t \geq 0\}$ reaches the level b for the first time, provided that absolute ruin has not occurred in the meantime. Then the function $L(u; b)$ can be expressed as

$$L(u; b) = \mathbb{E}^u[e^{-\beta T_b} \mathbf{1}(T_b < \tau)] \tag{3.1}$$

for $-c/\delta < u < b$. By the strong Markov property of the process $\{R(t), t \geq 0\}$, we have

$$L(u_1; u_3) = L(u_1; u_2) L(u_2; u_3), \quad -c/\delta < u_1 \leq u_2 \leq u_3 \leq b.$$

This identity has also been pointed out by Gerber et al. [22] where $L(u_1; u_3)$ is replaced by $C(u_1, u_3)$ there.

Let $f(u)$ be a nonnegative increasing function such that

$$L(u; b) = \frac{f(u)}{f(b)}, \quad -c/\delta < u \leq b. \tag{3.2}$$

As in [22] we can get $V_b(u) = L(u; b) V_b(b)$ for $-c/\delta < u < b$ and the left derivative of $V_b(u)$ at $u = b$ is $V_b'(b-) = 1$. Thus $1 = L'(b-; b) V_b(b)$. For $-c/\delta < u \leq b$, $V_b(u) = \frac{f(u)}{f(b)} \cdot \frac{1}{L'(b-; b)} = \frac{f(u)}{f'(b)}$. Then we obtain

$$V_b(u) = \begin{cases} \frac{f(u)}{f'(b)}, & -c/\delta < u \leq b, \\ u - b + V_b(b), & u > b. \end{cases} \tag{3.3}$$

It can be seen that $f(u)$ plays an important role in studying the optimal dividend problems. Our main result in Sect. 4 relies on the properties of $f(u)$. First, we prove the smoothness of $f(u)$ in Theorem 3.3.

The result of Theorem 3.3 relies on the following lemmas.

Lemma 3.1 *If* $-c/\delta < u < b$, *then* $\lim_{\varepsilon\to 0+} \frac{\mathbb{P}^u(T_b\le\varepsilon)}{\varepsilon} = 0$, *that is,* $\mathbb{P}(T_b \le \varepsilon | R(0) = u) = \mathbb{P}^u(T_b \le \varepsilon) = o(\varepsilon)$ $(\varepsilon \to 0+)$.

Proof For $0 < \varepsilon < \frac{b-u}{2c}$, it follows from the definition of T_b that

$$\begin{aligned} 0 \le \mathbb{P}^u(T_b \le \varepsilon) &\le \mathbb{P}^u\left(\max_{0\le t\le\varepsilon}\left\{\int_0^t C(R(s))ds + \sigma W(t)\right\} \ge b-u\right)\\ &\le \mathbb{P}^u\left(\max_{0\le t\le\varepsilon}\int_0^t C(R(s))ds + \max_{0\le t\le\varepsilon}\sigma W(t) \ge b-u\right)\\ &\le \mathbb{P}^u\left(\max_{0\le t\le\varepsilon}\sigma W(t) \ge b-u-c\varepsilon\right)\\ &= \mathbb{P}^u\left(\max_{0\le t\le\varepsilon} W(t) \ge \frac{b-u-c\varepsilon}{\sigma}\right). \end{aligned} \tag{3.4}$$

It can be seen that

$$\lim_{\varepsilon\to 0+}\frac{\mathbb{P}^u\left(\max_{0\le t\le\varepsilon} W(t) \ge \frac{b-u-c\varepsilon}{\sigma}\right)}{\varepsilon} \le \lim_{\varepsilon\to 0+}\frac{\mathbb{P}^u\left(\max_{0\le t\le\varepsilon} W(t) \ge \frac{b-u}{2\sigma}\right)}{\varepsilon}. \tag{3.5}$$

Since

$$\mathbb{P}^u\left(\max_{0\le t\le\varepsilon} W(t) \ge \frac{b-u}{2\sigma}\right) = \frac{2}{\sqrt{2\pi}}\int_{\frac{b-u}{2\sigma\sqrt{\varepsilon}}}^{\infty} \mathrm{e}^{-x^2/2}\,\mathrm{d}x,$$

using L'Hospital's rule, we get

$$\lim_{\varepsilon\to 0+}\frac{\mathbb{P}^u\left(\max_{0\le t\le\varepsilon} W(t) \ge \frac{b-u}{2\sigma}\right)}{\varepsilon} = 0. \tag{3.6}$$

From (3.4), (3.5) and (3.6), the result $\lim_{\varepsilon\to 0+}\frac{\mathbb{P}^u(T_b\le\varepsilon)}{\varepsilon} = 0$ follows. □

From the general theory of Markov processes, see for example Dynkin [23], we get the following lemma.

Lemma 3.2 *If* $\{X(t), t \ge 0\}$ *is a right continuous Feller process, then* $\{X(t), t \ge 0\}$ *is a strong Markov process. Moreover, suppose that* A *is its weak infinitesimal operator and* $\mathscr{D}(A)$ *is the domain of* A. *If* $h(x) \in \mathscr{D}(A)$, *then* $Ah(x)$ *is continuous.*

Theorem 3.3 *Assume that the distribution function* $P(x)$ *has a continuous density function* $p(x)$ *on* $[0, +\infty)$, *then* $f(u)$ *is twice continuously differentiable on* $(-c/\delta, +\infty)$ *and satisfies the integro-differential equation*

$$\frac{\sigma^2}{2}f''(u) + C(u)f'(u) + \lambda\int_0^{u+c/\delta} f(u-y)p(y)\,dy - (\lambda+\beta)f(u) = 0. \tag{3.7}$$

Proof Select an arbitrary $b > 0$, then $f(u) = f(b)L(u;b)$, $-c/\delta < u < b$. If we can prove that $L(u;b)$ is twice continuously differentiable in u on $(-c/\delta, b)$ and satisfies the integro-differential equation (3.7), then the results for $f(u)$ follows since b is arbitrary.

Let

$$Y(t)=\begin{cases}R(t), & t<\tau\wedge T_b,\\ \partial_1, & t\geq \tau\wedge T_b,\end{cases}\quad Z(t)=\begin{cases}Y(t), & t<\xi,\\ \partial_2, & t\geq \xi.\end{cases}$$

where ∂_1 and ∂_2 are two death states, ξ is an exponential random variable with parameter β and is independent of the stochastic process $\{Y(t), t\geq 0\}$. $\{Z(t), t\geq 0\}$ is a twice killing process for $\{R(t), t\geq 0\}$. The process $\{Z(t), t\geq 0\}$ is a homogeneous strong Markov process, see Chapter 3 of Blumenthal and Getoor [24]. Let $\widetilde{\mathscr{A}}$ denote the weak infinitesimal generator of the process $\{Z(t), t\geq 0\}$ and $\mathscr{D}(\widetilde{\mathscr{A}})$ be the domain of $\widetilde{\mathscr{A}}$. If $h(u)\in\mathscr{D}(\widetilde{\mathscr{A}})$, we have

$$\begin{aligned}\widetilde{\mathscr{A}}h(u)&=\lim_{t\to 0}\frac{\mathbb{E}^u[h(Z(t))]-h(u)}{t}\\ &=\frac{\sigma^2}{2}h''(u)+C(u)h'(u)+\lambda\int_0^{u+c/\delta}h(u-y)p(y)\,\mathrm{d}y-(\lambda+\beta)h(u).\end{aligned}$$

for $-c/\delta<u<b$.

Considering an infinitesimal time interval $[0,\mathrm{d}t]$, from Lemma 3.1 and the convention that $L(\partial_i,b)=0$, $i=1,2$, we get

$$\begin{aligned}\mathbb{E}^u[L(R_b(\mathrm{d}t);b)]&=\mathbb{E}^u[L(R_b(\mathrm{d}t);b)\mathbf{1}(T_b\leq \mathrm{d}t)]+\mathbb{E}^u[L(R_b(\mathrm{d}t);b)\mathbf{1}(T_b>\mathrm{d}t)]\\ &=\mathbb{E}^u[L(R(\mathrm{d}t);b)\mathbf{1}(T_b>\mathrm{d}t)]+o(\mathrm{d}t)\\ &=\mathbb{E}^u[L(R(\mathrm{d}t);b)]-\mathbb{E}^u[L(R(\mathrm{d}t);b)\mathbf{1}(T_b\leq \mathrm{d}t)]+o(\mathrm{d}t)\\ &=\mathbb{E}^u[L(R(\mathrm{d}t);b)]+o(\mathrm{d}t).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathbb{E}^u[L(R(\mathrm{d}t);b)]&=\mathbb{E}^u[L(R(\mathrm{d}t);b)\mathbf{1}(T_b\leq \mathrm{d}t)]+\mathbb{E}^u[L(R(\mathrm{d}t);b)\mathbf{1}(T_b>\mathrm{d}t)]\\ &=\mathbb{E}^u[L(Y(\mathrm{d}t);b)\mathbf{1}(T_b>\mathrm{d}t)]+o(\mathrm{d}t)\\ &=\mathbb{E}^u[L(Y(\mathrm{d}t);b)]-\mathbb{E}^u[L(Y(\mathrm{d}t);b)\mathbf{1}(T_b\leq \mathrm{d}t)]+o(\mathrm{d}t)\\ &=\mathbb{E}^u[L(Y(\mathrm{d}t);b)]+o(\mathrm{d}t).\end{aligned}\tag{3.8}$$

Since

$$\mathbb{E}^u[L(R(dt);b)]=\mathrm{e}^{\beta dt}L(u;b)+o(\mathrm{d}t),$$

we conclude from (3.8) that

$$\mathbb{E}^u[L(Y(\mathrm{d}t);b)]=\mathrm{e}^{\beta dt}L(u;b)+o(\mathrm{d}t).\tag{3.9}$$

Denote

$$h_L(u)=\lim_{dt\to 0}\frac{\mathbb{E}^u[L(Z(\mathrm{d}t);b)]-L(u;b)}{\mathrm{d}t}.$$

It follows from (3.9) that

$$\lim_{dt\to 0}\frac{\mathbb{E}^u[L(Z(\mathrm{d}t);b)]-L(u;b)}{\mathrm{d}t}=\lim_{dt\to 0}\frac{\mathbb{E}^u[\mathrm{e}^{-\beta dt}L(Y(\mathrm{d}t);b)]-L(u;b)}{\mathrm{d}t}=0.$$

So we have $h_L(u)=0$, that is, $h_L(u)$ is bounded and continuous. We also have

$$\lim_{dt\to 0}\mathbb{E}^u[h_L(Z(\mathrm{d}t))]=h_L(u).$$

By the definition of the weak infinitesimal generator of Markov processes (see, for example, Dynkin [23]), we know that $L(u; b) \in \mathscr{D}(\widetilde{\mathscr{A}})$ and

$$\widetilde{\mathscr{A}} L(u; b) = \frac{\sigma^2}{2} L''_u(u; b) + C(u) L'_u(u; b) + \lambda \int_0^{u+c/\delta} L(u-y; b) p(y)\,\mathrm{d}y - (\lambda+\beta) L(u; b) \tag{3.10}$$

for $-c/\delta < u < b$. It follows from Lemma 3.2 that $\widetilde{\mathscr{A}} L(u; b)$ is continuous, that is, $L(u; b)$ is twice continuously differentiable in u on $(-c/\delta, b)$.

From (3.10) and $h_L(u) = 0$, we get for $-c/\delta < u < b$ that

$$\frac{\sigma^2}{2} L''_u(u; b) + C(u) L'_u(u; b) + \lambda \int_0^{u+c/\delta} L(u-y; b) p(y)\,\mathrm{d}y - (\lambda+\beta) L(u; b) = 0.$$

This ends the proof. □

Now let us define the operator

$$\mathscr{A} g(u) = \frac{\sigma^2}{2} g''(u) + C(u) g'(u) + \lambda \int_0^{u+c/\delta} g(u-y) p(y)\,\mathrm{d}y - \lambda g(u).$$

Remark 3.4 Using the operator $\mathscr{A}$, Eq. (3.7) can be rewritten as

$$(\mathscr{A} - \beta) f(u) = 0, \ \ u > -c/\delta.$$

Remark 3.5 From (3.2) we know that $V_b(u) = \frac{f(u)}{f'(b)}$ for $-c/\delta < u < b$. It follows from Theorem 3.3 that $V_b(u)$ is twice continuously differentiable and satisfies

$$\frac{\sigma^2}{2} V''_b(u) + C(u) V'_b(u) + \lambda \int_0^{u+c/\delta} V_b(u-y) p(y)\,\mathrm{d}y - (\lambda+\beta) V_b(u) = 0. \tag{3.11}$$

for $-c/\delta < u < b$.

By Remark 3.4, Eq. (3.11) can be written as

$$(\mathscr{A} - \beta) V_b(u) = 0, \quad -c/\delta < u < b. \tag{3.12}$$

Remark 3.6 For $u > -c/\delta$, let $b \to \infty$, then we can see that $V_b(u) \to 0$. It can be seen from (3.3) that $f'(b) \to \infty$ as $b \to \infty$.

Put $\mathcal{B} = \{b \geq 0,\ f'(b) \leq f'(u) \text{ for all } u \geq 0\}$. For $b_1, b_2 \in \mathcal{B}$, it can be verified that $V_{b_1}(u) \leq V_{b_2}(u)$ if $b_1 \leq b_2$. Denote $b^* = \sup \mathcal{B}$, then b^* is the optimal barrier under the barrier dividend strategies. From Remark 3.6 we know that $b^* < \infty$.

π_{b^*} is a candidate of the optimal dividend strategy among the admissible dividend strategies set Π. In next section, we will prove that it is indeed optimal among Π if some conditions are satisfied. Throughout the rest of this paper, we assume that the distribution function $P(x)$ has a continuous density function $p(x)$ on $[0, +\infty)$.

4 Optimality of barrier strategy

In this section, we will prove that π_{b^*} is optimal among all admissible strategies in Π under certain conditions. Our main tool is from the stochastic control theory. Using the dynamic programming approach described in Fleming and Soner [25], we see that if $V(u)$ is twice continuously differentiable, then V satisfies the Hamilton–Jacobi–Bellman (HJB) equation

$$\max\{(\mathscr{A}-\beta)V(u),\ 1-V'(u)\}=0,\quad u>-c/\delta. \tag{4.1}$$

Lemma 4.1 *If $f(u)$ is convex for $u\geq b^*$, then $(\mathscr{A}-\beta)V_{b^*}(u)\leq 0$, $u\geq b^*$.*

Proof For any $u>b^*,\ b^*<z<u$, similarly as in (3.12), we get

$$\frac{\sigma^2}{2}V_u''(z)+C(u)V_u'(z)+\lambda\int_0^{z+c/\delta}V_u(z-y)p(y)\,\mathrm{d}y-(\lambda+\beta)V_u(z)=0. \tag{4.2}$$

Then

$$\begin{aligned}
&\frac{\sigma^2}{2}V_{b^*}''(z)+C(u)V_{b^*}'(z)+\lambda\int_0^{z+c/\delta}V_{b^*}(z-y)p(y)\,\mathrm{d}y-(\lambda+\beta)V_{b^*}(z)\\
&=\frac{\sigma^2}{2}(V_{b^*}''-V_u'')(z)+C(u)(V_{b^*}'-V_u')(z)\\
&\quad-\lambda\int_0^{z+c/\delta}(V_{b^*}-V_u)(z-y)p(y)\,\mathrm{d}y-(\lambda+\beta)(V_{b^*}-V_u)(z)\\
&=\frac{\sigma^2}{2}(V_{b^*}''-V_u'')(z)+C(u)(V_{b^*}'-V_u')(z)\\
&\quad-\lambda\int_0^{\infty}[(V_{b^*}-V_u)(z)-(V_{b^*}-V_u)(z-y)]p(y)\,\mathrm{d}y\\
&\quad-\beta(V_{b^*}-V_u)(z)
\end{aligned} \tag{4.3}$$

Since $f(u)$ is convex for $u\geq b^*$, we get

$$(V_{b^*}'-V_u')(y)=1-\frac{f'(y)}{f'(u)}\geq 0,\quad b^*\leq y\leq u. \tag{4.4}$$

If $y<b^*$, then

$$(V_{b^*}'-V_u')(y)=\frac{f'(y)}{f'(b^*)}-\frac{f'(y)}{f'(u)}\geq 0,\quad y<b^*. \tag{4.5}$$

In fact, $f(u)$ is increasing in u, so $f'(u)\geq 0$. This fact together with the definition of b^* gives (4.5).

From (4.4) and (4.5), we know that $(V_{b^*}-V_u)(z)-(V_{b^*}-V_u)(z-y)\geq 0$.

It follows from the optimality of π_{b^*} that $(V_{b^*}-V_u)(z)\geq 0$. Then from (4.3), we get

$$\begin{aligned}
&\frac{\sigma^2}{2}V_{b^*}''(z)+C(z)V_{b^*}'(z)+\lambda\int_0^{z+c/\delta}V_{b^*}(z-y)p(y)\,\mathrm{d}y-(\lambda+\beta)V_{b^*}(z)\\
&\leq\frac{\sigma^2}{2}(V_{b^*}''-V_u'')(z)+C(z)(V_{b^*}'-V_u')(z)
\end{aligned} \tag{4.6}$$

Letting $z\uparrow u$ in both sides of (4.6), we get

$$\begin{aligned}
&\frac{\sigma^2}{2}V_{b^*}''(z)+C(z)V_{b^*}'(z)+\lambda\int_0^{z+c/\delta}V_{b^*}(z-y)p(y)\,\mathrm{d}y-(\lambda+\beta)V_{b^*}(z)\\
&\leq\lim_{z\uparrow u}[\frac{\sigma^2}{2}(V_{b^*}''-V_u'')(z)+C(z)(V_{b^*}'-V_u')(z)]\leq 0.
\end{aligned}$$

This ends the proof. □

Lemma 4.2 *If $f(u)$ is convex for $u \geq b^*$, then V_{b^*} is a classical solution to the HJB equation* (4.1).

Proof From Remark 3.5, we know $V_{b^*}(u)$ is twice continuously differentiable on $(-c/\delta, b^*)$. By the optimality of π_{b^*}, we get $V'_{b^*}(b^*-) = V'_{b^*}(b^*+) = 1$ and $V''_{b^*}(b^*-) = V''_{b^*}(b^*+) = 0$. So $V_{b^*}(u)$ is twice continuously differentiable on $(-c/\delta, \infty)$.

It follows from Remark 3.5 that

$$(\mathscr{A} - \beta)V_{b^*}(u) = 0, \quad -c/\delta < u < b^*. \tag{4.7}$$

By the definition of b^*, we get

$$V'_{b^*}(u) = \frac{f'(u)}{f'(b^*)} \leq 1, \quad -c/\delta < u < b^*. \tag{4.8}$$

It follows from (4.7) and (4.8) that $V_{b^*}(u)$ is a classical solution to the HJB equation (4.1) when $-c/\delta < u < b^*$.

For $u \geq b^*$, we get from Lemma 4.1 that

$$(\mathscr{A} - \beta)V_{b^*}(u) \leq 0, \quad u \geq b^*. \tag{4.9}$$

It can be easily verified that

$$V'_{b^*}(u) = 1, \quad u \geq b^*. \tag{4.10}$$

It follows from (4.9) and (4.10) that $V_{b^*}(u)$ is a classical solution to the HJB equation (4.1) when $u \geq b^*$.

From the above discussions, we know that $V_{b^*}(u)$ is a classical solution to the HJB equation (4.1) when $u > -c/\delta$. □

Theorem 4.3 *If $f(u)$ is convex for $u \geq b^*$, we have $V_{b^*}(u) = V(u)$ for $u > -c/\delta$ and hence π_{b^*} is the optimal dividend strategy in Π.*

Proof Let $\pi \in \Pi$ be any admissible strategy. Applying the Itô formula to $\mathrm{e}^{-\beta(t\wedge\tau^\pi)}V_{b^*}(R^\pi_{t\wedge\tau^\pi})$, we get

$$\begin{aligned}
&\mathrm{e}^{-\beta(t\wedge\tau^\pi)}V_{b^*}(R^\pi_{t\wedge\tau^\pi}) - V_{b^*}(u)\\
&= \int_0^{t\wedge\tau^\pi} \mathrm{e}^{-\beta s}[\frac{\sigma^2}{2}V''_{b^*}(R^\pi_{s-}) + C(R^\pi_{s-})V'_{b^*}(R^\pi_{s-}) - \beta V_{b^*}(R^\pi_{s-})]\,\mathrm{d}s\\
&\quad + \sigma\int_0^{t\wedge\tau^\pi} \mathrm{e}^{-\beta s}V'_{b^*}(R^\pi_{s-})dW_s\\
&\quad - \int_0^{t\wedge\tau^\pi} \mathrm{e}^{-\beta s}V'_{b^*}(R^\pi_{s-})d\widetilde{L}^\pi_s\\
&\quad + \sum_{0<s\leq t\wedge\tau^\pi,\ R^\pi_{s-}\neq R^\pi_s} \mathrm{e}^{-\beta s}[V_{b^*}(R^\pi_s) - V_{b^*}(R^\pi_{s-})]\\
&\quad + \sum_{0<s\leq t\wedge\tau^\pi,\ R^\pi_s\neq R^\pi_{s+}} \mathrm{e}^{-\beta s}[V_{b^*}(R^\pi_{s+}) - V_{b^*}(R^\pi_s)].
\end{aligned} \tag{4.11}$$

Note that $R^{\pi}_{s-} \neq R^{\pi}_{s}$ occurs due to the jump of a claim and

$$\sum_{0<s\le t\wedge\tau^{\pi},\ R^{\pi}_{s-}\neq R^{\pi}_{s}} \mathrm{e}^{-\beta s}[V_{b^*}(R^{\pi}_{s}) - V_{b^*}(R^{\pi}_{s-})] - \lambda \int_0^{t\wedge\tau^{\pi}} \mathrm{e}^{-\beta s} \times \left(\int_0^{R^{\pi}_{s}+c/\delta} V_{b^*}(R^{\pi}_{s-} - y) p(y)\,\mathrm{d}y - V_{b^*}(R^{\pi}_{s-}) \right) \mathrm{d}s$$

is a martingale with expectation zero.

We know that $R^{\pi}_{s} \neq R^{\pi}_{s+}$ occurs due to the jump of dividend pay-out, then $R^{\pi}_{s+} - R^{\pi}_{s} = -(L^{\pi}_{s+} - L^{\pi}_{s})$ and

$$\sum_{0<s\le t\wedge\tau^{\pi},\ R^{\pi}_{s+}\neq R^{\pi}_{s}} \mathrm{e}^{-\beta s}[V_{b^*}(R^{\pi}_{s+}) - V_{b^*}(R^{\pi}_{s})] = - \sum_{0<s\le t\wedge\tau^{\pi},\ L^{\pi}_{s+}\neq L^{\pi}_{s}} \mathrm{e}^{-\beta s} \int_0^{L^{\pi}_{s+}-L^{\pi}_{s}} V'_{b^*}(R^{\pi}_{s} - y)\,\mathrm{d}y$$

It follows from $V'_{b^*} \geq 1$ that

$$\begin{aligned} &-\int_0^{t\wedge\tau^{\pi}} \mathrm{e}^{-\beta s} V'_{b^*}(R^{\pi}_{s-}) d\widetilde{L}^{\pi}_{s} + \sum_{0<s\le t\wedge\tau^{\pi},\ R^{\pi}_{s}\neq R^{\pi}_{s+}} \mathrm{e}^{-\beta s}[V_{b^*}(R^{\pi}_{s+}) - V_{b^*}(R^{\pi}_{s})] \\ &\le -\int_0^{t\wedge\tau^{\pi}} \mathrm{e}^{-\beta s} d\widetilde{L}^{\pi}_{s} - \sum_{0<s\le t\wedge\tau^{\pi},\ L^{\pi}_{s+}\neq L^{\pi}_{s}} \mathrm{e}^{-\beta s}(L^{\pi}_{s+} - L^{\pi}_{s}) \\ &= -\int_0^{t\wedge\tau^{\pi}} \mathrm{e}^{-\beta s}\,\mathrm{d}L^{\pi}_{s}. \end{aligned} \tag{4.12}$$

Plugging (4.12) into (4.11), we get

$$\begin{aligned} &\mathrm{e}^{-\beta(t\wedge\tau^{\pi})} V_{b^*}(R^{\pi}_{t\wedge\tau^{\pi}}) - V_{b^*}(u) \\ &\quad\le \int_0^{t\wedge\tau^{\pi}} \mathrm{e}^{-\beta s} \left[\frac{\sigma^2}{2} V''_{b^*}(R^{\pi}_{s-}) + C(R^{\pi}_{s-}) V'_{b^*}(R^{\pi}_{s-}) - \beta V_{b^*}(R^{\pi}_{s-}) \right] \mathrm{d}s \\ &\qquad + \sigma \int_0^{t\wedge\tau^{\pi}} \mathrm{e}^{-\beta s} V'_{b^*}(R^{\pi}_{s-}) dW_s - \int_0^{t\wedge\tau^{\pi}} \mathrm{e}^{-\beta s}\,\mathrm{d}L^{\pi}_{s} \\ &\qquad + \sum_{0<s\le t\wedge\tau^{\pi},\ R^{\pi}_{s-}\neq R^{\pi}_{s}} \mathrm{e}^{-\beta s}[V_{b^*}(R^{\pi}_{s}) - V_{b^*}(R^{\pi}_{s-})]. \end{aligned} \tag{4.13}$$

Therefore taking expectations on both sides of (4.13), we obtain

$$\begin{aligned} &\mathbb{E}^u[\mathrm{e}^{-\beta(t\wedge\tau^{\pi})} V_{b^*}(R^{\pi}_{t\wedge\tau^{\pi}})] \\ &\le V_{b^*}(u) - \mathbb{E}^u \int_0^{t\wedge\tau^{\pi}} \mathrm{e}^{-\beta s}\,\mathrm{d}L^{\pi}_{s} \\ &\quad + \mathbb{E}^u \int_0^{t\wedge\tau^{\pi}} \mathrm{e}^{-\beta s} \left[\frac{\sigma^2}{2} V''_{b^*}(R^{\pi}_{s-}) + C(R^{\pi}_{s-}) V'_{b^*}(R^{\pi}_{s-}) \right. \end{aligned}$$

$$-(\lambda+\beta)V_{b^*}(R^{\pi}_{s-})+\lambda\int_0^{R^{\pi}_s+c/\delta}V_{b^*}(R^{\pi}_{s-}-y)p(y)\,\mathrm{d}y\Bigg]\,\mathrm{d}s$$

$$=V_{b^*}(u)+\mathbb{E}^u\int_0^{t\wedge T^{\pi}_{ab}}\mathrm{e}^{-\beta s}(\mathscr{A}-\beta)V_{b^*}(R^{\pi}_{s-})\,\mathrm{d}s-\mathbb{E}^u\int_0^{t\wedge T^{\pi}_{ab}}\mathrm{e}^{-\beta s}\,\mathrm{d}L^{\pi}_s. \tag{4.14}$$

It follows from Lemma 4.2 that $V_{b^*}(u)$ is a classical solution to the HJB equation (4.1) and then the second term on the right-hand side in (4.14) is non-positive. The left-hand side of (4.14) is also non-positive, so

$$\mathbb{E}^u\int_0^{t\wedge\tau^{\pi}}\mathrm{e}^{-\beta s}\,\mathrm{d}L^{\pi}_s\le V_{b^*}(u). \tag{4.15}$$

Letting $t\to\infty$ in (4.15),

$$\mathbb{E}^u\int_0^{\tau^{\pi}}\mathrm{e}^{-\beta s}\,\mathrm{d}L^{\pi}_s\le V_{b^*}(u).$$

This ends the proof. □

Remark 4.4 From Theorem 3.3, we know that $f(u)$ is twice continuously differentiable on $(-c/\delta,+\infty)$ once $p(x)$ is continuous on $[0,+\infty)$. Moreover, if $f(x)$ is convex for $u\ge b^*$ then we can guarantee the optimality of barrier strategy.

Remark 4.5 When $\delta\to 0$ then $-c/\delta\to-\infty$. Then ruin or absolute ruin will never occur and this is not a realistic case. When $\delta\to+\infty$, then $-c/\delta\to 0$. Then our risk model reduces to the classical risk model perturbed by diffusion which belong to the classes of spectrally negative Lévy process. In this case, $f(u)$ in Theorem 4.3 play the role of scale function in Loeffen [20].

5 Numerical examples

In this section, we present two numerical examples to give some intuitive interpretations of the results, and make numerical comparisons with the results of the classical risk model perturbed by diffusion.

Example 5.1 Assume that the claim sizes are exponentially distributed with parameter $\rho>0$, that is, $P(y)=1-\mathrm{e}^{-\rho y},\ y>0$. When $-c/\delta<u<0$, Eq. (3.7) becomes

$$\frac{\sigma^2}{2}f''(u)+(c+\delta u)f'(u)+\lambda\int_0^{u+c/\delta}f(u-y)\rho\mathrm{e}^{-\rho y}\,\mathrm{d}y-(\lambda+\beta)f(u)=0. \tag{5.1}$$

When $u\ge 0$, Eq. (3.7) becomes

$$\frac{\sigma^2}{2}f''(u)+cf'(u)+\lambda\int_0^{u+c/\delta}f(u-y)\rho\mathrm{e}^{-\rho y}\,\mathrm{d}y-(\lambda+\beta)f(u)=0. \tag{5.2}$$

Applying the operator $(\frac{d}{du}+\rho)$ to Eq. (5.1), we get

$$\frac{\sigma^2}{2}f'''(u)+\left(\frac{\sigma^2\rho}{2}+c+\delta u\right)f''(u)+[\rho(c+\delta u)+\delta-\lambda-\beta]f'(u)-\beta\rho f(u)=0. \tag{5.3}$$

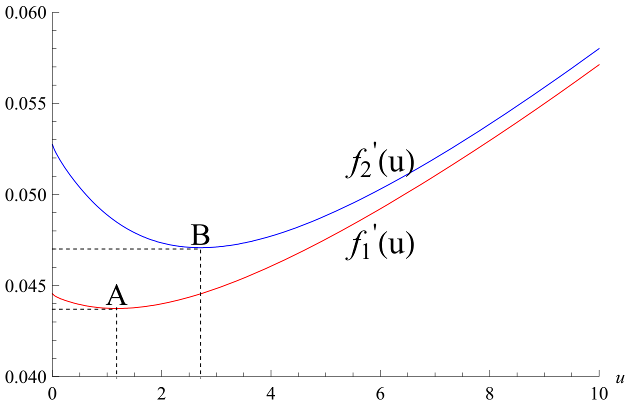

Fig. 1 $f_1'(u)$ when $\delta = 0.05$ (red line) and $f_2'(u)$ when $\delta = 0.08$ (blue line). (Color figure online)

With the boundary conditions $f(-c/\delta) = 0,\ f(0) = 1,\ f''(-c/\delta) = 0$, Eq. (5.3) can be determined.

Applying the operator $(\frac{d}{du} + \rho)$ to Eq. (5.2), we get

$$\frac{\sigma^2}{2} f'''(u) + (\frac{\sigma^2 \rho}{2} + c) f''(u) + (\rho c - \lambda - \beta) f'(u) - \beta \rho f(u) = 0. \tag{5.4}$$

It can be verified that there exist three characteristic roots of Eq. (5.4) λ_1, λ_2 and λ_3. So the general solution of Eq. (5.4) can be expressed as

$$f(u) = k_1 e^{\lambda_1 u} + k_2 e^{\lambda_2 u} + k_3 e^{\lambda_3 u}, \quad u \geq 0. \tag{5.5}$$

From Theorem 3.3 we know that $f(u)$ is twice continuously differentiable on $(-c/\delta, \infty)$, so $k_1,\ k_2$ and k_3 can be determined by the conditions: $f(0+) = f(0-) = 1$, $f'(0+) = f'(0-)$ and $f''(0+) = f''(0-)$.

Let $c = 1, \lambda = 0.5, \rho = 1, \beta = 0.02, \sigma = 0.3$. We consider two cases: $\delta = 0.05$ and $\delta = 0.08$. $b^* = 1.18$ when $\delta = 0.05$ and $b^* = 2.71$ when $\delta = 0.08$. Intuitively, the higher the debit interest, the higher the optimal dividend barrier. In Fig. 1, we plot the function $f'(u)$, where we denote $f_1'(u)$ in the case $\delta = 0.05$ and $f_2'(u)$ in the case $\delta = 0.08$. The abscissa of point A is 1.18 and the abscissa of point B is 2.71. From Fig. 1 we can see that the function $f_1'(u)$ is increasing in u after A and $f_2'(u)$ is increasing in u after B. So we conclude that the barrier dividend strategy at $b^* = 1.18$ in the case $\delta = 0.05$ and the barrier dividend strategy at $b^* = 2.71$ in the case $\delta = 0.08$ are both optimal among all dividend strategies.

When $\delta \to +\infty$, then $-c/\delta \to 0$ and the risk model becomes the classical risk model perturbed by diffusion. The barrier dividend barrier b^* is equal to 9.44733, and is larger than that of the jump-diffusion risk model with debit interest. In Table 1, we give the numerical values of $V_{b^*}(u)$ for the case $\delta = 0.05$, $\delta = 0.08$ and $\delta = +\infty$ ($+\infty$ means the classical risk model perturbed by diffusion). It can be seen that $V_{b^*}(u)$ is increasing as u increases and becomes smaller as the debit interest becomes larger. The function $V_{b^*}(u)$ of the classical risk model perturbed by diffusion is smaller than that of the jump-diffusion risk model with debit interest.

Table 1 The numerical values of $V_{b^*}(u)$ for the exponentially distributed claim sizes

$\delta\backslash u$	-3	-2.5	-2.0	-1.5	-1.0	-0.5	0	0.5	1.0
0.05	19.3826	20.0343	20.6547	21.2464	21.8117	22.3531	22.8726	23.3678	23.8689
0.08	16.7159	17.6656	18.5267	19.3065	20.0134	20.6558	21.2416	21.7875	22.314
$+\infty$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	4.0930	7.3538
$\delta\backslash u$	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
0.05	24.3689	24.8689	25.3689	25.8689	26.3689	26.8689	27.3689	27.8689	28.3689
0.08	22.8276	23.333	23.8342	24.3342	24.8342	25.3342	25.8342	26.3342	26.8342
$+\infty$	9.9641	12.0715	13.7903	15.2095	16.3978	17.4089	18.2841	19.0557	19.7489

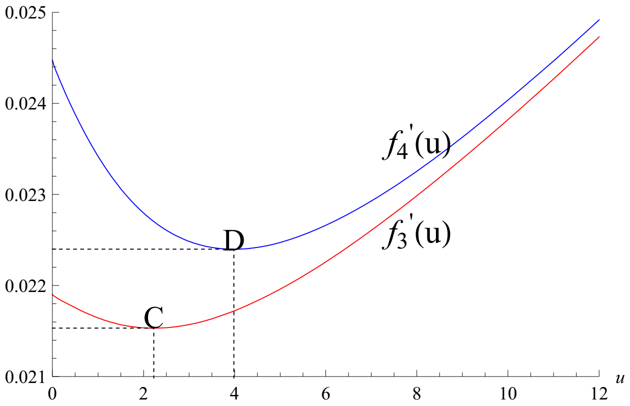

Fig. 2 $f_3'(u)$ when $\delta = 0.05$ (red line) and $f_4'(u)$ when $\delta = 0.08$ (blue line). (Color figure online)

Example 5.2 Assume that the claim size distribution is Erlang(2) distributed with parameter $\eta > 0$, that is, $p(y) = \eta^2 y e^{-\eta y},\ \ y > 0$. When $-c/\delta < u < 0$, Eq. (3.7) becomes

$$\frac{\sigma^2}{2} f''(u) + (c + \delta u) f'(u) + \lambda \int_0^{u+c/\delta} f(u - y)\eta^2 y e^{-\eta y}\, dy - (\lambda + \beta) f(u) = 0. \quad (5.6)$$

When $u \geq 0$, Eq. (3.7) becomes

$$\frac{\sigma^2}{2} f''(u) + c f'(u) + \lambda \int_0^{u+c/\delta} f(u - y)\eta^2 y e^{-\eta y}\, dy - (\lambda + \beta) f(u) = 0. \quad (5.7)$$

Applying the operator $\eta^2 \mathbf{I} + 2\eta \frac{d}{du} + \frac{d^2}{du^2}$ to Eq. (5.6), we get

$$\frac{\sigma^2}{2} f^{(4)}(u) + (\eta\sigma^2 + c + \delta u) f'''(u) + \left[\frac{\eta^2\sigma^2}{2} + 2\eta(c + \delta u) + 2\delta - (\lambda + \beta)\right] f''(u)$$
$$+ [\eta^2(c + \delta u) + 2\eta\delta - 2\eta(\lambda + \beta)] f'(u) - \eta^2 \beta f(u) = 0. \quad (5.8)$$

With the boundary conditions $f(-c/\delta) = 0$, $f(0) = 1$, $f''(-c/\delta) = 0$, $\frac{\sigma^2}{2} f'''(-c/\delta) + (\delta - \lambda - \beta) f'(-c/\delta) = 0$, Eq. (5.8) can be determined.

Applying the operator $\eta^2 \mathbf{I} + 2\eta \frac{d}{du} + \frac{d^2}{du^2}$ to Eq. (5.7), we get

$$\frac{\sigma^2}{2} f^{(4)}(u) + (\eta\sigma^2 + c) f'''(u) + \left[\frac{\eta^2\sigma^2}{2} + 2\eta c - (\lambda + \beta)\right] f''(u)$$
$$+ [\eta^2 c - 2\eta(\lambda + \beta)] f'(u) - \eta^2 \beta f(u) = 0. \quad (5.9)$$

It can be verified there exist four characteristic roots of Eq. (5.9) λ_1, λ_2, λ_3 and λ_4. So the general solution of Eq. (5.9) can be expressed as

$$f(u) = k_1 e^{\lambda_1 u} + k_2 e^{\lambda_2 u} + k_3 e^{\lambda_3 u} + k_4 e^{\lambda_4 u}, \quad u \geq 0. \quad (5.10)$$

From Theorem 3.3 we know $f(u)$ is twice continuously differentiable on $(-c/\delta, \infty)$, so k_1, k_2 and k_3 can be determined by the conditions: $f(0+) = f(0-) = 1$, $f'(0+) = f'(0-)$, $f''(0+) = f''(0-)$ and $f'''(0+) = f'''(0-) + \frac{2\delta}{\sigma^2} f'(0-)$.

Let $c = 2$, $\lambda = 0.5$, $\eta = 1$, $\beta = 0.02$, $\sigma = 0.3$. We also consider two cases: $\delta = 0.05$ and $\delta = 0.08$. $b^* = 2.22$ when $\delta = 0.05$ and $b^* = 3.98$ when $\delta = 0.08$. Intuitively, the higher the

Table 2 The numerical values of $V_{b^*}(u)$ for the Erlang(2) distributed claim sizes

$\delta \backslash u$	−3	−2.5	−2.0	−1.5	−1.0	−0.5	0	0.5	1.0
0.05	44.1691	44.7392	45.2966	45.8421	46.3763	46.8999	46.4431	46.9498	47.4537
0.08	40.8843	41.6088	42.2942	42.9437	43.5603	44.1469	44.646	45.1852	45.7129
$+\infty$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	22.2653	24.894
$\delta \backslash u$	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
0.05	47.9554	48.4558	48.9558	49.4558	49.9558	50.4558	50.9558	51.4558	51.9558
0.08	46.2315	46.7432	47.2499	47.753	48.2541	48.7543	49.2543	49.7543	50.2543
$+\infty$	27.3296	29.5139	31.4441	33.1407	34.6323	35.9483	37.1161	38.1597	39.0999

debit interest, the higher the optimal dividend barrier. In Fig. 1, we plot the function $f'(u)$, where we denote $f_3'(u)$ in the case $\delta = 0.05$ and $f_4'(u)$ in the case $\delta = 0.08$. The abscissa of the point C is 2.22 and the abscissa of the point D is 3.98. From Fig. 2 we can see that the function $f_3'(u)$ is increasing in u after C and $f_4'(u)$ is increasing in u after D. So we conclude that the barrier dividend strategy at $b^* = 1.18$ in the case $\delta = 0.05$ and the barrier dividend strategy at $b^* = 2.71$ in the case $\delta = 0.08$ are both optimal among all dividend strategies.

The barrier dividend barrier b^* of the classical risk model perturbed by diffusion is equal to 13.5895, and is larger than that of the jump-diffusion risk model with debit interest. In Table 2, we give the numerical values of $V_{b^*}(u)$ for the case $\delta = 0.05$, $\delta = 0.08$ and $\delta = +\infty$. It can be seen that $V_{b^*}(u)$ is increasing as u increases. Note that $V_{b^*}(u)$ becomes smaller as the debit interest becomes larger and $V_{b^*}(u)$ of the classical risk model perturbed by diffusion is the smallest. The conclusion is the same as that in Example 5.1.

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