Periodica Mathematica Hungarica (2020) 80:280–288
https://doi.org/10.1007/s10998-020-00337-y

# Convergence of linking Baskakov-type operators

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Published online: 26 March 2020


**Abstract**
In this paper we consider a link $B_{n,\rho}$ between Baskakov type operators $B_{n,\infty}$ and genuine Baskakov–Durrmeyer type operators $B_{n,1}$ depending on a positive real parameter $\rho$. The topic of the present paper is the pointwise limit relation $\left(B_{n,\rho} f\right)(x) \rightarrow \left(B_{n,\infty} f\right)(x)$ as $\rho \rightarrow \infty$ for $x \geq 0$. As a main result we derive uniform convergence on each compact subinterval of the positive real axis for all continuous functions $f$ of polynomial growth.

**Keywords** Approximation by positive operators · Rate of convergence · Degree of approximation

**Mathematics Subject Classification** 41A36 · 41A25

## 1 Introduction

The so-called Baskakov type operators depending on a parameter $c \in \mathbb{R}$ were defined by Baskakov [1]. They are suitable for the approximation of functions being continuous on the underlying interval. It is well known that for the special choice of $c = -1$, $c = 0$ and $c = 1$, respectively, one derives the classical Bernstein, Szász–Mirakjan and Baskakov operators, respectively.

In the context of approximation of integrable functions the Baskakov–Durrmeyer type operators play an important role. The construction is based on the corresponding definition of Bernstein–Durrmeyer operators (see [3,6,8,9,14] for further definitions). Although they have outstanding properties, i.e., they commute and are self-adjoint, they don't interpolate at finite endpoints of the interval and they only preserve constants. If the functions have

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supplemental finite limits at finite endpoints of the interval a modification of the Baskakov–Durrmeyer type operators leads to the genuine variants which, like the classical operators, preserve all linear functions and interpolate at finite endpoints of the interval.

During the last years the question arose how the classical operators and their genuine counterparts are connected. This investigation was initiated by Păltănea [10,11] by introducing a non-trivial link between Bernstein and Szász–Mirakjan operators, respectively, and their genuine Durrmeyer modifications (see also [4,5,12]).

The topic of the present paper is the investigation of convergence properties of linking Baskakov–Durrmeyer type operators for $c > 0$ when the linking parameter $\rho$ tends to infinity.

Let $c, n, \nu, \rho \in \mathbb{R}$, such that $n > c \geq 0$, $\rho > 0$, $\nu \geq 0$, and $x \in [0, \infty)$. Then the basis functions are given by

$$p_{n,\nu}^{[c]}(x) = \begin{cases} \dfrac{(nx)^{\nu}}{\nu!} e^{-nx}, & \text{if } c = 0, \\[2ex] \dfrac{\Gamma\left(\frac{n}{c} + \nu\right)}{\Gamma\left(\frac{n}{c}\right)\Gamma(\nu + 1)} (cx)^{\nu} (1 + cx)^{-n/c-\nu}, & \text{if } c > 0. \end{cases}$$

It can be shown that $\lim_{c \to 0^+} p_{n,\nu}^{[c]}(x) = p_{n,\nu}^{[0]}(x)$ for each $x \geq 0$. Using the Euler beta function $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ for $x, y > 0$, we obtain in case $c > 0$ the representation

$$(n - c) \cdot p_{n,\nu}^{[c]}(x) = \frac{c}{B(n/c - 1, \nu + 1)} (cx)^{\nu} (1 + cx)^{-n/c-\nu}.$$

For the sake of brevity and in order to be consistent with notations in several references, we put throughout the paper

$$\mu_{n,\nu,\rho}^{[c]}(x) = (n\rho + c)\, p_{n\rho+2c,\nu\rho-1}^{[c]}(x)$$

for $\nu \geq 1/\rho$. For $c > 0$, we have the explicit representation

$$\mu_{n,\nu,\rho}^{[c]}(x) = \frac{c}{B\left(\frac{n\rho}{c} + 1, \nu\rho\right)} (cx)^{\nu\rho-1} (1 + cx)^{-\frac{n\rho}{c}-\nu\rho-1}.$$

It is well known that $(n - c) \int_0^{\infty} p_{n,\nu}^{[c]}(t)\, dt = 1$ for all $\nu \geq 0$, so, obviously

$$\int_0^{\infty} \mu_{n,\nu,\rho}^{[c]}(t)\, dt = 1 \qquad (\nu \geq 1/\rho).$$

In the following definition we assume that $f : [0, \infty) \to \mathbb{R}$ is given in such a way that the corresponding integrals and series are convergent.

**Definition 1** *Let $c \geq 0$. The operators of Baskakov-type are defined by*

$$\left(B_{n,\infty}^{[c]} f\right)(x) = \sum_{\nu=0}^{\infty} p_{n,\nu}^{[c]}(x)\, f\left(\frac{\nu}{n}\right), \tag{1.1}$$

*and the genuine Baskakov–Durrmeyer type operators are denoted by*

$$\left(B_{n,1}^{[c]} f\right)(x) = p_{n,0}^{[c]}(x)\, f(0) + \sum_{\nu=1}^{\infty} p_{n,\nu}^{[c]}(x) \int_0^{\infty} \mu_{n,\nu,1}^{[c]}(t)\, f(t)\, dt. \tag{1.2}$$

*Depending on a parameter* $\rho > 0$ *the linking operators are given by*

$$\left(B_{n,\rho}^{[c]} f\right)(x) = p_{n,0}^{[c]}(x)\, f(0) + \sum_{\nu=1}^{\infty} p_{n,\nu}^{[c]}(x) \int_0^{\infty} \mu_{n,\nu,\rho}^{[c]}(t)\, f(t)\, dt. \tag{1.3}$$

Note that the genuine Baskakov–Durrmeyer type operators (1.2) are usually defined in the more explicit form

$$\left(B_{n,1}^{[c]} f\right)(x) = p_{n,0}^{[c]}(x)\, f(0) + (n+c) \sum_{\nu=1}^{\infty} p_{n,\nu}^{[c]}(x) \int_0^{\infty} p_{n+2c,\nu-1}^{[c]}(t)\, f(t)\, dt.$$

Setting $c = 0$ in (1.2) leads to the Phillips operators [13], the case $c > 0$ was investigated in [15]. To the best of our knowledge, the case $c = 0$ in (1.3) was first considered in [11] and the general case in [7].

The purpose of this paper is to prove the limit $\lim_{\rho\to\infty}\left(B_{n,\rho}^{[c]} f\right)(x) = \left(B_{n,\infty}^{[c]} f\right)(x)$ for all continuous functions $f$ on $[0,\infty)$ of polynomial growth. In the following theorem we state our main result.

**Theorem 1.1** *Let* $c, \gamma > 0$*. Assume that* $f \in C[0,\infty) \to \mathbb{R}$ *satisfies the growth condition* $f(t) = O(t^{\gamma})$ *as* $t \to \infty$*. Then, for any* $b > 0$*, there is a constant* $\rho_0 > 0$ *such that* $B_{n,\rho} f$ *exists for all* $\rho \geq \rho_0$ *and*

$$\lim_{\rho\to\infty}\left(B_{n,\rho}^{[c]} f\right)(x) = \left(B_{n,\infty}^{[c]} f\right)(x)$$

*uniformly for* $x \in [0, b]$*.*

The case $c = 0$ was solved by Păltănea [12, Theorem 4] for functions $f \in C[0,\infty)$ satisfying $f(t) = O\left(e^{\gamma t}\right)$ as $t \to \infty$, where $\gamma$ is an arbitrary positive constant.

Further results concerning the limit of the operators $B_{n,\rho}$ as $\rho \to \infty$ for $c \geq 0$ were proved in [2,7]. From the explicit representations of the images of polynomials for all operators $B_{n,\rho}$ it was possible to derive immediately that for $c \geq 0$

$$\lim_{\rho\to\infty}\left(B_{n,\rho}^{[c]} p\right)(x) = \left(B_{n,\infty}^{[c]} p\right)(x)$$

uniformly on every compact subinterval of $[0,\infty)$ for each polynomial $p$ (see [7, Theorem 1, Theorem 2, Corollary 1]).

A different function space was considered in [2] for the case $c \geq 0$. For $f \in C^2[0,\infty)$ with $\|f''\|_{\infty} < \infty$

$$\lim_{\rho\to\infty}\left(B_{n,\rho}^{[c]} f\right)(x) = \left(B_{n,\infty}^{[c]} f\right)(x)$$

uniformly on every compact subinterval of $[0,\infty)$ (see [2, Lemma 5, Corollary 3]).

## 2 Auxiliary results

In this section we present several lemmata which are needed for the proof of our main result in Sect. 3.

Let $W[0,\infty)$ be the class of all locally integrable functions on $[0,\infty)$ of polynomial growth, which are bounded on each compact subinterval of $[0,\infty)$. Obviously, for every $c > 0$, each function $f$ of this class satisfies an estimate of the type $|f(x)| \leq M(1+cx)^q$ for $x \geq 0$ with certain constants $M, q > 0$. We start with the following observation.

**Lemma 2.1** *Let $I \subset [0,\infty)$ be a compact interval. For $f \in W[0,\infty)$, there are constants $M, q > 0$, such that*

$$|f(x) - f(y)| \leq M \cdot (1+cx)^q$$

*for all $x \geq 0$ and $y \in I$.*

***Proof*** For $f \in W[0,\infty)$, there are constants $M_1, q > 0$ such that $|f(x)| \leq M_1(1+cx)^q$ for $x \geq 0$. In particular, with $M_2 := \max_{t\in I} |f(t)|$, we have $|f(x) - f(y)| \leq M_1(1+cx)^q + M_2 \leq M(1+cx)^q$ for $x \geq 0$ and $y \in I$ with a certain positive constant $M$. □

The following lemma guarantees the convergence of

$$\sum_{\nu=1}^{\infty} p_{n,\nu}^{[c]}(x) \int_0^{\infty} \mu_{n,\nu,\rho}^{[c]}(t)\ f(t)\ dt$$

in (1.3), for all functions $f \in W[0,\infty)$.

**Lemma 2.2** *Let $c > 0$ and $n, \nu, q \in \mathbb{N}$. If $f \in W[0,\infty)$ satisfies the estimate $|f(t)| \leq M(1+ct)^q$ for $t \in [0,\infty)$, then, for sufficiently large $\rho > 0$,*

$$\left|\int_0^{\infty} \mu_{n,\nu,\rho}^{[c]}(t)\ f(t)\ dt\right| \leq M \cdot \left(2\frac{n+c\nu}{n}\right)^q.$$

***Proof*** As

$$\mu_{n,\nu,\rho}^{[c]}(t)(1+ct)^q = \frac{\Gamma\left(\frac{n}{c}\rho+1+\nu\rho\right)\Gamma\left(\frac{n}{c}\rho+1-q\right)}{\Gamma\left(\frac{n}{c}\rho+1-q+\nu\rho\right)\Gamma\left(\frac{n}{c}\rho+1\right)}(n\rho+c-qc)\,p_{n\rho+2c-q,\nu\rho-1}^{[c]}(x),$$

we obtain

$$\left|\int_0^{\infty} \mu_{n,\nu,\rho}^{[c]}(t)\ f(t)\ dt\right| \leq M\frac{\Gamma\left(\frac{n}{c}\rho+1+\nu\rho\right)\Gamma\left(\frac{n}{c}\rho+1-q\right)}{\Gamma\left(\frac{n}{c}\rho+1-q+\nu\rho\right)\Gamma\left(\frac{n}{c}\rho+1\right)}.$$

Observing that

$$\frac{\Gamma\left(\frac{n}{c}\rho+1+\nu\rho\right)\Gamma\left(\frac{n}{c}\rho+1-q\right)}{\Gamma\left(\frac{n}{c}\rho+1-q+\nu\rho\right)\Gamma\left(\frac{n}{c}\rho+1\right)} \leq \left(\frac{\frac{n}{c}\rho+\nu\rho}{\frac{n}{c}\rho-q}\right)^q = \left(\frac{\frac{n}{c}+\nu}{\frac{n}{c}-q/\rho}\right)^q \leq \left(2\frac{n+c\nu}{n}\right)^q,$$

for sufficiently large $\rho > 0$, leads to the desired estimate.

**Lemma 2.3** *Let $c > 0$, $\rho > 0$ and $n \in \mathbb{N}$. Then*

$$\frac{\sqrt{2\pi}}{B\left(\frac{n}{c}\rho+1,\nu\rho\right)} \leq \left(1+\frac{n}{c\nu}\right)^{\nu\rho}\left(1+\frac{c\nu}{n}\right)^{\frac{n}{c}\rho+1/2}(\nu\rho)^{1/2}\exp\left(\frac{1}{12\rho}\right).$$

***Proof*** From the definition of the beta function it follows that

$$\frac{1}{B\left(\frac{n}{c}\rho+1,\nu\rho\right)} = \frac{\nu\rho\cdot\Gamma\left(\nu\rho+\frac{n}{c}\rho+1\right)}{\Gamma(\nu\rho+1)\,\Gamma\left(\frac{n}{c}\rho+1\right)}.$$

By Stirling's formula the gamma function satisfies $\Gamma(z+1) = \sqrt{2\pi}z^{z+1/2}\exp(-z+\vartheta(z))$ with $0 < \vartheta(z) < 1/(12z)$, for $z > 0$. Hence,

$$\frac{\sqrt{2\pi}}{B\left(\frac{n}{c}\rho+1,\nu\rho\right)} = \frac{\left(\nu\rho+\frac{n}{c}\rho\right)^{\nu\rho+\frac{n}{c}\rho+1/2}}{(\nu\rho)^{\nu\rho-1/2}\left(\frac{n}{c}\rho\right)^{\frac{n}{c}\rho+1/2}}\exp\left(\vartheta\left(\nu\rho+\frac{n}{c}\rho\right)-\vartheta(\nu\rho)-\vartheta\left(\frac{n}{c}\rho\right)\right)$$

which implies the desired estimate. □

**Lemma 2.4** *Let $f \in W[0,\infty)$, $n, N \in \mathbb{N}$ and $1 \le \nu \le N$. For each $\varepsilon > 0$ there exist positive constants $T$, $\rho_0$ such that for all $\nu \in \{1,\ldots,N\}$,*

$$\int_T^\infty \mu_{n,\nu,\rho}^{[c]}(t) \;\left| f(t) - f\left(\frac{\nu}{n}\right)\right| dt \le \varepsilon \qquad (\rho \ge \rho_0).$$

***Proof*** Let $1 \le \nu \le N$. By Lemma 2.1 there are positive constants $M$ and $q$ such that

$$B\left(\frac{n}{c}\rho+1,\nu\rho\right)\cdot\mu_{n,\nu,\rho}^{[c]}(t) \;\left| f(t) - f\left(\frac{\nu}{n}\right)\right| \le M(1+ct)^q\, c^{\nu\rho} t^{\nu\rho-1}(1+ct)^{-(n/c+\nu)\rho-1}.$$

Choose $T$ larger than the maximum point $t = \frac{\nu+q/\rho}{n-cq/\rho}$ (we can assume that $\rho$ is sufficiently large such that $cq/\rho < 1$) of the unimodal function

$$(ct)^{\nu\rho}\, t^q\, (1+ct)^{-(n/c+\nu)\rho}.$$

Then

$$\begin{aligned} &B\left(\frac{n}{c}\rho+1,\nu\rho\right)\int_T^\infty \mu_{n,\nu,\rho}^{[c]}(t) \;\left| f(t) - f\left(\frac{\nu}{n}\right)\right| dt \\ &\le M\cdot T^q\,(cT)^{\nu\rho}\,(1+cT)^{-(n/c+\nu)\rho}\int_T^\infty \frac{(1+ct)^{q-1}}{t^{q+1}}\,dt. \end{aligned}$$

Since the integral has a finite value, we derive by applying the estimate in Lemma 2.3 for $1 \le \nu \le N$

$$\frac{1}{B\left(\frac{n}{c}\rho+1,\nu\rho\right)} \le \sqrt{\frac{1}{2\pi}N\left(1+\frac{cN}{n}\right)\rho}\left(1+\frac{n}{c}\right)^{N\rho}\left(1+\frac{cN}{n}\right)^{\frac{n}{c}\rho}\exp\left(\frac{1}{12\rho}\right).$$

This entails

$$\begin{aligned} &\int_T^\infty \mu_{n,\nu,\rho}^{[c]}(t) \;\left| f(t) - f\left(\frac{\nu}{n}\right)\right| dt \\ &= O\left(\sqrt{\rho}\right)\left(\left(1+\frac{n}{c}\right)^N\left(1+\frac{cN}{n}\right)^{\frac{n}{c}}\left(\frac{cT}{1+cT}\right)^\nu(1+cT)^{-n/c}\right)^\rho \\ &= O\left(e^{-\alpha\rho}\right) \qquad (\rho\to\infty) \end{aligned}$$

with a positive constant $\alpha$ if $T$ is sufficiently large. □

**Lemma 2.5** *Let $f \in W[0,\infty)$, $n, N \in \mathbb{N}$, $\delta > 0$, $T > N/n$ and $1 \le \nu \le N$. For each $\varepsilon > 0$ there exists a positive constant $\rho_0$ such that for all $\nu \in \{1,\ldots,N\}$,*

$$\int_{\frac{\nu}{n}+\delta}^T \mu_{n,\nu,\rho}^{[c]}(t) \;\left| f(t) - f\left(\frac{\nu}{n}\right)\right| dt \le \varepsilon \qquad (\rho \ge \rho_0).$$

***Proof*** Let $1 \le \nu \le N$. As in the above proof we have

$$\begin{aligned} &B\left(\frac{n}{c}\rho+1,\nu\rho\right)\int_{\frac{\nu}{n}+\delta}^T \mu_{n,\nu,\rho}^{[c]}(t) \;\left| f(t) - f\left(\frac{\nu}{n}\right)\right| dt \\ &\le M_2\left(c\left(\frac{\nu}{n}+\delta\right)\right)^{\nu\rho}\left(1+c\left(\frac{\nu}{n}+\delta\right)\right)^{-(n/c+\nu)\rho}\cdot T\max_{\frac{1}{n}+\delta\le t\le T}\frac{(1+ct)^{q-1}}{t}. \end{aligned}$$

Put

$$\sigma_0 = \sigma_0(\nu) := \frac{\nu}{n} \le \frac{\nu}{n}+\delta =: \sigma(\nu) = \sigma.$$

For a better readability we don't indicate that $\sigma_0$ and $\sigma$ depend on $\nu$. Then

$$B\left(\frac{n}{c}\rho+1,\nu\rho\right)\int_{\frac{\nu}{n}+\delta}^{T}\mu_{n,\nu,\rho}^{[c]}(t)\ \left|f(t)-f\left(\frac{\nu}{n}\right)\right|dt \leq M_2 nT(1+cT)^{q-1}(c\sigma)^{n\sigma_0\rho}(1+c\sigma)^{-(n/c+n\sigma_0)\rho}.$$

From Lemma 2.3 we infer that for $1\leq\nu\leq N$

$$\frac{1}{B\left(\frac{n}{c}\rho+1,\nu\rho\right)}\leq\sqrt{\frac{1}{2\pi}N\left(1+\frac{cN}{n}\right)\rho}\left(1+\frac{1}{c\sigma_0}\right)^{n\sigma_0\rho}(1+c\sigma_0)^{\frac{n}{c}\rho}\exp\left(\frac{1}{12\rho}\right).$$

Hence,

$$\begin{aligned}&\int_{\frac{\nu}{n}+\delta}^{T}\mu_{n,\nu,\rho}^{[c]}(t)\ \left|f(t)-f\left(\frac{\nu}{n}\right)\right|dt\\ &\leq O\left(\sqrt{\rho}\right)\left(1+\frac{1}{c\sigma_0}\right)^{n\sigma_0\rho}(1+c\sigma_0)^{\frac{n}{c}\rho}(c\sigma)^{n\sigma_0\rho}(1+c\sigma)^{-(n/c+n\sigma_0)\rho}\\ &=O\left(\sqrt{\rho}\right)\left(\frac{1+\frac{1}{c\sigma_0}}{1+c\sigma}c\sigma\right)^{n\sigma_0\rho}\left(\frac{1+c\sigma_0}{1+c\sigma}\right)^{\frac{n}{c}\rho}\qquad(\rho\to\infty).\end{aligned}$$

Now, $\frac{1+\frac{1}{c\sigma_0}}{1+c\sigma}c\sigma=1+\frac{1}{\sigma_0}\frac{\delta}{1+c\sigma}$ and $\frac{1+c\sigma_0}{1+c\sigma}=1-c\frac{\delta}{1+c\sigma}$. This entails,

$$\int_{\frac{\nu}{n}+\delta}^{T}\mu_{n,\nu,\rho}^{[c]}(t)\ \left|f(t)-f\left(\frac{\nu}{n}\right)\right|dt=O\left(\sqrt{\rho}\right)\left[\left(1+\frac{z}{\sigma_0}\right)^{\sigma_0}(1-cz)^{\frac{1}{c}}\right]^{n\rho}\tag{2.1}$$

as $\rho\to\infty$, where we put $z:=\frac{\delta}{1+c\sigma}$. Furthermore, put $G(z)=\log\left[\left(1+\frac{z}{\sigma_0}\right)^{\sigma_0}(1-cz)^{\frac{1}{c}}\right]$. Note that $cz=\frac{c\delta}{1+c\sigma}<c\delta<1$. The derivative $G'$ satisfies

$$\begin{aligned}G'(z)&=\frac{d}{dz}\left(\sigma_0\log\left(1+\frac{z}{\sigma_0}\right)+\frac{1}{c}\log(1-cz)\right)\\ &=\frac{-(1+\sigma_0 c)z}{(z+\sigma_0)(1-cz)}<0\quad\text{for } z\in\left(0,\frac{1}{c}\right).\end{aligned}$$

Therefore, $G$ is monotonically decreasing on $[0,1/c)$. Because $G(0)=0$, we have $G(z)<0$ for $z\in(0,1/c)$. Hence,

$$\max_{1\leq\nu\leq N}\left(1+\frac{z}{\sigma_0}\right)^{\sigma_0}(1-cz)^{1/c}<1.$$

Now the lemma follows by (2.1). □

**Lemma 2.6** *Let $f\in W[0,\infty)$, $n,N\in\mathbb{N}$, $\delta>0$ and $1\leq\nu\leq N$. For each $\varepsilon>0$ there exists a positive constant $\rho_0$ such that for all $\nu\in\{1,\ldots,N\}$,*

$$\int_0^{\frac{\nu}{n}-\delta}\mu_{n,\nu,\rho}^{[c]}(t)\ \left|f(t)-f\left(\frac{\nu}{n}\right)\right|dt\leq\varepsilon\qquad(\rho\geq\rho_0).$$

***Proof*** Let $1 \le \nu \le N$. As in the proof of Lemma 2.4, we have

$$\begin{aligned}
& B\left(\frac{n}{c}\rho+1,\nu\rho\right)\int_0^{\frac{\nu}{n}-\delta}\mu_{n,\nu,\rho}^{[c]}(t)\ \left|f(t)-f\left(\frac{\nu}{n}\right)\right|\,dt\\
&\le M_2 c^{\nu\rho}\int_0^{\frac{\nu}{n}-\delta} t^{\nu\rho-1}(1+ct)^{-(n/c+\nu)\rho-1+q}\,dt\\
&\le M_2 c^{\nu\rho}\int_0^{\frac{\nu}{n}-\delta} t^{\nu\rho-1}(1+ct)^{-(n/c+\nu)\rho}\,dt\cdot\max_{0\le t\le\frac{\nu}{n}-\delta}(1+ct)^{q-1}\\
&\le M_2\left(1+c\frac{N}{n}\right)^{q-1} c^{\nu\rho}\int_0^{\frac{\nu}{n}-\delta} t^{\nu\rho-1}(1+ct)^{-(n/c+\nu)\rho}\,dt.
\end{aligned}$$

Choose $\rho$ so large that the maximum point $\frac{\nu-1/\rho}{n+c/\rho}$ of $t^{\nu\rho-1}(1+ct)^{-(n/c+\nu)\rho}$ is larger than $\frac{\nu}{n}-\delta$. Put

$$\tau=\tau(\nu):=\frac{\nu}{n}-\delta\le\frac{\nu}{n}=:\tau_0(\nu)=\tau_0.$$

For a better readability we don't indicate that $\tau$ and $\tau_0$ depend on $\nu$. Then

$$\int_0^{\frac{\nu}{n}-\delta}\mu_{n,\nu,\rho}^{[c]}(t)\ \left|f(t)-f\left(\frac{\nu}{n}\right)\right|\,dt\le O(1)\,\frac{c^{\nu\rho}}{B\left(\frac{n}{c}\rho+1,\nu\rho\right)}\tau\cdot\tau^{n\tau_0\rho-1}(1+c\tau)^{-(1/c+\tau_0)n\rho}$$

as $\rho\to\infty$. By Lemma 2.3 we have

$$\frac{1}{B\left(\frac{n}{c}\rho+1,\nu\rho\right)}\le O\left(\sqrt{\rho}\right)\left(1+\frac{1}{c\tau_0}\right)^{n\tau_0\rho}(1+c\tau_0)^{\frac{n}{c}\rho}\exp\left(\frac{1}{12\rho}\right),$$

as $\rho\to\infty$, since $1\le\nu\le N$. Therefore,

$$\begin{aligned}
& \int_0^{\frac{\nu}{n}-\delta}\mu_{n,\nu,\rho}^{[c]}(t)\ \left|f(t)-f\left(\frac{\nu}{n}\right)\right|\,dt\\
&\le O\left(\sqrt{\rho}\right)\left[\left(1+\frac{1}{c\tau_0}\right)^{\tau_0}(1+c\tau_0)^{\frac{1}{c}}(c\tau)^{\tau_0}(1+c\tau)^{-(1/c+\tau_0)}\right]^{n\rho}\\
&= O\left(\sqrt{\rho}\right)\left[\left(\frac{1+\frac{1}{c\tau_0}}{1+c\tau}c\tau\right)^{\tau_0}\left(\frac{1+c\tau_0}{1+c\tau}\right)^{\frac{1}{c}}\right]^{n\rho}
\end{aligned}$$

as $\rho\to\infty$. Now, $\frac{1+\frac{1}{c\tau_0}}{1+c\tau}c\tau=1-\frac{1}{\tau_0}\frac{\delta}{1+c\tau}$ and $\frac{1+c\tau_0}{1+c\tau}=1+c\frac{\delta}{1+c\tau}$. Hence,

$$\int_0^{\frac{\nu}{n}-\delta}\mu_{n,\nu,\rho}^{[c]}(t)\ \left|f(t)-f\left(\frac{\nu}{n}\right)\right|\,dt=O\left(\sqrt{\rho}\right)\left[\left(1-\frac{z}{\tau_0}\right)^{\tau_0}(1+cz)^{\frac{1}{c}}\right]^{n\rho}\tag{2.2}$$

as $\rho\to\infty$, where we put $z:=\frac{\delta}{1+c\tau}$. Furthermore, put $H(z)=\log\left[\left(1-\frac{z}{\tau_0}\right)^{\tau_0}(1+cz)^{\frac{1}{c}}\right]$. Note that $z=\frac{\delta}{1+c\tau}<\delta<\frac{1}{n}\le\frac{\nu}{n}=\tau_0$. The derivative $H'$ satisfies

$$\begin{aligned}
H'(z)&=\frac{d}{dz}\left(\tau_0\log\left(1-\frac{z}{\tau_0}\right)+\frac{1}{c}\log(1+cz)\right)\\
&=\frac{-(1+\tau_0 c)\,z}{(\tau_0-z)(1+cz)}<0,\qquad\text{for } z\in(0,\tau_0).
\end{aligned}$$

Therefore, $H$ is monotonically decreasing on $[0, \tau_0)$. Because $H(0) = 0$, we have $H(z) < 0$ for $z \in (0, \tau_0)$. Thus,

$$\max_{1 \le \nu \le N} \left(1 - \frac{z}{\tau_0}\right)^{\tau_0} (1 + cz)^{1/c} < 1.$$

Now the lemma follows by (2.2). □

## 3 Proof of the main result

***Proof of Theorem 1.1*** Since $\gamma$ is arbitrary, we can assume that there exist a real constant $M > 0$ and an integer $q \ge 0$ such that $|f(t)| \le M(1 + ct)^q$ for $t \in [0, \infty)$. For $\nu > bn$ the function $p_{n,\nu}^{[c]}$ is monotonically increasing on $[0, b]$. In particular, $0 \le p_{n,\nu}^{[c]}(x) \le p_{n,\nu}^{[c]}(b)$, and Lemma 2.2 implies that

$$\left| p_{n,\nu}^{[c]}(x) \int_0^\infty \mu_{n,\nu,\rho}^{[c]}(t)\ f(t)\ dt \right| \le p_{n,\nu}^{[c]}(b) \cdot M \left(2 \frac{c\nu + n}{n}\right)^q$$

for $x \in [0, b]$. Since $p_{n,0}^{[c]}(x) \le 1$, we have

$$\left|\left(B_{n,\rho}^{[c]} f\right)(x)\right| \le |f(0)| + 2^q M \sum_{\nu=1}^{\infty} p_{n,\nu}^{[c]}(b) \left(1 + c\frac{\nu}{n}\right)^q < \infty.$$

The uniform and absolute convergence of the series $B_{n,\rho}^{[c]} f$ allows us to choose an integer $N > bn$ such that

$$\left| \sum_{\nu=N+1}^{\infty} p_{n,\nu}^{[c]}(x) \int_0^\infty \mu_{n,\nu,\rho}^{[c]}(t)\ f(t)\ dt \right| < \varepsilon$$

for sufficiently large $\rho > 0$. For the same reason, we can achieve the inequality

$$\left| \sum_{\nu=N+1}^{\infty} p_{n,\nu}^{[c]}(x)\, f\left(\frac{\nu}{n}\right) \right| < \varepsilon$$

for all $x \in [0, b]$ and $\rho \ge \rho_0$. Therefore,

$$\left|\left(B_{n,\rho}^{[c]} f\right)(x) - \left(B_{n,\infty}^{[c]} f\right)(x)\right| < \sum_{\nu=1}^{N} p_{n,\nu}^{[c]}(x) \int_0^\infty \mu_{n,\nu,\rho}^{[c]}(t)\ \left| f(t) - f\left(\frac{\nu}{n}\right)\right| dt + 2\varepsilon$$

for all $x \in [0, b]$ and $\rho \ge \rho_0$. From the continuity of the function $f$ there is $\delta > 0$ such that $\delta < 1/n$ and $\left|f(t) - f\left(\frac{\nu}{n}\right)\right| < \varepsilon$ if $\left|t - \frac{\nu}{n}\right| < \delta$. It follows

$$\sum_{\nu=1}^{N} p_{n,\nu}^{[c]}(x) \int_{\frac{\nu}{n}-\delta}^{\frac{\nu}{n}+\delta} \mu_{n,\nu,\rho}^{[c]}(t)\ \left| f(t) - f\left(\frac{\nu}{n}\right)\right| dt$$
$$\le \varepsilon \sum_{\nu=1}^{N} p_{n,\nu}^{[c]}(x) \int_0^\infty \mu_{n,\nu,\rho}^{[c]}(t)\ dt \le \varepsilon.$$

This inequality and application of the Lemmata 2.4–2.6 leads to the final estimate

$$\left|\left(B_{n,\rho}^{[c]} f\right)(x) - \left(B_{n,\infty}^{[c]} f\right)(x)\right| < 6\varepsilon$$

for all $x \in [0, b]$ and $\rho \ge \rho_0$. This completes the proof of the theorem. □

**Acknowledgements** Open Access funding provided by Projekt DEAL.



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