Periodica Mathematica Hungarica (2020) 80:103–107
https://doi.org/10.1007/s10998-019-00307-z

On the distribution of square-full and cube-full primitive roots

Teerapat Srichan[1]

Published online: 2 January 2020

Abstract

A positive integer n is called an r-full integer if for all primes $p \mid n$ we have $p^r \mid n$. Let p be an odd prime. For $\gcd(n, p) = 1$, the smallest positive integer f such that $n^f \equiv 1 \pmod p$ is called the exponent of n modulo p. If $f = p - 1$ then n is called a primitive root modulo p. Let $T_r(n)$ be the characteristic function of the r-full primitive roots modulo p. In this paper we derive the asymptotic formula for the following sums

$$\sum_{n \le x} T_2(n), \quad \sum_{n \le x} T_3(n),$$

by using properties of character sums.

Keywords Character sums · Cube-full integers · Primitive roots · Square-full integers

Mathematics Subject Classification 11N25 · 11B50

1 Introduction and results

The problem of counting the primitive roots that are square-full is a topic in analytic number theory. In 1983 Shapiro [3] investigated square-full primitive roots and showed that

$$\sum_{n \le x} T_2(n) = \frac{\phi(p-1)}{p-1} \left(c\sqrt{x} + O(x^{1/3} p^{1/6} (\log p)^{1/3} 2^{\omega(p-1)}) \right), \tag{1.1}$$

where $\phi(n)$ is Euler's function, $\omega(n)$ denotes the number of distinct prime divisors of n, and

$$c = 2\left(1 - \frac{1}{p}\right) \sum_{(q|p)=-1} \frac{\mu^2(q)}{q^{3/2}}.$$

Very recently, Munsch and Trudgian [5] improved on the error term in (1.1) and showed that

✉ Teerapat Srichan
fscitrp@ku.ac.th

1 Department of Mathematics, Faculty of Science, Kasetsart University, Bangkok, Thailand

$$\sum_{n\le x} T_2(n) = \frac{\phi(p-1)}{p-1}\Big(\frac{1}{\zeta(3)}\left(1+\frac{1}{p}+\frac{1}{p^2}\right)^{-1} C_p x^{1/2} + O(x^{1/3}(\log x)p^{1/9}(\log p)^{1/6}2^{\omega(p-1)})\Big), \tag{1.2}$$

where $C_p \gg p^{-\frac{1}{8\sqrt{\epsilon}}}$. Munsch and Trudgian used the character estimate of Burgess (see [1]) and Lemma 1.3 of [4] to prove (1.2). They only consider the contribution of the principal and the quadratic characters. It would be interesting to see whether their method could be improved to consider the cubic characters.

In this paper we shall improve the result in (1.2) by using character sums. The considered characters are the principal, quadratic and cubic characters. The essential lemmas follow from the proof in Theorem 2.1 in [6]. We obtain the following theorem.

Theorem 1.1 *For a given odd prime $p \le x^{1/5}$,*

$$\sum_{n\le x} T_2(n) = \frac{\phi(p-1)}{p}\left(\frac{L(3/2,\chi_0)-L(3/2,\chi_1)}{L(3,\chi_0)}\right)x^{1/2} + \frac{\phi(p-1)}{p}\left(\frac{L(2/3,\chi_0)-L(2/3,\chi_2^2)}{L(2,\chi_0)}\right)x^{1/3} + O\left(x^{1/6}\phi(p-1)3^{\omega_{1,3}(p-1)}p^{1/2+\epsilon}\right),$$

here χ_0, $\chi_1 \ne \chi_0$, and $\chi_2 \ne \chi_0$ denote respectively the principal, quadratic and cubic character modulo p. The terms with the cubic characters only occur if $3|p-1$. Finally, $\omega_{1,3}(n)$ denotes the number of distinct prime $q \equiv 1 \pmod 3$ and q are divisors of n.

Remark 1.2 The result in Theorem 1.1 improves on (1.2) when $p < x^{3/25}$.

Remark 1.3 Munsch and Trudgian's result shows that for all sufficiently large p there is a positive square-full primitive root less than p. Cohen and Trudgian [2] conjectured that this may in fact hold for $p > 1052041$. It would be interesting to see whether the result in Theorem 1.1 could prove this conjecture.

It is natural to try study cube-full primitive roots modulo p. By the similar way, we obtain the following theorem.

Theorem 1.4 *For a given odd prime $p \le x^{1/7}$,*

$$\sum_{n\le x} T_3(n) = \alpha_1 x^{1/3} + \alpha_2 x^{1/4} + \alpha_3 x^{1/5} + O\left(x^{7/46}5^{\omega_{1,5}(p-1)}\phi(p-1)p^{35/92+\epsilon}\right),$$

where

$$\alpha_1 = \frac{\phi(p-1)}{p}(L(4/3,\chi_0)L(5/3,\chi_0)J(1/3,\chi_0) - L(4/3,\chi_1)L(5/3,\chi_1^2)J(1/3,\chi_1)),$$

$$\alpha 2 = \frac{\phi(p-1)}{p}(L(3/4,\chi_0)L(5/4,\chi_0)J(1/4,\chi_0) - L(3/4,\chi_2^3)L(5/4,\chi_2)J(1/4,\chi_2)),$$

$$\alpha 3 = \frac{\phi(p-1)}{p}(L(3/5,\chi_0)L(4/5,\chi_0)J(1/5,\chi_0) - L(3/5,\chi_3^3)L(4/5,\chi_3^4)J(1/5,\chi_3)),$$

here χ_0, $\chi_1 \ne \chi_0$, $\chi_2 \ne \chi_0$, and $\chi_3 \ne \chi_0$ denote respectively the principal, cubic, quartic, quintic character modulo p. Finally, $\omega_{1,5}(n)$ denotes the number of distinct primes dividing n that are congruent to 1 $\pmod 5$.

2 Preliminary results

The following lemmas will be used to prove Theorems 1.1 and 1.4.

Lemma 2.1 (see Lemma 8.5.1 in [3]) *For a given odd prime p, the characteristic function of the primitive root modulo p is*

$$\frac{\phi(p-1)}{p-1}\sum_{d\mid p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\Gamma_d}\chi(n)=\begin{cases}1 & \text{if } n \text{ is a primitive root mod } p\\ 0 & otherwise,\end{cases}$$

where Γ_d denotes the set of characters of the character group modulo p that are of order d.

The following lemmas give us the contribution of the principal, quadratic and cubic characters modulo p, which will be used to improve the result (1.2).

Lemma 2.2 *Let χ be a Dirichlet character modulo p, χ_0 denotes the principal character, $L(s,\chi)$ be the associated Dirichlet L-function and ϵ be a fixed positive number. Then, for $p \le x^{1/5}$, we have*

$$\sum_{\substack{m\le x\\ m \text{ square-full}}}\chi(m)=\begin{cases}\frac{p-1}{p}\left(\frac{L(3/2,\chi_0)}{L(3,\chi_0)}x^{1/2}+\frac{L(2/3,\chi_0)}{L(2,\chi_0)}x^{1/3}\right)\\ +O(x^{1/6}p^{1/2+\epsilon}), & \text{if } \chi=\chi_0,,\\ \frac{p-1}{p}\left(\frac{L(3/2,\chi)}{L(3,\chi_0)}x^{1/2}\right)+O(x^{1/6}p^{3/2+\epsilon}), & \text{if } \chi\ne\chi_0,\ \chi^2=\chi_0,\\ \frac{p-1}{p}\left(\frac{L(2/3,\chi^2)}{L(2,\chi_0)}x^{1/3}\right)+O(x^{1/6}p^{3/2+\epsilon}), & \text{if } \chi\ne\chi_0,\ \chi^3=\chi_0,\\ O(x^{1/6}p^{3/2+\epsilon}), & \text{if } \chi^2\ne\chi_0 \text{ and } \chi^3\ne\chi_0.\end{cases}$$

Proof See the proof of Theorem 2.1 in [6]. □

Next lemmas will be used to prove Theorem 1.4.

Lemma 2.3 *Let χ be a Dirichlet character modulo p, χ_0 denotes the principal character, $L(s,\chi)$ be the associated Dirichlet L-function and ϵ be a fixed positive number. Let*

$$J(s,\chi)=\prod_q\left(1-\frac{\chi^8(q)}{q^{8s}}-\frac{\chi^9(q)}{q^{9s}}-\frac{\chi^{10}(q)}{q^{10s}}+\frac{\chi^{13}(q)}{q^{13s}}+\frac{\chi^{14}(q)}{q^{14s}}\right),\quad \Re(s)>\frac{1}{8},$$

where $\prod_q$ denotes the product over all primes q. Then, for $p\le x^{1/7}$, we have

$$\begin{aligned}\frac{p}{p-1}\sum_{\substack{m\le x\\ m \text{ cube-full}}}\chi_0(m)&=L(4/3,\chi_0)L(5/3,\chi_0)J(1/3,\chi_0)x^{1/3}\\ &\quad+L(3/4,\chi_0)L(5/4,\chi_0)J(1/4,\chi_0)x^{1/4}\\ &\quad+L(4/5,\chi_0)L(3/5,\chi_0)J(1/5,\chi_0)x^{1/5}+O(x^{7/46}p^{35/92+\epsilon}),\end{aligned}$$

if there exist characters $\chi_1\ne\chi_0 \mod p$ such that $\chi_1^3=\chi_0$, then

$$\frac{p}{p-1}\sum_{\substack{m\le x\\ m \text{ cube-full}}}\chi_1(m)=L(4/3,\chi_1)L(5/3,\chi_1^2)J(1/3,\chi_1)x^{1/3}+O(x^{7/46}p^{127/92+\epsilon}),$$

if there exist characters $\chi_2 \neq \chi_0 \mod p$ *such that* $\chi_2^4 = \chi_0$, *then*

$$\frac{p}{p-1} \sum_{\substack{m \leq x \\ m \text{ cube-full}}} \chi_2(m) = L(3/4, \chi_2^3) L(5/4, \chi_2) J(1/4, \chi_2) x^{1/4} + O(x^{7/46} p^{127/92+\epsilon}),$$

if there exist characters $\chi_3 \neq \chi_0 \mod p$ *such that* $\chi_3^5 = \chi_0$, *then*

$$\frac{p}{p-1} \sum_{\substack{m \leq x \\ m \text{ cube-full}}} \chi_3(m) = L(4/5, \chi_3^4) L(3/5, \chi_3^3) J(1/5, \chi_3) x^{1/5} + O(x^{7/46} p^{127/92+\epsilon}),$$

if there exist characters χ *such that* $\chi^3 \neq \chi_0$, $\chi^4 \neq \chi_0$ *and* $\chi^5 \neq \chi_0$, *then*

$$\sum_{\substack{m \leq x \\ m \text{ cube-full}}} \chi(m) \ll x^{7/46} p^{127/92+\epsilon}.$$

Proof See the proof of Theorem 2.3 in [6]. □

3 Proof of Theorems 1.1 and 1.4

Proof In view of Lemma 2.1, for a given odd prime p, we have

$$\sum_{n \leq x} T_2(n) = \frac{\phi(p-1)}{p-1} \left\{ \sum_{\substack{m \leq x \\ m \text{ square-full}}} \chi_0(m) - \sum_{\substack{m \leq x \\ m \text{ square-full} \\ d=2}} \chi(m) - \sum_{\substack{m \leq x \\ m \text{ square-full} \\ d=3}} \chi(m) \right\}$$
$$+ \frac{\phi(p-1)}{p-1} \sum_{\substack{d \mid p-1 \\ d>3}} \frac{\mu(d)}{\phi(d)} \sum_{\chi \in \Gamma_d} \sum_{\substack{m \leq x \\ m \text{ square-full}}} \chi(m).$$

We bound the last sum by using the last case in Lemma 2.2. Thus

$$\left| \sum_{\substack{d \mid p-1 \\ d>3}} \frac{\mu(d)}{\phi(d)} \sum_{\chi \in \Gamma_d} \sum_{\substack{m \leq x \\ m \text{ square-full}}} \chi(m) \right| \ll 3^{\omega_{1,3}(p-1)} x^{1/6} p^{3/2+\epsilon}.$$

We use the first three cases in Lemma 2.2 to regroup the main term of size $x^{1/2}$ and $x^{1/3}$ for the first three sums and obtain

$$\frac{p-1}{p} \left(\frac{L(3/2, \chi_0) - L(3/2, \chi_1)}{L(3, \chi_0)} \right) x^{1/2} + \frac{p-1}{p} \left(\frac{L(2/3, \chi_0) - L(2/3, \chi_2^2)}{L(2, \chi_0)} \right) x^{1/3}.$$

In view of Lemma 2.2, the error terms of the first three sums do not domonate $3^{\omega_{1,3}(p-1)} x^{1/6} p^{3/2+\epsilon}$, which establishes the formula.

The proof of Theorem 1.4 follows by the same method as in the proof of Theorem 1.1, with Lemma 2.2 replaced by Lemma 2.3 respectively. □

4 Concluding remarks

We consider here only square-full and cube-full integers since our method makes use of two auxiliary lemmas (Lemmas 2.2 and 2.3) which treat only these two cases. The author is now working out the case of 4-full integers belonging to an arithmetic progression by the same method in [6]; even in this case the calculation become much more complex.

References

1. D.A. Burgess, On character sums and primitive roots. Proc. Lond. Math. Soc. **3**(1), 179–192 (1962)
2. S.D. Cohen, T. Trudgian, On the least square-free primitive root modulo p. J. Number Theory **170**, 10–16 (2017)
3. H.N. Shapiro, *Introduction to the theory of number. Pure and Applied Mathematics*. (Wiley, New York, 1983)
4. M. Munsch, Character sums over squarefree and squarefull numbers. Archiv der Mathematik **102**(6), 555–563 (2014)
5. M. Munsch, T. Trudgian, Square-full primitive roots. Int. J. Number Theory **14**(04), 1013–1021 (2018)
6. Teerapat Srichan, Square-full and cube-full numbers in arithmetic progressions. Siauliai Math. Seminar **8**(16), 223–248 (2013)