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Complete monotonicity of some entropies

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Abstract It is well-known that the Shannon entropies of some parameterized probability distributions are concave functions with respect to the parameter. In this paper we consider a family of such distributions (including the binomial, Poisson, and negative binomial distributions) and investigate their Shannon, Rényi, and Tsallis entropies with respect to complete monotonicity.

Keywords Entropies · Concavity · Complete monotonicity · Inequalities

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1 Introduction

Let $c \in \mathbb{R}$, $I_c := \left[0, -\frac{1}{c}\right]$ if $c < 0$, and $I_c := [0, +\infty)$ if $c \geq 0$.

As usual, for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}_0$ the binomial coefficients are defined by

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \quad \text{if } k \in \mathbb{N}, \text{ and } \binom{\alpha}{0} := 1.$$

Let $n > 0$ be a real number such that $n > c$ if $c \geq 0$, or $n = -cl$ with some $l \in \mathbb{N}$ if $c < 0$.

For $k \in \mathbb{N}_0$ and $x \in I_c$ define

$$p_{n,k}^{[c]}(x) := (-1)^k \binom{-\frac{n}{c}}{k} (cx)^k (1+cx)^{-\frac{n}{c}-k}, \quad \text{if } c \neq 0,$$

$$p_{n,k}^{[0]}(x) := \lim_{c \to 0} p_{n,k}^{[c]}(x) = \frac{(nx)^k}{k!} e^{-nx}.$$

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Details and historical notes concerning these functions can be found in [3,7,21] and the references therein. In particular,

$$\frac{d}{dx} p_{n,k}^{[c]}(x) = n \left(p_{n+c,k-1}^{[c]}(x) - p_{n+c,k}^{[c]}(x) \right). \tag{1.1}$$

Moreover,

$$\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1; \tag{1.2}$$

$$\sum_{k=0}^{\infty} k p_{n,k}^{[c]}(x) = nx, \tag{1.3}$$

so that $\left(p_{n,k}^{[c]}(x) \right)_{k \geq 0}$ is a parameterized probability distribution. Its associated Shannon entropy is

$$H_{n,c}(x) := - \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) \log p_{n,k}^{[c]}(x),$$

while the Rényi entropy of order 2 and the Tsallis entropy of order 2 are given, respectively, by (see [18,20])

$$R_{n,c}(x) := - \log S_{n,c}(x); \quad T_{n,c}(x) := 1 - S_{n,c}(x),$$

where

$$S_{n,c}(x) := \sum_{k=0}^{\infty} \left(p_{n,k}^{[c]}(x) \right)^2, \quad x \in I_c.$$

The cases $c = -1$, $c = 0$, $c = 1$ correspond, respectively, to the binomial, Poisson, and negative binomial distributions. For other details see also [15,16].

In this paper we investigate the above entropies with respect to the complete monotonicity.

2 Shannon entropy

2.1 Let's start with the case $c < 0$.

$H_{n,-1}$ is a concave function; this is a special case of the results of [19]; see also [6,8,9] and the references therein.

Here we shall determine the signs of all the derivatives of $H_{n,c}$.

Theorem 2.1 *Let $c < 0$. Then, for all $k \geq 0$,*

$$H_{n,c}^{(2k+2)}(x) \leq 0, \quad x \in \left(0, -\frac{1}{c} \right), \tag{2.1}$$

$$H_{n,c}^{(2k+1)}(x) = \begin{cases} \geq 0 & x \in (0, -\frac{1}{2c}], \\ \leq 0 & x \in [-\frac{1}{2c}, -\frac{1}{c}). \end{cases} \tag{2.2}$$

Proof We have $n = -cl$ with $l \in \mathbb{N}$. As in [10], let us represent $\log(l!)$ by integrals:

$$\log(l!) = \int_0^{\infty} \left(l - \frac{1 - e^{-ls}}{1 - e^{-s}} \right) \frac{e^{-s}}{s} ds = \int_0^1 \left(\frac{1 - (1 - t)^l}{t} - l \right) \frac{dt}{\log(1 - t)}. \tag{2.3}$$

Now using (1.2), (1.3) and (2.3) we get

$$H_{n,c}(x) = H_{l,-1}(-cx) = -l\left[(-cx)\log(-cx) + (1+cx)\log(1+cx)\right] + \int_0^1 \frac{-t}{\log(1-t)} \frac{(1+cxt)^l + (1-t-cxt)^l - 1 - (1-t)^l}{t^2} dt.$$

It is a matter of calculus to prove that

$$H''_{n,c}(x) = cl\left(\frac{1}{x} - \frac{c}{1+cx}\right) + c^2 l(l-1)\int_0^1 \frac{-t}{\log(1-t)}\left[(1+cxt)^{l-2} + (1-t-cxt)^{l-2}\right] dt,$$

and for $k \geq 0$

$$H^{(2k+2)}_{n,c}(x) = cl(2k)!\left(\frac{1}{x^{2k+1}} - \left(\frac{c}{1+cx}\right)^{2k+1}\right) + l(l-1)\ldots(l-2k-1)c^{2k+2} \times \int_0^1 \frac{-t}{\log(1-t)}\left[(1+cxt)^{l-2k-2} + (1-t-cxt)^{l-2k-2}\right] t^{2k} dt.$$

For $0 < t < 1$ we have

$$0 < \frac{-t}{\log(1-t)} < 1, \tag{2.4}$$

so that

$$H^{(2k+2)}_{n,c}(x) \leq cl(2k)!\left(\frac{1}{x^{2k+1}} - \left(\frac{c}{1+cx}\right)^{2k+1}\right) + l(l-1)\ldots(l-2k-1)c^{2k+2} \times \int_0^1 \left[(1+cxt)^{l-2k-2} + (1-t-cxt)^{l-2k-2}\right] t^{2k} dt. \tag{2.5}$$

Repeated integration by parts yields

$$\int_0^1 (1+cxt)^{l-2k-2} t^{2k} dt \leq \frac{(2k)!}{(l-2)(l-3)\ldots(l-2k-1)(cx)^{2k}} \int_0^1 (1+cxt)^{l-2} dt,$$

and so

$$\int_0^1 (1+cxt)^{l-2k-2} t^{2k} dt \leq \frac{(2k)!\left[(1+cx)^{l-1} - 1\right]}{(l-1)(l-2)\ldots(l-2k-1)(cx)^{2k+1}}. \tag{2.6}$$

Replacing x by $-\frac{1}{c} - x$ we obtain

$$\int_0^1 (1-t-cxt)^{l-2k-2} t^{2k} dt \leq \frac{(2k)!\left[1 - (-cx)^{l-1}\right]}{(l-1)(l-2)\ldots(l-2k-1)(1+cx)^{2k+1}}. \tag{2.7}$$

From (2.5), (2.6) and (2.7) it follows that

$$H^{(2k+2)}_{n,c}(x) \leq cl(2k)!\left[\frac{(1+cx)^{l-1}}{x^{2k+1}} - \frac{c^{2k+1}(-cx)^{l-1}}{(1+cx)^{2k+1}}\right] \leq 0,$$

and this proves (2.1).

It is easy to verify that $H_{n,c}^{(2k+1)}\left(-\frac{1}{2c}\right)=0$. Since $H_{n,c}^{(2k+2)}\le 0$, it follows that $H_{n,c}^{(2k+1)}$ is decreasing, and this implies (2.2). □

2.2 Consider the case $c = 0$

$H_{n,0}$ is the Shannon entropy of the Poisson distribution. The derivative of this function is completely monotonic: see, e.g., [2, p. 2305]. For the sake of completeness we insert here a short proof.

Theorem 2.2 *$H'_{n,0}$ is completely monotonic, i.e.,*

$$(-1)^k H_{n,0}^{(k+1)}(x)\ge 0,\quad k\ge 0,\quad x>0. \tag{2.8}$$

Proof Note that $H_{n,0}(y)=H_{1,0}(ny)$; so it suffices to investigate the derivatives of $H_{1,0}(x)$.

According to [10, (2.5)],

$$\begin{aligned}H_{1,0}(x)&=x-x\log x+\int_0^\infty \frac{e^{-t}}{t}\left(x-\frac{1-\exp\left(x(e^{-t}-1)\right)}{1-e^{-t}}\right)dt\\&=x-x\log x-\int_0^1\left(x-\frac{1-e^{-sx}}{s}\right)\frac{ds}{\log(1-s)}.\end{aligned}$$

It follows that

$$H'_{1,0}(x)=-\log x-\int_0^1\left(1-e^{-sx}\right)\frac{ds}{\log(1-s)}$$

and for $k\ge 1$,

$$H_{1,0}^{(k+1)}(x)=(-1)^k\left(\frac{(k-1)!}{x^k}+\int_0^1 s^k e^{-sx}\frac{ds}{\log(1-s)}\right). \tag{2.9}$$

By using (2.4) we get

$$\begin{aligned}\int_0^1\frac{s^k e^{-sx}}{\log(1-s)}ds&\ge-\int_0^1 s^{k-1}e^{-sx}ds\\&=-\int_0^x\frac{t^{k-1}}{x^k}e^{-t}dt\ge-\int_0^\infty\frac{1}{x^k}t^{k-1}e^{-t}dt\\&=-\frac{(k-1)!}{x^k}.\end{aligned}$$

Combined with (2.9), this proves (2.8) for $k\ge 1$. In particular, we see that $H_{n,0}$ is concave and non-negative on $[0,+\infty)$; it follows that $H'_{n,0}\ge 0$ and so (2.8) is completely proved. □

2.3 Let now $c > 0$

Theorem 2.3 *For $c>0$, $H'_{n,c}$ is completely monotonic.*

Proof Since $H_{m,c}(y)=H_{\frac{m}{c},1}(cy)$, it suffices to study the derivatives of $H_{n,1}(x)$.

By using (1.2), (1.3) and

$$\log A=\int_0^\infty\frac{e^{-x}-e^{-Ax}}{x}dx,\quad A>0,$$

we get

$$H_{n,1}(x) = n\left((1+x)\log(1+x) - x\log x\right) + \int_0^\infty \frac{e^{-ns}-e^{-s}}{s(1-e^{-s})}\left(1-(1+x-xe^{-s})^{-n}\right)ds$$
$$= n\left((1+x)\log(1+x) - x\log x\right) + \int_0^1 \frac{1-(1-t)^{n-1}}{t\log(1-t)}\left(1-(1+tx)^{-n}\right)dt.$$

It follows that, for $j \geq 1$,

$$\frac{1}{n}H_{n,1}^{(j+1)}(x) = (-1)^{j-1}(j-1)!\left((x+1)^{-j} - x^{-j}\right)$$
$$+(-1)^{j-1}(n+1)(n+2)\dots(n+j)$$
$$\times\int_0^1 \frac{-t}{\log(1-t)}\left[1-(1-t)^{n-1}\right](1+xt)^{-n-j-1}t^{j-1}dt.$$

Using again (2.4), we get

$$(-1)^{j-1}\frac{1}{n}H_{n,1}^{(j+1)}(x) \leq (j-1)!\left((x+1)^{-j}-x^{-j}\right) + (n+1)(n+2)\dots(n+j)$$
$$\times\int_0^1\left[1-(1-t)^{n-1}\right](1+xt)^{-n-j-1}t^{j-1}dt$$
$$= u(x)+v(x),$$

where

$$u(x) := \frac{(j-1)!}{(x+1)^j} - (n+1)(n+2)\dots(n+j)\int_0^1 t^{j-1}(1-t)^{n-1}(1+xt)^{-n-j-1}dt,$$
$$v(x) := (n+1)(n+2)\dots(n+j)\int_0^1 t^{j-1}(1+xt)^{-n-j-1}dt - \frac{(j-1)!}{x^j}.$$

We shall prove that $u(x) \leq 0$ and $v(x) \leq 0$, $x > 0$. Let us remark that

$$\int_0^1 t^{j-1}(1-t)^{n-1}(1+xt)^{-n-j-1}dt \geq \int_0^1 t^{j-1}(1-t)^n(1+xt)^{-n-j-1}dt, \tag{2.10}$$

and integration by parts yields

$$\int_0^1 \frac{t^{j-1}(1-t)^n}{(1+xt)^{n+j+1}}dt = \frac{j-1}{(n+1)(x+1)}\int_0^1 \frac{t^{j-2}(1-t)^{n+1}}{(1+xt)^{n+j+1}}dt.$$

Applying repeatedly this formula we obtain

$$\int_0^1 \frac{t^{j-1}(1-t)^n}{(1+xt)^{n+j+1}}dt = \frac{(j-1)!}{(n+1)(n+2)\dots(n+j)}\frac{1}{(x+1)^j}. \tag{2.11}$$

Now (2.10) and (2.11) imply $u(x) \leq 0$.
Using again integration by parts we get

$$\int_0^1 t^{j-1}(1+xt)^{-n-j-1}dt \leq \frac{j-1}{(n+j)x}\int_0^1 t^{j-2}(1+xt)^{-n-j}dt$$
$$\leq \dots \leq \frac{(j-1)!}{(n+1)(n+2)\dots(n+j)}\frac{1}{x^j},$$

which shows that $v(x) \leq 0$.

We conclude that

$$(-1)^{j-1} H_{n,1}^{(j+1)}(x) \leq 0, \quad j \geq 1, x > 0. \tag{2.12}$$

In particular, (2.12) shows that $H_{n,1}$ is concave on $[0, +\infty)$; it is also non-negative, which means that $H'_{n,1} \geq 0$. Combined with (2.12), this shows that $H'_{n,1}$ is completely monotonic, and the proof is finished. □

Remark 2.4 (2.11) can be obtained alternatively by using the change of variables $y = (1 - t)/(1 + xt)$ and the properties of the Beta function. An alternative proof of the inequality $v(x) \leq 0$ follows from

$$\int_0^1 t^{j-1}(1+xt)^{-n-j-1} dt \leq \frac{1}{x^{j-1}} \int_0^{\infty} \frac{(xt)^{j-1}}{(1+xt)^{n+j+1}} dt$$
$$= \frac{1}{x^j} \int_0^{\infty} \frac{s^{j-1}}{(1+s)^{j+n+1}} ds = \frac{1}{x^j} B(j, n+1) = \frac{1}{x^j} \frac{(j-1)!n!}{(n+j)!}.$$

Corollary 2.5 *The following inequalities are valid for $x > 0$ and $c \geq 0$:*

$$\log \frac{x}{cx+1} \leq \sum_{k=0}^{\infty} p_{n+c,k}^{[c]}(x) \log \frac{k+1}{ck+n} \leq \log \frac{nx+1}{ncx+n}. \tag{2.13}$$

In particular, for $c = 0$ and $n = 1$,

$$\log x \leq \sum_{k=0}^{\infty} e^{-x} \frac{x^k}{k!} \log (k+1) \leq \log (x+1).$$

Proof We have seen that $H'_{n,c}(x) \geq 0$. An application of (1.1) yields

$$H'_{n,c}(x) = n \left(\log \frac{1+cx}{x} + \sum_{k=0}^{\infty} p_{n+c,k}^{[c]}(x) \log \frac{k+1}{n+ck} \right).$$

This proves the first inequality in (2.13); the second is a consequence of Jensen's inequality applied to the concave function $\log t$. □

3 Rényi entropy and Tsallis entropy

The following conjecture was formulated in [13]:

Conjecture 3.1 $S_{n,-1}$ *is convex on* $[0, 1]$.

Th. Neuschel [11] proved that $S_{n,-1}$ is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$. The conjecture and Neuschel's result can also be found in [5].

A proof of the conjecture was given by G. Nikolov [12], who related it to some new inequalities involving Legendre polynomials. Another proof can be found in [4].

Using the important results of Elena Berdysheva [3], the following extension was obtained in [17]:

Theorem 3.2 [17, Theorem 9] *For $c < 0$, $S_{n,c}$ is convex on* $\left[0, -\frac{1}{c}\right]$.

A stronger conjecture was formulated in [14] and [17]:

Conjecture 3.3 *For $c \in \mathbb{R}$, $S_{n,c}$ is logarithmically convex, i.e., $\log S_{n,c}$ is convex.*

This was validated for $c \geq 0$ by U. Abel, W. Gawronski and Th. Neuschel [1], who proved a stronger result:

Theorem 3.4 [1] *For $c \geq 0$, the function $S_{n,c}$ is completely monotonic, i.e.,*

$$(-1)^m \left(\frac{d}{dx}\right)^m S_{n,c}(x) > 0, \quad x \geq 0, m \geq 0.$$

Consequently, for $c \geq 0$, $S_{n,c}$ is logarithmically convex, and hence convex.

Summing up, for the Rényi entropy $R_{n,c} = -\log S_{n,c}$ and Tsallis entropy $T_{n,c} = 1 - S_{n,c}$ we have the following

Corollary 3.5 (i) *Let $c \geq 0$. Then $R_{n,c}$ is increasing and concave, while $T'_{n,c}$ is completely monotonic on* $[0, +\infty)$.
(ii) *$T_{n,c}$ is concave for all $c \in \mathbb{R}$.*

Proof (i) Apply Theorem 3.4.
(ii) For $c < 0$, apply Theorem 3.2. For $c \geq 0$, Theorem 3.4 shows that $S_{n,c}$ is convex, so that $T_{n,c}$ is concave.

□

Remark 3.6 As far as we know, Conjecture 3.3 is still open for $c < 0$, so that the concavity of $R_{n,c}$, $c < 0$, remains to be investigated.

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