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CR-structures of codimension 2 on tangent bundles in Riemann–Finsler geometry

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Abstract We determine a 2-codimensional CR-structure on the slit tangent bundle T_0M of a Finsler manifold (M, F) by imposing a condition on the almost complex structure Ψ associated to F when restricted to the structural distribution of a framed f-structure. This condition is satisfied when (M, F) is of scalar flag curvature (particularly flat). In the Riemannian case (M, g) this last condition means that g is of constant curvature. This CR-structure is finally generalized by using one positive parameter but under more difficult conditions.

Keywords CR-structure · Metric framed f-structure · Finsler geometry · Scalar flag curvature · Space form

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1 Introduction

Finsler geometry is very rich in remarkable tensor fields φ of (1, 1)-type and associated structures. More precisely, there are: an (almost) tangent structure ($\varphi^2 = 0$), an almost complex one ($\varphi^2 = -I$) and also an almost product structure ($\varphi^2 = I$). In [1] another well-known type of structures, namely an f-structure ($\varphi^3 + \varphi = 0$) is obtained in this geometry. In fact, this f-structure belongs to a very interesting particular case which is called *framed f-structure* and has, in addition to φ, a set of vector fields and differential 1-forms interrelated. Moreover, a conformal deformation of the Sasaki type metric can be added in order to obtain

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a *metric framed f-structure*. This metric framed f-structure of M. Anastasiei was recently generalized in [8,15].

The present note concerns yet another kind of structures, namely the *CR-structures*, with an important rôle at the border between differential geometry and complex analysis, as it is pointed out in [7]. We restrict ourselves to the real case; more precisely, based on a relationship between framed f-structures and CR-structure established in [2, p. 130] we found a CR-structure on the slit tangent bundle T_0M of a Finsler manifold (M, F). This CR-structure is constructed with the above almost complex structure denoted by Ψ_F in Sect. 3 and its existence is constrained by one condition expressing the vanishing of the Nijenhuis tensor of Ψ_F on the structural distribution of the framed f-structure from [1]. The above condition is expressed as a relation between the curvature of the Cartan nonlinear connection and the Jacobi endomorphism and is satisfied in dimension two or if (M, F) is of scalar flag curvature which in the particular case of Riemannian geometry (M, g) means that the metric g has a constant curvature. Several important classes of Finsler manifolds with scalar flag curvature are discussed in Chapter 7 of [5].

Inspired by [15] we generalize this CR-structure using a real parameter $\beta > \frac{1}{2}$ but with more difficult conditions. More precisely, we take into account the same vector fields and 1-forms as in the previous framed f-structure but deform the metric and the almost complex structure on both horizontal and vertical directions. At $\beta = 1$ we recover the previous CR-structure.

Finally, let us note that our CR-structures are of codimension 2 and the (complex) geometry of these structures was studied in [11,12] and recently in [9,10]. But for the Riemannian case the only studies until now are on hypersurfaces of Sasakian manifolds [13,14] and not on (slit) tangent bundle. The para-CR version of this study is the paper [6].

2 CR-structures from framed f-structures

Framed f-structures constitute a particular case of f-structures. A detailed study of this class of tensor fields of (1, 1)-type, especially from a local point of view, can be found in [16].

Let N be a smooth $(2n + s)$-dimensional manifold with $n, s \geq 1$ and fix a distribution D of dimension $2n$ on N. Considering D as a vector bundle over N let $\Gamma(D)$ be the module of its sections. Supposing D is endowed with a morphism $J : D \to D$ of vector bundles satisfying $J^2 = -I$ where I is the identity (Kronecker) morphism on D, the pair (D, J) is called *almost complex distribution*.

The first main notion is given in [2, p.128].

Definition 2.1 If for all $X, Y \in \Gamma(D)$ we have

$$\begin{cases} [JX, JY] - [X, Y] \in \Gamma(D) \\ N_J(X, Y) := [JX, JY] - [X, Y] - J([X, JY] + [JX, Y]) = 0, \end{cases} \tag{2.1}$$

then (D, J) is a *CR-structure* on N and the triple (N, D, J) is a *CR-manifold*.

A second main notion is that of a framed f-structure.

Definition 2.2 Let φ be a tensor field of (1, 1)-type and s pairs (ξ_a, η^a), $1 \leq a \leq s$ of vector fields and 1-forms on N. If

(i) $\varphi^3 + \varphi = 0, rank\ \varphi = 2n,$

(ii) $\varphi^2 = -I + \sum_{a=1}^{s} \eta^a \otimes \xi_a$, $\varphi(\xi_a) = 0$, $\eta^a(\xi_b) = \delta_b^a$, $\eta^a \circ \varphi = 0$,
then the data (φ, ξ_a, η^a) is called a *framed f-structure*.

Following [2, p. 130] we associate to a framed f-structure

(1) the $(1, 2)$-type *torsion tensor field*

$$S = N_{\varphi} + 2\sum_{a=1}^{s} d\eta^a \otimes \xi_a, \tag{2.2}$$

(2) *the structural distribution*

$$D = \{X \in \Gamma(TM); \eta^1(X) = ... = \eta^s(X) = 0\} = \cap_{a=1}^{s} \ker \eta^a. \tag{2.3}$$

For a 1-form η we use the differential

$$2d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]). \tag{2.4}$$

These notions lead to

Definition 2.3 The framed f-structure is called *D-normal* if S vanishes on D i.e. $S(X, Y) = 0$ for all $X, Y \in \Gamma(D)$.

The relationship between the above structures was pointed out by A. Bejancu in Proposition 1.1 of [2, p. 130].

Proposition 2.4 *If (φ, ξ_a, η^a) is a D-normal framed f-structure, then $(D, J = \varphi|_D)$ is a CR-structure.*

Proof The restriction J of φ to D is obviously an almost complex structure. Conditions (2.1) result from the fact that for $X, Y \in \Gamma(D)$ we have

$$S(X, Y) = 0 = [JX, JY] + \varphi^2([X, Y]) - \varphi([X, JY] + [JX, Y]) - \sum_{a=1}^{s} \eta^a([X, Y])\xi_a. \tag{2.5}$$

For other details see the cited reference. □

3 A metric framed f-structure on the tangent bundle of a Finsler manifold

Let M be now a smooth m-dimensional manifold with $m \geq 2$ and $\pi : TM \to M$ its tangent bundle. Let $x = (x^i) = (x^1, ..., x^m)$ be local coordinates on M and $(x, y) = (x^i, y^i) = (x^1, ..., x^m, y^1, ..., y^m)$ the induced local coordinates on TM. Denote by O the null-section of π.

Recall after [5] that a *Finsler fundamental function* on M is a map $F : TM \to \mathbb{R}_+$ with the following properties:

(F1) F is smooth on the slit tangent bundle $T_0M := TM \setminus O$ and continuous on O,
(F2) F is positive homogeneous of degree 1: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$,
(F3) the matrix $(g_{ij}) = \left(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right)$ is invertible and its associated quadratic form is positive definite.

The tensor field $g = \{g_{ij}(x, y); 1 \leq i, j \leq m\}$ is called *the Finsler metric* and the homogeneity of F implies:

$$F^2(x, y) = g_{ij}y^i y^j = y_i y^i, \tag{3.1}$$

where $y_i = g_{ij}y^j$. The pair (M, F) is called *Finsler manifold.*

On T_0M we have two distributions:

(i) $V(TM) := \ker \pi_*$, called *the vertical distribution* and not depending of F. It is integrable and has the basis $\left\{\frac{\partial}{\partial y^i}; 1 \leq i \leq m\right\}$. A remarkable section of it is *the Liouville vector field* $\Gamma = y^i \frac{\partial}{\partial y^i}$.

(ii) $H(TM)$ with the basis $\left\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial y^i} - N^j_i \frac{\partial}{\partial y^j}\right\}$, where

$$N^i_j = \frac{1}{2}\frac{\gamma^i_{00}}{\partial y^j} \tag{3.2}$$

with $\gamma^i_{00} = \gamma^i_{jk}y^j y^k$ built from the usual Christoffel symbols

$$\gamma^i_{jk} = \frac{1}{2}g^{ia}\left(\frac{\partial g_{ak}}{\partial x^j} + \frac{\partial g_{ja}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^a}\right). \tag{3.3}$$

$H(TM)$ is often called the *Cartan* (or canonical) *nonlinear connection* of the geometry (M, F) and a remarkable section of it is *the geodesic spray*

$$S_F = y^i \frac{\delta}{\delta x^i}. \tag{3.4}$$

In particular, if g does not depend on y, we recover Riemannian geometry.

The dual basis of the above local basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ of $\Gamma(T_0M)$ is $(dx^i, \delta y^i = dy^i + N^i_j dx^j)$. On T_0M we have a Riemannian metric of Sasaki type

$$G_F = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j. \tag{3.5}$$

Another Finslerian object is the tensor field of $(1, 1)$-type $\Psi_F : \Gamma(T_0M) \to \Gamma(T_0M)$

$$\Psi_F\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}, \quad \Psi_F\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}. \tag{3.6}$$

It results that Ψ_F is an almost complex structure and the pair (Ψ_F, G_F) is an almost Kähler structure on T_0M.

In order to obtain a framed f-structure on T_0M associated to the Finslerian function F, the following objects are considered in [1]

$$\begin{cases} \xi_1 = S_F, \xi_2 = \Gamma, \\ \eta^1 = \frac{1}{F^2}y_i dx^i, \quad \eta^2 = \frac{1}{F^2}y_i\delta y^i, \\ \varphi = \Psi_F + \eta^1 \otimes \xi_2 - \eta^2 \otimes \xi_1, \\ G = \frac{1}{F^2}G_F. \end{cases} \tag{3.7}$$

Then the main result of [1] is that the data $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2)$ is a framed f-structure on T_0M with η^a the G-dual of ξ_a, $1 \leq a \leq 2$ and, moreover

$$G(\varphi\cdot, \varphi\cdot) = G - \eta^1 \otimes \eta^1 - \eta^2 \otimes \eta^2. \tag{3.8}$$

Also, ξ_a are unitary vector fields with respect to G and $(G, \varphi, \xi_a, \eta^a)$ is a *metric framed f-structure*.

4 Putting all together

The last paragraph of the previous section provides the ingredients of Sect. 2 with $N = T_0M$, $s = 2$ and $n = m - 1$, which motivates our choice $m \geq 2$. Then the structural distribution is

$$D_F = \ker \eta^1 \cap \ker \eta^2 = \{\xi_1\}^{\perp G} \cap \{\xi_2\}^{\perp G} = \{\xi_1\}^{\perp G_F} \cap \{\xi_2\}^{\perp G_F}, \tag{4.1}$$

where $\{X\}^{\perp G}$ is the G-orthogonal complement of $span\{X\}$. We have $D_F = (span\{\xi_1, \xi_2\})^{\perp G_F}$ and this implies that D_F has dimension $2m - 2$. For a geometrical meaning of the distribution $span\{\xi_1, \xi_2\}$ in [1] is defined the differential 2-form ω_F, naturally associated to the metric framed f-structure

$$\omega_F = G(\cdot, \varphi\cdot), \tag{4.2}$$

and it follows that $span\{\xi_1, \xi_2\}$ is the kernel of ω_F. Also, the homogeneity of F implies the homogeneity of $S_F = \xi_1$, which means

$$[\Gamma, S_F] = [\xi_2, \xi_1] = \xi_1, \tag{4.3}$$

and thus $span\{\xi_1, \xi_2\}$ is an integrable distribution; see also Theorem 3.15 of [3, p. 236].

A concrete expression of D_F appears in [4, p. 11]. More precisely, consider after the cited paper

(i) the horizontal vector fields

$$h_i = \frac{\delta}{\delta x^i} - \frac{1}{F^2} y_i S_F, \tag{4.4}$$

and the corresponding $(m-1)$-distribution $\mathcal{H}_{m-1} = span\{h_i; 1 \leq i \leq m\}$,

(ii) the vertical vector fields

$$v_i = \frac{\partial}{\partial y^i} - \frac{1}{F^2} y_i \Gamma, \tag{4.5}$$

and also the corresponding $(m-1)$-distribution $\mathcal{V}_{m-1} = span\{v_i; 1 \leq i \leq m\}$.

We have

$$D_F = \mathcal{H}_{m-1} \oplus \mathcal{V}_{m-1}, \tag{4.6}$$

and the same Theorem 3.15 of [3, p. 236] proves the integrability of $\mathcal{V}_{m-1}$; see also [4, p. 12].

Regarding the integrability of the nonlinear connection $H(TM)$ we have

$$\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right] = R^i_{jk} \frac{\partial}{\partial y^i}, \tag{4.7}$$

where

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}. \tag{4.8}$$

The tensor field $R = \{R^i_{jk}(x, y); 1 \leq i, j, k \leq m\}$ is called *the curvature* of the Cartan nonlinear connection and

$$R^i_j := R^i_{kj} y^k \tag{4.9}$$

are the components of *the Jacobi endomorphism* $\Phi = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j$, [4, p. 5]. Now we are ready for the first main result:

Theorem 4.1 *If the curvature tensor of* (M, F) *has the form*

$$R^i_{jk} = \lambda \left(X^i_k y_j - X^i_j y_k \right) \tag{4.10}$$

with λ *a smooth function on* T_0M *and the tensor field* $\{X^i_j(x, y); 1 \leq i, j \leq m\}$ *satisfying*

$$y_i X^i_j = y_j \tag{4.11}$$

for all $i, j \in \{1, ..., m\}$*, then the pair* $(D_F, J_F = \Psi_F|_{D_F})$ *is a CR-structure on* T_0M*.*

Proof We express the Nijenhuis tensor field of Ψ_F as

$$\begin{aligned} N_{\Psi_F}(X, Y) = [\Psi_F X, \Psi_F Y] - [X, Y] - \Psi_F(A(X, Y)) = B(X, Y) \\ -\Psi_F(A(X, Y)) \end{aligned} \tag{4.12}$$

with $A(X, Y) := [X, \Psi_F Y] + [\Psi_F X, Y]$ and $B(X, Y) = [\Psi_F X, \Psi_F Y] - [X, Y]$. It follows that $B(X, Y) = A(\Psi_F X, Y)$ and then

$$N_{\Psi_F}(X, Y) = A(\Psi_F X, Y) - \Psi_F \circ A(X, Y). \tag{4.13}$$

We prove firstly that A is a D_F-valued (0, 2)-tensor field. From (4.7) and

$$\left[\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k} \right] = \frac{\partial N^i_j}{\partial y^k} \frac{\partial}{\partial y^i} = \frac{\partial^2 \gamma^i_{00}}{\partial y^j \partial y^k} \frac{\partial}{\partial y^i} \tag{4.14}$$

we obtain

$$A \left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \right) = A \left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k} \right) = 0, \quad A \left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k} \right) = R^i_{jk} \frac{\partial}{\partial y^i}, \tag{4.15}$$

which means that $\eta^1 \circ A = 0$ and

$$A = R^i_{jk} dx^j \wedge \delta y^k \otimes \frac{\partial}{\partial y^i}. \tag{4.16}$$

A main identity in Finsler geometry is

$$y_i R^i_{ab} = 0, \tag{4.17}$$

and then $\eta^2 \circ A = 0$, which conclude the first part of the proof.

Secondly, we search for the framework of Proposition 2.4. The torsion tensor S on D_F is

$$S(X, Y) = N_\varphi(X, Y) - \eta^1([X, Y])\xi_1 - \eta^2([X, Y])\xi_2$$

with

$$N_\varphi(X, Y) = [\Psi_F X, \Psi_F Y] + \varphi^2([X, Y]) - \varphi \circ A(X, Y).$$

Since φ is an element of a framed f-structure, we get

$$N_\varphi(X, Y) = [\Psi_F X, \Psi_F Y] - [X, Y] + \eta^1([X, Y])\xi_1 + \eta^2([X, Y])\xi_2 - \varphi \circ A(X, Y)$$

and from the definition (3.7_3) of φ it follows

$$\begin{aligned} S(X, Y) &= [\Psi_F X, \Psi_F Y] - [X, Y] - (\Psi_F + \eta^1 \otimes \xi_2 - \eta^2 \otimes \xi_1) \circ A(X, Y) \\ &= N_{\Psi_F}(X, Y). \end{aligned} \tag{4.18}$$

In local coordinates we have

$$N_{\Psi_F} = R^i_{jk}\delta y^j \wedge \delta y^k \otimes \frac{\partial}{\partial y^i}, \tag{4.19}$$

and then N_{Ψ_F} has components only when applied on the pair (v_a, v_b). A long but straightforward computation yields

$$N_{\Psi_F}(v_a, v_b) = 2\left[R^i_{ab} + \frac{1}{F^2}(R^i_a y_b - R^i_b y_a)\right]\frac{\partial}{\partial y^i}, \tag{4.20}$$

and therefore the normality condition is

$$F^2 R^i_{ab} = R^i_b y_a - R^i_a y_b, \tag{4.21}$$

which can be expressed as

$$N_{\Psi_F} = \eta^2 \wedge \left(R^i_k \delta y^k \otimes \frac{\partial}{\partial y^i}\right). \tag{4.22}$$

Relation (4.10) yields

$$R^i_k = \lambda\left(F^2 X^i_k - y^a X^i_a y_k\right) \tag{4.23}$$

and then both sides of (4.21) are equal to $\lambda F^2(X^i_k y_j - X^i_j y_k)$, which gives the final conclusion. Condition (4.11) corresponds to relation (4.17).

Let us also point out that condition (4.10) gives the following expression for the Nijenhuis tensor

$$N_{\Psi_F} = 2\lambda F^2 \eta^2 \wedge \left(X^i_j \delta y^j \otimes \frac{\partial}{\partial y^i}\right), \tag{4.24}$$

which yields again the vanishing of N_{Ψ_F} on D_F due to the presence of η^2. Concerning the tensor field A we have

$$A = \lambda F^2\left[\eta^1 \wedge \left(X^i_j \delta y^j \otimes \frac{\partial}{\partial y^i}\right) - \left(X^i_j dx^j \otimes \frac{\partial}{\partial y^i}\right) \wedge \eta^2\right], \tag{4.25}$$

which proves the relations $\eta^1 \circ A = \eta^2 \circ A = 0$. □

Example 4.2 Recall that in dimension 2 the Nijenhuis tensor field of any almost complex structure vanishes. Then every 2-dimensional Finsler manifold (M^2, F) satisfies the condition of Theorem 4.1. Let $V(TM)$ be spanned by the vector fields Γ and V respectively, $H(TM)$ be spanned by the vector fields S_F and H. Then D_F is spanned by V and H and

$$J_F(H) = -V, \quad J_F(V) = H. \tag{4.26}$$

We have that H is a linear combination of h_1 and h_2 while V is a linear combination of v_1 and v_2. □

In order to consider examples in any dimension we remark that a solution of condition (4.11) is

$$X^i_j = \mu\delta^i_j + (1-\mu)\frac{y^i y_j}{F^2} \tag{4.27}$$

again with μ a smooth function on T_0M.

Example 4.3 If $\mu = 1$ then $X^i_j = \delta^i_j$ and the Finsler manifold (M, F) is *of scalar flag curvature* λ since

$$R^i_{jk} = \lambda\left(\delta^i_k y_j - \delta^i_j y_k\right), \tag{4.28}$$

and then

$$R^i_k = \lambda\left(\delta^i_k F^2 - y^i y_k\right). \tag{4.29}$$

Corollary 4.4 *If (M, F) is of scalar flag curvature, then $(D_F = (span\{S_F, \Gamma\})^{\perp G_F}, J_F)$ is a CR-structure on T_0M.*

Remark also that the hypothesis of scalar flag curvature yields

$$N_{\Psi_F} = 2\lambda F^2\eta^2 \wedge \pi_{V(TM)}, \tag{4.30}$$

where $\pi_{V(TM)}$ is the projector on the vertical part in the G_F-orthogonal decomposition $T(T_0M) = H(TM) \oplus V(TM)$ i.e $\pi_{V(TM)} = \delta y^i \otimes \frac{\partial}{\partial y^i}$. However, Ψ_F is integrable only in the flat case (i.e. $\lambda = 0$) since $N_{\Psi_F}(\Gamma, v_a) = 2\lambda F^2 v_a$. The integrability of Ψ_F as a tensor field of $(1, 1)$-type is equivalent with the integrability of the Cartan nonlinear connection of (M, F) and then (T_0M, Ψ_F, G_F) is a Kähler manifold.

Particular case 4.5 (Riemannian space) Let $g = (g_{ij}(x))$ be a Riemannian metric on M. Then $\gamma^i_{jk}(x, y) = \Gamma^i_{jk}(x)$ are the Riemannian Christoffel symbols and

$$R^i_{jk}(x, y) = R^i_{jka}(x)y^a \tag{4.31}$$

where $R_g = (R^i_{jka})$ is the Riemannian curvature tensor of g. It results that a Riemannian geometry $(M, F = (g_{ij}(x)y^i y^j)^{\frac{1}{2}})$ is of scalar flag curvature if and only if g is of constant curvature. Therefore on the slit tangent bundle of a space form (M, g) there exists a CR-structure on the distribution complementary (with respect to the Sasaki lift of g) to the distribution generated by the Liouville vector field and the geodesic spray S_g. □

Example 4.6 Returning to the general non-Riemannian case (4.27) with $\mu = 0$ we get

$$X^i_j = \frac{y^i y_j}{F^2}, \tag{4.32}$$

and then $R^i_{jk} = 0$, which means that (M, F) is flat, a situation belonging also to Example 4.3 for vanishing scalar curvature. □

For the general μ we have

$$N_{\Psi_F} = 2\lambda F^2\eta^2 \wedge \left[\mu\pi_{V(TM)} + (1-\mu)\eta^2 \otimes \Gamma\right] = 2\lambda\mu F^2\eta^2 \wedge \mu\pi_{V(TM)}. \tag{4.33}$$

5 A 1-parametric generalization

Let $\alpha > 0$ and $\beta > 0$ be two positive numbers. Following the approach of [15], let $v : TM \to \mathbb{R}$ be a function of the form $v = \bar{v} \circ \tau$ where $\tau = F^2$ and $\bar{v} : [0, +\infty) \to \mathbb{R}$ is a smooth function. Supposing that

$$\alpha + 2t\bar{v}(t) > 0 \tag{5.1}$$

for any $t \in (0, +\infty)$, in the cited paper, the smooth functions $\bar{w} : [0, +\infty) \to \mathbb{R}$, $w : TM \to \mathbb{R}$

$$\bar{w}(t) = -\frac{\beta \bar{v}(t)}{\alpha + t\bar{v}(t)} \quad \text{and} \quad w = \bar{w} \circ \tau, \tag{5.2}$$

and the Riemannian metric on T_0M

$$\bar{G} = G_{ij} dx^i \otimes dx^j + H_{ij} \delta y^i \otimes \delta y^j \tag{5.3}$$

are defined, where

$$\begin{cases} G_{ij} = \frac{1}{\beta} g_{ij} + \frac{v}{\alpha\beta} y_i y_j \\ H_{ij} = \beta g_{ij} + w \circ \tau y_i y_j. \end{cases} \tag{5.4}$$

Inspired by [15] we define also

$$\begin{cases} \bar{\xi}_1 = (\beta + w\tau) S_F, & \bar{\xi}_2 = \Gamma = \xi_2, \\ \bar{\eta}^1 = \frac{1}{\tau} y_i dx^i = \eta^1, & \bar{\eta}^2 = (\frac{\beta}{\tau} + w) y_i \delta y^i, \\ \bar{\Psi}_F(\frac{\delta}{\delta x^i}) = -G_i^a \frac{\partial}{\partial y^a}, & \bar{\Psi}_F(\frac{\partial}{\partial y^i}) = H_i^a \frac{\delta}{\delta x^a}, \end{cases} \tag{5.5}$$

where the lift of indices in the third line is constructed with $g^{-1} = (g^{ab})$. In fact, the only difference between us and [15] is with respect to 1-form $\bar{\eta}^i$; in order to reobtain that of Sect. 3 we divide with τ the 1-forms of Peyghan–Zhong. With a computation similar to that of Theorem 4.8 of Peyghan–Zhong we derive that $(\bar{G}, \bar{\varphi}, \bar{\xi}_a, \bar{\eta}^a)$ with

$$\bar{\varphi} = \bar{\Psi}_F + \bar{\eta}^1 \otimes \bar{\xi}_2 - \bar{\eta}^2 \otimes \bar{\xi}_1 \tag{5.6}$$

is a metric framed f-structure on T_0M if and only if

$$\beta + t\bar{w}(t) = 1. \tag{5.7}$$

From this condition we get that $\bar{\xi}_a = \xi_a$ and $\bar{\eta}^a = \eta^a$. From (5.2) and (5.7) we obtain

$$\bar{v}(t) = \frac{\alpha(\beta - 1)}{t}, \quad \bar{w}(t) = \frac{1 - \beta}{t}. \tag{5.8}$$

In the particular case $\alpha = \beta = 1$ we recover the metric framed f-structure of Anastasiei since $\bar{v} = \bar{w} \equiv 0$.

Now, under condition (5.7) we have the same structural distribution D_F but the expression of the tensor field

$$\bar{A}(X, Y) := [X, \bar{\Psi}_F Y] + [\bar{\Psi}_F X, Y] \tag{5.9}$$

is more complicated. More detailed

$$\begin{cases} \bar{A}(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}) = \left(\frac{\delta G_j^v}{\delta x^k} - \frac{\delta G_k^v}{\delta x^j} + G_j^u \frac{\partial N_k^v}{\partial y^u} - G_k^u \frac{\partial N_j^v}{\partial y^u}\right) \frac{\partial}{\partial y^v} \\ \bar{A}(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) = \left(\frac{\partial H_k^v}{\partial y^j} - \frac{\partial H_j^v}{\partial y^k}\right) \frac{\delta}{\delta x^v} + \left(H_j^u \frac{\partial N_u^v}{\partial y^k} - H_k^u \frac{\partial N_u^v}{\partial y^j}\right) \frac{\partial}{\partial y^v}, \\ \bar{A}(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}) = \frac{\delta H_k^v}{\delta x^j} \frac{\delta}{\delta x^v} + \left(H_k^u R_{ju}^v + \frac{\partial G_j^v}{\partial y^k}\right) \frac{\partial}{\partial y^v}, \end{cases} \tag{5.10}$$

where, with (5.7)

$$\begin{cases} G_{ij} = \frac{1}{\beta} g_{ij} + \frac{\beta - 1}{\beta\tau} y_i y_j, & H_{ij} = \beta g_{ij} + \frac{1-\beta}{\tau} y_i y_j \\ G_j^a = \frac{1}{\beta} \delta_j^a + \frac{\beta - 1}{\beta\tau} y^a y_j, & H_j^a = \beta \delta_j^a + \frac{1-\beta}{\tau} y^a y_j \\ \bar{\Psi}_F(\frac{\delta}{\delta x^i}) = -\frac{1}{\beta} \frac{\partial}{\partial y^i} + \frac{1-\beta}{\beta\tau} y_i \Gamma, & \bar{\Psi}_F(\frac{\partial}{\partial y^i}) = \beta \frac{\delta}{\delta x^i} + \frac{1-\beta}{\tau} y_i S_F. \end{cases} \tag{5.11}$$

It results that α disappears and this motivates the title of this section, namely 1-parametric generalization and not 2-parametric. Note that $\bar{\Psi}_F(h_i) = -\frac{1}{\beta}v_i$ and $\bar{\Psi}_F(v_i) = \beta h_i$.

Then

$$\begin{cases} \bar{A}(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}) = \frac{\beta-1}{\beta\tau}\left[\frac{\delta}{\delta x^k}\left(y_j y^v\right) - \frac{\delta}{\delta x^j}\left(y_k y^v\right)\right]\frac{\partial}{\partial y^v} \\ \bar{A}(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}) = (1-\beta)\left[\frac{\partial}{\partial y^j}(\frac{y_k y^v}{\tau}) - \frac{\partial}{\partial y^k}(\frac{y_j y^v}{\tau})\right]\frac{\delta}{\delta x^v} \\ \bar{A}(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}) = \frac{1-\beta}{\tau}\frac{\delta}{\delta x^j}\left(y_k y^v\right)\frac{\delta}{\delta x^v} + \left[\beta R^v_{jk} + \frac{1-\beta}{\tau}y_k y^u R^v_{ju} + \frac{\beta-1}{\beta}\frac{\partial}{\partial y^k}\left(\frac{y_j y^v}{\tau}\right)\right]\frac{\partial}{\partial y^v}. \end{cases} \tag{5.12}$$

Choosing $\alpha = 1$ the second main result is

Theorem 5.1 *Let $\beta > \frac{1}{2}$ and the smooth functions $\bar{v}(t) = -\bar{w}(t) = \frac{\beta-1}{t}$. If for any $X, Y \in D_F$ we have*

(1) $\bar{A}(X, Y) \in D_F$,
(2) $N_{\bar{\Psi}_F}(X, Y) = 0$, then $(D_F, \bar{J}_F = \bar{\Psi}_F|_{D_F})$ is a CR-structure on T_0M.

Proof The condition in β follows from (5.1). Exactly as in the proof of Theorem 4.1 we have

$$S(X, Y) = N_{\bar{\Psi}_F}(X, Y) - \eta^1(\bar{A}(X, Y))\xi_2 + \eta^2(\bar{A}(X, Y))\xi_1. \tag{5.13}$$

and the conclusion follows directly. Let us note that 1) corresponds to condition (2.1_1) while 2) corresponds to condition (2.1_2). □

Let us remark that

$$\beta\eta^2 \circ \bar{A}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) = \eta^1 \circ \bar{A}\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k}\right) - \eta^1 \circ \bar{A}\left(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}\right), \tag{5.14}$$

and then the vanishing of $\eta^1 \circ \bar{A}\left(\frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^b}\right)$ implies the vanishing of $\eta^2 \circ \bar{A}\left(\frac{\delta}{\delta x^u}, \frac{\delta}{\delta x^v}\right)$. The vanishing of the former expression means that y_k is an eigenvector for $\frac{\delta}{\delta x^j}$

$$\frac{\delta y_k}{\delta x^j} = \left(-\frac{N^a_j y_a}{F^2}\right)y_k \tag{5.15}$$

and then y_k is an eigenvector for the geodesic spray

$$S_F(y_k) = \left(-\frac{N^a_j y^j y_a}{F^2}\right)y_k. \tag{5.16}$$

Such condition holds in the Euclidian space $(\mathbb{R}^m, g_{ij} = \delta_{ij})$ but here the expression $\eta^2 \circ \bar{A}(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^k})$ is non-vanishing since

$$y_v\frac{\partial}{\partial y^k}\left(\frac{y_j y^v}{F^2}\right) = \delta_{jk} - \frac{y_j y^k}{F^2} \neq 0 \tag{5.17}$$

and then it remains an open problem to find Riemannian and/or Finsler manifolds satisfying the conditions of Theorem 5.1 with $\beta \neq 1$.

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