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Generalized Fibonacci numbers of the form $wx^2 + 1$

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Abstract Let $P \geq 3$ be an integer and let (U_n) and (V_n) denote generalized Fibonacci and Lucas sequences defined by $U_0 = 0, U_1 = 1$; $V_0 = 2, V_1 = P$, and $U_{n+1} = PU_n - U_{n-1}$, $V_{n+1} = PV_n - V_{n-1}$ for $n \geq 1$. In this study, when P is odd, we solve the equation $U_n = wx^2 + 1$ for $w = 1, 2, 3, 5, 6, 7, 10$. After then, we solve some Diophantine equations utilizing solutions of these equations.

Keywords Generalized Fibonacci numbers · Generalized Lucas numbers · Congruences · Diophantine equation

Mathematics Subject Classifications 11B37 · 11B39 · 11B50 · 11B99 · 11D41

1 Introduction

Let P and Q be nonzero integers. Generalized Fibonacci sequence (U_n) and Lucas sequence (V_n) are defined by $U_0(P, Q) = 0, U_1(P, Q) = 1$; $V_0(P, Q) = 2, V_1(P, Q) = P$, and $U_{n+1}(P, Q) = PU_n(P, Q) + QU_{n-1}(P, Q)$, $V_{n+1}(P, Q) = PV_n(P, Q) + QV_{n-1}(P, Q)$ for $n \geq 1$. $U_n(P, Q)$ and $V_n(P, Q)$ are called n-th generalized Fibonacci number and n-th generalized Lucas number, respectively. Generalized Fibonacci and Lucas numbers for negative subscripts are defined as $U_{-n}(P, Q) = -(-Q)^{-n}U_n(P, Q)$ and $V_{-n}(P, Q) = (-Q)^{-n}V_n(P, Q)$, respectively.

Since

$$U_n(-P, Q) = (-1)^{n-1}U_n(P, Q) \text{ and } V_n(-P, Q) = (-1)^n V_n(P, Q),$$

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it will be assumed that $P \geq 1$. Moreover, we will assume that $P^2+4Q > 0$. For $P = Q = 1$, we have classical Fibonacci and Lucas sequences (F_n) and (L_n). For $P = 2$ and $Q = 1$, we have Pell and Pell-Lucas sequences (P_n) and (Q_n). For more information about generalized Fibonacci and Lucas sequences one can consult [1–4].

Generalized Fibonacci and Lucas numbers of the form kx^2 have been investigated since 1962. In [5], the authors solved $U_n = x^2$, $V_n = x^2$, $U_n = 2x^2$, and $V_n = 2x^2$ for odd relatively prime integers P and Q. The reader can consult [6] or [7] for a brief discussion of the subject.

In [8], the authors showed that when $a \neq 0$ and b are integers, then the equation $U_n(P, \pm 1) = ax^2 + b$ has only a finite number of solutions n. In [9], Keskin solved the equations $V_n(P, -1) = wx^2 + 1$ and $V_n(P, -1) = wx^2 - 1$ for $w = 1, 2, 3, 6$ when P is odd. In [10], Karaatlı and Keskin solved the equations $V_n(P, -1) = 5x^2 \pm 1$ and $V_n(P, -1) = 7x^2 \pm 1$. Similar equations are tackled in [11] by using very different methods (see also [12–14]). In this study, we solve the equation $U_n(P, -1) = wx^2 + 1$ for $w = 1, 2, 3, 5, 6, 7, 10$.

We will use the Jacobi symbol throughout this study. Our method is elementary and used by Cohn, Ribenboim and McDaniel in [15] and [16] , respectively.

2 Preliminaries

From now on, instead of $U_n(P, -1)$ and $V_n(P, -1)$, we sometimes write U_n and V_n, respectively. Moreover, we will assume that $P \geq 3$.

The following lemmas can be proved by induction.

Lemma 2.1 *If n is a positive integer, then $U_{2n} \equiv n(-1)^{n+1} P \pmod{P^2}$ and $U_{2n+1} \equiv (-1)^n \pmod{P^2}$.*

Lemma 2.2 *If n is a positive integer, then $V_{2n} \equiv 2(-1)^n \pmod{P^2}$ and $V_{2n+1} \equiv (-1)^n(2n+1)P \pmod{P^2}$.*

The following theorems are given in [9].

Theorem 2.3 *Let P be odd. If $V_n = kx^2$ for some $k|P$ with $k > 1$, then $n = 1$.*

Theorem 2.4 *Let P be odd. If $V_n = 2kx^2$ for some $k|P$ with $k > 1$, then $n = 3$.*

Theorem 2.5 *Let P be odd. If $U_n = kx^2$ for some $k|P$ with $k > 1$, then $n = 2$ or $n = 6$ and $3|P$.*

Theorem 2.6 *Let P be odd. If $k|P$ with $k > 1$, then the equation $U_n = 2kx^2$ has no solutions.*

Theorem 2.7 *Let P be odd. If $k|P$ with $k > 1$, then the equation $U_n = kx^2 + 1$ has only the solution $n = 1$.*

The following theorem is given in [10].

Theorem 2.8 *Let P be odd. If $V_n = 7kx^2$ for some $k|P$ with $k > 1$, then $n = 1$.*

Now we give some known theorems from [5], which will be useful for solving the equation $U_n = wx^2 + 1$. We use a theorem from [17] while solving $V_n = 2x^2$.

Theorem 2.9 *Let P be odd. If $V_n = x^2$ for some integer x, then $n = 1$. If $V_n = 2x^2$ for some integer x, then $n = 3$, $P = 3, 27$.*

Theorem 2.10 *Let P be odd. If $U_n = x^2$ for some integer x, then $n = 1$ or $n = 2$, $P = \square$ or $n = 6$, $P = 3$. If $U_n = 2x^2$ for some integer x, then $n = 3$.*

The following lemma is a consequence of a theorem given in [18].

Lemma 2.11 *All positive integer solutions of the equation $3x^2 - 2y^2 = 1$ are given by $(x, y) = (U_n(10, -1) - U_{n-1}(10, -1), U_n(10, -1) + U_{n-1}(10, -1))$ with $n \geq 1$.*

The proof of the following lemma is easy and will be omitted.

Lemma 2.12 *All positive integer solutions of the equation $x^2 - 7y^2 = 2$ are given by $(x, y) = (3\,(U_{m+1}(16 - 1) - U_m(16, -1))\,, 17U_m(16, -1) - U_{m-1}(16, -1))$ with $m \geq 0$.*

The following theorems are well known (see [19–22]).

Lemma 2.13 *All positive integer solutions of the equation $x^2 - (P^2 - 4)y^2 = 4$ are given by $(x, y) = (V_n(P, -1), U_n(P, -1))$ with $n \geq 1$.*

Lemma 2.14 *All positive integer solutions of the equation $x^2 - Pxy + y^2 = 1$ are given by $(x, y) = (U_n(P, -1), U_{n-1}(P, -1))$ with $n \geq 2$.*

The following two theorems are given in [23].

Theorem 2.15 *Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$ and m be a nonzero integer. Then*

$$U_{2mn+r} \equiv U_r \pmod{U_m}. \tag{2.1}$$

Theorem 2.16 *Let $n \in \mathbb{N} \cup \{0\}$ and $m, r \in \mathbb{Z}$. Then*

$$U_{2mn+r} \equiv (-1)^n\, U_r \pmod{V_m}. \tag{2.2}$$

If $n = 2 \cdot 2^k a + r$ with a odd, then we get

$$U_n = U_{2 \cdot 2^k a + r} \equiv -U_r \pmod{V_{2^k}}. \tag{2.3}$$

by (2.2).

When P is odd, since $8|U_3$, using (2.1) we get

$$U_{6q+r} \equiv U_r \pmod{U_3} \tag{2.4}$$

and therefore

$$U_{6q+r} \equiv U_r \pmod{8}. \tag{2.5}$$

From Lemma 2.1 and Lemma 2.2, it follows that if $q|P$ with $q > 1$, then

$$q|V_n \Leftrightarrow n \text{ is odd and } q|U_n \Leftrightarrow n \text{ is even}. \tag{2.6}$$

$$\text{If } P^2 \equiv -1 \pmod{5}, \quad \text{then } 5|U_5. \tag{2.7}$$

$$\text{If } P \text{ is odd, then } 2|V_n \Leftrightarrow 2|U_n \Leftrightarrow 3|n. \tag{2.8}$$

Now we give some identities concerning generalized Fibonacci and Lucas numbers:

$$U_{-n} = -U_n \text{ and } V_{-n} = V_n,$$
$$U_{2n+1} - 1 = U_n V_{n+1}, \tag{2.9}$$
$$U_{2n} = U_n V_n, \tag{2.10}$$
$$V_n^2 - (P^2 - 4)U_n^2 = 4, \tag{2.11}$$
$$V_{2n} = V_n^2 - 2. \tag{2.12}$$

Let $m = 2^a k$, $n = 2^b l$, k and l odd, $a, b \geq 0$, and $d = (m, n)$. Then (see [24])

$$(U_m, V_n) = \begin{cases} V_d & \text{if } a > b, \\ 1 \text{ or } 2 & \text{if } a \leq b. \end{cases} \tag{2.13}$$

From (2.11) and Lemma 2.2, it follows that

$$5|V_n \Leftrightarrow 5|P \text{ and } n \text{ is odd.} \tag{2.14}$$

An induction method shows that

$$V_{2^k} \equiv 7 \pmod 8$$

and thus

$$\left(\frac{2}{V_{2^k}}\right) = 1 \tag{2.15}$$

and

$$\left(\frac{-1}{V_{2^k}}\right) = -1 \tag{2.16}$$

for all $k \geq 1$.

Lemma 2.17 *Let P be odd. Then*

$$\left(\frac{P-1}{V_{2^k}}\right) = \left(\frac{P+1}{V_{2^k}}\right) = 1 \tag{2.17}$$

for all $k \geq 1$. Moreover, if $3 \nmid P$, then

$$\left(\frac{3}{V_{2^k}}\right) = 1 \tag{2.18}$$

for all $k \geq 1$.

Proof If $3 \nmid P$, then $P^2 \equiv 1 \pmod 3$ and therefore $V_2 = P^2 - 2 \equiv -1 \pmod 3$. An induction method shows that $V_{2^k} \equiv -1 \pmod 3$ since $V_{2^k} = \left(V_{2^{k-1}}\right)^2 - 2$ by (2.12). Thus

$$\left(\frac{3}{V_{2^k}}\right) = -\left(\frac{V_{2^k}}{3}\right) = -\left(\frac{-1}{3}\right) = 1.$$

Since $V_2 = P^2 - 2 \equiv -1 \pmod{P^2 - 1}$, it follows that $V_{2^k} \equiv -1 \pmod{P^2 - 1}$. Thus $V_{2^k} \equiv -1 \pmod{P - 1}$ and $V_{2^k} \equiv -1 \pmod{P + 1}$. Let $P - 1 = 2^t a$ with a odd. Then we get

$$\left(\frac{P-1}{V_{2^k}}\right) = \left(\frac{2^t a}{V_{2^k}}\right) = \left(\frac{2}{V_{2^k}}\right)^t \left(\frac{a}{V_{2^k}}\right) = \left(\frac{a}{V_{2^k}}\right) = (-1)^{\frac{a-1}{2}} \left(\frac{V_{2^k}}{a}\right) \tag{2.19}$$

since $\left(\frac{2}{V_{2^k}}\right) = 1$ by (2.15). By using the fact that $V_{2^k} \equiv -1 \pmod a$, we get $\left(\frac{P-1}{V_{2^k}}\right) = (-1)^{\frac{a-1}{2}} \left(\frac{-1}{a}\right) = 1$ by (2.19). Similarly, it is seen that $\left(\frac{P+1}{V_{2^k}}\right) = 1$. □

When P is odd, it can be shown that

$$\left(\frac{5}{V_{2^k}}\right)=\begin{cases}-1 & \text{if } P^2\equiv -1 \,(\text{mod}5),\\ 1 & \text{if } P^2\equiv 1\,(\text{mod}5)\end{cases} \tag{2.20}$$

and

$$\left(\frac{7}{V_{2^k}}\right)=\begin{cases}-1 & \text{if } P^2\equiv 4 \,(\text{mod}7),\\ 1 & \text{if } P^2\equiv 1\,(\text{mod}7)\end{cases} \tag{2.21}$$

for all $k\geq 1$.

3 Main theorems

From now on, we will assume that n is a positive integer and P is an odd integer.

Theorem 3.1 *If $U_n=2kx^2+1$ with $k|P$ and $k>1$, then $n=1$ or $n=5$.*

Proof Assume that $U_n=2kx^2+1$ for some integer x. Then n is odd by Lemma 2.1. It is clear that $n=1$ is a solution. Assume that $n>1$. Then we have $n=2m+1$ with $m\geq 1$. Thus, we get $U_mV_{m+1}=U_{2m+1}-1=2kx^2$ by (2.9). It can be seen that m is even by (2.6). Thus, $(U_m,V_{m+1})=P$ by (2.13). Then it follows that

$$U_m=k_1Pa^2 \quad\text{and}\quad V_{m+1}=2k_2Pb^2$$

or

$$U_m=2k_1Pa^2 \quad\text{and}\quad V_{m+1}=k_2Pb^2$$

for some natural numbers a and b with $k=k_1k_2$. Thus, it is seen that

$$U_m=ut^2 \text{ and } V_{m+1}=2vs^2 \tag{3.1}$$

or

$$U_m=2ut^2 \quad\text{and}\quad V_{m+1}=vs^2 \tag{3.2}$$

for some natural numbers u,v,s,t with $u|P$ and $v|P$. Assume that (3.1) is satisfied. By using Theorems 2.4 and 2.10, it is seen that $m=2$. Therefore $n=5$. The identity (3.2) is impossible by Theorems 2.6 and 2.10.

Theorem 3.2 *Let $w=1,2,3,6$. If $U_n=wx^2+1$ for some integer x, then $(w,n)=$ $(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(6,1),(6,2),(6,5)$.*

Proof Assume that $U_n=wx^2+1$ for some integer x. Let $n>3$. Then $n=4q+r$ for some $q>0$ with $0\leq r\leq 3$. Then $n=2\cdot 2^ka+r$ with a odd and $k\geq 1$. Thus,

$$wx^2=-1+U_n\equiv -1-U_r\,(\text{mod}V_{2^k})$$

by (2.3). This shows that

$$wx^2\equiv -1,-2,-(P+1),-P^2\,(\text{mod}V_{2^k}).$$

Since $\left(\frac{2}{V_{2^k}}\right)=1$, $\left(\frac{-1}{V_{2^k}}\right)=-1$, and $\left(\frac{P+1}{V_{2^k}}\right)=1$ by (2.15), (2.16), and (2.17), respectively, we get

$$\left(\frac{w}{V_{2^k}}\right)=-1. \tag{3.3}$$

If $w = 1, 2$, then (3.3) is impossible. Let $w = 3, 6$. If $3 \nmid P$, then again (3.3) is impossible since $\left(\frac{3}{V_{2^k}}\right) = 1$ by (2.18). Therefore $n \le 3$ in case $3 \nmid P$ and $w = 3, 6$. But $n = 3$ is not a solution in this case. If $w = 6$ and $3|P$, then by Theorem 3.1, we get $n = 1$ or $n = 5$. Thus, $n = 1, 5$ for the case $w = 6$ and $3|P$. If $w = 3$ and $3|P$, then by Theorem 2.7, we get $n = 1$. □

Theorem 3.3 *If $U_n = 5x^2 + 1$ for some integer x, then $n = 1$ or $n = 2$.*

Proof Assume that $U_n = 5x^2 + 1$ for some integer x. If $5|P$, then by Theorem 2.7, $n = 1$. Assume that $5 \nmid P$. Let $n > 2$ and n be even. Now we divide the proof into two cases.

Case I. Let $P^2 \equiv 1 \pmod 5$. Since n is even, $n = 4q + r$ for some positive integer q with $r = 0, 2$. Thus, $n = 2 \cdot 2^k a + r$ with a odd and $k \ge 1$. Then

$$5x^2 = -1 + U_n \equiv -1 - U_r \pmod{V_{2^k}}$$

by (2.3). This shows that

$$5x^2 \equiv -1, -(P+1) \pmod{V_{2^k}},$$

which is impossible since $\left(\frac{-1}{V_{2^k}}\right) = -1$, $\left(\frac{P+1}{V_{2^k}}\right) = 1$, and $\left(\frac{5}{V_{2^k}}\right) = 1$ by (2.16), (2.17), and (2.20), respectively.

Case II. Let $P^2 \equiv -1 \pmod 5$. We get $5x^2 \equiv -1 \pmod P$ since $P|U_n$ when n is even. This shows that

$$-1 = \left(\frac{P}{5}\right) = \left(\frac{5}{P}\right) = \left(\frac{-1}{P}\right),$$

which implies that $P \equiv 3, 7 \pmod 8$. Since n is even, we get $n = 6q + r$, $r = 0, 2, 4$. Then $5x^2 + 1 \equiv U_r \pmod 8$ by (2.5). If $r = 0$, then we get $5x^2 \equiv -1 \pmod 8$, which is impossible. Let $r = 2$. Then $5x^2 + 1 \equiv U_2 \pmod 8$ by (2.5), which shows that $5x^2 + 1 \equiv P \pmod 8$. But this is impossible since $P \equiv 3, 7 \pmod 8$. Let $r = 4$. Then $n = 12t + 4$ or $n = 12t + 10$ for some integer t. Let $n = 12t + 10$. Then $n = 12q_1 - 2$ with $q_1 > 0$. Thus, $n = 2 \cdot 2^k a - 2$ with a odd and $k \ge 1$. Then it follows that

$$5x^2 = -1 + U_n \equiv -1 - U_{-2} \pmod{V_{2^k}}$$

by (2.3), which implies that

$$5x^2 \equiv P - 1 \pmod{V_{2^k}}.$$

This is impossible since $\left(\frac{P-1}{V_{2^k}}\right) = 1$ and $\left(\frac{5}{V_{2^k}}\right) = -1$ by (2.17) and (2.20), respectively. Let $n = 12t + 4$. Since $16|U_6$, we get $5x^2 + 1 = U_n \equiv U_4 \pmod{16}$ by (2.1). A simple computation shows that $5x^2 + 1 \equiv 1, 5, 6, 14 \pmod{16}$ and therefore $U_4 \equiv 1, 5, 6, 14 \pmod{16}$. Moreover, we have $5x^2 + 1 = U_n \equiv U_4 \equiv -P \pmod 8$ by (2.5). Using the fact that $5x^2 + 1 \equiv 1, 5, 6, 14 \pmod 8$, we see that $P \equiv 3, 7 \pmod 8$. Since $P \equiv 3, 7 \pmod 8$ and $P^3 - 2P = U_4 \equiv 1, 5, 6, 14 \pmod{16}$, it is seen that $P \equiv 3, 15 \pmod{16}$. Let $P \equiv 3 \pmod{16}$ and $P \equiv 3 \pmod 5$. Since n is even, $n = 10q + r$, $r \in \{0, 2, 4, 6, 8\}$. Using $5|U_5$, we get $5x^2 + 1 = U_n \equiv U_r \pmod 5$ by (2.1). A simple computation shows that $r = 4$. Since $n = 10q + 4$ and $n = 12t + 4$, we get $n = 60k + 4$ for some natural number k. Thus, by using (2.2), it is seen that

$$U_n = U_{60k+4} \equiv U_4 \pmod{V_5},$$

which implies that

$$5x^2 \equiv P^3 - 2P - 1 \,(\bmod P^4 - 5P^2 + 5)$$

since $V_5 = P(P^4 - 5P^2 + 5)$. This shows that

$$\left(\frac{5}{P^4-5P^2+5}\right) = \left(\frac{P^3-2P-1}{P^4-5P^2+5}\right) = \left(\frac{(P^3-2P-1)/4}{P^4-5P^2+5}\right).$$

By using the facts that $(P^3 - 2P - 1)/4 \equiv 1 \pmod 4$, $P^4 - 5P^2 + 5 \equiv 1 \pmod 5$, $P^4 - 5P^2 + 5 \equiv 9 \pmod{16}$, and $-3P^2 + P + 5 \equiv 13 \pmod{16}$, we get

$$\begin{aligned} 1 &= \left(\frac{5}{P^4-5P^2+5}\right) = \left(\frac{P^4-5P^2+5}{(P^3-2P-1)/4}\right) = \left(\frac{-3P^2+P+5}{(P^3-2P-1)/4}\right) \\ &= \left(\frac{(P^3-2P-1)/4}{-3P^2+P+5}\right) = \left(\frac{P^3-2P-1}{-3P^2+P+5}\right) = \left(\frac{9(P^3-2P-1)}{-3P^2+P+5}\right) \\ &= \left(\frac{-2(P+2)}{-3P^2+P+5}\right) = \left(\frac{-2}{-3P^2+P+5}\right)\left(\frac{P+2}{-3P^2+P+5}\right) \\ &= -\left(\frac{P+2}{-3P^2+P+5}\right) = -\left(\frac{-3P^2+P+5}{P+2}\right) = -\left(\frac{-1}{P+2}\right) = -1, \end{aligned}$$

a contradiction. Let $P \equiv 3 \pmod{16}$ and $P \equiv 2 \pmod 5$. Then $(P-1)/2 \equiv 3 \pmod 5$ and $(P-1)/2 \equiv 1 \pmod 8$ and therefore

$$\left(\frac{5}{(P-1)/2}\right) = -1 \text{ and } \left(\frac{-2}{(P-1)/2}\right) = 1. \tag{3.4}$$

Moreover,

$$5x^2 = -1 + U_n = -1 + U_{12t+4} \equiv -1 + U_4 \equiv P^3 - 2P - 1 \,(\bmod U_3)$$

by (2.1). This implies that

$$5x^2 \equiv P^3 - 2P - 1 \,(\bmod P - 1).$$

This shows that

$$5x^2 \equiv -2 \,(\bmod (P-1)/2),$$

which is impossible by (3.4). Let $P \equiv 15 \pmod{16}$. Then $(P^2 - 3)/2 \equiv 3 \pmod 5$ and $(P^2 - 3)/2 \equiv 7 \pmod 8$. Moreover, we get

$$5x^2 = -1 + U_n = -1 + U_{12t+4} \equiv -1 + U_4 \equiv P^3 - 2P - 1 \,(\bmod V_3)$$

by (2.2). This shows that

$$5x^2 \equiv P - 1 \,(\bmod (P^2-3)/2)$$

and therefore

$$\left(\frac{5}{(P^2-3)/2}\right) = \left(\frac{P-1}{(P^2-3)/2}\right).$$

This is impossible since

$$\begin{aligned} -1 &= \left(\frac{(P^2-3)/2}{5}\right) = \left(\frac{5}{(P^2-3)/2}\right) = \left(\frac{(P-1)/2}{(P^2-3)/2}\right)\left(\frac{2}{(P^2-3)/2}\right) \\ &= \left(\frac{(P-1)/2}{(P^2-3)/2}\right) = -\left(\frac{(P^2-3)/2}{(P-1)/2}\right) = -\left(\frac{-1}{(P-1)/2}\right) = 1. \end{aligned}$$

Now assume that $n > 3$ and n is odd. Then $n = 2m + 1$ with $m > 1$. Therefore $U_{2m+1} = 5x^2 + 1$, which implies that $5x^2 = U_{2m+1} - 1 = U_m V_{m+1}$ by (2.9). Let m be odd. Then $(U_m, V_{m+1}) = 1$ by (2.13) and (2.8). Thus,

$$U_m = a^2 \quad \text{and} \quad V_{m+1} = 5b^2 \tag{3.5}$$

or

$$U_m = 5a^2 \quad \text{and} \quad V_{m+1} = b^2 \tag{3.6}$$

for some integers a and b. The identities (3.5) and (3.6) are impossible by (2.14) and Theorem 2.9, respectively. Let m be even. Then $(U_m, V_{m+1}) = P$ by (2.13). Thus,

$$U_m = Pa^2 \text{ and } V_{m+1} = 5Pb^2 \tag{3.7}$$

or

$$U_m = 5Pa^2 \text{ and } V_{m+1} = Pb^2 \tag{3.8}$$

for some integers a and b. The identities (3.7) and (3.8) are impossible by (2.14) and Theorem 2.3, respectively. Therefore $n \leq 3$. If $n = 3$, we get $P^2 - 1 = U_3 = 5x^2 + 1$, which implies that $P^2 \equiv 2 \pmod 5$. This is impossible. Thus, $n = 1$ or $n = 2$. □

Theorem 3.4 *If $U_n = 7x^2 + 1$ for some integer x, then $n = 1, 2, 3$.*

Proof Assume that $U_n = 7x^2 + 1$ for some integer x. If $7|P$, then by Theorem 2.7, $n = 1$. Assume that $7 \nmid P$. Let $n > 2$ and n be even. Then $7x^2 + 1 \equiv 0 \pmod P$. This shows that $\left(\frac{7}{P}\right) = \left(\frac{-1}{P}\right)$, which implies that $\left(\frac{P}{7}\right) = 1$. Therefore $P \equiv 1, 2, 4 \pmod 7$. Now we distinguish three cases.

Case I. Let $P \equiv 1 \pmod 7$. Since n is even, $n = 4q + r$ for some $q > 0$ with $r = 0, 2$. Thus, $n = 2 \cdot 2^k a + r$ with a odd and $k \geq 1$. Then we get

$$7x^2 = -1 + U_n \equiv -1 - U_r \pmod{V_{2^k}}$$

by (2.3), which implies that

$$7x^2 \equiv -1, -(P + 1) \pmod{V_{2^k}}.$$

This is impossible since $\left(\frac{-1}{V_{2^k}}\right) = -1$, $\left(\frac{P+1}{V_{2^k}}\right) = 1$, and $\left(\frac{7}{V_{2^k}}\right) = 1$ by (2.16), (2.17), and (2.21), respectively.

Case II. Let $P \equiv 4 \pmod 7$. Then $7|V_2$ and

$$7x^2 = -1 + U_n = -1 + U_{4q+r} \equiv -1 \pm U_r \pmod{V_2}$$

by (2.2). This is impossible since $7 \nmid (-1 \pm U_r)$ for $r = 0, 2$.

Case III. Let $P \equiv 2 \pmod 7$. If $n = 4q + 2$, then $n = 4(q + 1) - 2 = 2 \cdot 2^k a - 2$ with a odd and $k \geq 1$. Thus, we get

$$7x^2 = -1 + U_n \equiv -1 - U_{-2} \pmod{V_{2^k}}$$

by (2.3), which implies that

$$7x^2 \equiv P - 1 \pmod{V_{2^k}}.$$

But this is impossible since $\left(\frac{P-1}{V_{2^k}}\right)=1$ and $\left(\frac{7}{V_{2^k}}\right)=-1$ by (2.17) and (2.21), respectively. Let $n=4q$. Then $n=12t+r$ with $r=0,4,8$. Assume that $n=12t$. Since P is odd, we can write $P^2-1=2^m a$ with a odd. Thus,

$$7x^2=-1+U_n\equiv -1+U_0\ (\mathrm{mod}\,U_3)$$

by (2.1), which implies that

$$7x^2\equiv -1\ (\mathrm{mod}\,a).$$

This shows that $\left(\frac{7}{a}\right)=\left(\frac{-1}{a}\right)$ and therefore $\left(\frac{a}{7}\right)=1$. Thus,

$$1=\left(\frac{a}{7}\right)=\left(\frac{2^m a}{7}\right)=\left(\frac{P^2-1}{7}\right)=\left(\frac{3}{7}\right)=-1,$$

a contradiction. Assume that $n=12t+4$. Since $16|U_6$, we get $U_n\equiv U_4\ (\mathrm{mod}16)$ by (2.1). This shows that $7x^2+1\equiv P^3-2P\ (\mathrm{mod}16)$. Since $7x^2+1\equiv 0,1,8,13\ (\mathrm{mod}16)$, a simple computation shows that $P\equiv 11,15\ (\mathrm{mod}16)$. Let $P\equiv 11\ (\mathrm{mod}16)$. Then

$$7x^2=-1+U_n=-1+U_{12t+4}\equiv -1+U_4\equiv P^3-2P-1\ (\mathrm{mod}\,U_3)$$

by (2.1), which shows that $7x^2\equiv -2\ (\mathrm{mod}\,P-1)$. Thus, we get $\left(\frac{7}{(P-1)/2}\right)=\left(\frac{-2}{(P-1)/2}\right)$ and therefore $\left(\frac{(P-1)/2}{7}\right)=\left(\frac{2}{(P-1)/2}\right)$. But this is impossible since $(P-1)/2\equiv 5\ (\mathrm{mod}8)$ and $(P-1)/2\equiv 4\ (\mathrm{mod}7)$. Let $P\equiv 15\ (\mathrm{mod}16)$. By using a similar argument, it is seen that $7x^2\equiv P-1\ (\mathrm{mod}\,P^2-3)$. This shows that

$$\left(\frac{7}{(P^2-3)/2}\right)=\left(\frac{(P-1)/2}{(P^2-3)/2}\right)\left(\frac{2}{(P^2-3)/2}\right).$$

Since $(P^2-3)/2\equiv 4\ (\mathrm{mod}7)$, $(P^2-3)/2\equiv 7\ (\mathrm{mod}8)$, and $(P-1)/2\equiv 7\ (\mathrm{mod}8)$, we get

$$-1=\left(\frac{7}{(P^2-3)/2}\right)=\left(\frac{(P-1)/2}{(P^2-3)/2}\right)=-\left(\frac{(P^2-3)/2}{(P-1)/2}\right)=-\left(\frac{-1}{(P-1)/2}\right)=1,$$

a contradiction. Assume that $n=12t+8$. Then we can write $n=12m-4$. A simple computation shows that $P\equiv 1,5\ (\mathrm{mod}16)$ in this case. Let $P\equiv 1\ (\mathrm{mod}16)$. Then

$$7x^2=-1+U_n=-1+U_{12m-4}\equiv -1+U_{-4}\equiv -(P^3-2P+1)\ (\mathrm{mod}\,U_3),$$

which implies that $7x^2\equiv -2\ (\mathrm{mod}\,P+1)$. Thus, we get

$$\left(\frac{7}{(P+1)/2}\right)=\left(\frac{-2}{(P+1)/2}\right).$$

Therefore by using the facts that $(P+1)/2\equiv 1\ (\mathrm{mod}8)$ and $(P+1)/2\equiv 5\ (\mathrm{mod}7)$, we get

$$-1=\left(\frac{(P+1)/2}{7}\right)=\left(\frac{2}{(P+1)/2}\right)=1,$$

a contradiction. Let $P\equiv 5\ (\mathrm{mod}16)$. Then

$$7x^2=-1+U_n=-1+U_{12m-4}\equiv -1+U_{-4}\equiv -(P^3-2P+1)\ (\mathrm{mod}\,V_3)$$

by (2.2), which implies that $7x^2 \equiv -(P+1) \pmod{P^2-3}$. By using the facts that $(P^2-3)/2 \equiv 4 \pmod 7$, $(P^2-3)/2 \equiv 3 \pmod 8$, and $(P+1)/2 \equiv 3 \pmod 8$, we get

$$\begin{aligned} 1 &= \left(\frac{7}{(P^2-3)/2}\right)\left(\frac{-1}{(P^2-3)/2}\right)\left(\frac{(P+1)/2}{(P^2-3)/2}\right)\left(\frac{2}{(P^2-3)/2}\right) \\ &= -\left(\frac{(P^2-3)/2}{7}\right)\left(\frac{(P+1)/2}{(P^2-3)/2}\right) = -\left(\frac{(P+1)/2}{(P^2-3)/2}\right) \\ &= \left(\frac{(P^2-3)/2}{(P+1)/2}\right) = \left(\frac{-1}{(P+1)/2}\right) = -1, \end{aligned}$$

a contradiction. Thus, we conclude that $n \leq 2$. Now assume that n is odd. Then $n = 2m+1$ with $m \geq 0$. Thus, $U_{2m+1} = 7x^2 + 1$, which implies that $7x^2 = U_{2m+1} - 1 = U_m V_{m+1}$ by (2.9). Let m be odd. Then $(U_m, V_{m+1}) = 1$ by (2.13) and (2.8). Thus,

$$U_m = a^2 \quad \text{and} \quad V_{m+1} = 7b^2 \tag{3.9}$$

or

$$U_m = 7a^2 \quad \text{and} \quad V_{m+1} = b^2 \tag{3.10}$$

for some integers a and b. Assume that (3.9) is satisfied. Then by Theorem 2.10, we get $m = 1$ and therefore $n = 3$. The identity (3.10) is impossible by Theorem 2.9. Let m be even. Then $(U_m, V_{m+1}) = P$ by (2.13). This implies that

$$U_m = Pa^2 \text{ and } V_{m+1} = 7Pb^2 \tag{3.11}$$

or

$$U_m = 7Pa^2 \text{ and } V_{m+1} = Pb^2. \tag{3.12}$$

for some integers a and b. By using Theorems 2.3 and 2.8, we have in both cases that $m+1 = 1$ and therefore $n = 1$. Consequently, we have $n = 1, 2, 3$. □

Theorem 3.5 *If $U_n = 10x^2 + 1$ for some integer x, then $n = 1, 2$.*

Proof If $5|P$, then by Theorem 3.1, $n = 1$ or $n = 5$. Assume that $5 \nmid P$. Let $n > 2$ and n be even. Then $10x^2 + 1 \equiv 0 \pmod P$ since $P|U_n$ when n is even. Therefore

$$\left(\frac{5}{P}\right) = \left(\frac{-2}{P}\right).$$

If $P \equiv \pm 1 \pmod 5$, then $P \equiv 1, 3 \pmod 8$. If $P \equiv \pm 2 \pmod 5$, then $P \equiv 5, 7 \pmod 8$. The remainder of the proof is split into two cases.

Case I. Let $P \equiv \pm 1 \pmod 5$. Since n is even, we get $n = 4q + r$ for some positive integer q with $r = 0, 2$. Thus, $n = 2 \cdot 2^k a + r$ with a odd and $k \geq 1$. Then

$$10x^2 = -1 + U_n \equiv -1 - U_r \pmod{V_{2^k}}$$

by (2.3). This shows that

$$10x^2 \equiv -1, -(P+1) \pmod{V_{2^k}},$$

which is impossible since $\left(\frac{2}{V_{2^k}}\right) = 1$, $\left(\frac{-1}{V_{2^k}}\right) = -1$, $\left(\frac{P+1}{V_{2^k}}\right) = 1$, and $\left(\frac{5}{V_{2^k}}\right) = 1$ by (2.15), (2.16), (2.17), and (2.20), respectively.

Case II. Let $P \equiv \pm 2 \pmod 5$. Since n is even, we get $n = 6q + r$ with $r = 0, 2, 4$. Then $10x^2 + 1 \equiv U_r \pmod 8$ by (2.5). If $r = 0$, then we get $10x^2 \equiv -1 \pmod 8$, which is impossible. Let $r=2$. Then $10x^2+1\equiv U_2$ (mod8), which shows that $10x^2+1\equiv P$ (mod8), which is impossible since $P\equiv 5,7$ (mod8). Let $r=4$. Then either $n=12t+10$ or $n=12t+4$ for some nonnegative integer t. Assume that $n=12t+10$. Then $n=12q_1-2$ with $q_1>0$. Thus, $n=2\cdot 2^k a-2$ with a odd and $k\geq 1$. This shows that

$$10x^2\equiv -1+U_n\equiv -1-U_{-2}\,(\mathrm{mod}\,V_{2^k})$$

by (2.3), which shows that

$$10x^2\equiv P-1\,(\mathrm{mod}\,V_{2^k}).$$

This is impossible since $\left(\frac{2}{V_{2^k}}\right)=1$, $\left(\frac{P-1}{V_{2^k}}\right)=1$, and $\left(\frac{5}{V_{2^k}}\right)=-1$ by (2.15), (2.17), and (2.20), respectively. Assume that $n=12t+4$. It can be seen that $U_n=10x^2+1\equiv 1,9,11$ (mod16). Moreover, we get $U_n\equiv U_4$ (mod16) by (2.1) since $16|U_6$. A simple computation shows that $P\equiv 7,13,15$ (mod16) since $P\equiv 5,7$ (mod8) and $U_4=P^3-2P$. Let $P\equiv 7,15$ (mod16). Then $(P^2-3)/2\equiv 3$ (mod5) and $(P^2-3)/2\equiv 7$ (mod8). Moreover, we get

$$10x^2=-1+U_n=-1+U_{12t+4}\equiv -1+U_4\equiv P^3-2P-1\,(\mathrm{mod}\,V_3)$$

by (2.2). This shows that

$$10x^2\equiv P-1\,(\mathrm{mod}(P^2-3))$$

and therefore

$$5x^2\equiv (P-1)/2\,(\mathrm{mod}(P^2-3)/2).$$

Then we get

$$\left(\frac{5}{(P^2-3)/2}\right)=\left(\frac{(P-1)/2}{(P^2-3)/2}\right).$$

This is impossible since

$$\begin{aligned}-1&=\left(\frac{(P^2-3)/2}{5}\right)=\left(\frac{5}{(P^2-3)/2}\right)=\left(\frac{(P-1)/2}{(P^2-3)/2}\right)\\&=-\left(\frac{(P^2-3)/2}{(P-1)/2}\right)=-\left(\frac{-1}{(P-1)/2}\right)=1.\end{aligned}$$

Let $P\equiv 13$ (mod16) and $P\equiv 2$ (mod5). Then

$$(P-1)/4\equiv 4\,(\mathrm{mod}5)\text{ and }(P-1)/4\equiv 3\,(\mathrm{mod}4)$$

and therefore

$$\left(\frac{5}{(P-1)/4}\right)=1\text{ and }\left(\frac{-1}{(P-1)/4}\right)=-1. \tag{3.13}$$

Moreover,

$$10x^2=-1+U_n=-1+U_{12t+4}\equiv -1+U_4\equiv P^3-2P-1\,(\mathrm{mod}\,U_3)$$

by (2.1). This implies that

$$10x^2\equiv P^3-2P-1\,(\mathrm{mod}\,P-1).$$

This shows that

$$10x^2\equiv -2\,(\mathrm{mod}(P-1)/2)$$

and therefore

$$5x^2 \equiv -1 \,(\mathrm{mod}(P-1)/4),$$

which is impossible by (3.13). Let $P \equiv 13$ (mod16) and $P \equiv 3$ (mod5). Since n is even, $n = 10q + r$ with $r \in \{0, 2, 4, 6, 8\}$. Since $5|U_5$ by (2.7), we get $10x^2 + 1 = U_n \equiv U_r$ (mod5) by (2.1). A simple computation shows that $r = 4$. Since $n = 10q+4$ and $n = 12t+4$, we get $n = 60k + 4$ for some natural number k. Thus, by using (2.2), it is seen that

$$U_n = U_{60k+4} \equiv U_4 \,(\mathrm{mod}\, V_5),$$

which implies that

$$10x^2 \equiv P^3 - 2P - 1 \,(\mathrm{mod}\, P^4 - 5P^2 + 5)$$

since $V_5 = P(P^4 - 5P^2 + 5)$. This shows that

$$5x^2 \equiv (P^3 - 2P - 1)/2 \,(\mathrm{mod}\, P^4 - 5P^2 + 5)$$

and therefore

$$\left(\frac{5}{P^4 - 5P^2 + 5}\right) = \left(\frac{(P^3 - 2P - 1)/2}{P^4 - 5P^2 + 5}\right).$$

Since $(P^3 - 2P - 1)/2 \equiv 5$ (mod8), $P^4 - 5P^2 + 5 \equiv 1$ (mod5), $P^4 - 5P^2 + 5 \equiv 9$ (mod16), and $-3P^2 + P + 5 \equiv 7$ (mod16), we get

$$\begin{aligned}
1 &= \left(\frac{P^4 - 5P^2 + 5}{5}\right) = \left(\frac{5}{P^4 - 5P^2 + 5}\right) = \left(\frac{(P^3 - 2P - 1)/2}{P^4 - 5P^2 + 5}\right) \\
&= \left(\frac{P^4 - 5P^2 + 5}{(P^3 - 2P - 1)/2}\right) = \left(\frac{-3P^2 + P + 5}{(P^3 - 2P - 1)/2}\right) = \left(\frac{(P^3 - 2P - 1)/2}{-3P^2 + P + 5}\right) \\
&= \left(\frac{P^3 - 2P - 1}{-3P^2 + P + 5}\right)\left(\frac{2}{-3P^2 + P + 5}\right) = \left(\frac{P^3 - 2P - 1}{-3P^2 + P + 5}\right) \\
&= \left(\frac{9(P^3 - 2P - 1)}{-3P^2 + P + 5}\right) = \left(\frac{-2(P+2)}{-3P^2 + P + 5}\right) \\
&= \left(\frac{-2}{-3P^2 + P + 5}\right)\left(\frac{P+2}{-3P^2 + P + 5}\right) \\
&= -\left(\frac{P+2}{-3P^2 + P + 5}\right) = \left(\frac{-3P^2 + P + 5}{P+2}\right) = \left(\frac{-1}{P+2}\right) = -1,
\end{aligned}$$

a contradiction. Now assume that $n > 1$ and n is odd. Then $n = 2m + 1$ with $m \geq 1$. Therefore $U_{2m+1} = 10x^2 + 1$, which implies that $10x^2 = U_{2m+1} - 1 = U_m V_{m+1}$ by (2.9). Let m be odd. Then $(U_m, V_{m+1}) = 1$ by (2.13) and (2.8). Thus,

$$U_m = a^2 \text{ and } V_{m+1} = 10b^2, \tag{3.14}$$

$$U_m = 10a^2 \text{ and } V_{m+1} = b^2, \tag{3.15}$$

$$U_m = 2a^2 \text{ and } V_{m+1} = 5b^2, \tag{3.16}$$

or

$$U_m = 5a^2 \text{ and } V_{m+1} = 2b^2 \tag{3.17}$$

for some integers a and b. The identity (3.15) is impossible by Theorem 2.9. The identities (3.14) and (3.16) are impossible by (2.14), and (3.17) is impossible by Theorem 2.9. Let m

be even. Then $(U_m, V_{m+1}) = P$ by (2.13). Thus,

$$U_m = Pa^2 \text{ and } V_{m+1} = 10Pb^2, \tag{3.18}$$
$$U_m = 10Pa^2 \text{ and } V_{m+1} = Pb^2, \tag{3.19}$$
$$U_m = 2Pa^2 \text{ and } V_{m+1} = 5Pb^2, \tag{3.20}$$

or

$$U_m = 5Pa^2 \text{ and } V_{m+1} = 2Pb^2 \tag{3.21}$$

for some integers a and b. The identities (3.18) and (3.20) are impossible by (2.14), and (3.19) is impossible by Theorem 2.3. Assume that (3.21) is satisfied. Then by Theorem 2.4, we get $m = 2$ and therefore $n = 5$. Consequently, we have $n = 1, 2, 5$. But it can be seen that 5 is not a solution and therefore $n = 1, 2$. □

By using MAGMA [25], it can be shown that the equation $2Px^2+1 = U_5 = P^4-3P^2+1$ has only the solution $P = 3$. Therefore we can give the following corollary by using Theorem 3.1 and Lemmas 2.13 and 2.14.

Corollary 3.6 *The equations $x^2-(P^2-4)(2Py^2+1)^2 = 4$ and $(2Px^2+1)^2 - P(2Px^2+1)y + y^2 = 1$ have positive integer solutions only when $P = 3$. The solutions are given by $(x, y) = (123, 3)$ and $(x, y) = (3, 21)$, respectively.*

Corollary 3.7 *Let $k = 1, 2, 3, 5, 10$. The equations $x^2 - (P^2 - 4)(ky^2 + 1)^2 = 4$ and $(kx^2+1)^2 - P(kx^2+1)y + y^2 = 1$ have positive integer solutions only when $P = ka^2+1$ for some integer a.*

Corollary 3.8 *The equations $x^2 - (P^2 - 4)(6y^2 + 1)^2 = 4$ and $(6x^2 + 1)^2 - P(6x^2 + 1)y + y^2 = 1$ have positive integer solutions only when $P = 6a^2 + 1$ for some integer a or $P = 3(U_m(10, -1) + U_{m-1}(10, -1))$ for some $m \geq 1$ and there is only one solution in each case.*

Proof In order to prove the corollary we must solve the equation $6x^2 + 1 = U_5 = P^4 - 3P^2 + 1$. Since $6x^2 + 1 = P^4 - 3P^2 + 1$, it is seen that $P = 3a$ and $x = 3b$ for some integers a and b. Then we get $a^2(3a^2 - 1) = 2b^2$, which implies that $3a^2 - 1 = 2v^2$. This shows that $3a^2 - 2v^2 = 1$. Thus, by Lemma 2.11, we get $a = U_m(10, -1) - U_{m-1}(10, -1)$ for some $m \geq 1$. Since $P = 3a$, the proof follows. □

From Theorem 3.4 and Lemma 2.12, we can give the following corollary easily.

Corollary 3.9 *The equations $x^2 - (P^2 - 4)(7y^2 + 1)^2 = 4$ and $(7x^2 + 1)^2 - P(7x^2 + 1)y + y^2 = 1$ have positive integer solutions only when $P = 7a^2 + 1$ for some integer a or $P = 3\,(U_{m+1}(16 - 1) - U_m(16, -1))$ for some $m \geq 1$ and there is only one solution in each case.*

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