Period Math Hung (2016) 73:43–61
DOI 10.1007/s10998-016-0124-5

A Munn type representation of abundant semigroups with a multiplicative ample transversal

Shoufeng Wang[1]

Published online: 24 March 2016

Abstract The celebrated construction by Munn of a fundamental inverse semigroup T_E from a semilattice E provides an important tool in the study of inverse semigroups and ample semigroups. Munn's semigroup T_E has the property that a semigroup is a fundamental inverse semigroup (resp. a fundamental ample semigroup) with a semilattice of idempotents isomorphic to E if and only if it is embeddable as a full inverse subsemigroup (resp. a full subsemigroup) into T_E. The aim of this paper is to extend Munn's approach to a class of abundant semigroups, namely abundant semigroups with a multiplicative ample transversal. We present here a semigroup $T_{(I,\Lambda,E^\circ,P)}$ from a so-called *admissible quadruple* (I, Λ, E°, P) that plays for abundant semigroups with a multiplicative ample transversal the role that T_E plays for inverse semigroups and ample semigroups. More precisely, we show that a semigroup is a fundamental abundant semigroup (resp. fundamental regular semigroup) having a multiplicative ample transversal (resp. multiplicative inverse transversal) whose admissible quadruple is isomorphic to (I, Λ, E°, P) if and only if it is embeddable as a full subsemigroup (resp. full regular subsemigroup) into $T_{(I,\Lambda,E^\circ,P)}$. Our results generalize and enrich some classical results of Munn on inverse semigroups and of Fountain on ample semigroups.

Keywords The Munn semigroup of an admissible quadruple · Multiplicative ample transversal · Fundamental abundant semigroup

Mathematics Subject Classification 20M10

1 Introduction

Let S be a semigroup. We denote the set of all idempotents of S by $E(S)$ and the set of all inverses of $x \in S$ by $V(x)$. Recall that $V(x) = \{a \in S | xax = x, axa = a\}$ for all

✉ Shoufeng Wang
wsf1004@163.com

1 Department of Mathematics, Yunnan Normal University, Kunming 650500, Yunnan, People's Republic of China

$x \in S$. A semigroup S is called *regular* if $V(x) \neq \emptyset$ for any $x \in S$, and a regular semigroup S is called *inverse* if $E(S)$ is a commutative subsemigroup (i.e. a subsemilattice) of S, or equivalently, the cardinal of $V(x)$ is equal to 1 for all $x \in S$.

Recall that a regular semigroup S is *fundamental* if the largest congruence contained in $\mathcal{H}$ on S is the identity congruence. Structure theorems for certain important subclasses of the class of fundamental regular semigroups are already known. The first initiating the work in this direction is due to Munn [19]. He proved that given a semilattice E, the *Munn semigroup* T_E of all isomorphisms of principal ideals of E is "maximal" in the class of all fundamental inverse semigroups whose semilattices of idempotents are E, that is, every semigroup belonging to this class is isomorphic to a full inverse subsemigroup of T_E. Further from Munn [19] if S is an inverse semigroup such that $E(S)$ is isomorphic to a given semilattice E, then there exists a homomorphism $f : S \to T_E$ and the kernel of f is the largest congruence contained in $\mathcal{H}$ on S.

The pioneering work of Munn was generalized first by Hall in 1971 to orthodox semigroups (i.e. regular semigroups whose idempotents form subsemigroups) in [17] in which the Hall semigroup W_B of a band B was constructed. Recall that *a band* is a semigroup in which every element is idempotent. The Hall semigroup W_B has properties analogous to those described above for T_E (see Hall [17] for details). As another direction, Fountain [10] generalized Munn's result to a class of non-regular semigroup, namely *adequate* semigroups, by considering Green's $*$-relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on semigroups. Let S be a semigroup and $a, b \in S$. Then a and b are $\mathcal{L}^*$-related if and only if they are $\mathcal{L}$-related in an oversemigroup of S; the relation $\mathcal{R}^*$ can be defined dually. It is obvious that $\mathcal{L}^*$ and $\mathcal{R}^*$ are a left congruence and a right congruence, respectively. We denote the $\mathcal{L}^*$-class (resp. $\mathcal{R}^*$-class) of S containing $a \in S$ by $L^*_a(S)$ (resp. $R^*_a(S)$). A semigroup S is *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$–class of S contains an idempotent and an abundant semigroup S is called *adequate* if $E(S)$ is subsemilattice of S. If S is adequate, the $\mathcal{L}^*$-class (resp. $\mathcal{R}^*$-class) of $a \in S$ contains a unique idempotent, denoted by a^* (resp. a^+). From Proposition 1.6 in [10], if S is adequate, then for all $a, b \in S$, we have $a\mathcal{L}^*b$ (resp. $a\mathcal{R}^*b$) if and only if $a^* = b^*$ (resp. $a^+ = b^+$), moreover,

$$(ab)^* = (a^*b)^*, \quad (ab)^+ = (ab^+)^+, \quad a^+(ab)^+ = (ab)^+, \quad (ab)^*b^* = (ab)^*. \tag{1.1}$$

If S is a regular semigroup, then $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$. Obviously, regular semigroups are abundant and inverse semigroups are adequate. Moreover, for an inverse semigroup S and $a \in S$, we have $a^* = a^{-1}a$ and $a^+ = aa^{-1}$. For an abundant semigroup, let $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$ and let $\mathcal{D}^*$ be the smallest equivalence containing $\mathcal{L}^*$ and $\mathcal{R}^*$. An abundant semigroup S is called *fundamental* if the largest congruence μ_S contained in $\mathcal{H}^*$ is the identity congruence. From Proposition 2.1 in [7], for an abundant semigroup S and $a, b \in S$, we have

$$a\mu_S b \quad \text{if and only if } ea\mathcal{L}^*eb \quad \text{and} \quad ae\mathcal{R}^*be \quad \text{for all } e \in E(S). \tag{1.2}$$

In [10], Fountain shows that if S is an adequate semigroup and satisfies the following condition

$$(\forall a \in S)(\forall e \in E(S)) \;\; ea = a(ea)^*, \quad ae = (ae)^+a, \tag{1.3}$$

then there is a homomorphism $f : S \to T_{E(S)}$ whose kernel is the largest congruence contained in $\mathcal{H}^*$ on S. Such a semigroup is called *type A* by Fountain in [10] and called *ample* by Gould in [14]. Obviously, an inverse semigroup is ample.

Following the above direction, Fountain et al. [11] and Gomes and Gould [12] investigate other classes of non-regular semigroups having a semilattice of idempotents by using Munn's approach. More recent developments in this area can be found in the survey articles of Gould

[15] and Hollings [18]. Furthermore El-Qallali et al. [9], Gomes and Gould [13] and Wang [21] go a step further to extend Hall's approach for orthodox semigroups to some classes of non-regular semigroups having a band of idempotents. From the above texts, we can see that the approaches of Munn and Hall can be extended to some generalizations of inverse semigroups and orthodox semigroups, respectively.

On the other hand, Blyth and McFadden [3] introduced the concept of inverse transversals for regular semigroups. A subsemigroup S° of a regular semigroup S is called an *inverse transversal* of S if $V(x) \cap S^{\circ}$ contains exactly one element for all $x \in S$. Clearly, in this case, S° is an inverse subsemigroup of S. Since an inverse semigroup can be regarded as an inverse transversal of itself, the class of regular semigroups with inverse transversals contains the class of inverse semigroups as a proper subclass. Regular semigroups with inverse transversals are investigated extensively by many authors (see Blyth [4] and Tang [20] for details), and some generalizations of inverse transversals are proposed (see [5,8,22,23]). In particular, El-Qallali introduced adequate transversals and ample transversals for abundant semigroups in [8]. It is well known that both adequate semigroups (resp. ample semigroups) and regular semigroups with inverse transversals are abundant semigroups having adequate transversals (resp. ample transversals). Adequate transversals of abundant semigroups have been studied by several researchers and some meaningful results are obtained, see [1,2,6,16] and their references.

Inspired by the above facts, the following problem is natural: Can we study abundant semigroups with an adequate transversal by Munn's approach? In this paper, we shall initiate the investigation of the above question by extending Munn's approach to abundant semigroups with a multiplicative ample transversal. We present here a semigroup $T_{(I,\Lambda,E^{\circ},P)}$ from an *admissible quadruple* $(I, \Lambda, E^{\circ}, P)$ (see Definition 3.1) that plays for abundant semigroups with a multiplicative ample transversal the role that the Munn semigroup T_E plays for inverse semigroups and ample semigroups. More precisely, for a given admissible quadruple $(I, \Lambda, E^{\circ}, P)$, we show that a semigroup is a fundamental abundant semigroup (resp. fundamental regular semigroup) having a multiplicative ample transversal (resp. multiplicative inverse transversal) whose admissible quadruple is isomorphic to $(I, \Lambda, E^{\circ}, P)$ if and only if it is embeddable as a full subsemigroup (resp. full regular subsemigroup) into $T_{(I,\Lambda,E^{\circ},P)}$. Moreover, some further properties of admissible quadruples are also explored.

2 Preliminaries

This section gives some useful results related to ample transversals which will be used throughout the paper. We begin with the following alternative description of $\mathcal{L}^*$, which may be found in Fountain [10].

Lemma 2.1 *Elements a, b of a semigroup S are $\mathcal{L}^*$-related if and only if, for all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.*

Let S be an abundant semigroup and U an abundant subsemigroup of S. If there exist an idempotent $e \in L_a^*(S) \cap U$ and an idempotent $f \in R_a^*(S) \cap U$ for all $a \in U$, then U is called a $*$-*subsemigroup* of S. It is well known that U is a $*$-subsemigroup if and only if

$$\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U), \quad \mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U).$$

It is obvious that a regular subsemigroup of a regular semigroup S is always a $*$-subsemigroup of S. From El-Qallali [8], an adequate $*$-subsemigroup S° of an abundant semigroup S is called an *adequate transversal* of S if for each element $a \in S$, there are a unique element $\overline{a}$ in S° and $u, v \in E(S)$ such that

$$a = u\overline{a}v, \quad \text{where } u\mathcal{L}\overline{a}^+, v\mathcal{R}\overline{a}^* \quad \text{and} \quad \overline{a}^+, \overline{a}^* \in E(S^\circ). \tag{2.1}$$

In this case, u, v are uniquely determined by a and so we denote them by u_a and v_a, respectively. An adequate transversal S° of S is called an *ample* transversal if S° is also an ample semigroup. Let S be an abundant semigroup with an ample transversal S°. Denote

$$I^S = \{u_a | a \in S\}, \quad \Lambda^S = \{v_a | a \in S\}.$$

The following lemma characterizes the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on an abundant semigroups having an ample transversal.

Lemma 2.2 (Proposition 2.3 in [6]) *Let S be an abundant semigroup with an ample transversal S° and $a, b \in S$. Then $a\mathcal{R}^*b$ (resp. $a\mathcal{L}^*b$) if and only if $u_a = u_b$ (resp. $v_a = v_b$).*

Recall that a subsemigroup U of a semigroup S is *full* if $E(S) \subseteq U$.

Lemma 2.3 *Let S be an abundant semigroup with an ample transversal S° and U a full subsemigroup of S. Then U is an abundant semigroup with an ample transversal $S^\circ \cap U$. Moreover, we have $I^S = I^U$ and $\Lambda^S = \Lambda^U$.*

Proof Since S is abundant, S° is an ample $*$-subsemigroup of S and U is full, by Lemma 2.1 and its dual, we can easily show that U is abundant and $S^\circ \cap U$ is an ample $*$-subsemigroup of U. Now, let $a \in U$. Since U is full, $a = u_a\overline{a}v_a$ and $u_a, v_a, \overline{a}^*, \overline{a}^+ \in E(S)$, we have

$$\overline{a} = \overline{a}^+\overline{a}\,\overline{a}^* = \overline{a}^+u_a\overline{a}v_a\overline{a}^* = \overline{a}^+a\overline{a}^* \in U \tag{2.2}$$

by (2.1). This yields that the equality $a = u_a\overline{a}v_a$ holds in U, whence $S^\circ \cap U$ is an ample transversal of U and $I^S = I^U$, $\Lambda^S = \Lambda^U$. □

An ample transversal S° of an abundant semigroup S is *multiplicative* if $fg \in E(S^\circ)$ for all $f \in \Lambda^S$ and $g \in I^S$, or equivalently, if $v_au_b \in E(S^\circ)$ for all $a, b \in S$.

Corollary 2.4 *Let S be an abundant semigroup with an ample transversal S° and U a full subsemigroup of S.*

(1) *If S° is multiplicative, then $S^\circ \cap U$ is also multiplicative.*
(2) *If S° is fundamental, then U is fundamental.*

Proof Item (1) follows from the fact $I^S = I^U$ and $\Lambda^S = \Lambda^U$ obtained by Lemma 2.3. Now we prove the item (2). Let $a, b \in U$ and $a\mu_U b$. Then $a\mathcal{H}^*b$ in U. By Lemma 2.2 and (2.1), we have $\overline{a}^+\mathcal{L}u_a = u_b\mathcal{L}\overline{b}^+$ and $\overline{a}^*\mathcal{R}v_a = v_b\mathcal{R}\overline{b}^*$ whence $\overline{a}^+ = \overline{b}^+$ and $\overline{a}^* = \overline{b}^*$. Since $a\mu_U b$, we have

$$\overline{a} = (\overline{a}^+a\overline{a}^*)\mu_U(\overline{b}^+b\overline{b}^*) = \overline{b}$$

by (2.2). In view of the fact (1.2), we have $e\overline{a}\mathcal{L}^*e\overline{b}$ and $\overline{a}e\mathcal{R}^*\overline{b}e$ in U for all $e \in E(U)$. Since U is full, we have $E(S^\circ) \subseteq E(U)$ and so $e\overline{a}\mathcal{L}^*e\overline{b}$ and $\overline{a}e\mathcal{R}^*\overline{b}e$ in U for all $e \in E(S^\circ)$. This implies that $e\overline{a}\mathcal{L}^*e\overline{b}$ and $\overline{a}e\mathcal{R}^*\overline{b}e$ in $S^\circ \cap U$ for all $e \in E(S^\circ)$ since $S^\circ \cap U$ is a $*$-subsemigroup of U. Observe that $S^\circ \cap U$ is an ample subsemigroup of the ample semigroup S°, it follows that $e\overline{a}\mathcal{L}^*e\overline{b}$ and $\overline{a}e\mathcal{R}^*\overline{b}e$ in S° for all $e \in E(S^\circ)$. In view of the fact (1.2) again, we have $\overline{a}\mu_{S^\circ}\overline{b}$. This gives $\overline{a} = \overline{b}$ since S° is fundamental. This implies that $a = u_a\overline{a}v_a = u_b\overline{b}v_b = b$. Thus, U is fundamental. □

To give further properties of abundant semigroups with an ample transversal, we need the following notion. Let S be an abundant semigroup and B the subsemigroup generated by $E(S)$. For $e \in E(S)$, we denote the subsemigroup of eBe generated by the idempotents of eBe by $\langle e \rangle$. From El-Qallali and Fountain [7], we say S is *idempotent-connected* (IC for short) if for all $u \in R_a^*(S) \cap E(S)$ and $v \in L_a^*(S) \cap E(S)$, there is an isomorphism $\alpha : \langle u \rangle \to \langle v \rangle$ satisfying $xa = a(x\alpha)$ for all $x \in \langle u \rangle$. On IC abundant semigroups, we have the following.

Lemma 2.5 *Let S be an IC abundant semigroup and $a \in S$. If $u, v \in E(S)$ and $u\mathcal{R}^*a\mathcal{L}^*v$, then there exists a unique isomorphism α from $\langle u \rangle$ onto $\langle v \rangle$ such that $a(x\alpha) = xa$ for all $x \in \langle u \rangle$. This isomorphism is called the idempotent-connected isomorphism from $\langle u \rangle$ onto $\langle v \rangle$.*

Proof Let α and β be two isomorphisms satisfying the conditions given in the lemma. Then for all $x \in \langle u \rangle$, we have $a(x\alpha) = xa = a(x\beta)$. Since $a\mathcal{L}^*v$ and $x\alpha, x\beta \in \langle v \rangle$, we have $x\alpha = v(x\alpha) = v(x\beta) = x\beta$ by Lemma 2.1. □

Combining Lemmas 2.1 and 4.1 in El-Qallali [8] and Proposition 3.1, Lemma 6.12 in Guo [16], we have the following lemma.

Lemma 2.6 ([8,16]) *Let S be an abundant semigroup with a multiplicative ample transversal S°. Then S is an IC abundant semigroup. Moreover, for $a, b \in S$ and $x \in E(S)$, we have*

(1) $u_a\mathcal{R}^*a\mathcal{L}^*v_a$.
(2) $u_{ab} = u_a(\overline{a}v_au_b)^+, \overline{ab} = \bar{a}v_au_b\bar{b}, v_{ab} = (v_au_b\overline{b})^*v_b, \overline{x} = v_xu_x \in E(S^\circ)$.
(3) $u_{u_a} = u_a, \overline{u_a} = \overline{a}^+ = v_{u_a}, u_{v_a} = \overline{a}^* = \overline{v_a}, v_{v_a} = v_a$.

Corollary 2.7 *Let S be an abundant semigroup with a multiplicative ample transversal S° and $x \in E(S)$. Then $x = u_xv_x$.*

Proof If $x \in E(S)$, then by Lemma 2.6(2), $\overline{x} \in E(S^\circ)$. This implies that $\overline{x}^+ = \overline{x}$. Therefore $x = u_x\overline{x}v_x = u_x\overline{x}^+v_x = u_xv_x$ by (2.1). □

Recall that a band B is called *left normal* (resp. *right normal, normal*) if $efg = egf$ (resp. $efg = feg, efge = egfe$) for all $e, f, g \in B$.

Lemma 2.8 (Lemma 2.1 and Proposition 2.6 in [6]) *Let S be an abundant semigroup with a multiplicative ample transversal S°. Then*

(1) $I^S \cap \Lambda^S = E(S^\circ)$.
(2) $I^S = \{e \in E(S) |$ *there exists a unique* $e^\circ \in E(S^\circ)$ *such that* $e\mathcal{L}e^\circ\}$, $\Lambda^S = \{f \in E(S) |$ *there exists a unique* $f^\circ \in E(S^\circ)$ *such that* $f\mathcal{R}f^\circ\}$.
(3) I^S *(resp.* Λ^S*) is a left normal band (resp. a right normal band).*

To end this section, we explore the relationship between transversals. As usual, if S is a regular semigroup with an inverse transversal S°, then we denote the unique element in $V_{S^\circ}(a) = V(a) \cap S^\circ$ by a° for all $a \in S$, moreover, let $(a^\circ)^\circ = a^{\circ\circ}$. From the remarks before Example 2.2 in [8] and Corollary 2.7 in [22], we have the fact below.

Lemma 2.9 ([8,22]) *Let S be a regular semigroup and S° a subsemigroup of S. Then S° is an inverse transversal of S if and only if S° is an ample transversal of S. In this case, $u_a = aa^\circ, \overline{a} = a^{\circ\circ}$ and $v_a = a^\circ a$ for all $a \in S$. As consequence, we have $I^S = \{aa^\circ | a \in S\}$ and $\Lambda^S = \{a^\circ a | a \in S\}$.*

For multiplicative inverse transversals of bands, we have the following result.

Lemma 2.10 ([3]) *Let B° be an inverse transversal of a band B. Then B° is multiplicative if and only if B is normal.*

3 The Munn semigroups of admissible quadruples

In this section, a generalization of the Munn semigroup of a semilattice, namely *the Munn semigroup of an admissible quadruple*, is constructed. Moreover, we show that this semigroup is a fundamental regular semigroup with a multiplicative inverse transversal. We first introduce admissible quadruples, which is inspired by Lemma 2.8.

Definition 3.1 Let I (resp. Λ) be a left normal band (resp. a right normal band), $E^\circ = I \cap \Lambda$ a subsemilattice of I and Λ, and $P = (P_{f,g})_{\Lambda \times I}$ be a $\Lambda \times I$-matrix over E°. The quadruple (I, Λ, E°, P) is called *admissible* if for all $g \in I$ and $f \in \Lambda$, there exist $g^\circ, f^\circ \in E^\circ$ such that $g\mathcal{L}g^\circ$, $f\mathcal{R}f^\circ$ and for all $i, j \in E^\circ$,

$$iP_{f,g} = P_{if,g}, \quad P_{f,g}j = P_{f,gj}, \quad P_{f,j} = fj, \quad P_{i,g} = ig. \tag{3.1}$$

Remark 3.2 On admissible quadruples, we have the following remarks.

(1) Since E° is a subsemilattice, the elements g° and f° in Definition 3.1 are uniquely determined by g and f, respectively. In particular, $i \in E^\circ$ if and only if $i^\circ = i$. Thus, $P^\circ_{f,g} = P_{f,g}$ for all $f \in \Lambda$ and $g \in I$.
(2) If S is an abundant semigroup with a multiplicative ample transversal S°, then it is obvious that $(I^S, \Lambda^S, E(S^\circ), P^S)$ is an admissible quadruple by Lemma 2.8, where $P^S_{f,g}$ is equal to the product of f and g in S for all $f \in \Lambda$ and $g \in I$. In this case, $(I^S, \Lambda^S, E(S^\circ), P^S)$ is called *the admissible quadruple* of S. If U is a full subsemigroup of S, then by Lemma 2.3 and Corollary 2.4(1), $S^\circ \cap U$ is a multiplicative ample transversal of U and the admissible quadruples of S and U are equal.

To construct the Munn semigroup of an admissible quadruple, we need some preliminaries. First, we have the following basic facts on admissible quadruples.

Lemma 3.3 *Let (I, Λ, E°, P) be an admissible quadruple and $e, g \in I$, $f, h \in \Lambda$. Then*

$$eg = eg^\circ, \quad (eg)^\circ = e^\circ g^\circ, \quad fh = f^\circ h, \quad (fh)^\circ = f^\circ h^\circ.$$

Moreover, we have $eE^\circ e = eE^\circ$ and $fE^\circ f = E^\circ f$, which are subsemilattices of I and Λ, respectively.

Proof Since $g\mathcal{L}g^\circ$ and I is a left normal band, we have $eg = egg^\circ = eg^\circ g = eg^\circ$. This implies that $e^\circ g = e^\circ g^\circ$, and so $eg\mathcal{L}e^\circ g = e^\circ g^\circ \in E^\circ$ by the fact that $e\mathcal{L}e^\circ$. This yields that $(eg)^\circ = e^\circ g^\circ$. Finally, it follows that $eE^\circ e = eE^\circ$ by the fact that I is a left normal band. Moreover, for $i, j \in E^\circ$, we have

$$(ei)(ej) = (eie)j = eij = eji = (eje)i = (ej)(ei),$$

whence eE° is a subsemilattice of I. The remaining facts of this lemma can be proved by symmetry. □

Remark 3.4 Let I (resp. Λ) be a left normal band (resp. a right normal band), and $E^\circ = I \cap \Lambda$ be a subsemilattice of I and Λ. Suppose that for each $e \in I$ and $f \in \Lambda$ there exist $e^\circ, f^\circ \in E^\circ$ such that $e\mathcal{L}e^\circ$ and $f\mathcal{R}f^\circ$, respectively. Then by Lemma 3.3 it is easy to see that (I, Λ, E°, Q) forms an admissible quadruple, where $Q_{f,g} = f^\circ g^\circ$ for all $f \in \Lambda$ and $g \in I$. This admissible quadruple is called the *normal admissible quadruple determined by I, Λ and E°*.

Let (I, Λ, E°, P) be an admissible quadruple and $e \in I$, $f \in \Lambda$. If eE° is isomorphic to $E^\circ f$, we write $eE^\circ \cong E^\circ f$, and denote the set of isomorphisms from eE° to $E^\circ f$ by $T_{e,f}$. Moreover, we denote

$$\mathcal{U} = \{(e, f) \in I \times \Lambda | eE^\circ \cong E^\circ f\}, \quad T_{(I,\Lambda,E^\circ,P)} = \bigcup_{(e,f)\in\mathcal{U}} T_{e,f}.$$

Obviously, the elements in the Munn semigroup T_{E° of the semilattice E° are contained in $T_{(I,\Lambda,E^\circ,P)}$. The following proposition provides some other elements in $T_{(I,\Lambda,E^\circ,P)}$. As usual, we use ι_M to denote the identity transformation on the non-empty set M.

Proposition 3.5 *Let (I, Λ, E°, P) be an admissible quadruple and $g \in I$, $f \in \Lambda$. Define*

$$\pi_{f,g} : gP_{f,g}E^\circ \to E^\circ P_{f,g} f, \quad x \mapsto x^\circ P_{f,g} f.$$

Then $\pi_{f,g} \in T_{gP_{f,g}, P_{f,g}f}$ and the inverse mapping of $\pi_{f,g}$ is

$$\pi_{f,g}^{-1} : E^\circ P_{f,g} f \to gP_{f,g}E^\circ, \quad y \mapsto gP_{f,g}y^\circ.$$

In particular, we have

$$\pi_{g^\circ,g}: gE^\circ \to E^\circ g^\circ, \quad x \mapsto x^\circ g^\circ, \quad \pi_{f,f^\circ} : f^\circ E^\circ \to E^\circ f, \quad x \mapsto x^\circ f$$

and $\pi_{i,j} = \iota_{ijE^\circ}$, $\pi_{i,i} = \iota_{iE^\circ}$ for all $i, j \in E^\circ$.

Proof Clearly, $\pi_{f,g}$ is well defined. Let $x \in gP_{f,g}E^\circ$ and $y \in E^\circ P_{f,g} f$. Then by condition (3.1) and Lemma 3.3, we have

$$gP_{f,g}(x^\circ P_{f,g} f)^\circ = gP_{f,g}x^\circ P_{f,g} f^\circ = gf^\circ P_{f,g}x^\circ = gP_{f^\circ f,g}x^\circ = gP_{f,g}x^\circ = gP_{f,g}x = x.$$

Dually, we can obtain that $(gP_{f,g}y^\circ)^\circ P_{f,g} f = y$. Moreover, for $x_1, x_2 \in gP_{f,g}E^\circ$, by Lemma 3.3 and $f\mathcal{R}f^\circ$, we have $(x_1x_2)\pi_{f,g} = (x_1x_2)^\circ P_{f,g} f = x_1^\circ x_2^\circ P_{f,g} f$ and

$$\begin{aligned}(x_1\pi_{f,g})(x_2\pi_{f,g}) &= (x_1^\circ P_{f,g} f)(x_2^\circ P_{f,g} f) = x_1^\circ P_{f,g}(f x_2^\circ) P_{f,g} f \\ &= x_1^\circ P_{f,g}(f^\circ x_2^\circ) P_{f,g} f = x_1^\circ x_2^\circ P_{f,g}(f^\circ f) = x_1^\circ x_2^\circ P_{f,g} f = (x_1x_2)\pi_{f,g}.\end{aligned}$$

This implies that the first part of this lemma holds. The remaining part follows from the fact that

$$P_{g^\circ,g} = g^\circ g = g^\circ, \quad P_{f,f^\circ} = ff^\circ = f^\circ, \quad P_{i,j} = ij, x^\circ = x$$

for all $g \in I$, $f \in \Lambda$ and $i, j, x \in E^\circ$. □

The following results give some simple but useful properties of the elements in the set $T_{(I,\Lambda,E^\circ,P)}$.

Lemma 3.6 *Let (I, Λ, E°, P) be an admissible quadruple, $\alpha \in T_{e,f}$ and $x \in eE^\circ$, $y \in E^\circ f$.*

(1) *$e\alpha = f$ and α^{-1} is an isomorphism from $E^\circ f$ onto eE°.*
(2) *$(xE^\circ)\alpha = E^\circ(x\alpha)$ and $(E^\circ y)\alpha^{-1} = (y\alpha^{-1})E^\circ$.*
(3) *$(x\alpha)^\circ = (x\alpha)f^\circ$ and $x\alpha = (x\alpha)^\circ f$.*
(4) *$(y\alpha^{-1})^\circ = e^\circ(y\alpha^{-1})$ and $y\alpha^{-1} = e(y\alpha^{-1})^\circ$.*

Proof (1) This is obvious.

(2) Let $xi \in xE^\circ, i \in E^\circ$. Since I is a left normal band, we have $xix = xi$ and so

$$(xi)\alpha = (xix)\alpha = (xi)\alpha \cdot (x\alpha) = ((xi)\alpha)^\circ(x\alpha) \in E^\circ(x\alpha)$$

by Lemma 3.3. Conversely, let $u' \in E^\circ(x\alpha)$. Since $x\alpha \in E^\circ f$, we obtain that $u' \in E^\circ f$ whence $u' = u\alpha$ for some $u \in eE^\circ$. Observe that I is a left normal band and Λ is a right normal band, it follows that

$$u\alpha = (u\alpha)(x\alpha) = (x\alpha)(u\alpha)(x\alpha) = (xux)\alpha = (xu)\alpha.$$

Noticing that α is injective, we get $u = xu = xu^\circ \in xE^\circ$ by Lemma 3.3. This implies that $u' = u\alpha \in (xE^\circ)\alpha$. The other identity can be proved by similar methods.

(3) Since $x\alpha \in E^\circ f \subseteq \Lambda$, we have $x\alpha = (x\alpha)f = (x\alpha)^\circ f$ whence

$$(x\alpha)^\circ = ((x\alpha)f)^\circ = (x\alpha)^\circ f^\circ = (x\alpha)f^\circ$$

by Lemma 3.3.

(4) This is the dual of item (3). □

Now, let (I, Λ, E°, P) be an admissible quadruple, $\alpha \in T_{e,f}, \beta \in T_{g,h}$. Consider the composition $\alpha\pi_{f,g}^{-1}\beta$ in the symmetric inverse semigroup on the set $I \cup \Lambda$. Since

$$\mathrm{dom}(\pi_{f,g}^{-1}\beta) = (gP_{f,g}E^\circ \cap gE^\circ)\pi_{f,g} = (gP_{f,g}E^\circ)\pi_{f,g} = E^\circ P_{f,g}f,$$

it follows that

$$\mathrm{dom}(\alpha\pi_{f,g}^{-1}\beta) = (E^\circ f \cap E^\circ P_{f,g}f)\alpha^{-1} = (E^\circ P_{f,g}f)\alpha^{-1} = (P_{f,g}f)\alpha^{-1}E^\circ$$

and

$$\mathrm{ran}(\alpha\pi_{f,g}^{-1}\beta) = (E^\circ P_{f,g}f)\pi_{f,g}^{-1}\beta = (gP_{f,g}E^\circ)\beta = E^\circ(gP_{f,g})\beta$$

by Lemma 3.6(2). Thus, we have

$$\alpha\pi_{f,g}^{-1}\beta \in T_{j,k}, \quad j = (P_{f,g}f)\alpha^{-1}, \quad k = (gP_{f,g})\beta. \tag{3.2}$$

In view of the above discussions, we can define a multiplication "$\circ$" on $T_{(I,\Lambda,E^\circ,P)}$ as follows: For $\alpha \in T_{e,f}, \beta \in T_{g,h}$,

$$\alpha \circ \beta = \alpha\pi_{f,g}^{-1}\beta,$$

where $\pi_{f,g}^{-1}$ is defined as in Proposition 3.5.

Lemma 3.7 *The set $T_{(I,\Lambda,E^\circ,P)}$ forms a semigroup with respect to the multiplication "$\circ$" defined above.*

Proof Now, let $\alpha \in T_{e,f}, \beta \in T_{g,h}, \gamma \in T_{s,t}$ and

$$\alpha \circ \beta \in T_{j,k}, \quad (\alpha \circ \beta) \circ \gamma \in T_{m,n}, \quad \beta \circ \gamma \in T_{p,q}, \quad \alpha \circ (\beta \circ \gamma) \in T_{a,b},$$

where

$$j = (P_{f,g}f)\alpha^{-1}, \quad k = (gP_{f,g})\beta, \quad p = (P_{h,s}h)\beta^{-1}, \quad q = (sP_{h,s})\gamma,$$
$$m = (P_{k,s}k)(\alpha \circ \beta)^{-1}, \quad n = (sP_{k,s})\gamma, \quad a = (P_{f,p}f)\alpha^{-1}, \quad b = (pP_{f,p})(\beta \circ \gamma).$$

On one hand, in view of the fact $k \in E^\circ h$ and Lemma 3.3, we have $k = kh = k^\circ h$. By condition (3.1) and $h^\circ \mathcal{R} h$,

$$P_{k,s} = P_{k^\circ h,s} = k^\circ P_{h,s} = k^\circ P_{h^\circ h,s} = k^\circ h^\circ P_{h,s} = k^\circ P_{h,s} h^\circ. \tag{3.3}$$

Since $h^\circ k = hk$ (by Lemma 3.3), $k^\circ \mathcal{R} k$ and Λ is a right normal band, this implies that

$$P_{k,s} k = k^\circ P_{h,s} h^\circ k = k^\circ P_{h,s} hk = k^\circ (p\beta)k = (p\beta)k^\circ k = (p\beta)k = (pgP_{f,g})\beta.$$

By Lemma 3.3, condition (3.1) and the fact that $g\mathcal{L}g^\circ$, we have

$$\begin{aligned} m &= (P_{k,s}k)(\alpha \circ \beta)^{-1} = (P_{k,s}k)\beta^{-1}\pi_{f,g}\alpha^{-1} \\ &= ((pgP_{f,g})\beta)\beta^{-1}\pi_{f,g}\alpha^{-1} = (pgP_{f,g})\pi_{f,g}\alpha^{-1} = ((pgP_{f,g})^\circ P_{f,g} f)\alpha^{-1} \\ &= (p^\circ g^\circ P_{f,g} P_{f,g} f)\alpha^{-1} = (p^\circ P_{f,g} g^\circ f)\alpha^{-1} = (p^\circ P_{f,gg^\circ} f)\alpha^{-1} = (p^\circ P_{f,g} f)\alpha^{-1}. \end{aligned}$$

On the other hand, by Lemma 3.3 and the fact $p \in gE^\circ$, we have

$$gp^\circ = gp = p, \quad P_{f,p} = P_{f,gp^\circ} = P_{f,g} p^\circ = p^\circ P_{f,g} \tag{3.4}$$

by condition (3.1), which implies that $P_{f,p} f = p^\circ P_{f,g} f$ and so

$$a = (P_{f,p} f)\alpha^{-1} = (p^\circ P_{f,g} f)\alpha^{-1} = m.$$

Dually, we can obtain that $n = b$.

Take $x \in aE^\circ = mE^\circ$ and denote $y = x\alpha$. On one hand,

$$x(\alpha \circ \beta) = (x\alpha)\pi_{f,g}^{-1}\beta = y\pi_{f,g}^{-1}\beta = (gP_{f,g} y^\circ)\beta. \tag{3.5}$$

Since both k and $x(\alpha \circ \beta)$ are in Λ, and $\operatorname{ran}(\alpha \circ \beta) = E^\circ k$, by Lemma 3.3 we have

$$k^\circ (x(\alpha \circ \beta))^\circ = (x(\alpha \circ \beta) \cdot k)^\circ = (x(\alpha \circ \beta))^\circ \tag{3.6}$$

Combining the identities (3.3), (3.6) and (3.5), we obtain

$$\begin{aligned} x[(\alpha \circ \beta) \circ \gamma] &= (x(\alpha \circ \beta))\pi_{k,s}^{-1}\gamma = (sP_{k,s} \cdot (x(\alpha \circ \beta))^\circ)\gamma \\ &= (sk^\circ P_{h,s} \cdot (x(\alpha \circ \beta))^\circ)\gamma = (sP_{h,s} \cdot k^\circ (x(\alpha \circ \beta))^\circ)\gamma \\ &= (sP_{h,s} \cdot (x(\alpha \circ \beta))^\circ)\gamma = (sP_{h,s} \cdot ((gP_{f,g} y^\circ)\beta)^\circ)\gamma. \end{aligned}$$

On the other hand,

$$x[\alpha \circ (\beta \circ \gamma)] = (x\alpha)\pi_{f,p}^{-1}\beta\pi_{h,s}^{-1}\gamma = y\pi_{f,p}^{-1}\beta\pi_{h,s}^{-1}\gamma = (pP_{f,p} y^\circ)\beta\pi_{h,s}^{-1}\gamma.$$

Observe that $gp^\circ = p$ (by (3.4)) and $y^\circ p = y^\circ p^\circ$ (by Lemma 3.3), it follows by the identity (3.4) that

$$\begin{aligned} (pP_{f,p} y^\circ)\beta &= (gp^\circ P_{f,g} y^\circ)\beta = (gP_{f,g} y^\circ p^\circ)\beta \\ &= (gP_{f,g} y^\circ p)\beta = (gP_{f,g} y^\circ)\beta(p\beta) = (gP_{f,g} y^\circ)\beta \cdot P_{h,s} h. \end{aligned}$$

By Lemma 3.3 and condition (3.1),

$$\begin{aligned} x[\alpha \circ (\beta \circ \gamma)] &= ((pP_{f,p} y^\circ)\beta)\pi_{h,s}^{-1}\gamma = ((gP_{f,g} y^\circ)\beta \cdot P_{h,s} h)\pi_{h,s}^{-1}\gamma \\ &= (sP_{h,s} \cdot ((gP_{f,g} y^\circ)\beta \cdot P_{h,s} h)^\circ)\gamma = (sP_{h,s} \cdot h^\circ P_{h,s} \cdot ((gP_{f,g} y^\circ)\beta)^\circ)\gamma \\ &= (sP_{h,s} \cdot P_{h^\circ h,s} \cdot ((gP_{f,g} y^\circ)\beta)^\circ)\gamma = (sP_{h,s} \cdot ((gP_{f,g} y^\circ)\beta)^\circ)\gamma. \end{aligned}$$

Hence, $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ and so $T_{(I,\Lambda,E^\circ,P)}$ forms a semigroup with respect to the multiplication "$\circ$". $\square$

Let $(I, \Lambda, E^{\circ}, P)$ be an admissible quadruple. If $I = \Lambda = E^{\circ}$, then it is easy to check that the semigroup $T_{(I,\Lambda,E^{\circ},P)}$ coincides with the Munn semigroup $T_{E^{\circ}}$ since $\pi_{f,g}^{-1} = \iota_{fgE^{\circ}}$ for all $f \in \Lambda$ and $g \in I$ in the case. Thus, the semigroup $T_{(I,\Lambda,E^{\circ},P)}$ can be regarded as a generalization of the Munn semigroup of a semilattice and will be called the *Munn semigroup of the admissible quadruple* $(I, \Lambda, E^{\circ}, P)$.

Theorem 3.8 *The semigroup $T_{(I,\Lambda,E^{\circ},P)}$ is a regular semigroup with a multiplicative inverse transversal*

$$T_{E^{\circ}} = \{\alpha \in T_{(I,\Lambda,E^{\circ},P)} | \alpha \in T_{p,q},\ p, q \in E^{\circ}\}.$$

Proof For $\alpha \in T_{e,f}$, let $\alpha^{\circ} = \pi_{f,f^{\circ}}\alpha^{-1}\pi_{e^{\circ},e}$, where $\pi_{f,f^{\circ}}$ and $\pi_{e^{\circ},e}$ are defined as in Proposition 3.5. It is routine to check that $\alpha^{\circ} \in T_{f^{\circ},e^{\circ}}$ and so $\alpha^{\circ} \in T_{E^{\circ}}$. Furthermore, we have

$$\alpha \circ \alpha^{\circ} = \alpha\pi_{f,f^{\circ}}^{-1}\alpha^{\circ} = \alpha\pi_{f,f^{\circ}}^{-1}\pi_{f,f^{\circ}}\alpha^{-1}\pi_{e^{\circ},e} = \pi_{e^{\circ},e}. \tag{3.7}$$

Similarly, we can prove that $\pi_{e^{\circ},e} \circ \alpha = \alpha$ and $\alpha^{\circ} \circ \pi_{e^{\circ},e} = \alpha^{\circ}$, which implies that α° is an inverse of α in $T_{E^{\circ}}$.

Now, let $\beta \in T_{p,q}$ $(p, q \in E^{\circ})$ be an inverse of $\alpha \in T_{e,f}$ in $T_{E^{\circ}}$. We have to show that $\beta = \alpha^{\circ}$. Let $\alpha \circ \beta \in T_{j,k}$ and $\beta \circ \alpha \in T_{u,v}$. Then $\alpha \circ \beta \circ \alpha = \alpha \in T_{e,f}$. Moreover,

$$j = (P_{f,p}f)\alpha^{-1} \in \mathrm{dom}\alpha = eE^{\circ}, \quad k = (pP_{f,p})\beta \in \mathrm{ran}\beta = E^{\circ}q \subseteq E^{\circ}.$$

Similarly, we can obtain that $v \in \mathrm{ran}\alpha = E^{\circ}f$ and $u \in pE^{\circ} \subseteq E^{\circ}$. It follows that

$$e = (P_{f,u}f)\alpha^{-1} = (fuf)\alpha^{-1} = (uf)\alpha^{-1}, \quad f = (eP_{k,e})\alpha = (eke)\alpha = (ek)\alpha$$

by condition (3.1) and the fact that I is a left normal band and Λ is a right normal band, respectively. This implies that $uf = e\alpha = f$ and $ek = f\alpha^{-1} = e$. Since $u \in pE^{\circ}$ and $k \in E^{\circ}q$, we have $pu = u$ and $kq = k$ whence $puf = uf = f$ and $ekq = ek = e$. This yields that $pf = f$ and $eq = e$. Consider $\beta \circ \alpha \circ \beta = \beta \in T_{p,q}$. Then we have

$$p = (P_{q,j}q)\beta^{-1} = (qjq)\beta^{-1} = (qj)\beta^{-1}, \quad q = (pP_{v,p})\beta = (pvp)\beta = (vp)\beta.$$

This implies that $qj = p\beta = q$ and $vp = q\beta^{-1} = p$. Observe that $j \in eE^{\circ}$ and I is a left normal band, it follows that $j = ej = eje$ whence $je = j$ and $qe = qje = qj = q$. Similar discussion gives $fp = p$. Therefore $f\mathcal{R}p$ and $e\mathcal{L}q$. Observe that $p, q \in E^{\circ}$, it follows that $p = f^{\circ}$ and $q = e^{\circ}$ by the definition of admissible quadruple, whence $\beta \in T_{f^{\circ},e^{\circ}}$. Moreover, by simple calculations, we can see that

$$\beta = \beta\pi_{e^{\circ},e}^{-1}\pi_{e^{\circ},e} = \beta \circ \pi_{e^{\circ},e} = \beta \circ \alpha \circ \alpha^{\circ}$$

by the identity (3.7) and so

$$\alpha^{\circ} = \alpha^{\circ} \circ \alpha \circ \beta \circ \alpha \circ \alpha^{\circ} = \alpha^{\circ} \circ \alpha \circ \beta.$$

This implies that $\alpha^{\circ}\mathcal{L}\beta$ in $T_{(I,\Lambda,E^{\circ},P)}$. Dually, we can obtain $\alpha^{\circ}\mathcal{R}\beta$ in $T_{(I,\Lambda,E^{\circ},P)}$ and so $\alpha^{\circ}\mathcal{H}\beta$ in $T_{(I,\Lambda,E^{\circ},P)}$. However, both α° and β are the inverses of α, and hence $\beta = \alpha^{\circ}$.

By the above discussions, we have shown that $T_{E^{\circ}}$ is an inverse transversal of $T_{(I,\Lambda,E^{\circ},P)}$. To see $T_{E^{\circ}}$ is multiplicative, we take $\alpha \in T_{e,f}$ and $\beta \in T_{g,h}$. Then by the identity (3.7) and its dual, and $v_{\alpha} = \alpha^{\circ} \circ \alpha$, $u_{\beta} = \beta \circ \beta^{\circ}$ (by Lemma 2.9), we have

$$v_{\alpha} \circ u_{\beta} = \alpha^{\circ} \circ \alpha \circ \beta \circ \beta^{\circ} = \pi_{f,f^{\circ}} \circ \pi_{g^{\circ},g} \in T_{j,k}$$

where

$$j = (P_{f,g}f)\pi^{-1}_{f,f^\circ} \in \mathrm{ran}\pi^{-1}_{f,f^\circ} = f^\circ E^\circ \subseteq E^\circ$$

and

$$k = (gP_{f,g})\pi_{g^\circ,g} \in \mathrm{ran}\pi_{g^\circ,g} = E^\circ g^\circ \subseteq E^\circ,$$

whence $v_\alpha \circ u_\beta \in T_{E^\circ}$. This implies that T_{E° is a multiplicative inverse transversal of $T_{(I,\Lambda,E^\circ,P)}$. □

The following corollary characterizes the idempotents in $T_{(I,\Lambda,E^\circ,P)}$.

Corollary 3.9 *Let $\alpha \in T_{e,f}$. Then $\alpha \in E(T_{(I,\Lambda,E^\circ,P)})$ if and only if*

$$f^\circ = P_{f,e} = e^\circ \quad \text{and} \quad e(x\alpha)^\circ = x \quad \text{for all } x \in eE^\circ.$$

Proof Let $\alpha \in E(T)$. Then $\alpha \circ \alpha = \alpha$, whence $\mathrm{dom}(\alpha \circ \alpha) = \mathrm{dom}\alpha$ and $\mathrm{ran}(\alpha \circ \alpha) = \mathrm{ran}\alpha$. This implies that $(P_{f,e}f)\alpha^{-1} = e$ and $(eP_{f,e})\alpha = f$, which gives $P_{f,e}f = e\alpha = f$ and $eP_{f,e} = f\alpha^{-1} = e$. Moreover, by Lemma 3.3 and condition (3.1) we have

$$fP_{f,e} = f^\circ P_{f,e} = P_{f^\circ f,e} = P_{f,e}$$

and

$$P_{f,e}e = P_{f,e}e^\circ = P_{f,ee^\circ} = P_{f,e}.$$

Therefore $e\mathcal{L}P_{f,e} \in E^\circ$ and $f\mathcal{R}P_{f,e} \in E^\circ$ and so $f^\circ = P_{f,e} = e^\circ$. On the other hand, for $x \in eE^\circ$, we have

$$x\alpha = x(\alpha \circ \alpha) = ((x\alpha)\pi^{-1}_{f,e})\alpha.$$

Since α is bijective, it follows that

$$x = (x\alpha)\pi^{-1}_{f,e} = (eP_{f,e})(x\alpha)^\circ = ee^\circ(x\alpha)^\circ = e(x\alpha)^\circ.$$

Conversely, if the given condition in the corollary holds, then we can deduce that $\alpha \in E(T_{(I,\Lambda,E^\circ,P)})$ by the above discussions. □

Corollary 3.10 *If (I, Λ, E°, Q) is the normal admissible quadruple determined by I, Λ and E°, then the Munn semigroup $NT = T_{(I,\Lambda,E^\circ,Q)}$ is an orthodox semigroup with a multiplicative inverse transversal T_{E° such that $E(NT)$ forms a normal band.*

Proof Let $\alpha, \beta \in E(NT)$ where $\alpha \in T_{e,f}$ and $\beta \in T_{g,h}$. Then by Corollary 3.9, we have

$$f^\circ = e^\circ \quad \text{and} \quad e(x\alpha)^\circ = x \quad \text{for all } x \in eE^\circ \tag{3.8}$$

and

$$h^\circ = g^\circ \quad \text{and} \quad g(y\beta)^\circ = y \quad \text{for all } y \in gE^\circ \tag{3.9}$$

Denote $\alpha \circ \beta \in T_{j,k}$, where

$$j = (Q_{f,g}f)\alpha^{-1} = (f^\circ g^\circ f)\alpha^{-1} = (g^\circ f)\alpha^{-1}$$

and $k = (gQ_{f,g})\beta = (gf^\circ g^\circ)\beta = (gf^\circ)\beta$. Take $x = (g^\circ f)\alpha^{-1}$ in (3.8). Then by Lemma 3.6(4), we have

$$j = (g^\circ f)\alpha^{-1} = e(((g^\circ f)\alpha^{-1})\alpha)^\circ = e(g^\circ f)^\circ = ef^\circ g^\circ = ee^\circ g^\circ = eg^\circ, \tag{3.10}$$

which implies that $j^\circ = (eg^\circ)^\circ = e^\circ g^\circ$. Similarly, we can show that $k^\circ = e^\circ g^\circ$ by (3.8) and (3.9). This gives that $j^\circ = Q_{k,j} = j^\circ k^\circ = k^\circ$. On the other hand, for $x \in jE^\circ = eg^\circ E^\circ \subseteq eE^\circ$, we have

$$\begin{aligned} x(\alpha \circ \beta) &= (x\alpha)\pi_{f,g}^{-1}\beta = (gQ_{f,g}(x\alpha)^\circ)\beta = (gf^\circ g^\circ(x\alpha)^\circ)\beta \\ &= (gg^\circ f^\circ(x\alpha)^\circ)\beta = (g \cdot f^\circ(x\alpha)^\circ)\beta = (g((x\alpha)f)^\circ)\beta = (g(x\alpha)^\circ)\beta. \end{aligned}$$

Since $eg^\circ = eg$ (by Lemma 3.3), $ege = eg$ (as I is a left normal band) and $jx = x$, this implies that

$$\begin{aligned} j(x(\alpha \circ \beta))^\circ &= eg^\circ((g(x\alpha)^\circ)\beta)^\circ = e \cdot g((g(x\alpha)^\circ)\beta)^\circ \\ &= eg(x\alpha)^\circ = ege(x\alpha)^\circ = egx = eg^\circ x = jx = x \end{aligned}$$

by (3.10), (3.9) and (3.8). Again by Corollary 3.9, we have $\alpha \circ \beta \in E(NT)$. This implies that NT is orthodox, and so $E(T_{E^\circ})$ is a multiplicative inverse transversal of $E(NT)$ by Theorem 3.8. In view of Lemma 2.10, $E(NT)$ is a normal band. □

The following example illustrates Theorem 3.8 and Corollary 3.10.

Example 3.11 Let $I = \{0, e, g\}$ and $\Lambda = \{0, e, f\}$ be a left normal band and a right normal band, respectively, and their multiplication tables are:

I	0	e	g
0	0	0	0
e	0	e	e
g	0	g	g

Λ	0	e	f
0	0	0	0
e	0	e	f
f	0	e	f

.

Denote $E^\circ = I \cap \Lambda = \{0, e\}$ and define a $\Lambda \times I$-matrix P over E° by

P	0	e	g
0	0	0	0
e	0	e	e
f	0	e	0

Then it is routine to check that (I, Λ, E°, P) is an admissible quadruple and

$$0E^\circ = \{0\}, \quad eE^\circ = \{0, e\}, \quad gE^\circ = \{0, g\}; \quad E^\circ 0 = \{0\}, \quad E^\circ e = \{0, e\}, \quad E^\circ f = \{0, f\}.$$

This implies that

$$\mathcal{U} = \{(0, 0), (e, e), (e, f), (g, e), (g, f)\}$$

and $|T_{i,\lambda}| = 1$ for all $i \in I$ and $\lambda \in \Lambda$. Moreover, if we denote the unique element in $T_{i,\lambda}$ by $\alpha_{i,\lambda}$, then we have the multiplication table of $T_{(I,\Lambda,E^\circ,P)}$

$\circ$	$\alpha_{0,0}$	$\alpha_{e,e}$	$\alpha_{e,f}$	$\alpha_{g,e}$	$\alpha_{g,f}$
$\alpha_{0,0}$	$\alpha_{0,0}$	$\alpha_{0,0}$	$\alpha_{0,0}$	$\alpha_{0,0}$	$\alpha_{0,0}$
$\alpha_{e,e}$	$\alpha_{0,0}$	$\alpha_{e,e}$	$\alpha_{e,f}$	$\alpha_{e,e}$	$\alpha_{e,f}$
$\alpha_{e,f}$	$\alpha_{0,0}$	$\alpha_{e,e}$	$\alpha_{e,f}$	$\alpha_{0,0}$	$\alpha_{0,0}$
$\alpha_{g,e}$	$\alpha_{0,0}$	$\alpha_{g,e}$	$\alpha_{g,f}$	$\alpha_{g,e}$	$\alpha_{g,f}$
$\alpha_{g,f}$	$\alpha_{0,0}$	$\alpha_{g,e}$	$\alpha_{g,f}$	$\alpha_{0,0}$	$\alpha_{0,0}$

.

In this case, we have

$$V_{T_{E^\circ}}(\alpha_{0,0}) = \{\alpha_{0,0}\}, \quad V_{T_{E^\circ}}(\alpha_{e,e}) = V_{T_{E^\circ}}(\alpha_{e,f}) = V_{T_{E^\circ}}(\alpha_{g,e}) = V_{T_{E^\circ}}(\alpha_{g,f}) = \{\alpha_{e,e}\}$$

and

$$I^T = \{\alpha_{0,0}, \alpha_{e,e}, \alpha_{g,e}\}, \quad \Lambda^T = \{\alpha_{0,0}, \alpha_{e,e}, \alpha_{e,f}\}, \quad \Lambda^T I^T = T_{E^\circ},$$

this shows that $T_{E^\circ} = \{\alpha_{0,0}, \alpha_{e,e}\}$ is a multiplicative inverse transversal of $T_{(I,\Lambda,E^\circ,P)}$. Observe that $E(T_{(I,\Lambda,E^\circ,P)})$ is not a band (since $\alpha_{g,e} \circ \alpha_{e,f} = \alpha_{g,f} \notin E(T_{(I,\Lambda,E^\circ,P)})$).

Now, we consider the Munn semigroup $T_{(I,\Lambda,E^\circ,Q)}$ of the normal admissible quadruple determined by I, Λ and E°, where the $\Lambda \times I$-matrix Q over E° is defined by

Q	0	e	g
0	0	0	0
e	0	e	e
f	0	e	e

.

Then we can obtain the multiplication table of $T_{(I,\Lambda,E^\circ,Q)}$:

$\circ$	$\alpha_{0,0}$	$\alpha_{e,e}$	$\alpha_{e,f}$	$\alpha_{g,e}$	$\alpha_{g,f}$
$\alpha_{0,0}$	$\alpha_{0,0}$	$\alpha_{0,0}$	$\alpha_{0,0}$	$\alpha_{0,0}$	$\alpha_{0,0}$
$\alpha_{e,e}$	$\alpha_{0,0}$	$\alpha_{e,e}$	$\alpha_{e,f}$	$\alpha_{e,e}$	$\alpha_{e,f}$
$\alpha_{e,f}$	$\alpha_{0,0}$	$\alpha_{e,e}$	$\alpha_{e,f}$	$\alpha_{e,e}$	$\alpha_{e,f}$
$\alpha_{g,e}$	$\alpha_{0,0}$	$\alpha_{g,e}$	$\alpha_{g,f}$	$\alpha_{g,e}$	$\alpha_{g,f}$
$\alpha_{g,f}$	$\alpha_{0,0}$	$\alpha_{g,e}$	$\alpha_{g,f}$	$\alpha_{g,e}$	$\alpha_{g,f}$

.

In this case, $T_{(I,\Lambda,E^\circ,Q)}$ forms a normal band with a multiplicative inverse transversal $T_{E^\circ} = \{\alpha_{0,0}, \alpha_{e,e}\}$.

The following corollary gives some useful information about the semigroup $T_{(I,\Lambda,E^\circ,P)}$ which will be used in the next sections frequently.

Corollary 3.12 *Let $\alpha \in T_{e,f}$, $\beta \in T_{g,h}$ and $T = T_{(I,\Lambda,E^\circ,P)}$.*

(1) $\alpha^\circ = \pi_{f,f^\circ}\alpha^{-1}\pi_{e^\circ,e} \in T_{f^\circ,e^\circ}$, $\overline{\alpha} = \alpha^{\circ\circ} = \pi^{-1}_{e^\circ,e}\alpha\pi^{-1}_{f,f^\circ} \in T_{e^\circ,f^\circ}$.
(2) $u_\alpha = \alpha \circ \alpha^\circ = \pi_{e^\circ,e}$, $v_\alpha = \alpha^\circ \circ \alpha = \pi_{f,f^\circ}$ *and so* $I^T = \{\pi_{e^\circ,e} | e \in I\}$ *and* $\Lambda^T = \{\pi_{f,f^\circ} | f \in \Lambda\}$.
(3) $\alpha\mathcal{R}\beta$ *(resp. $\alpha\mathcal{L}\beta$) in T if and only if $e = g$ (resp. $f = h$).*

Proof Item (1) follows directly from the proof of Theorem 3.8 and Lemma 2.9, and item (2) follows from Lemma 2.9 and the identity (3.7) and its dual. Item (3) follows from Lemma 2.2, Lemma 2.9, item (2) above and the fact that $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$ on a regular semigroup. □

We say that two admissible quadruples (I, Λ, E°, P) and (J, Π, F°, R) are *isomorphic* if there exist an isomorphism φ from I onto J and an isomorphism ψ from Λ onto Π such that

$$\varphi|_{E^\circ} = \psi|_{E^\circ}, \quad E^\circ\varphi = F^\circ, \quad P_{f,g}\varphi = R_{f\varphi,g\psi}$$

for all $f \in \Lambda$ and $g \in I$. If this is the case, then one can easily show that $T_{(I,\Lambda,E^\circ,P)}$ is isomorphic to $T_{(J,\Pi,F^\circ,R)}$. Moreover, we have the following.

Corollary 3.13 *Let (I, Λ, E°, P) be an admissible quadruple. Then (I, Λ, E°, P) is isomorphic to the admissible quadruple of $T = T_{(I,\Lambda,E^\circ,P)}$. In particular, if (I, Λ, E°, Q) is the normal admissible quadruple determined by I, Λ and E°, then (I, Λ, E°, Q) is isomorphic to the admissible quadruple of the normal band $E(NT)$, where NT is the Munn semigroup $T_{(I,\Lambda,E^\circ,Q)}$.*

Proof By Corollary 3.12(2), we can define the mappings

$$\varphi : I \to I^T, \quad e \mapsto \pi_{e^\circ, e}, \quad \psi : \Lambda \to \Lambda^T, \quad f \mapsto \pi_{f, f^\circ}.$$

By Lemma 3.3, condition (3.1) and Proposition 3.5, it is routine to check that the above mappings are isomorphisms such that

$$\varphi|_{E^\circ} = \psi|_{E^\circ}, \quad E^\circ\varphi = E(T_{E^\circ}), \quad P_{f,g}\psi = P^T_{f\varphi, g\psi}.$$

For the normal admissible quadruple (I, Λ, E°, Q), $E(NT)$ is a normal band by Corollary 3.10 and so $E(NT)$ is a full subsemigroup of NT. By Remark 3.2(2), $E(T_{E^\circ}) = T_{E^\circ} \cap E(NT)$ is an inverse transversal of $E(NT)$ and the admissible quadruples of NT and $E(NT)$ are equal. By the first part of this corollary, (I, Λ, E°, Q) is isomorphic to the admissible quadruple of $E(NT)$. □

Remark 3.14 The above Corollary 3.13 shows that admissible quadruples come from regular semigroups with multiplicative inverse transversals and normal admissible quadruples come from normal bands with inverse transversals, respectively.

Now, we are a position to give the main result of this section.

Theorem 3.15 *Let (I, Λ, E°, P) be an admissible quadruple and U a full subsemigroup of $T_{(I,\Lambda,E^\circ,P)}$. Then U is a fundamental abundant semigroup with a multiplicative ample transversal $T_{E^\circ} \cap U$ whose admissible quadruple is isomorphic to (I, Λ, E°, P). In particular, if U is also regular, then U is a fundamental regular semigroup with a multiplicative inverse transversal $T_{E^\circ} \cap U$. As a direct consequence, $T_{(I,\Lambda,E^\circ,P)}$ itself is fundamental.*

Proof By Theorem 3.8 and Lemma 2.9, T_{E° is a multiplicative ample transversal of $T_{(I,\Lambda,E^\circ,P)}$. Since the Munn semigroup T_{E° of the semilattice E° is fundamental, it follows that U is a fundamental abundant semigroup with a multiplicative ample transversal $T_{E^\circ} \cap U$ whose admissible quadruple is isomorphic to (I, Λ, E°, P) by Lemma 2.3, Corollary 2.4, Remark 3.2(2) and Corollary 3.13. The remaining result now follows from Lemma 2.9. □

4 A Munn type representation of abundant semigroups with a multiplicative ample transversal

In this section, we always assume that S is an abundant semigroup with a multiplicative ample transversal S°. By the previous section, we have the Munn semigroup $T_{(I^S,\Lambda^S,E(S^\circ),P^S)}$ of the admissible quadruple of S, where $P^S_{f,g}$ is equal to the product of f and g in S for all $f \in \Lambda^S$ and $g \in I^S$ (see Lemma 2.8 and Remark 3.2). The aim of this section is to show that there is a homomorphism $\rho : S \to T_{(I^S,\Lambda^S,E(S^\circ),P^S)}$ whose kernel is μ_S. For simplicity, we write $E(S^\circ)$ as E°. To accommodate with the notations of Sect. 3, we use the notations in Sect. 3 for the admissible quadruple $(I^S, \Lambda^S, E(S^\circ), P^S)$ throughout this section.

In view of Lemmas 2.5 and 2.6, S is IC and for every $a \in S$, there exists a unique idempotent-connected isomorphism from $\langle u_a \rangle$ onto $\langle v_a \rangle$. We denote this isomorphism by λ_a in the sequel. For all $a \in S$, denote the restriction of λ_a to $u_a E^\circ$ by ρ_a, that is, $\rho_a = \lambda_a|_{u_a E^\circ}$. Recall that $a(x\lambda_a) = xa$ for all $a \in S$ and $x \in \langle u_a \rangle$. By the definition of ρ_a, we have $a(x\rho_a) = xa$ for all $a \in S$ and $x \in u_a E^\circ$.

Lemma 4.1 $\rho_a \in T_{u_a, v_a}$ *for all $a \in S$.*

Proof Clearly, we have $u_aE^\circ = u_aE^\circ u_a \subseteq \langle u_a\rangle = \mathrm{dom}\lambda_a$. Take $x = u_ai \in u_aE^\circ, i \in E^\circ$. Then $a(x\lambda_a) = xa$. Since $a = u_a\overline{a}v_a$, we have $u_a\overline{a}v_a(x\lambda_a) = u_aiu_a\overline{a}v_a$. Observe that $u_a\mathcal{L}\overline{a}^+\mathcal{R}^*\overline{a}$ and $\overline{a}^+i = i\overline{a}^+$, it follows that

$$\overline{a}v_a(x\lambda_a) = \overline{a}^+u_a\overline{a}v_a(x\lambda_a) = \overline{a}^+u_aiu_a\overline{a}v_a = i\overline{a}v_a.$$

Because S° is ample, we have $i\overline{a} = \overline{a}(i\overline{a})^*$, $(i\overline{a})^* \in E^\circ$ by the identity (1.3). This implies that $\overline{a}v_a(x\lambda_a) = \overline{a}(i\overline{a})^*v_a$. By Lemma 2.1 and the fact that $\overline{a}\mathcal{L}^*\overline{a}^*\mathcal{R}v_a$ (see (2.1)) and $x\lambda_a \in \langle v_a\rangle$, we obtain that

$$x\lambda_a = v_a(x\lambda_a) = \overline{a}^*v_a(x\lambda_a) = \overline{a}^*(i\overline{a})^*v_a = (i\overline{a})^*\overline{a}^*v_a = (i\overline{a})^*v_a \in E^\circ v_a. \tag{4.1}$$

Dually, we can see that $\lambda_a^{-1}|_{E^\circ v_a}$ is a mapping from $E^\circ v_a$ to u_aE°. Thus, $\rho_a \in T_{u_a,v_a}$. □

Lemma 4.2 *$\rho_a \circ \rho_b = \rho_{ab}$ for all $a, b \in S$.*

Proof Since $\rho_a \in T_{u_a,v_a}$ and $\rho_b \in T_{u_b,v_b}$, we can assume that $\rho_a \circ \rho_b \in T_{j,k}$ where

$$j = (P^S_{v_a,u_b}v_a)\rho_a^{-1} = (v_au_bv_a)\rho_a^{-1}, \quad k = (u_bP^S_{v_a,u_b})\rho_b = (u_bv_au_b)\rho_b.$$

We first show that $j = u_{ab}$ and $k = v_{ab}$. In fact, by Lemma 2.6(2), we have $u_{ab} = u_a(\bar{a}v_au_b)^+ \in u_aE^\circ$. Since S° is an ample semigroup and $v_au_b \in E^\circ$, we get $(\overline{a}v_au_b)^+\overline{a} = \overline{a}v_au_b$ by (1.3). In view of the identity (4.1), it follows that

$$u_{ab}\rho_a = ((\overline{a}v_au_b)^+\overline{a})^*v_a = (\overline{a}v_au_b)^*v_a = (\overline{a}^*v_au_b)^*v_a = (v_au_b)^*v_a = v_au_bv_a$$

by the identity (1.1) and the fact that $v_a\mathcal{R}^*\overline{a}^*$ and $v_au_b \in E^\circ$. This implies that $j = (v_au_bv_a)\rho_a^{-1} = u_{ab}$. Dually, we can prove that $k = v_{ab}$.

Finally, let $x \in \mathrm{dom}\rho_{ab}$. On one hand,

$$x(\rho_a \circ \rho_b) = x\rho_a\pi^{-1}_{v_a,u_b}\rho_b = (u_bP^S_{v_a,u_b}(x\rho_a)^\circ)\rho_b = (u_bv_au_b(x\rho_a)^\circ)\rho_b.$$

On the other hand, since $v_au_b \in E^\circ \subseteq \Lambda^S$, $x\rho_a \in \Lambda^S$, we have

$$(x\rho_a)^\circ \in E^\circ, \quad (x\rho_a)^\circ \cdot v_au_b = v_au_b \cdot (x\rho_a)^\circ, \quad (x\rho_a)(v_au_b) = (x\rho_a)^\circ(v_au_b)$$

by Lemma 3.3. Observe that Λ^S is a right normal band, it follows that $v_au_b(x\rho_a)v_au_b = (x\rho_a)v_au_b$, whence

$$\begin{aligned}ab \cdot x(\rho_a \circ \rho_b) &= ab \cdot (u_bv_au_b(x\rho_a)^\circ)\rho_b = a \cdot [b \cdot (u_bv_au_b(x\rho_a)^\circ)\rho_b]\\ &= a \cdot u_bv_au_b(x\rho_a)^\circ b = a \cdot u_b(x\rho_a)^\circ v_au_bb = av_a \cdot u_b(x\rho_a)^\circ v_au_bb\\ &= a \cdot v_au_b(x\rho_a)v_au_bb = a(x\rho_a) \cdot v_au_bb = xa \cdot v_au_bb = xab = ab \cdot (x\rho_{ab}).\end{aligned}$$

Since $ab\mathcal{L}^*v_{ab}$ and

$$x(\rho_a \circ \rho_b), \quad x\rho_{ab} \in \mathrm{ran}\rho_{ab} = E^\circ v_{ab} = v_{ab}E^\circ v_{ab},$$

we have

$$x(\rho_a \circ \rho_b) = v_{ab} \cdot x(\rho_a \circ \rho_b) = v_{ab} \cdot x\rho_{ab} = x\rho_{ab}$$

by Lemma 2.1. Thus $\rho_a \circ \rho_b = \rho_{ab}$. □

Lemma 4.3 $\mu_S = \{(a, b) \in S \times S | \rho_a = \rho_b\}$.

Proof Denote $\delta = \{(a, b) \in S \times S | \rho_a = \rho_b\}$. By Lemma 4.2, δ is a congruence on S. If $\rho_a = \rho_b$, then domρ_a = domρ_b and ranρ_a = ranρ_b, which implies that $a\mathcal{R}^*u_a = u_b\mathcal{R}^*b$ and $a\mathcal{L}^*v_a = v_b\mathcal{L}^*b$ by Lemma 2.6, and so $a\mathcal{H}^*b$. Thus $\delta \subseteq \mathcal{H}^*$.

On the other hand, let σ be a congruence on S such that $\sigma \subseteq \mathcal{H}^*$ and $(a, b) \in \sigma$. Then $(a, b) \in \mathcal{H}^*$ and so

$$v_a\mathcal{L}^*a\mathcal{L}^*b\mathcal{L}^*v_b, \quad u_a\mathcal{R}^*a\mathcal{R}^*b\mathcal{R}^*u_b$$

by Lemma 2.6. Since I^S is a left normal band and Λ^S is a right normal band, we have $\overline{a}^+\mathcal{L}u_a = u_b\mathcal{L}\overline{b}^+$ and $\overline{a}^*\mathcal{R}v_a = v_b\mathcal{R}\overline{b}^*$ by (2.1), and so $\overline{a}^+ = \overline{b}^+$ and $\overline{a}^* = \overline{b}^*$. Since $a\sigma b$, it follows that $\overline{a} = \overline{a}^+a\overline{a}^*\sigma\overline{b}^+b\overline{b}^* = \overline{b}$ by (2.2). Let

$$x \in \mathrm{dom}\rho_a = \mathrm{dom}\rho_b = u_aE^\circ = u_bE^\circ.$$

Observe that $a(x\rho_a) = xa$, $b(x\rho_b) = xb$ and $a\sigma b$, it follows that $a(x\rho_a) = xa\ \sigma\ xb = b(x\rho_b)$ and so $\overline{a}^+a(x\rho_a)\sigma\overline{b}^+b(x\rho_b)$. Because

$$x\rho_a, x\rho_b \in E^\circ v_a = E^\circ v_b = \overline{a}^*E^\circ v_a = \overline{b}^*E^\circ v_b, \overline{a}\sigma\overline{b},$$

we have

$$\overline{a}(x\rho_a) = \overline{a}^+a\overline{a}^*(x\rho_a) = \overline{a}^+a(x\rho_a)\sigma\overline{b}^+b(x\rho_b) = \overline{b}^+b\overline{b}^*(x\rho_b) = \overline{b}(x\rho_b)\sigma\overline{a}(x\rho_b),$$

which implies that $\overline{a}(x\rho_a)\sigma\overline{a}(x\rho_b)$ and so $\overline{a}(x\rho_a)\mathcal{L}^*\overline{a}(x\rho_b)$ by $\sigma \subseteq \mathcal{H}^*$. Since $\overline{a}\mathcal{L}^*\overline{a}^*$, we have

$$x\rho_a = \overline{a}^*(x\rho_a)\mathcal{L}\overline{a}^*(x\rho_b) = x\rho_b.$$

Observe that Λ^S is a right normal band, it follows that $x\rho_a = x\rho_b$. This implies that $\rho_a = \rho_b$. Thus, δ is the largest congruence contained in $\mathcal{H}^*$ on S. That is to say, $\delta = \mu_S$. □

Theorem 4.4 *Define* $\rho : S \to T = T_{(I^S,\Lambda^S,E(S^\circ),P^S)}$, $a \mapsto \rho_a$. *Then* ρ *is a homomorphism whose kernel is* μ_S. *Moreover,* ρ *satisfies the following conditions*:

(1) $\rho|_{I^S}$ *(resp.* $\rho|_{\Lambda^S}$*) is an isomorphism from* I^S *onto* I^T *(resp.* Λ^T*).*
(2) $S^\circ\rho \subseteq T_{E^\circ}$ *and* $\rho|_{E^\circ}$ *is an isomorphism from* E° *onto* $E(T_{E^\circ})$.
(3) $\rho|_{E(S)}$ *is a bijection from* $E(S)$ *onto* $E(T)$.

Proof The first part follows from Lemmas 4.1, 4.2 and 4.3.

(1) By Corollary 3.12, $I^T = \{\pi_{e^\circ,e} | e \in I^S\}$. We first show that $\rho_e = \pi_{e^\circ,e}$ for all $e \in I^S$. In fact, since $e = ee^\circ e^\circ$ and $e\mathcal{L}e^\circ\mathcal{R}e^\circ \in E^\circ$, we have $u_e = e$, $v_e = e^\circ$ by (2.1), and so $\rho_e \in T_{u_e,v_e} = T_{e,e^\circ} \ni \pi_{e^\circ,e}$. Let $x \in \mathrm{dom}\rho_e = \mathrm{dom}\pi_{e^\circ,e} = eE^\circ$. Then we have $e(x\rho_e) = xe$, $x\rho_e \in \mathrm{ran}\rho_e = E^\circ e^\circ$ and so

$$x\rho_e = e^\circ(x\rho_e) = e^\circ e(x\rho_e) = e^\circ xe.$$

Since I^S is a left normal band, it follows that

$$e^\circ xe = e^\circ ex = e^\circ x = e^\circ x^\circ = x^\circ e^\circ = x\pi_{e^\circ,e}$$

by Lemma 3.3 and Proposition 3.5. This yields that $\rho_e = \pi_{e^\circ,e}$ for all $e \in I^S$. By the above discussions and Lemma 4.2, $\rho|_{I^S}$ is a homomorphism from I^S onto I^T. Since the kernel of ρ is μ_S, it follows that $\rho|_{I^S}$ is injective. The result for $\rho|_{\Lambda^S}$ can be proved by symmetry.

(2) If $a \in S^\circ$, then $u_a, v_a \in E^\circ$. This implies that $\rho_a \in T_{u_a,v_a} \subseteq T_{E^\circ}$ by Lemma 4.1 and Theorem 3.8. The remaining result follows from item (1) by considering the restriction of $\rho|_{I^S}$ to E°.

(3) Since ρ is a homomorphism whose kernel is μ_S, we have $E(S)\rho \subseteq E(T)$ and $\rho|_{E(S)}$ is injective. Let $\alpha \in T_{e,f}$, $e \in I^S$, $f \in \Lambda^S$ and $\alpha \in E(T)$. Then $\alpha \circ \alpha = \alpha$, whence $\mathrm{dom}(\alpha \circ \alpha) = \mathrm{dom}\alpha$. This implies that $(P^S_{f,e} f)\alpha^{-1} = e$, which gives $fef = P^S_{f,e} f = e\alpha = f$ by Lemma 3.6(1). Thus, $(ef)^2 = e(fef) = ef \in E(S)$. Moreover, by Corollaries 2.7, 3.12, the fact that $\rho_e = \pi_{e^\circ,e}$, $\rho_f = \pi_{f,f^\circ}$ and Lemma 4.2, we have

$$\alpha = u_\alpha \circ v_\alpha = \pi_{e^\circ,e} \circ \pi_{f,f^\circ} = \rho_e \circ \rho_f = \rho_{ef},$$

whence $\rho|_{E(S)}$ is also surjective. □

Combining Theorem 3.15 and Theorem 4.4, we obtain the main result of this paper.

Theorem 4.5 *Let (I, Λ, E°, P) be a given admissible quadruple. Then a semigroup S is a fundamental abundant semigroup (resp. fundamental regular semigroup) having a multiplicative ample transversal (resp. multiplicative inverse transversal) whose admissible quadruple is isomorphic to (I, Λ, E°, P) if and only if it is isomorphic to a full subsemigroup (resp. full regular subsemigroup) of $T_{(I,\Lambda,E^\circ,P)}$.*

5 Properties of some special admissible quadruples

In this section, we consider some special admissible quadruples. An admissible quadruple (I, Λ, E°, P) is called *rigid* if $|T_{e,f}| = 1$ for all $(e, f) \in \mathcal{U}$.

Proposition 5.1 *An admissible quadruple (I, Λ, E°, P) is rigid if and only if $\mathcal{H}^*$ is a congruence on every abundant semigroup with a multiplicative ample transversal whose admissible quadruple is isomorphic to (I, Λ, E°, P).*

Proof Let S be an abundant semigroup with a multiplicative ample transversal S° whose admissible quadruple $(I^S, \Lambda^S, E(S^\circ), P^S)$ is rigid. By Theorem 4.4,

$$\rho : S \to T_{(I^S,\Lambda^S,E(S^\circ),P^S)}, \quad a \mapsto \rho_a$$

is a homomorphism whose kernel is μ_S. If $(a, b) \in \mathcal{H}^*$, then $u_a = u_b$ and $v_a = v_b$ by Lemma 2.2. This implies that $\rho_a, \rho_b \in T_{u_a,v_a} = T_{u_b,v_b}$ and $\rho_a = \rho_b$ since $(I^S, \Lambda^S, E(S^\circ), P^S)$ is rigid. This gives $\mathcal{H}^* \subseteq \mu_S$ and so $\mathcal{H}^* = \mu_S$ is a congruence on S.

On the other hand, by Theorem 3.8, Lemma 2.9 and Corollary 3.13, $T_{(I,\Lambda,E^\circ,P)}$ is an abundant semigroup with a multiplicative ample transversal whose admissible quadruple is isomorphic to (I, Λ, E°, P). If the relation $\mathcal{H}^*$ on $T_{(I,\Lambda,E^\circ,P)}$ is a congruence, then we have $\mu_S = \mathcal{H}^*$. Observe that $T_{(I,\Lambda,E^\circ,P)}$ is fundamental by Theorem 3.15, it follows that $\mathcal{H}^*$ is the identity congruence on $T_{(I,\Lambda,E^\circ,P)}$. This gives $|T_{e,f}| = 1$ for all $(e, f) \in \mathcal{U}$ by Corollary 3.12(3). That is, (I, Λ, E°, P) is rigid. □

We call an admissible quadruple (I, Λ, E°, P) *uniform* if $(e, f) \in \mathcal{U}$ for all $e \in I$ and $f \in \Lambda$. On uniform admissible quadruples, we have the following.

Proposition 5.2 *Let S be an abundant semigroup with a multiplicative ample transversal S°. If S is $\mathcal{D}^*$-simple (i.e. any two elements in S are $\mathcal{D}^*$-related), then its admissible quadruple is uniform. On the other hand, if an admissible quadruple (I, Λ, E°, P) is uniform, then $T_{(I,\Lambda,E^\circ,P)}$ is $\mathcal{D}$-simple (and also $\mathcal{D}^*$-simple since $T_{(I,\Lambda,E^\circ,P)}$ is regular).*

Proof Let $u_a \in I^S$ and $v_b \in \Lambda^S$ where $a, b \in S$. If S is $\mathcal{D}^*$-simple, then $u_a \mathcal{D}^* v_b$ and so there exist $c_1, c_2, \ldots, c_n \in S$ such that $u_a \mathcal{R}^* c_1 \mathcal{L}^* c_2 \mathcal{R}^* c_3 \cdots c_n \mathcal{R}^* v_b$. By Lemma 2.2 and Lemma 2.6(3), we have

$$u_a = u_{u_a} = u_{c_1}, \quad v_{c_1} = v_{c_2}, \quad u_{c_2} = u_{c_3}, \ldots, v_{c_{n-1}} = v_{c_n}, \quad u_{c_n} = u_{v_b}, \quad v_{v_b} = v_b.$$

This implies that $\rho_{c_1} \rho_{c_2}^{-1} \rho_{c_3} \cdots \rho_{c_n}^{-1} \rho_{v_b}$ is an isomorphism from $u_a E^\circ$ onto $E^\circ v_b$ by Lemma 4.1 and so $(u_a, v_b) \in \mathcal{U}$. Thus, the admissible quadruple of S is uniform.

On the other hand, let (I, Λ, E°, P) be uniform and $\alpha \in T_{e,f}$ and $\beta \in T_{g,h}$ be two elements in $T_{(I,\Lambda,E^\circ,P)}$. Since (I, Λ, E°, P) is uniform, we can take $\gamma \in T_{e,h}$. Then we have $\alpha \mathcal{R} \gamma \mathcal{L} \beta$ and so $\alpha \mathcal{D} \beta$ by Corollary 3.12(3). Thus $T_{(I,\Lambda,E^\circ,P)}$ is $\mathcal{D}$-simple. □

Finally, an admissible quadruple (I, Λ, E°, P) is called *left anti-uniform* (resp. *right anti-uniform*) if $(e, f) \in \mathcal{U}$ implies that $f = e^\circ$ (resp. $e = f^\circ$) for all $e \in I$ and $f \in \Lambda$. To give some properties of left anti-uniform and right anti-uniform admissible quadruples, we need some notions and facts. Recall the an abundant semigroup S is called *superabundant* if every $\mathcal{H}^*$-class of S contains an idempotent. We call a superabundant semigroup S is a *left normal* (resp. *right normal*) superabundant semigroup if $E(S)$ forms a left normal band (resp. right normal band).

Lemma 5.3 *Let S be an abundant semigroup with a multiplicative ample transversal S° and $a \in S$. If S is a left normal (resp. right normal) superabundant semigroup, then $a \mathcal{H}^* u_a$ (resp. $a \mathcal{H}^* v_a$).*

Proof Since $E(S)$ is a left normal band and $v_a \mathcal{R} \overline{a}^*$ by (2.1), we have $v_a = \overline{a}^* v_a = \overline{a}^* v_a \overline{a}^* = \overline{a}^*$. On the other hand, since S is superabundant, it follows that $a \mathcal{H}^* e$ for some $e \in E(S)$. Since $\overline{a}^+ \mathcal{L} u_a \mathcal{R}^* a \mathcal{L}^* v_a = \overline{a}^*$ by (2.1) and Lemma 2.6(2), we have $\overline{a}^+ \mathcal{L} u_a \mathcal{R} e \mathcal{L} v_a = \overline{a}^*$. This implies that $\overline{a}^* \mathcal{R} \overline{a}^* u_a \mathcal{L} u_a \mathcal{L} \overline{a}^+$. Observe that $\overline{a}^* u_a \in E(S^\circ) I^S \subseteq I^S \subseteq E(S)$ by Lemma 2.8, it follows that $\overline{a}^* \mathcal{L} \overline{a}^+ \overline{a}^* \mathcal{R} \overline{a}^+$ whence $\overline{a}^+ = \overline{a}^*$. Thus, $u_a \mathcal{R}^* a \mathcal{L}^* v_a = \overline{a}^* = \overline{a}^+ \mathcal{L} u_a$, which gives $a \mathcal{H}^* u_a$. Dually, we can prove that $a \mathcal{H}^* v_a$ if S is a right normal superabundant semigroup. □

Proposition 5.4 *An admissible quadruple (I, Λ, E°, P) is left anti-uniform (resp. right anti-uniform) if and only if every abundant semigroup with a multiplicative ample transversal whose admissible quadruple is isomorphic to (I, Λ, E°, P) is a left normal (resp. right normal) superabundant semigroup.*

Proof Let (I, Λ, E°, P) be left anti-uniform and S be an abundant semigroup with a multiplicative ample transversal whose admissible quadruple is isomorphic to (I, Λ, E°, P). Then $(I^S, \Lambda^S, E(S^\circ), P^S)$ is left anti-uniform. By Lemma 4.1, for every $a \in S$, we have $(u_a, v_a) \in \mathcal{U}$ and $v_a = u_a^\circ = \overline{a}^+$ since $u_a \mathcal{L} \overline{a}^+$ by (2.1). This implies that $u_a \mathcal{R}^* a \mathcal{L}^* v_a = u_a^\circ = \overline{a}^+ \mathcal{L} u_a$ by (2.1) and Lemma 2.6(1), and hence $a \mathcal{H}^* u_a$. Therefore S is superabundant. Moreover, for $e \in E(S)$, we have $\overline{e} \in E(S^\circ)$ by Lemma 2.6(2), and so $v_e = \overline{e}^+ = \overline{e}$. This implies that $e = u_e \overline{e} v_e = u_e \overline{e} \in I^S$. Thus $E(S) = I^S$ is a left normal band by Lemma 2.8.

Conversely, let (I, Λ, E°, P) be not left anti-uniform. Then there exist $e \in I$ and $f \in \Lambda$ such that $(e, f) \in \mathcal{U}$ and $f \neq e^\circ$. Thus there exists $\alpha \in T_{e,f}$ such that $u_\alpha = \alpha \circ \alpha^\circ = \pi_{e^\circ,e} \in T_{e,e^\circ}$ in $T_{(I,\Lambda,E^\circ,P)}$ by Corollary 3.12(2). Since $\alpha \in T_{e,f}$, $u_\alpha \in T_{e,e^\circ}$ and $f \neq e^\circ$, it follows that α and u_α are not $\mathcal{L}^*$-related in $T_{(I,\Lambda,E^\circ,P)}$ by Corollary 3.12(3). In view of Lemma 5.3, $T_{(I,\Lambda,E^\circ,P)}$ is not a left normal superabundant semigroup. The case for right anti-uniform admissible quadruples can be proved by symmetry. □

Acknowledgments The author expresses his profound gratitude to the referee for the valuable comments, which improve greatly the presentation of this article. In particular, the author shortens the original proof of Theorem 4.4 according to the referee's suggestions. As the referee has pointed out, ample semigroups are generalized to restriction semigroups now, and a generalized Munn representation for restriction semigroups is also explored in the literature (see [11,15,18]). It is likely that the results of this paper could be extended by similar methods to some classes of E-semiabundant semigroups. Thanks also go to Professor Maria B. Szendrei for the timely communications. This paper is supported jointly by a Nature Science Foundation of Yunnan Province (2012FB139) and a Nature Science Foundation of China (11301470).

References

1. J. Al-Bar, J. Renshaw, On adequate transversals. Commun. Algebra **37**(7), 2309–2324 (2009)
2. J. Al-Bar, J. Renshaw, Adequate transversals of quasi-adequate semigroups. Commun. Algebra **40**(3), 905–930 (2012)
3. T.S. Blyth, R. McFadden, Regular semigroups with a multiplicative inverse transversal. Proc. R. Soc. Edinb. Sect. A **92**(3–4), 253–270 (1982)
4. T. S. Blyth, Inverse transversals–a guided tour. Semigroups (Braga, 1999), (World Sci. Publ., River Edge, NJ, 2000), 26–43
5. J.F. Chen, On regular semigroups with orthodox transversals. Commun. Algebra **27**(9), 4275–4288 (1999)
6. J.F. Chen, Abundant semigroups with adequate transversals. Semigroup Forum **60**(1), 67–79 (2000)
7. A. El-Qallali, J. B. Fountain, Idempotent-connected abundant semigroups. Proc. Roy. Soc. Edinburgh Sect. A **91**(1-2), 79–90 (1981/1982)
8. A. El-Qallali, Abundant semigroups with a multiplicative type A transversal. Semigroup Forum **47**(3), 327–340 (1993)
9. A. El-Qallali, J.B. Fountain, V. Gould, Fundamental representations for classes of semigroups containing a band of idempotents. Commun. Algebra **36**(8), 2998–3031 (2008)
10. J.B. Fountain, Adequate semigroups. Proc. Edinb. Math. Soc. **22**(2), 113–125 (1979)
11. J.B. Fountain, G.M.S. Gomes, V. Gould, A Munn type representation for a class of E-semiadequate semigroups. J. Algebra **218**(2), 693–714 (1999)
12. G.M.S. Gomes, V. Gould, Fundamental Ehresmann semigroups. Semigroup Forum **63**(1), 11–33 (2001)
13. G.M.S. Gomes, V. Gould, Fundamental semigroups having a band of idempotents. Semigroup Forum **77**(2), 279–299 (2008)
14. V. Gould, Graph expansions of right cancellative monoids. Int. J. Algebra Comput. **6**(6), 713–733 (1996)
15. V. Gould, Restriction and Ehresmann semigroups. In: Proceedings of the International Conference on Algebra 2010, (World Science Publication, Hackensack, 2012), pp. 265–288
16. X.J. Guo, Abundant semigroups with a multiplicative adequate transversal. Acta Math. Sin. (Engl. Ser.) **18**(2), 229–244 (2002)
17. T.E. Hall, Orthodox semigroups. Pacific J. Math. **39**(3), 677–686 (1971)
18. C. Hollings, From right PP monoids to restriction semigroups: a survey. Eur. J. Pure Appl. Math. **2**(1), 21–57 (2009)
19. W.D. Munn, Fundamental inverse semigroups. Quart. J. Math. Oxf. Ser. **2**(21), 157–170 (1970)
20. X.L. Tang, Regular semigroups with inverse transversals. Semigroup Forum **55**(1), 24–32 (1997)
21. Y.H. Wang, Hall-type representations for generalised orthogroups. Semigroup Forum **89**(3), 518–545 (2014)
22. R.H. Zhang, S.F. Wang, On generalized adequate transversals. Commun. Algebra **34**(7), 2419–2436 (2006)
23. R.H. Zhang, S.F. Wang, On generalized inverse transversals. Acta Math. Sin. (Engl. Ser.) **24**(7), 1193–1204 (2008)