Period Math Hung (2016) 72:151–157
DOI 10.1007/s10998-016-0115-6

Some Ricci-flat (α, β)-metrics

Esra Sengelen Sevim[1] · **Semail Ülgen**[2]

Published online: 14 March 2016

Abstract In this paper, we study a special class of Finsler metrics, (α, β)-metrics, defined by $F = \alpha\phi(\beta/\alpha)$, where α is a Riemannian metric and β is a 1-form. We find an equation that characterizes Ricci-flat (α, β)-metrics under the condition that the length of β with respect to α is constant.

Keywords Ricci curvature · Einstein metrics · (α, β)-metrics

1 Introduction

Riemannian metrics on a manifold are quadratic metrics, while Finsler metrics are those without restriction on the quadratic property. The Riemannian curvature in Riemannian geometry can be extended to Finsler metrics as a family of linear transformations on the tangent spaces. The Ricci curvature is the trace of the Riemann curvature. It is a natural problem to study Finsler metrics with isotropic Ricci curvature $Ric = Ric(x, y)$ and

$$Ric = (n-1)\tau F^2 \tag{1.1}$$

where $\tau = \tau(x)$ is a scalar function on the n-dimensional manifold and $F(x, y)$ is a Finsler metric. Such metrics are called Einstein Finsler metrics.

In this paper, we consider Einstein metrics defined by a Riemannian metric α and 1-form β in the following form:

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \tag{1.2}$$

✉ Semail Ülgen
sulgen@antalya.edu.tr

Esra Sengelen Sevim
esra.sengelen@bilgi.edu.tr

1 Department of Mathematical Sciences, İstanbul Bilgi University, Eski Silahtaraga Elektrik Santrali, Kazim Karabekir Cad. No: 2/13, Eyüp, 34060 Istanbul, Turkey

2 Antalya International University, Üniversite Cad. No: 2 07190, Dösemealti, Antalya, Turkey

where $\phi = \phi(s)$ is a positive smooth function. Finsler metrics defined in (1.2) are called (α, β)-metrics.

The simplest (α, β)-metrics are Randers metrics also defined by $F = \alpha + \beta$. In [1], Bao–Robles find equations on α and β that characterize Randers metrics of constant Ricci curvature. There are many Randers metrics of constant Ricci curvature. Thus one just needs to focus on Ricci-flat (α, β)-metrics. In [4] and [5], the authors obtained equations on α, β and ϕ that characterize Ricci-flat (α, β)-metrics of Douglas type. In [6], the authors obtained equations on α, β and ϕ that characterize Ricci-flat (α, β)-metrics which is not of Douglas type. In this paper, we show that there are some more Ricci-flat (α, β)-metrics.

In this paper, we prove the following theorem.

Theorem 1.1 *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be an (α, β)-metric on an n-dimensional manifold M where $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric, $\beta = b_i y^i$ is a 1-form and $\phi = \phi(s)$ is a positive C^∞ function. Suppose that α, β and ϕ satisfy the following conditions:*

(a) ${}^{\alpha}\mathbf{Ric} = (n-1)(c_1\alpha^2 + c_2\beta^2)\tau$,
(b) $r_{ij} = 0$,
(c) $s_j = 0$,
(d) $t_{ij} = (c_1 + c_2 b^2)(b_i b_j - a_{ij} b^2)\tau$,
(e) ϕ *satisfies*

$$0=(c_1+c_2s^2)+(c_1+c_2b^2)\left\{2\frac{(s^2-b^2)}{(n-1)}\left(Q'-Q^2+sQQ'\right)+Q^2b^2+2Qs\right\}, \quad (1.3)$$

where $b := \sqrt{a^{ij} b_i b_j}$, c_1 and c_2 are constants, $\tau := \tau(x)$ is a scalar function, $t_{ij} := s_{im} s^m_j$ and

$$Q := \frac{\phi'}{\phi - s\phi'}.$$

Then F is Ricci-flat.

The equation (1.3) is an ordinary differential equation. It is of first order in Q and second order in ϕ. According to the ODE theory, the local solution of (1.3) exists nearby $s = 0$ for any given initial conditions. But we are unable to express it in terms of elementary functions and we are unable to show that the solution is defined on an interval containing $[-b, b]$. Thus the (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ defined by ϕ might be singular. We can give the following example taking $c_2 = 0$ in Theorem 1.1, then α, β satisfies Theorem 1.1 (a)–(e). Then for any $\phi = \phi(s)$ satisfying (1.3), we obtain a (possibly singular) Ricci-flat (α, β)-metrics.

Example 1.2 Let $F = \alpha + \beta$ be the family of Randers metrics on S^3 constructed in [2] (see also [7]). It is shown that $r_{ij} = 0$ and $s_j = 0$. Thus for any C^∞ positive function $\phi = \phi(s)$ satisfying (2.2), the (α, β)-metric $F = \alpha\phi(\beta/\alpha)$ has vanishing S-curvature.

2 Preliminaries

A Finsler metric on a manifold M is a nonnegative scalar function $F = F(x, y)$ on the tangent bundle TM, where x is a point in M and $y \in T_x M$ is a tangent vector at x. In local coordinates, the geodesics of a Finsler metric $F = F(x, y)$ are characterized by

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i := \frac{1}{4} g^{il}(x, y) \Big\{ [F^2]_{x^k y^l}(x, y) y^k - [F^2]_{x^l}(x, y) \Big\}, \tag{2.1}$$

and $g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$. The local functions G^i on TM define a global vector field

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

The vector field G is called the *spray* of F and the local functions $G^i = G^i(x, y)$ are called *spray coefficients* of F.

For any $x \in M$ and $y \in T_x M \backslash \{0\}$, the Riemann curvature $\mathbf{R}_y : T_x M \to T_x M$ is defined by $\mathbf{R}_y(u) = R^i{}_k(x, y) u^k \frac{\partial}{\partial x^i}|_x$, where

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m} \frac{\partial G^m}{\partial y^k}.$$

Then the Ricci curvature is given by

$$\mathbf{Ric} = 2\frac{\partial G^i}{\partial x^i} - \frac{\partial^2 G^i}{\partial x^m \partial y^i} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^i} - \frac{\partial G^i}{\partial y^m} \frac{\partial G^m}{\partial y^i}.$$

An (α, β)-metric on a manifold M is a scalar function on TM defined by

$$F := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a C^∞ function on $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form with $b(x) := \|\beta_x\|_\alpha < b_0$. It can be shown that for any Riemannian metric α and any 1-form β on M with $b(x) < b_0$ the function $F = \alpha\phi(\beta/\alpha)$ is a (positive definite) Finsler metric if and only if ϕ satisfies

$$\phi(s) > 0, \qquad \phi(s) - s\phi'(s) + (\rho^2 - s^2)\phi''(s) > 0, \qquad (|s| \le \rho < b_0). \tag{2.2}$$

Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_j := b^i r_{ij}, \qquad s_j := b^i s_{ij},$$

where "|" denotes the covariant derivative with respect to the Levi-Civita connection of α. By (2.1), the spray coefficients G^i of F are given by the following Lemma.

Lemma 2.1 [3] *For an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, the spray coefficients of F are given by*

$$G^i = {}^\alpha G^i + \alpha Q s^i_0 + \Theta \{r_{00} - 2Q\alpha s_0\} \frac{y^i}{\alpha} + \Psi \{r_{00} - 2Q\alpha s_0\} b^i, \tag{2.3}$$

where ${}^{\alpha}G^i$ are the spray coefficients of α,

$$Q := \frac{\phi'}{\phi - s\phi'},$$
$$\Theta := \frac{Q - sQ'}{2\Delta},$$
$$\Psi := \frac{Q'}{2\Delta},$$
$$\Delta := 1 + sQ + (b^2 - s^2)Q'$$

and $s^i{}_j := a^{ik}s_{kj}$, $s_{ij} := a_{ih}s^h_j$. *The index "0" means contracting with* y, *for example,* $s^i{}_0 := s^i{}_j y^j$, $s_0 := s_i y^i$, $s_{ij}y^j := s_{i0}$, $s_{ij}y^i := s_{0i}$, $r_{00} := r_{ij}y^i y^j$.

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. Throughout this section, we assume that the dimension is greater than two. First we give the following Lemma.

Lemma 3.1 *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an n-dimensional manifold M, $n \geq 3$. Suppose that $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$ satisfy the conditions of Theorem 1.1 (b) and (c), then the following equations are satisfied:*

$$s^m_{0|m} = (n-1)(c_1 + c_2 b^2)\tau\beta \tag{3.1}$$

where $\tau = \tau(x)$ is a scalar function and c_1 and c_2 are constants.

Proof By Ricci identities, we have

$$b_{i|j|k} - b_{i|k|j} = b^{m}\,{}^{\alpha}\mathrm{R}_{imjk},$$
$$-b_{k|i|j} + b_{k|j|i} = -b^{m}\,{}^{\alpha}\mathrm{R}_{kmij},$$
$$b_{j|k|i} - b_{j|i|k} = b^{m}\,{}^{\alpha}\mathrm{R}_{jmki}.$$

On the other hand,

$$b_{i|k|j} + b_{k|i|j} = 2r_{ik|j},$$
$$-b_{k|j|i} - b_{j|k|i} = -2r_{kj|i}.$$

Adding all the equations above, we get

$$s_{ij|k} = \frac{1}{2}(b_{i|j|k} + b_{j|i|k}) = -b^{m}\,{}^{\alpha}\mathrm{R}_{kmij} + r_{ik|j} - r_{kj|i}. \tag{3.2}$$

The condition (b) in Theorem (1.1) helps one to rewrite the above equation as follows:

$$s_{ij|k} = \frac{1}{2}(b_{i|j|k} + b_{j|i|k}) = -b^{m}\,{}^{\alpha}\mathrm{R}_{kmij}. \tag{3.3}$$

Hence,

$$s^m_{0|m} = b^{m}\,{}^{\alpha}\mathtt{Ric}_{m0} + r^m_{m|0} - r^m_{0|m}. \tag{3.4}$$

The condition (b) in Theorem 1.1, (3.4) implies the following:

$$s^m_{0|m} = b^{m}\,{}^{\alpha}\mathtt{Ric}_{m0}. \tag{3.5}$$

The condition (a) in Theorem (1.1) implies the following

$$
\begin{aligned}
{}^{\alpha}\mathtt{Ric}_{ij} &= \Big(\frac{1}{2}{}^{\alpha}\mathbf{Ric}\Big)_{y^i y^j} \\
&= (n-1)\Big(c_1 a_{ij} + c_2 b_i b_j\Big)\tau,
\end{aligned} \tag{3.6}
$$

and we obtain:

$$
b^{m\,\alpha}\mathtt{Ric}_{m0} = (n-1)(c_1 + c_2 b^2)\tau\beta. \tag{3.7}
$$

Hence, the equation in (3.1) follows from equations (3.5) and (3.7). □

Next, we compute the Ricci curvature of the (α, β)-metric under the conditions $(a)-(e)$ of Theorem 1.1. By Lemma 2.1, the spray coefficients of F can be written as

$$
G^i = {}^{\alpha}G + T^i, \tag{3.8}
$$

where

$$
T^i = \alpha Q s^i_0. \tag{3.9}
$$

It is well known [3] that the curvature tensor can be written as

$$
R^i_k =^{\alpha} R^i_k + H^i_k, \tag{3.10}
$$

where

$$
H^i_k := 2T^i_{|k} - T^i_{|j\cdot k} y^j + 2T^j T^i_{\cdot j\cdot k} - T^i_{\cdot j} T^j_{\cdot k}, \tag{3.11}
$$

and $''.''$ and $''|''$ mean vertical covariant derivative and horizontal covariant derivative with respect to α, respectively. Then

$$
\mathbf{Ric} = {}^{\alpha}\mathbf{Ric} + H^i_i, \tag{3.12}
$$

where ${}^{\alpha}\mathbf{Ric}$ denotes the Ricci curvature of α and

$$
H^i_i := 2T^i_{|i} - T^i_{|j\cdot i} y^j + 2T^j T^i_{\cdot j\cdot i} - T^i_{\cdot j} T^j_{\cdot i}. \tag{3.13}
$$

To compute the Ricci curvature under the conditions $r_{ij} = 0$ and $s_j = 0$, we need:

$$
\begin{aligned}
&b_{i|j} = s_{ij}, \quad y_i s^i_0 = 0, \quad y_i s^i_{0|j} = 0, \quad s_{ij} y^i y^j = 0, \\
&y_i s^i_{0|j} = 0, \quad b_i s^i_0 = 0, \quad b_i s^i_j = 0, \quad b_i s^i_{0|j} = -s_{ij} s^i_0.
\end{aligned} \tag{3.14}
$$

We also easily get

$$
\begin{aligned}
&s_{\cdot i} = \frac{b_i}{\alpha} - s\frac{y_i}{\alpha^2}, \quad s_{\cdot i} b^i = \frac{1}{\alpha}(b^2 - s^2), \\
&s_{\cdot i} y^i = 0, \quad s_{\cdot i} s^i_0 = 0, \quad s_{\cdot i} s^i_{0|j} = -\frac{s_{ij} s^i_0}{\alpha}.
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
&s_{\cdot j\cdot i} = -\frac{b_j y_i}{\alpha^3} - \frac{b_i y_j}{\alpha^3} + 3s\frac{y_i y_j}{\alpha^4} - s\frac{a_{ji}}{\alpha^2}, \\
&s_{\cdot j\cdot i} s^i_0 = -\frac{s}{\alpha^2} s_{j0}, \quad s_{\cdot j\cdot i} s^i_0 s^j_0 = -\frac{s}{\alpha^2} s_{j0} s^j_0.
\end{aligned} \tag{3.16}
$$

$$
\begin{aligned}
&s_{|i} = \frac{s_{0i}}{\alpha}, \quad s_{|i} y^i = 0, \quad s_{|i} b^i = 0, \quad s_{|j\cdot i} = \frac{s_{ij}}{\alpha} - \frac{s_{0j} y_i}{\alpha^3}, \\
&s_{|j\cdot i} s^i_0 = \frac{s_{ij} s^i_0}{\alpha}, \quad s_{\cdot j\cdot i} s^i_0 s^j_0 = -\frac{s}{\alpha^2} s_{j0} s^j_0.
\end{aligned} \tag{3.17}
$$

Using the above identities in (3.17), the equation $T^i_{|i} = \alpha Q' s_{|i} s^i_0 + \alpha Q s^i_{0|i}$ is simplified to

$$T^i_{|i} = Q' s_{0i} s^i_0 + \alpha Q s^i_{0|i}. \tag{3.18}$$

The identities in (3.16) and (3.17) are used in

$$\begin{aligned} T^i_{|j} &= \alpha Q' s_{|j} s^i_0 + \alpha Q s^i_{0|j}, \\ T^i_{|j\cdot i} &= \frac{y_i}{\alpha} Q' s_{|j} s^i_0 + \alpha Q'' s_{\cdot i} s_{|j} s^i_0 + \alpha Q' s_{|j\cdot i} s^i_0 + \alpha Q' s_{|j} s^i_i + \frac{y_i}{\alpha} Q s^i_{0|j} \\ &\quad + \alpha Q' s_{\cdot i} s^i_{0|j} + \alpha Q s^i_{i|j}, \end{aligned}$$

to get the following simplified equations:

$$\begin{aligned} T^i_{|j\cdot i} &= \alpha Q' \frac{s_{ij} s^i_0}{\alpha} - \alpha Q' \frac{s_{ij} s^i_0}{\alpha}, \\ T^i_{|j\cdot i} &= 0, \\ T^i_{|j\cdot i} y^j &= 0. \end{aligned} \tag{3.19}$$

We further have

$$\begin{aligned} T^i_{\cdot j} &= \frac{y_j}{\alpha} Q s^i_0 + \alpha Q' s_{\cdot j} s^i_0 + \alpha Q s^i_j, \\ T^i_{\cdot j\cdot i} &= \left(\frac{a_{ij}}{\alpha} - \frac{y_i y_j}{\alpha^3}\right) Q s^i_0 + \frac{y_j}{\alpha} Q' s_{\cdot i} s^i_0 + \frac{y_j}{\alpha} Q s^i_i + \frac{y_i}{\alpha} Q' s_{\cdot j} s^i_0 + \alpha Q'' s_{\cdot i} s_{\cdot j} s^i_0 \\ &\quad + \alpha Q' s_{\cdot j\cdot i} s^i_0 + \alpha Q' s_{\cdot j} s^i_i + \frac{y_i}{\alpha} Q s^i_j + \alpha Q' s_{\cdot i} s^i_j, \\ T^j T^i_{\cdot j\cdot i} &= \alpha Q s^j_0 \left\{ \left(\frac{a_{ij}}{\alpha} - \frac{y_i y_j}{\alpha^3}\right) Q s^i_0 + \frac{y_j}{\alpha} Q' s_{\cdot i} s^i_0 + \frac{y_j}{\alpha} Q s^i_i + \frac{y_i}{\alpha} Q' s_{\cdot j} s^i_0 + \alpha Q'' s_{\cdot i} s_{\cdot j} s^i_0 \right. \\ &\quad \left. + \alpha Q' s_{\cdot j\cdot i} s^i_0 + \alpha Q' s_{\cdot j} s^i_i + \frac{y_i}{\alpha} Q s^i_j + \alpha Q' s_{\cdot i} s^i_j \right\}. \end{aligned}$$

Using the identities in (3.14), (3.15), (3.16) and (3.17), we get:

$$T^j T^i_{\cdot j\cdot i} = Q^2 s_{i0} s^i_0 - s Q Q' s_{j0} s^j_0 + Q^2 s^j_0 s^0_j - s Q Q' s^j_0 s^0_j.$$

Using the fact that $s^0_j = -s_{j0}$, we obtain the following simple equation:

$$T^j T^i_{\cdot j\cdot i} = 0. \tag{3.20}$$

After multiplying the following equations:

$$\begin{aligned} T^i_{\cdot j} &= \frac{y_j}{\alpha} Q s^i_0 + \alpha Q' s_{\cdot j} s^i_0 + \alpha Q s^i_j, \\ T^j_{\cdot i} &= \frac{y_i}{\alpha} Q s^j_0 + \alpha Q' s_{\cdot i} s^j_0 + \alpha Q s^j_i, \end{aligned}$$

and then simplifying them we get

$$T^i_{\cdot j} T^j_{\cdot i} = 2Q^2 s_{0i} s^i_0 - 2s Q Q' s_{0i} s^i_0 + \alpha^2 Q^2 s^i_j s^j_i. \tag{3.21}$$

Plugging (3.18), (3.19), (3.20) and (3.21) into (3.13), we obtain

$$H^i_i = 2(Q' - Q^2 + s Q Q') t_{00} - \alpha^2 Q^2 t^m_m + 2\alpha Q s^i_{0|i}, \tag{3.22}$$

where $t_{00} = t_{ij} y^i y^j$, $t_{00} = (c_1 + c_2 b^2)(s^2 - b^2)\tau\alpha^2$. Hence H^i_i and also **Ric** are expressed as follows:

$$H^i_i = (c_1 + c_2 b^2)\tau\Big\{2(Q' - Q^2 + sQQ')(s^2 - b^2) + (n-1)Q^2 b^2 + 2(n-1)sQ\Big\}\alpha^2. \quad (3.23)$$

and

$$\mathbf{Ric} = {}^{\alpha}\mathbf{Ric} + \tau\alpha^2\Gamma \quad (3.24)$$

where

$$\Gamma := (c_1 + c_2 b^2)\Big\{2(Q' - Q^2 + sQQ')(s^2 - b^2) + (n-1)Q^2 b^2 + 2(n-1)sQ\Big\}. \quad (3.25)$$

Thus **Ric** $= 0$ if and only if

$$(n-1)(c_1 + c_2 s^2) + \Gamma = 0 \quad (3.26)$$

We can rewrite (3.26) as (1.3). □

Acknowledgments Authors are both supported in part by The Scientific and Technological Research Council of Turkey (TUBITAK), Grant (No. 113F311).

References

1. D. Bao, R.L. Bryant, S.-S. Chern, Z. Shen, *A Sampler of Finsler Geometry* (Cambridge University Press, Cambridge, 2004)
2. D. Bao, Z. Shen, Finsler metrics of constant flag curvature on the Lie group S^3. Lond. Math. Soc. **66**, 453–467 (2002)
3. S.S. Chern, Z. Shen, *Riemann Finsler Geometry* (World Scientific Publishers, Singapore, 2005)
4. X. Cheng, Z. Shen, Y. Tian, Ricci-flat Douglas (α, β)-metrics. Differ. Geom. Appl. **30**(1), 20–32 (2012)
5. E.S. Sevim, Z. Shen, L. Zhao, On a class of Ricci-flat Douglas metrics. Int. J. Math. **23**(6), 1250046 (2012)
6. E.S. Sevim, Z. Shen, L. Zhao, Some Ricci-flat Finsler metrics. Publ. Math. Debr. **83**(4), 617–623 (2013)
7. Z. Shen, *Lectures on Finsler Geometry* (World Scientific Publishers, Singapore, 2001)