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Approximation by modified Szász-Durrmeyer operators

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Abstract The main goal of this paper is to introduce Durrmeyer modifications for the generalized Szász–Mirakyan operators defined in (Aral et al., in Results Math 65:441–452, 2014). The construction of the new operators is based on a function ρ which is continuously differentiable ∞ times on $[0, \infty)$, such that $\rho(0) = 0$ and $\inf_{x\in[0,\infty)} \rho'(x) \geq 1$. Involving the weighted modulus of continuity constructed using the function ρ, approximation properties of the operators are explored: uniform convergence over unbounded intervals is established and a quantitative Voronovskaya theorem is given. Moreover, we obtain direct approximation properties of the operators in terms of the moduli of smoothness. Our results show that the new operators are sensitive to the rate of convergence to f, depending on the selection of ρ. For the particular case $\rho(x) = x$, the previous results for classical Szász-Durrmeyer operators are captured.

Keywords Szász-Durrmeyer operators · Weighted modulus of continuity · Quantitative Voronovskaya theorem

Mathematics Subject Classification 41A25 · 41A35 · 41A36

1 Introduction

Approximation theory has an important role in mathematical research, with a great potential for applications. Since Korovkin's famous theorem in 1950, the study of the linear methods of approximation given by sequences of positive and linear operators became a firmly entrenched part of approximation theory. Due to this fact, the well-known operators

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such as Bernstein, Szász, Baskakov etc. and their generalizations have been studied intensively. Recently Cárdenas-Morales et al. [8] introduced Bernstein-type operators defined for $f \in C[0,1]$ by $B_n\left(f \circ \tau^{-1}\right) \circ \tau$, B_n being the classical Bernstein operators and τ being any function that is continuously differentiable ∞ times on $[0,1]$, such that $\tau(0)=0$, $\tau(1)=1$ and $\tau'(x)>0$ for $x \in [0,1]$. They investigated the shape preserving and convergence properties, as well as the asymptotic behavior and saturation. A Durrmeyer type generalization of $B_n\left(f \circ \tau^{-1}\right) \circ \tau$ was also studied in [3]. The results of the aforementioned papers show that it is possible to obtain some improvements of the classical approximation by Bernstein and Bernstein-Durrmeyer operators in certain senses, simultaneously. Very recently Aral et al. [6] introduced similar modifications of the Szász–Mirakyan operators. Let us recall that construction.

Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\mathbb{R}^+$ be the positive real semi-axis $[0,\infty)$. Assume that ρ is any function satisfying the conditions:

(p_1) ρ is a continuously differentiable function on $\mathbb{R}^+$,
(p_2) $\rho(0)=0$, $\inf_{x\in[0,\infty)} \rho'(x) \geq 1$.

The generalized Szász–Mirakyan operators are defined by

$$S_n^{\rho}(f;x) = S_n\left(f \circ \rho^{-1}; \rho(x)\right) = \sum_{k=0}^{\infty} \left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right) \mathcal{P}_{n,\rho,k}(x), \tag{1.1}$$

where $\mathcal{P}_{n,\rho,k}(x) := \exp(-n\rho(x))(n\rho(x))^k/k!$. S_n are the classical Szász–Mirakyan operators and can be obtained from S_n^{ρ} as a particular case $\rho(x)=x$. The weighted uniform convergence of S_n^{ρ} to f, the rate of convergence with the aid of weighted modulus of continuity and some shape preserving properties of S_n^{ρ} were studied.

The aim of this article is to introduce Durrmeyer type modifications of the operators (1.1). A Durrmeyer type generalization of Szász–Mirakyan operators was introduced in [18]. Later on, further Durrmeyer type generalizations of the Szász–Mirakyan operators have been studied intensively. Among others, we refer the reader to [5,7,15] and references therein. The general integral modification of (1.1) to approximate Lebesgue integrable functions on $\mathbb{R}^+$ can be defined as

$$D_n^{\rho}(f;x) = n\sum_{k=0}^{\infty} \mathcal{P}_{n,\rho,k}(x) \int_0^{\infty} \left(f \circ \rho^{-1}\right)(t)\, p_{n,k}(t)\, dt, \tag{1.2}$$

where $n \in \mathbb{N}$, $p_{n,k}(t) = \exp(-nx)(nx)^k/k!$ and ρ is any function with the assumptions (ρ_1) and (ρ_2). The operators D_n^{ρ} are linear and positive and in the case of $\rho(x)=x$, the operators reduce to the classical ones. By considering the notion of ρ-convexity (a function $f \in C^k\left(\mathbb{R}^+\right)$ is said to be ρ-convex of order $k \in \mathbb{N}$ whenever $D_{\rho}^k f := D^k\left(f \circ \rho^{-1}\right) \circ \rho \geq 0$, where D is the differential operator). Note that the operators D_n^{ρ} map ρ-convex functions of order k onto ρ-convex functions of order k, so they are said to be ρ-convex of order k, which means that D_n^{ρ} transform the so called ρ-polynomials into polynomials of the same degree, that is, if we consider the set $\mathbb{P}_{\rho,k} := \left\{\rho^i : i=0,1,...k,\ k \in \mathbb{N}\right\}$, then $D_n^{\rho}\left(\mathbb{P}_{\rho,k}\right) \subset \mathbb{P}_{\rho,k}$.

We shall first show that the operators (1.2) are an approximation process for functions belonging to a weighted space, we shall prove uniform convergence of the operators and determine the degree of this uniform convergence as well. In the next section, we obtain local approximation properties. The last section is devoted to a Voronovskaya type theorem in quantitative form. Quantitative Voronovskaya theorems have been studied intensively in the last decade. This kind of results are very useful to describe the rate of point-wise convergence and error of approximation simultaneously. In the paper [19], a quantitative Voronovskaya type theorem was presented for Bernstein operators in terms of usual modulus of continuity and in terms of the least concave majorant of usual modulus of continuity in [13,14]. Since the modulus of continuity doesn't work on unbounded intervals, we obtain the corresponding theorem with the weighted modulus of continuity. Some quantitative form of Voronvskaya's theorem on bounded and unbounded intervals, we refer the readers to [1,2,4].

2 Preliminary results

In what follows, we give the moments and recurrence relation for the central moments of the operators without proofs since they are similar to the corresponding results for the Szász-Durrmeyer operators. Also they can be verified just by taking $\rho(x) = x$. We also recall the weighted modulus of continuity and its properties.

Lemma 2.1 *We have*

$$\mathcal{D}_n^\rho(1;x) = 1, \quad \mathcal{D}_n^\rho(\rho;x) = \rho(x) + \frac{1}{n}, \tag{2.1}$$

$$\mathcal{D}_n^\rho\left(\rho^2;x\right) = \rho^2(x) + \frac{4n\rho(x)+2}{n^2}, \tag{2.2}$$

$$\mathcal{D}_n^\rho\left(\rho^3;x\right) = \rho^3(x) + \frac{9n^2\rho^2(x)+18n\rho(x)+6}{n^3}. \tag{2.3}$$

Lemma 2.2 *If we define the central moment of degree m,*

$$\mu_{n,m}^\rho(x) = \mathcal{D}_n^\rho\left((\rho(t)-\rho(x))^m;x\right)$$

then we have

$$n\rho'(x)\,\mu_{n,m+1}^\rho(x) = \rho'(x)\left[(m+1)\mu_{n,m}^\rho(x) + 2m\rho(x)\,\mu_{n,m-1}^\rho(x)\right] + \rho(x)\,D\mu_{n,m}^\rho(x).$$

Also, using the above recurrence relation we get

$$\mu_{n,1}^\rho(x) = \frac{1}{n},$$

$$\mu_{n,2}^\rho(x) = \frac{2+2n\rho(x)}{n^2},$$

$$\mu_{n,3}^\rho(x) = \frac{6+12n\rho(x)}{n^3},$$

$$\mu_{n,4}^\rho(x) = \frac{24+72n\rho(x)+12n^2\rho^2(x)}{n^4},$$

$$\mu_{n,5}^\rho(x) = \frac{\left(180n^2+72n\right)\rho^2(x)+(408n)\rho(x)+120}{n^5},$$

$$\mu_{n,6}^\rho(x) = \frac{120n^3\rho^3(x)+\left(2160n^2+984n\right)\rho^2(x)+(2688n)\rho(x)+720}{n^6}.$$

Throughout the paper we shall consider the following class of functions. $C_B\left(\mathbb{R}^+\right)$ is the space of all real valued continuous and bounded functions f on $\mathbb{R}^+$. Let $\varphi(x) = 1 + \rho^2(x)$.

$$B_\varphi\left(\mathbb{R}^+\right) = \left\{f : \mathbb{R}^+ \to \mathbb{R},\ |f(x)| \le M_f \varphi(x), x \ge 0\right\},$$
$$C_\varphi\left(\mathbb{R}^+\right) = \left\{f \in B_\varphi\left(\mathbb{R}^+\right), f \text{ is continuous on } \mathbb{R}^+\right\},$$
$$C_\varphi^*\left(\mathbb{R}^+\right) = \left\{f \in C_\varphi\left(\mathbb{R}^+\right),\ \lim_{x\to\infty} f(x)/\varphi(x) = const.\right\},$$
$$U_\varphi\left(\mathbb{R}^+\right) = \left\{f \in C_\varphi\left(\mathbb{R}^+\right),\ f(x)/\varphi(x) \text{ is uniformly continuous on } \mathbb{R}^+\right\},$$

where M_f is a constant depending only on f. $C_B\left(\mathbb{R}^+\right)$ is the linear normed space with the norm $\|f\| = \sup_{x\in\mathbb{R}^+}|f(x)|$ and the other spaces are normed linear spaces with the norm $\|f\|_\varphi = \sup_{x\in\mathbb{R}^+}|f(x)|/\varphi(x)$.

The weighted modulus of continuity defined in [16] is given by

$$\omega_\rho(f;\delta) = \sup_{\substack{x,t\in\mathbb{R}^+ \\ |\rho(t)-\rho(x)|\le\delta}} \frac{|f(t)-f(x)|}{\varphi(t)+\varphi(x)}$$

for each $f \in C_\varphi\left(\mathbb{R}^+\right)$ and for every $\delta > 0$. We observe that $\omega_\rho(f;0) = 0$ for every $f \in C_\varphi\left(\mathbb{R}^+\right)$ and the function $\omega_\rho(f;\delta)$ is nonnegative and nondecreasing with respect to δ for $f \in C_\varphi\left(\mathbb{R}^+\right)$ and also $\lim_{\delta\to0}\omega_\rho(f;\delta) = 0$ for every $f \in U_\varphi\left(\mathbb{R}^+\right)$(For more details see [16]).

Let $\delta > 0$ and $\mathcal{W}_\infty^2 = \left\{g \in C_B[0,\infty); g', g'' \in C_B[0,\infty)\right\}$. The Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf\left\{\|f-g\| + \delta\|g\|_{\mathcal{W}_\infty^2}; g \in \mathcal{W}_\infty^2\right\},$$

where

$$\|f\|_{\mathcal{W}_\infty^2} := \|f\| + \left\|f'\right\| + \left\|f''\right\|.$$

It was shown in [9] that there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \le C\left\{\omega_2\left(f;\sqrt{\delta}\right) + \min(1,\delta)\|f\|\right\},$$

where the second order modulus of continuity is defined by

$$\omega_2\left(f,\sqrt{\delta}\right) = \sup_{0<h\le\sqrt{\delta}}\ \sup_{0\le x<\infty}|f(x+2h) - 2f(x+h) + f(x)|.$$

The usual modulus of continuity for $f \in C_B\left(\mathbb{R}^+\right)$ is given by

$$\omega(f,\delta) = \sup_{0<h\le\delta}\ \sup_{0\le x<\infty}|f(x+h) - f(x)|.$$

3 Uniform convergence of $\mathcal{D}_n^\rho$

In this section, we obtain the uniform convergence of the operators $\mathcal{D}_n^\rho$ in terms of the weighted Korovkin theorem [11,12] and we describe the rate of the corresponding uniform convergence. Foremost, we recall the weighted form of the Korovkin theorem.

Lemma 3.1 ([11]) *The positive linear operators L_n, $n \ge 1$, act from $C_\varphi\left(\mathbb{R}^+\right)$ to $B_\varphi\left(\mathbb{R}^+\right)$ if and only if the inequality*

$$|L_n(\varphi; x)| \le K_n \varphi(x),$$

holds, where K_n *is a positive constant depending on* n.

Theorem 3.2 ([11]) *Let the sequence of linear positive operators* (L_n), $n \ge 1$, *acting from* $C_\varphi(\mathbb{R}^+)$ *to* $B_\varphi(\mathbb{R}^+)$ *satisfy the following three conditions*

$$\lim_{n\to\infty} \left\| L_n \rho^\nu - \rho^\nu \right\|_\varphi = 0, \quad \nu = 0, 1, 2.$$

Then for any function $f \in C_\varphi^*(\mathbb{R}^+)$,

$$\lim_{n\to\infty} \| L_n f - f \|_\varphi = 0.$$

Therefore, we have the following result.

Theorem 3.3 *For each function* $f \in C_\varphi^*(\mathbb{R}^+)$

$$\lim_{n\to\infty} \left\| \mathcal{D}_n^\rho f - f \right\|_\varphi = 0.$$

Proof We first have to show that $\mathcal{D}_n^\rho : C_\varphi(\mathbb{R}^+) \to B_\varphi(\mathbb{R}^+)$. In fact, using (2.1)–(2.2) we have

$$\left|\mathcal{D}_n^\rho(\varphi; x)\right| = 1 + \rho^2(x) + \frac{4n\rho(x) + 2}{n^2}.$$

Since

$$\left|\mathcal{D}_n^\rho(\varphi; x)\right| \le \left(1 + \rho^2(x)\right)\left(\frac{n^2 + 4n + 2}{n^2}\right) \quad \text{whenever } \rho(x) \le 1,$$

and

$$\left|\mathcal{D}_n^\rho(\varphi; x)\right| \le \left(1 + \rho^2(x)\right)\left(\frac{n^2 + 4n}{n^2}\right) \quad \text{whenever } \rho(x) > 1,$$

we get

$$\left|\mathcal{D}_n^\rho(\varphi; x)\right| \le \left(1 + \rho^2(x)\right)\frac{2\left(n^2 + 4n + 2\right)}{n^2}$$

which verifies our assertion by Lemma 3.1. On the other hand, since

$$\begin{aligned} \left\|\mathcal{D}_n^\rho 1 - 1\right\|_\varphi &= 0, \quad \left\|\mathcal{D}_n^\rho \rho - \rho\right\|_\varphi = 1/n, \\ \left\|\mathcal{D}_n^\rho \rho^2 - \rho^2\right\|_\varphi &= \sup_{x\in\mathbb{R}^+} \left(4n\rho(x) + 2\right)/n^2\left(1 + \rho^2(x)\right) \le 6/n, \end{aligned} \tag{3.1}$$

we deduce

$$\lim_{n\to\infty} \left\|\mathcal{D}_n^\rho f - f\right\|_\varphi = 0$$

by Theorem 3.2. □

Let us describe the rate of the above convergence. To do this, we consider the following theorem proved in [16].

Theorem 3.4 ([16]) *Let* $L_n : C_\varphi(\mathbb{R}^+) \to B_\varphi(\mathbb{R}^+)$ *be a sequence of positive linear operators with*

$$\begin{aligned} \|L_n(1) - 1\|_{\varphi^0} &= a_n, \\ \|L_n(\rho) - \rho\|_{\varphi^{\frac{1}{2}}} &= b_n, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \left\|L_n\left(\rho^2\right) - \rho^2\right\|_\varphi &= c_n, \\ \left\|L_n\left(\rho^3\right) - \rho^3\right\|_{\varphi^{\frac{3}{2}}} &= d_n, \end{aligned} \tag{3.3}$$

where a_n, b_n, c_n *and* d_n *tend to zero as* $n \to \infty$. *Then*

$$\|L_n(f) - f\|_{\varphi^{\frac{3}{2}}} \le (7 + 4a_n + 2c_n)\,\omega_\rho(f;\delta_n) + \|f\|_\varphi a_n$$

for all $f \in C_\varphi(\mathbb{R}^+)$, *where*

$$\delta_n = 2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + a_n + 3b_n + 3c_n + d_n.$$

Theorem 3.5 *For all* $f \in C_\varphi(\mathbb{R}^+)$ *we have*

$$\left\|\mathcal{D}_n^\rho(f) - f\right\|_{\varphi^{\frac{3}{2}}} \le \left(7 + \frac{12}{n}\right)\omega_\rho\left(f; \frac{4\sqrt{2}}{\sqrt{n}} + \frac{54}{n}\right).$$

Proof In order to apply Theorem 3.4, we should calculate the sequences a_n, b_n, c_n and d_n. In light of (2.1) and (2.2) we obtain

$$\left\|\mathcal{D}_n^\rho(1) - 1\right\|_{\varphi^0} = a_n = 0$$

and

$$b_n = \left\|\mathcal{D}_n^\rho(\rho) - \rho\right\|_{\varphi^{\frac{1}{2}}} = \sup_{x\in\mathbb{R}^+} \frac{1}{n\sqrt{1+\rho^2(x)}} \le \frac{1}{n}.$$

Also by (3.1) we have

$$c_n = \left\|L_n\left(\rho^2\right) - \rho^2\right\|_\varphi = \sup_{x\in\mathbb{R}^+} \frac{4n\rho(x) + 2}{n^2\left(1 + \rho^2(x)\right)} \le \frac{6}{n}.$$

Finally using (2.3), we get

$$d_n = \left\|L_n\left(\rho^3\right) - \rho^3\right\|_{\varphi^{\frac{3}{2}}} = \sup_{x\in\mathbb{R}^+} \frac{9n^2\rho^2(x) + 18n\rho(x) + 6}{n^3\left(1 + \rho^2(x)\right)^{\frac{3}{2}}} \le \frac{33}{n}.$$

Since all conditions of Theorem 3.4 are satisfied, the desired result follows. □

4 Local approximation

Theorem 4.1 *Let* ρ *be a function satisfying the conditions* (p_1), (p_2) *and* $\|\rho''\|$ *be finite. If* $f \in C_B(\mathbb{R}^+)$, *then we have*

$$\left|\mathcal{D}_n^\rho(f;x) - f(x)\right| \le C\left\{\omega_2\left(f;\sqrt{\frac{4(1+n\rho(x))}{n^2}}\right) + \min\left(1, \frac{4(1+n\rho(x))}{n^2}\right)\|f\|\right\} + \omega\left(f, \frac{1}{n}\right),$$

where C *is a constant independent of* n.

Proof Let us consider the auxiliary operator

$$\hat{D}_n(f;x) = \mathcal{D}_n^\rho(f;x) + f(x) - \left(f \circ \rho^{-1}\right)\left(\rho(x) + \frac{1}{n}\right).$$

It is clear by Lemma 2.1 that

$$\hat{D}_n(1;x) = \mathcal{D}_n^\rho(1;x) = 1 \tag{4.1}$$

and

$$\hat{D}_n(\rho(t);x) = \mathcal{D}_n^\rho(\rho(t);x) + \rho(x) - \rho(x) - \frac{1}{n} = \rho(x). \tag{4.2}$$

The classical Taylor's expansion of $g \in \mathcal{W}_{\infty}^{2}$ yields for $t \in \mathbb{R}^{+}$ that

$$g(t) = \left(g \circ \rho^{-1}\right)(\rho(t)) = \left(g \circ \rho^{-1}\right)(\rho(x)) + D\left(g \circ \rho^{-1}\right)(\rho(x))(\rho(t) - \rho(x)) + \int_{\rho(x)}^{\rho(t)} (\rho(t) - u)\, D^{2}\left(g \circ \rho^{-1}\right)(u)\, du.$$

Using (4.1) and (4.2) we have

$$\hat{D}_n(g;x) - g(x) = \hat{D}_n\left(\int_{\rho(x)}^{\rho(t)} (\rho(t) - u)\, D^{2}\left(g \circ \rho^{-1}\right)(u)\, du; x\right). \tag{4.3}$$

On the other hand, with the change of variable $u = \rho(y)$ we get

$$\int_{\rho(x)}^{\rho(t)} (\rho(t) - u)\, D^{2}\left(g \circ \rho^{-1}\right)(u)\, du = \int_{x}^{t} (\rho(t) - \rho(y))\, D^{2}\left(g \circ \rho^{-1}\right)\rho(y)\, \rho'(y)\, dy.$$

Using the equality

$$D^{2}\left(g \circ \rho^{-1}\right)(\rho(y)) = \frac{1}{\rho'(y)} \frac{g''(y)\,\rho'(y) - g'(y)\,\rho''(y)}{(\rho'(y))^{2}}, \tag{4.4}$$

one can write

$$\begin{aligned}
&\int_{\rho(x)}^{\rho(t)} (\rho(t) - u)\, D^{2}\left(g \circ \rho^{-1}\right)(u)\, du \\
&= \int_{x}^{t} (\rho(t) - \rho(y)) \left(\frac{1}{\rho'(y)} \frac{g''(y)\,\rho'(y) - g'(y)\,\rho''(y)}{(\rho'(y))^{2}}\right) dy \\
&= \int_{\rho(x)}^{\rho(t)} (\rho(t) - u) \frac{g''\left(\rho^{-1}(u)\right)}{\left(\rho'\left(\rho^{-1}(u)\right)\right)^{3}} du - \int_{\rho(x)}^{\rho(t)} (\rho(t) - u) \frac{g'\left(\rho^{-1}(u)\right)\rho''\left(\rho^{-1}(u)\right)}{\left(\rho'\left(\rho^{-1}(u)\right)\right)^{3}} du.
\end{aligned}$$

So (4.3) can be written as

$$\begin{aligned}
\hat{D}_n(g;x) - g(x) &= \hat{D}_n\left(\int_{\rho(x)}^{\rho(t)} (\rho(t) - u) \frac{g''\left(\rho^{-1}(u)\right)}{\left(\rho'\left(\rho^{-1}(u)\right)\right)^{3}} du; x\right) \\
&\quad - \hat{D}_n\left(\int_{\rho(x)}^{\rho(t)} (\rho(t) - u) \frac{g'\left(\rho^{-1}(u)\right)\rho''\left(\rho^{-1}(u)\right)}{\left(\rho'\left(\rho^{-1}(u)\right)\right)^{3}} du; x\right)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{D}_n^{\rho}\left(\int\limits_{\rho(x)}^{\rho(t)} (\rho(t)-u)\,\frac{g''\left(\rho^{-1}(u)\right)}{\left(\rho'\left(\rho^{-1}(u)\right)\right)^3}du;x\right)\\
&\quad-\int\limits_{\rho(x)}^{\rho(x)+\frac{1}{n}}\left(\rho(x)+\frac{1}{n}-u\right)\frac{g''\left(\rho^{-1}(u)\right)}{\left(\rho'\left(\rho^{-1}(u)\right)\right)^3}du\\
&\quad-\mathcal{D}_n^{\rho}\left(\int\limits_{\rho(x)}^{\rho(t)} (\rho(t)-u)\,\frac{g'\left(\rho^{-1}(u)\right)\rho''\left(\rho^{-1}(u)\right)}{\left(\rho'\left(\rho^{-1}(u)\right)\right)^3}du;x\right)\\
&\quad+\int\limits_{\rho(x)}^{\rho(x)+\frac{1}{n}}\left(\rho(x)+\frac{1}{n}-u\right)\frac{g'\left(\rho^{-1}(u)\right)\rho''\left(\rho^{-1}(u)\right)}{\left(\rho'\left(\rho^{-1}(u)\right)\right)^3}du.
\end{aligned}
$$

Since ρ is strictly increasing on $[0,\infty)$ and with the condition (p_2), we get

$$
\begin{aligned}
\left|\hat{D}_n(g;x)-g(x)\right| &\le \left(M_{n,2}^{\rho}(x)+\frac{1}{n^2}\right)\left(\left\|g''\right\|+\left\|g'\right\|\left\|\rho''\right\|\right)\\
&\le \frac{4(1+n\rho(x))}{n^2}\left(\left\|g''\right\|+\left\|g'\right\|\left\|\rho''\right\|\right).
\end{aligned}
$$

Also, it is clear that

$$
\left\|\mathcal{D}_n^{\rho}\right\| \le \|f\|.
$$

Hence we have

$$
\begin{aligned}
\left|\mathcal{D}_n^{\rho}(f;x)-f(x)\right| &\le \left|\hat{D}_n(f;x)-f(x)+\left(f\circ\rho^{-1}\right)\left(\rho(x)+\frac{1}{n}\right)-f(x)\right|\\
&\le \left|\hat{D}_n(f-g;x)\right|+\left|\hat{D}_n(g;x)-g(x)\right|+|g(x)-f(x)|\\
&\quad+\left|\left(f\circ\rho^{-1}\right)\left(\rho(x)+\frac{1}{n}\right)-\left(f\circ\rho^{-1}\right)(\rho(x))\right|\\
&\le 4\|f-g\|+\frac{4(1+n\rho(x))}{n^2}(\|g''\|+\|g'\|\|\rho''\|)\\
&\quad+\omega\left(f\circ\rho^{-1},\frac{1}{n}\right)
\end{aligned}
$$

and choosing $C:=\max\left\{1,\|\rho''\|\right\}$ we have

$$
\begin{aligned}
\left|\mathcal{D}_n^{\rho}(f;x)-f(x)\right| &\le C\left\{\|f-g\|+\frac{4(1+n\rho(x))}{n^2}(\|g''\|+\left\|g'\right\|+\|g\|)\right\}\\
&\quad+\omega\left(f\circ\rho^{-1},\frac{1}{n}\right)\\
&= C\left\{\|f-g\|+\frac{4(1+n\rho(x))}{n^2}\|g\|_{\mathcal{W}_\infty^2}\right\}+\omega\left(f\circ\rho^{-1},\frac{1}{n}\right).
\end{aligned}
$$

Taking the infimum on the right hand side over all $g \in \mathcal{W}_{\infty}^{2}$ we obtain

$$\begin{aligned}\left|\mathcal{D}_n^{\rho}(f;x)-f(x)\right| &\leq CK_2\left(f;\frac{4(1+n\rho(x))}{n^2}\right)+\omega\left(f\circ\rho^{-1},\frac{1}{n}\right)\\ &\leq C\left\{\omega_2\left(f;\sqrt{\frac{4(1+n\rho(x))}{n^2}}\right)+\min\left(1,\frac{4(1+n\rho(x))}{n^2}\right)\|f\|\right\}\\ &\quad+\omega\left(f\circ\rho^{-1},\frac{1}{n}\right).\end{aligned}$$

Furthermore, for $x \in \mathbb{R}^{+}$ we have

$$\begin{aligned}\omega\left(f\circ\rho^{-1},t\right) &= \sup\left\{\left|f\left(\rho^{-1}(y)\right)-f\left(\rho^{-1}(x)\right)\right| : 0\leq y-x\leq t\right\}\\ &= \sup\left\{\left|f(\bar{y})-f(\bar{x})\right| : 0\leq\rho(\bar{y})-\rho(\bar{x})\leq t\right\}.\end{aligned}$$

Since $0 \leq \rho(\bar{y})-\rho(\bar{x}) \leq t$, then $0 \leq (\bar{y}-\bar{x})\rho'(u) \leq t$ for some $u \in (\bar{x},\bar{y})$, i.e., $0 \leq \bar{y}-\bar{x} \leq t/\rho'(u) \leq t$. Thus we have

$$\begin{aligned}\omega\left(f\circ\rho^{-1},t\right) &= \sup\left\{\left|f(\bar{y})-f(\bar{x})\right| : 0\leq\bar{y}-\bar{x}\leq t\right\}\\ &= \omega(f,t).\end{aligned}$$

Hence we have

$$\begin{aligned}\left|\mathcal{D}_n^{\rho}(f;x)-f(x)\right| \leq C\left\{\omega_2\left(f;\sqrt{\frac{4(1+n\rho(x))}{n^2}}\right)+\min\left(1,\frac{4(1+n\rho(x))}{n^2}\right)\|f\|\right\}\\ +\omega\left(f,\frac{1}{n}\right).\end{aligned}$$

□

5 Pointwise convergence of $\mathcal{D}_n^{\rho}$

In this section, we shall focus on pointwise convergence of $\mathcal{D}_n^{\rho}$ by obtaining the Voronovskaya theorem in quantitative form. We need the following lemma.

Lemma 5.1 ([16]) *For every* $f \in C_{\varphi}\left(\mathbb{R}^{+}\right)$, *for* $\delta > 0$ *and for all* $x, y \geq 0$,

$$\left|f(u)-f(x)\right| \leq \left(\varphi(u)+\varphi(x)\right)\left(2+\frac{\left|\rho(u)-\rho(x)\right|}{\delta}\right)\omega_{\rho}(f,\delta) \tag{5.1}$$

holds.

Theorem 5.2 *If the function* ρ *satisfies the conditions* (p_1), (p_2) *and* $f''/\left(\rho'\right)^2, f'.\rho''/\left(\rho'\right)^3 \in C_{\varphi}\left(\mathbb{R}^{+}\right)$, *then we have for any* $x \in [0,\infty)$ *that*

$$\begin{aligned}&\left|n\left[\mathcal{D}_n^{\rho}(f;x)-f(x)\right]-D\left(f\circ\rho^{-1}\right)(\rho(x))-\rho(x)D^2\left(f\circ\rho^{-1}\right)(\rho(x))\right|\\ &\quad\leq\frac{1}{n}+12\left(\rho^2(x)+\rho(x)+2\right)\frac{(1+n\rho(x))}{n}\left\{\omega_{\rho}\left(\frac{f''}{\left(\rho'\right)^2},\delta_n^{\rho}(x)\right)+\omega_{\rho}\left(\frac{f'\rho''}{\left(\rho'\right)^3},\delta_n^{\rho}(x)\right)\right\},\end{aligned}$$

where $\delta_n^{\rho}(x)=\left(144\frac{(1+n\rho(x))^2}{n^4}\right)^{\frac{1}{3}}$.

Proof By the Taylor expansion of $f \circ \rho^{-1}$ we can write

$$\left(f \circ \rho^{-1}\right)(\rho(t)) = \left(f \circ \rho^{-1}\right)(\rho(x)) + D\left(f \circ \rho^{-1}\right)(\rho(x))(\rho(t) - \rho(x)) + \frac{D^2\left(f \circ \rho^{-1}\right)(\rho(x))(\rho(t) - \rho(x))^2}{2} + h(t,x)(\rho(t) - \rho(x))^2, \tag{5.2}$$

where

$$h(t,x) = \frac{D^2\left(f \circ \rho^{-1}\right)(\xi) - D^2\left(f \circ \rho^{-1}\right)(\rho(x))}{2}$$

and ξ is a number between $\rho(x)$ and $\rho(t)$. Applying the operator $\mathcal{D}_n^{\rho}$ to both sides of equality (5.2), we immediately have

$$\begin{aligned}
&\left|n\left[\mathcal{D}_n^{\rho}(f;x) - f(x)\right] - D\left(f \circ \rho^{-1}\right)(\rho(x)) - \rho(x) D^2\left(f \circ \rho^{-1}\right)(\rho(x))\right| \\
&\leq \left|nD\left(f \circ \rho^{-1}\right)(\rho(x))\mu_{n,1}^{\rho}(x) - D\left(f \circ \rho^{-1}\right)(\rho(x))\right| \\
&\quad + \left|\frac{nD^2\left(f \circ \rho^{-1}\right)(\rho(x))}{2}\mu_{n,2}^{\rho}(x) - \rho(x) D^2\left(f \circ \rho^{-1}\right)(\rho(x))\right| \\
&\quad + n\mathcal{D}_n^{\rho}\left(|h(t,x)|(\rho(t) - \rho(x))^2; x\right).
\end{aligned}$$

Let us estimate $|h(t,x)|$. Using (4.4) and (5.1), respectively, we have

$$\begin{aligned}
|h(t,x)| &= \frac{1}{2}\left\{\frac{f''(\xi)}{(\rho'(\xi))^2} - \frac{f''(x)}{(\rho'(x))^2} + f'(x)\frac{\rho''(x)}{(\rho'(x))^3} - f'(\xi)\frac{\rho''(\xi)}{(\rho'(\xi))^3}\right\} \\
&\leq \frac{1}{2}(\varphi(t) + \varphi(x))\left(2 + \frac{|\rho(t) - \rho(x)|}{\delta}\right)\left\{\omega_{\rho}\left(\frac{f''}{(\rho')^2}, \delta\right) + \omega_{\rho}\left(\frac{f'\rho''}{(\rho')^3}, \delta\right)\right\}.
\end{aligned}$$

On the other hand, since $\varphi(t) + \varphi(x) \leq \delta^2 + 2\rho^2(x) + 2\rho(x)\delta + 2$ whenever $|\rho(t) - \rho(x)| \leq \delta$, we have

$$|h(t,x)| \leq \frac{3}{2}\left(\delta^2 + 2\rho^2(x) + 2\rho(x)\delta + 2\right)\left\{\omega_{\rho}\left(\frac{f''}{(\rho')^2}, \delta\right) + \omega_{\rho}\left(\frac{f'\rho''}{(\rho')^3}, \delta\right)\right\}$$

and since $\varphi(t) + \varphi(x) \leq \left(\frac{\rho(t)-\rho(x)}{\delta}\right)^2\left(\delta^2 + 2\rho^2(x) + 2\rho(x)\delta + 2\right)$ whenever $|\rho(t) - \rho(x)| > \delta$, we have

$$\begin{aligned}
|h(t,x)| &\leq \frac{3}{2}\left(\delta^2 + 2\rho^2(x) + 2\rho(x)\delta + 2\right)\frac{|\rho(t) - \rho(x)|^3}{\delta^3} \\
&\quad \times \left\{\omega_{\rho}\left(\frac{f''}{(\rho')^2}, \delta\right) + \omega_{\rho}\left(\frac{f'\rho''}{(\rho')^3}, \delta\right)\right\}.
\end{aligned}$$

Choosing $\delta < 1$ we deduce

$$\begin{aligned}
|h(u,x)| &\leq 3\left(\rho^2(x) + \rho(x) + 2\right)\left(\frac{|\rho(t) - \rho(x)|^3}{\delta^3} + 1\right) \\
&\quad \times \left\{\omega_{\rho}\left(\frac{f''}{(\rho')^2}, \delta\right) + \omega_{\rho}\left(\frac{f'\rho''}{(\rho')^3}, \delta\right)\right\}.
\end{aligned}$$

So using Lemma 2.2 and the Cauchy–Schwarz inequality we get

$$\begin{aligned}&\left|n\left[\mathcal{D}_n^{\rho}(f;x)-f(x)\right]-D\left(f\circ\rho^{-1}\right)(\rho(x))-\rho(x)\,D^2\left(f\circ\rho^{-1}\right)(\rho(x))\right|\\&\le\frac{1}{n}+3n\left(\rho^2(x)+\rho(x)+2\right)\left\{\omega_\rho\left(\frac{f''}{(\rho')^2},\delta\right)+\omega_\rho\left(\frac{f'\rho''}{(\rho')^3},\delta\right)\right\}\\&\quad\times\mu_{n,2}^{\rho}\left(1+\frac{1}{\delta^3}\frac{\sqrt{\mu_{n,4}(x)\,\mu_{n,6}(x)}}{\mu_{n,2}(x)}\right)\end{aligned}$$

and if we choose $\delta=\left(\sqrt{\frac{\mu_{n,4}(x)\mu_{n,6}(x)}{\mu_{n,2}(x)}}\right)^{\frac{1}{3}}$ we get

$$\begin{aligned}&\left|n\left[\mathcal{D}_n^{\rho}(f;x)-f(x)\right]-D\left(f\circ\rho^{-1}\right)(\rho(x))-\rho(x)\,D^2\left(f\circ\rho^{-1}\right)(\rho(x))\right|\\&\le\frac{1}{n}+6n\left(\rho^2(x)+\rho(x)+2\right)\mu_{n,2}^{\rho}\\&\quad\times\left\{\omega_\rho\left(\frac{f''}{(\rho')^2},\left(\sqrt{\frac{\mu_{n,4}(x)\,\mu_{n,6}(x)}{\mu_{n,2}(x)}}\right)^{\frac{1}{3}}\right)+\omega_\rho\left(\frac{f'\rho''}{(\rho')^3},\left(\sqrt{\frac{\mu_{n,4}(x)\,\mu_{n,6}(x)}{\mu_{n,2}(x)}}\right)^{\frac{1}{3}}\right)\right\}.\end{aligned}\tag{5.3}$$

On the other hand, straightforward calculations give

$$\mu_{n,2}^{\rho}(x)=2\frac{(1+n\rho(x))}{n^2},\quad \mu_{n,4}^{\rho}(x)\le 36\frac{(1+n\rho(x))^2}{n^4},\quad \mu_{n,6}^{\rho}(x)\le 1152\frac{(1+n\rho(x))^3}{n^6}.$$

Hence we have

$$\sqrt{\frac{\mu_{n,4}(x)\,\mu_{n,6}(x)}{\mu_{n,2}(x)}}\le 144\frac{(1+n\rho(x))^2}{n^4}=\delta_n^{\rho}(x).$$

If the above estimates are substituted in (5.3), we get the desired result. □

Corollary 5.3 *We have the following particular cases:*

(i) Suppose that $\rho(x)=x$. If $f''\in C_{x^2}\left(\mathbb{R}^+\right)$ (where $C_{x^2}\left(\mathbb{R}^+\right)$ is the analogues one of $C_{\varphi}\left(\mathbb{R}^+\right)$), then we have for any $x\in[0,\infty)$ that

$$\begin{aligned}&\left|n\left[\mathcal{D}_n(f;x)-f(x)\right]-f'(x)-xf''(x)\right|\\&\le\frac{1}{n}+12\left(x^2+x+2\right)\frac{(1+nx)}{n}\Omega\left(f'',\delta_n(x)\right),\end{aligned}$$

where $\Omega(f;\delta)$ is another weighted modulus of continuity defined in [17] and $\delta_n(x)=\left(144\frac{(1+nx)^2}{n^4}\right)^{\frac{1}{3}}$.

(ii) If $f''/\left(\rho'\right)^2,\ f'.\rho''/\left(\rho'\right)^3\in U_{\varphi}\left(\mathbb{R}^+\right)$, then we have for any $x\in[0,\infty)$ that

$$\lim_{n\to\infty}n\left[\mathcal{D}_n^{\rho}(f;x)-f(x)\right]=D\left(f\circ\rho^{-1}\right)(\rho(x))+\rho(x)\,D^2\left(f\circ\rho^{-1}\right)(\rho(x)).$$

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