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Diophantine equations with Appell sequences

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Abstract We consider the Diophantine equation $P_n(x) = g(y)$ in x, y where $P_n(x), g(x) \in \mathbb{Q}[x]$, $\deg g(x) \geq 3$ and $\{P_n(x)\}_{n\geq 0}$ is an Appell sequence. Under some reasonable assumptions on $P_n(x)$ we prove an ineffective finiteness result on the above equation.

Keywords Diophantine equations · Appell sequences · Decomposition

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1 Introduction

For $n \in \mathbb{N} \cup \{0\}$, let $P_n(x)$ be a polynomial with rational coefficients and with $\deg P_n(x) = n$. Further, let $P_0(x)$ be a non-zero constant. The sequence $\{P_n(x)\}_{n\geq 0}$ is called an *Appell sequence* (and $P_n(x)$ is called an *Appell polynomial*) if

$$P_n'(x) = nP_{n-1}(x) \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.1}$$

The history of such polynomials goes back to Appell's work [2] in 1880. There are several well-known examples of Appell sequences, such as the Bernoulli polynomials $B_n(x)$, the Euler polynomials $E_n(x)$, and the Hermite polynomials $H_n(x)$, respectively defined by the following generating series (see e.g. [12])

$$\frac{t\exp(tx)}{\exp(t) - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!};$$

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$$\frac{2\exp(xt)}{\exp(t)+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad (|t| < \pi);$$
$$\frac{\exp(tx)}{\exp(t^2/2)} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

The above defined Hermite polynomials $H_n(x)$ are sometimes denoted by $He_n(x)$, e.g. in Abramowitz and Stegun [1].

The following properties of Appell polynomials will often be used in the text, sometimes without special reference.

We recall the so-called *Appell Identity*:

$$P_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} P_k(x) y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} P_k(y) x^{n-k}, \tag{1.2}$$

which, by setting $y = 0$, implies that there exists a sequence of rational numbers $\{c_n\}_{n\geq 0}$ with $c_0 \neq 0$ such that

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} c_k x^{n-k}, \quad \text{where } c_k := P_k(0) \ (k \geq 0). \tag{1.3}$$

For the proofs of (1.2) and (1.3) see, for instance Roman [12].

Let $\mathbb{K}$ be an arbitrary field. We denote by $\mathbb{K}[x]$ the ring of polynomials in the variable x with coefficients from $\mathbb{K}$. A *decomposition* of a polynomial $F(x)$ over $\mathbb{K}$ is an equality of the following form

$$F(x) = G_1(G_2(x)) \quad (G_1(x), G_2(x) \in \mathbb{K}[x]),$$

which is nontrivial if

$$\deg G_1(x) > 1 \quad \text{and} \quad \deg G_2(x) > 1.$$

Two decompositions $F(x) = G_1(G_2(x))$ and $F(x) = H_1(H_2(x))$ are said to be *equivalent* if there exists a linear polynomial $\ell(x) \in \mathbb{K}[x]$ such that $G_1(x) = H_1(\ell(x))$ and $H_2(x) = \ell(G_2(x))$. The polynomial $F(x)$ is called *decomposable* over $\mathbb{K}$ if it has at least one nontrivial decomposition over $\mathbb{K}$; otherwise it is said to be *indecomposable*.

The decomposition of Bernoulli polynomials has been described by Bilu et al. in [6]. Decomposition properties of Euler polynomials were recently investigated by Rakaczki and Kreso [11]. These results can both be summarized as follows: the corresponding polynomial ($B_n(x)$ or $E_n(x)$) is indecomposable over $\mathbb{C}$ for all odd n, while, if n is even, then any nontrivial decomposition of the polynomial under consideration over $\mathbb{C}$ is equivalent to one of the form

$$\widehat{P}_{n/2}\left(\left(x - \frac{1}{2}\right)^2\right),$$

where $\widehat{P}_{n/2}(x)$ is a polynomial of degree $n/2$ which is indecomposable for every n. These results from [6] and [11] suggest the following notion. We say that an Appell sequence $\{P_n(x)\}_{n\geq 0}$ is of *special type* if $P_n(x)$ is indecomposable over $\mathbb{C}$ for all odd n, and, for even n, every nontrivial decomposition of $P_n(x)$ is equivalent to a decomposition of the form

$$P_n(x) = \widehat{P}_{n/2}\left(\left(x - \frac{1}{2}\right)^2\right), \tag{1.4}$$

with an indecomposable polynomial $\widehat{P}_{n/2}(x)$ over $\mathbb{C}$ of degree $n/2$. Clearly, the polynomials $\{B_n(x)\}_{n\geq 0}$ and $\{E_n(x)\}_{n\geq 0}$ are of special type.

The theory of polynomial decomposition is strongly connected to the theory of separable Diophantine equations since, in 2000, Bilu and Tichy [5] established their general ineffective finiteness criterion on equations of the form $f(x) = g(y)$. (See Proposition 2.1 below.)

In this paper we study the Diophantine equation

$$P_n(x) = g(y) \quad \text{in integers } x, y, \tag{1.5}$$

where $P_n(x)$ is from an Appell sequence of special type and $g(x) \in \mathbb{Q}[x]$, $\deg g(x) \geq 3$. For technical reasons, we restrict ourselves to Appell sequences $\{P_n(x)\}_{n\geq 0}$ for which

$$\frac{3P_2(-c_1/c_0)^2 - 2c_0P_4(-c_1/c_0)}{3P_2(-c_1/c_0)^2 - c_0P_4(-c_1/c_0)} \quad \text{is not a positive integer.} \tag{1.6}$$

Remark In the following table, we give the value of the constant from (1.6) for the case when $P_n(x)$ is a Bernoulli, Euler or an Hermite polynomial, respectively.

$B_n(x)$	$E_n(x)$	$H_n(x)$
$9/2$	$7/2$	undefined

For $P_n(x) = B_n(x)$, Rakaczki [10], and independently Kulkarni and Sury [9] characterized those pairs $(n, g(y))$ for which equation (1.5) has infinitely many integer solutions. Recently, Rakaczki and Kreso [11] proved an analogous result for the case when $P_n(x) = (E_n(0) \pm E_n(x))/2$ (which is not an Appell polynomial anymore). For further related results we refer to [7,8].

We prove the following.

Theorem 1.1 *Let $g(x) \in \mathbb{Q}[x]$ with $\deg g(x) \geq 3$, and suppose that $\{P_n(x)\}_{n\geq 0}$ is an Appell sequence of special type with property* (1.6). *Then for $n \geq 7$, equation* (1.5) *has only finitely many integer solutions x, y, apart from the following cases:*

(i) $g(x) = P_n(h(x))$, *where $h(x)$ is a polynomial over $\mathbb{Q}$.*
(ii) $g(x) = \gamma(\delta(x)^m)$, *where m is a positive integer.*
(iii) *n is even and* $g(x) = \widehat{P}_{n/2}(q(x)^2)$
(iv) *n is even and* $g(x) = \widehat{P}_{n/2}(\delta(x)q(x)^2)$
(v) *n is even and* $g(x) = \widehat{P}_{n/2}(c\delta(x)^t)$, *where $t \geq 3$ is an odd integer*
(vi) *n is even and* $g(x) = \widehat{P}_{n/2}((a\delta(x)^2 + b)q(x)^2)$

Here $a, b, c \in \mathbb{Q} \setminus \{0\}$, $\gamma(x), \delta(x) \in \mathbb{Q}[x]$ are linear polynomials and $q(x) \in \mathbb{Q}[x]$ is a non-zero polynomial.

We prove the above theorem by applying among other things the general finiteness criterion of Bilu and Tichy [5] for equation (1.5). Hence our finiteness result is ineffective.

Remark For $n \geq 7$, our main result is a common generalization of the aforementioned results of Rakaczki [10], Kulkarni and Sury [9] and Rakaczki and Kreso [11]. In the special cases $P_n(x) \in \{B_n(x), E_n(x)\}$, one can exclude the exceptional case (ii) by making use of some specific properties of the Bernoulli or Euler polynomials, respectively. (See [9–11])

2 Auxiliary results

Before proving Theorem 1.1, we collect the results that will be applied in the proof. First, we recall the finiteness criterion of Bilu and Tichy [5]. To do this, we need to define five kinds of so-called standard pairs of polynomials.

Let α, β be nonzero rational numbers, $\mu, \nu, q > 0$ and $r \geq 0$ be integers, and let $v(x) \in \mathbb{Q}[x]$ be a nonzero polynomial (which may be constant). Denote by $D_\mu(x, \delta)$ the μ-th Dickson polynomial, defined by the functional equation $D_\mu(z+\delta/z, \delta) = z^\mu + (\delta/z)^\mu$ or by the explicit formula

$$D_\mu(x, \delta) = \sum_{i=0}^{\lfloor \mu/2 \rfloor} d_{\mu,i} x^{\mu-2i} \qquad \text{with } \ d_{\mu,i} = \frac{\mu}{\mu - i}\binom{\mu - i}{i}(-\delta)^i.$$

Two polynomials $f_1(x)$ and $g_1(x)$ are said to form a *standard pair over* $\mathbb{Q}$ if one of the ordered pairs $(f_1(x), g_1(x))$ or $(g_1(x), f_1(x))$ belongs to the list below. The five kinds of standard pairs are then listed in the following table.

Kind	Explicit form of $\{f_1(x), g_1(x)\}$	Parameter restrictions
First	$(x^q, \alpha x^r v(x)^q)$	$0 \leq r < q, (r, q) = 1, \ r + \deg v(x) > 0$
Second	$(x^2, (\alpha x^2 + \beta)v(x)^2)$	–
Third	$(D_\mu(x, \alpha^\nu), D_\nu(x, \alpha^\mu))$	$(\mu, \nu) = 1$
Fourth	$(\alpha^{\frac{-\mu}{2}} D_\mu(x, \alpha), -\beta^{\frac{-\nu}{2}} D_\nu(x, \beta))$	$(\mu, \nu) = 2$
Fifth	$((\alpha x^2 - 1)^3, 3x^4 - 4x^3)$	–

The following proposition is a special case of the main result of [5].

Proposition 2.1 *Let $f(x), g(x) \in \mathbb{Q}[x]$ be nonconstant polynomials such that the equation $f(x) = g(y)$ has infinitely many solutions in rational integers x, y. Then $f = \varphi \circ f_1 \circ \lambda$ and $g = \varphi \circ g_1 \circ \mu$, where $\lambda(x), \mu(x) \in \mathbb{Q}[x]$ are linear polynomials, $\varphi(x) \in \mathbb{Q}[x]$, and $(f_1(x), g_1(x))$ is a standard pair over $\mathbb{Q}$.*

For $P(x) \in \mathbb{C}[x]$, a complex number c is said to be an *extremum* if $P(x) - c$ has multiple roots. The P*-type* of c is defined to be the tuple $(\alpha_1, \ldots, \alpha_s)$ of the multiplicities of the distinct roots of $P(x) - c$ in an increasing order. Obviously, $s < \deg P(x)$ and $\alpha_1 + \ldots + \alpha_s = \deg P(x)$.

Proposition 2.2 *For $a \neq 0$ and $k \geq 3$, $D_\mu(x, \alpha)$ has exactly two extrema $\pm 2\alpha^{\frac{\mu}{2}}$. If μ is odd, then both are of P-type $(1, 2, 2, \ldots, 2)$. If μ is even, then $2\alpha^{\frac{\mu}{2}}$ is of P-type $(1, 1, 2, \ldots, 2)$ and $-2\alpha^{\frac{\mu}{2}}$ is of P-type $(2, 2, \ldots, 2)$.*

Proof See, for instance [4, Proposition 3.3]. □

We end this section with two technical results. Let $d_1, e_1 \in \mathbb{Q}^*$ and $d_0, e_0 \in \mathbb{Q}$

Proposition 2.3 *Suppose that $\{P_n(x)\}_{n \geq 0}$ is an Appell sequence of special type. Then the polynomial $P_n(d_1 x + d_0)$ is not of the form $e_1 x^q + e_0$, with $q \geq 7$.*

Proof We assume the contrary, i.e., that we have

$$P_n(d_1x + d_0) = e_1x^q + e_0 \tag{2.1}$$

with $q \geq 7$. Obviously, we then have $n = q$.

We observe from (1.2) and (2.1) that

$$P_1(d_0) = P_2(d_0) = \ldots = P_{n-1}(d_0) = 0. \tag{2.2}$$

Since, by (1.3), $P_1(d_0) = c_0d_0 + c_1$, we get

$$d_0 = -\frac{c_1}{c_0}. \tag{2.3}$$

Further, since, by (1.1),

$$P_k(x) = \frac{k!}{(n-1)!} P_{n-1}^{(n-1-k)}(x), \quad k = 1, \ldots, n-1, \tag{2.4}$$

we infer that d_0 is a root of $P_{n-1}(x)$ of multiplicity $(n-1)$. Thus, in view of (2.3), we have $P_{n-1}(x) = c_0\,(x + c_1/c_0)^{n-1}$, which implies

$$P_n(x) = c_0\left(x + \frac{c_1}{c_0}\right)^n + C \quad \text{with} \quad C = P_n\left(-\frac{c_1}{c_0}\right). \tag{2.5}$$

First, *if* $n \geq 7$ *is even* , then, by (2.5), one can easily find the nontrivial decomposition $P_n(x) = Q(R(x))$ with

$$Q(x) = c_0x^2 + C, \quad \text{and} \quad R(x) = \left(x + \frac{c_1}{c_0}\right)^{n/2}. \tag{2.6}$$

Since $n \geq 7$, this nontrivial decomposition is obviously not equivalent to the one in (1.4), contradicting that $\{P_n(x)\}_{n\geq 0}$ is of special type.

Now, let $n \geq 7$ *be an odd positive integer.* If n is composite, then any divisor v of n with $1 < v < n$ leads to a nontrivial decomposition

$$P_n(x) = c_0\left(\left(x + \frac{c_1}{c_0}\right)^v\right)^{n/v} + C, \tag{2.7}$$

which again contradicts that $\{P_n(x)\}_{n\geq 0}$ is of special type (and in this case $P_n(x)$ is indecomposable). If n is a prime, then derivating both sides of (2.5) we obtain

$$P_{n-1}(x) = c_0\left(x - \frac{1}{2}\right)^{n-1}, \tag{2.8}$$

where of course the exponent $n-1$ is even. Similarly as above, this leads to a nontrivial decomposition not equivalent to (1.4) and thus to a contradiction. □

Proposition 2.4 *Suppose that* $\{P_n(x)\}_{n\geq 0}$ *is an Appell sequence which satisfies* (1.6). *Then the polynomial* $P_n(d_1x + d_0)$ *is not of the form* $e_1D_\mu(x,\delta) + e_0$, *where* $D_\mu(x,\delta)$ *the* μ*-th Dickson polynomial with* $\mu > 4$, $\delta \in \mathbb{Q}^*$.

Proof Suppose that the Appell sequence $\{P_n(x)\}_{n\geq 0}$ satisfies (1.6), and that we have

$$P_n(d_1x + d_0) = e_1D_\mu(x,\delta) + e_0. \tag{2.9}$$

Clearly, $n = \mu$. Comparing the leading coefficients of both sides we get

$$d_1^n c_0 = e_1, \tag{2.10}$$

where the numbers c_k ($k \geq 0$) are defined in (1.3). Similarly, from (1.2) and the equality of the coefficients of x^{n-1} on both sides we obtain

$$nd_1^{n-1}P_1(d_0) = 0, \tag{2.11}$$

which implies

$$d_0 = -\frac{c_1}{c_0}. \tag{2.12}$$

Again, by (1.2), comparing the coefficients of x^{n-2} gives

$$\binom{n}{2}d_1^{n-2}P_2(d_0) = -e_1 n\delta, \tag{2.13}$$

whence, together with (2.10) it follows that

$$d_1^2 = -\frac{(n-1)P_2(d_0)}{2c_0\delta} \tag{2.14}$$

Now we compare the coefficients of x^{n-4} on both sides of (2.9) and we obtain

$$\binom{n}{4}d_1^{n-4}P_4(d_0) = \frac{e_1 n(n-3)\delta^2}{2}, \tag{2.15}$$

which along with (2.10) leads to

$$d_1^4 = \frac{(n-1)(n-2)P_4(d_0)}{12c_0\delta^2}. \tag{2.16}$$

After substituting (2.14) into (2.16), we obtain

$$3(n-1)P_2(d_0)^2 = (n-2)c_0P_4(d_0), \tag{2.17}$$

whence, together with (2.12) it follows that

$$n = \frac{3P_2(-c_1/c_0)^2 - 2c_0P_4(-c_1/c_0)}{3P_2(-c_1/c_0)^2 - c_0P_4(-c_1/c_0)}. \tag{2.18}$$

This is a contradiction by (1.6). □

We note that Proposition 2.4 is a common generalization of Lemma 5.3 in [6], Lemma 2.4 in [3], and of the second statement of Lemma 12 in [11].

3 Proof of Theorem 1.1

Let $g(x) \in \mathbb{Q}[x]$ with $\deg g(x) \geq 3$. Suppose that equation (1.5) has infinitely many integer solutions x, y with an Appell sequence $\{P_n(x)\}_{n\geq 0}$ of special type satisfying (1.6) and with $n \geq 7$. Then by Proposition 2.1 it follows that there exist $\lambda(x), \mu(x), \varphi(x) \in \mathbb{Q}[x]$, $\deg \lambda(x) = \deg \mu(x) = 1$ such that

$$P_n(x) = \varphi(f_1(\lambda(x))) \quad \text{and} \quad g(x) = \varphi(g_1(\mu(x))), \tag{3.1}$$

where $(f_1(x), g_1(x))$ is a standard pair over $\mathbb{Q}$.

Let $\lambda^{-1}(x) = a_1x + a_0$, $\mu^{-1}(x) = b_1x + b_0$, where $a_0, a_1, b_0, b_1 \in \mathbb{Q}$ with $a_1b_1 \neq 0$. Then we can rewrite (3.1) as

$$P_n(a_1x + a_0) = \varphi(f_1(x)) \quad \text{and} \quad g(b_1x + b_0) = \varphi(g_1(x)), \tag{3.2}$$

Since $P_n(x)$ is of special type and $\deg P_n(x) = n$, we obtain that

$$\deg \varphi(x) \in \left\{1, \frac{n}{2}, n\right\}.$$

3.1 The case $\deg \varphi(x) = n$

If we assume that $\deg \varphi(x) = n$, then by (3.1), we have $\deg f_1(x) = 1$. Thus $P_n(x) = \varphi(t(x))$, where $t(x) \in \mathbb{Q}[x]$ is a linear polynomial. Clearly, $t^{-1}(x) \in \mathbb{Q}[x]$ is also linear. By (3.1), we obtain $P_n(t^{-1}(x)) = \varphi(t(t^{-1}(x))) = \varphi(x)$. Hence

$$g(x) = \varphi(g_1(\mu(x))) = P_n(t^{-1}(g_1(\mu(x)))) = P_n(q(x)), \tag{3.3}$$

where $q(x) = t^{-1}(g_1(\mu(x)))$. So, if, in our case, equation (1.5) has infinitely many solutions, then $g(x)$ is of the form as in Theorem 1.1 (i).

3.2 The case $\deg \varphi(x) = 1$

Let $\varphi(x) = \varphi_1 x + \varphi_0$, where $\varphi_1, \varphi_0 \in \mathbb{Q}$ and $\varphi_1 \neq 0$. We now study the five kinds of standard pairs.

In view of our assumptions on n and $\deg g(x)$, it follows that the standard pair $(f_1(x), g_1(x))$ cannot be of the second or fifth kind.

If it is of the third or fourth kind, we then have $P_n(a_1 x + a_0) = e_1 D_\mu(x, \delta) + e_0$ with $e_0 \in \mathbb{Q}$, $e_1, \delta \in \mathbb{Q}^*$. This contradicts Proposition 2.4.

If $(f_1(x), g_1(x))$ is a standard pair of the first kind, then we have either

(I) $P_n(a_1 x + a_0) = \varphi_1 x^q + \varphi_0$, or
(II) $P_n(a_1 x + a_0) = \varphi_1 \alpha x^p \nu(x)^q + \varphi_0$, where $0 \leq p < q$, $(p, q) = 1$ and $p + \deg \nu(x) > 0$.

The first case (I) is impossible by Proposition 2.3 since $n \geq 7$ by assumption.

Let us now consider the second case (II). Then we have $g(x) = \varphi_1 \mu(x)^q + \varphi_0 = \varphi(\mu(x)^q)$, where $q \geq 3$ and $\mu(x) \in \mathbb{Q}[x]$ is linear, which is case (ii) of Theorem 1.1.

3.3 The case $\deg \varphi(x) = n/2$

Clearly, n is then even, and from (3.1) we observe that $\deg f_1(x) = 2$. Hence it follows that, in (3.1), $(f_1(x), g_1(x))$ cannot be a standart pair of the fifth kind. Further, we obtain a nontrivial decomposition of $P_n(x)$, which, since $P_n(x)$ is of special type, implies that there exists a linear polynomial $\ell(x) = \ell_1 x + \ell_0$ over $\mathbb{Q}$ such that

$$\varphi(x) = \widehat{P}_{n/2}(\ell(x)) \quad \text{and} \quad \ell(f_1(\lambda(x))) = \left(x - \frac{1}{2}\right)^2. \tag{3.4}$$

Again, we study the remaining kinds of standard pairs.

First, we consider the case when, in (3.1), $(f_1(x), g_1(x))$is a standard pair of the first kind. If $f_1(x) = x^t$, then by $\deg f_1(x) = 2$, we have $(f_1(x), g_1(x)) = (x^2, \alpha x p(x)^2)$. Putting $\lambda(x) = \lambda_1 x + \lambda_0$, (3.4) takes the form $\ell((\lambda_1 x + \lambda_0)^2) = (x - 1/2)^2$, whence an easy calculation gives $\ell(x) = x/\lambda_1^2$. Substituting this into (3.1), we obtain

$$g(x) = \widehat{P}_{n/2}(\ell(g_1(\mu(x)))) = \widehat{P}_{n/2}\left(\frac{\alpha \mu(x) p(\mu(x))^2}{\lambda_1^2}\right) \tag{3.5}$$

So $g(x)$ is of the form (iv) with $\delta(x) = \alpha \mu(x)/\lambda_1^2$ and $q(x) = p(\mu(x))$.

In the switched case $(f_1(x), g_1(x)) = (\alpha x^r p(x)^t, x^t)$, where $0 \le r < t$, $(r, t) = 1$ and $r + \deg p(x) > 0$, $\deg f_1(x) = 2$ implies that one of the following cases occurs:

(A) $r = 0,\ t = 1$ and $\deg p(x) = 2$, or
(B) $r = 2,\ t > 2$ is odd and $p(x)$ is a constant polynomial.

In case (A) we have $g_1(x) = x$, whence from (3.1) and (3.4) we obtain

$$g(x) = \widehat{P}_{n/2}(\ell(g_1(\mu(x)))) = \widehat{P}_{n/2}(\ell(\mu(x))) = \widehat{P}_{n/2}(\delta(x)q(x)^2), \tag{3.6}$$

where $\delta(x) = \ell(\mu(x))$ and $q(x) \equiv 1$. Thus $g(x)$ is again of the form (iv).

In the second case (B), we can write $f_1(x) = \beta x^2$, with $\beta = \alpha p(x)^t \in \mathbb{Q}\setminus\{0\}$. Substituting this into (3.4), we deduce that $\ell(x) = x/(\beta\lambda_1^2)$, whence, by (3.1), we get

$$g(x) = \widehat{P}_{n/2}(\ell(g_1(\mu(x)))) = \widehat{P}_{n/2}\left(\frac{\mu(x)^t}{\beta\lambda_1^2}\right) = \widehat{P}_{n/2}(c\delta(x)^t), \tag{3.7}$$

where $c = 1/(\beta\lambda_1^2)$, $\delta(x) = \mu(x)$ and $t > 2$ is odd. This is option (v) in Theorem 1.1.

Next let $(f_1(x), g_1(x))$, in (3.1), be a standard pair of the second kind. If $(f_1(x), g_1(x)) = (x^2, (\alpha x^2 + \beta)v(x)^2)$, then a calculation from (3.4) yields $\ell(x) = x/\lambda_1^2$, and by (3.1) we have

$$\begin{aligned} g(x) &= \widehat{P}_{n/2}(\ell(g_1(\mu(x)))) = \\ &= \widehat{P}_{n/2}\left(\frac{(\alpha\mu(x)^2 + \beta)v(\mu(x))^2}{\lambda_1^2}\right) = \widehat{P}_{n/2}((\alpha\delta(x)^2 + \beta)q(x)^2), \end{aligned} \tag{3.8}$$

where $\delta(x) = \mu(x)$ and $q(x) = v(\mu(x))/\lambda_1$. So we are led to option (vi) of our theorem.

In the switched case $(f_1(x), g_1(x)) = ((\alpha x^2 + \beta)v(x)^2, x^2)$, since $\deg f_1(x) = 2$, $v(x)$ is a constant polynomial and

$$g(x) = \widehat{P}_{n/2}(\ell(g_1(\mu(x)))) = \widehat{P}_{n/2}((\ell_1\mu(x)^2 + \ell_0)q(x)^2), \tag{3.9}$$

where $q(x) \equiv 1$. Thus, we arrived again at option (vi) with $\delta(x) = \mu(x)$ and $a = \ell_1$, $b = \ell_0$.

Now, if the standard pair $(f_1(x), g_1(x))$ is of the third kind over $\mathbb{Q}$, then $(f_1(x), g_1(x)) = (D_2(x, \alpha^t), D_t(x, \alpha^2))$ with t being odd. Let us substitute $f_1(x) = x^2 - 2\alpha^t$ into (3.4) to deduce that $\ell(x) = (x + 2\alpha^t)/\lambda_1^2$, whence

$$g(x) = \widehat{P}_{n/2}(\ell(g_1(\mu(x)))) = \widehat{P}_{n/2}\left(\frac{D_t(\mu(x), \alpha^2) + 2\alpha^t}{\lambda_1^2}\right). \tag{3.10}$$

It follows from Proposition 2.2 that $-2\alpha^t/\lambda_1^2$ is an extremum of the polynomial $D_t(\mu(x), \alpha^2)/\lambda_1^2$, which is of P-type $(1, 2, \ldots, 2)$ as t is odd. Hence $(D_t(\mu(x), \alpha^2) + 2\alpha^t)/\lambda_1^2 = \delta(x)q(x)^2$ for some $\delta(x), q(x) \in \mathbb{Q}[x]$ with $\deg \delta(x) = 1$. We deduce, that $g(x)$ is of the form (iv).

Finally, consider the case when $(f_1(x), g_1(x))$ is a standard pair of the fourth kind over $\mathbb{Q}$. Then

$$(f_1(x), g_1(x)) = \left(\frac{D_2(x, \alpha)}{\alpha}, \frac{D_t(x, \beta)}{\beta^{(t/2)}}\right),$$

with an even t. Substituting this into (3.4), an easy calculation yields $\ell(x) = (\alpha x + 2\alpha)/\lambda_1^2$, whence, by (3.1), we obtain

$$g(x) = \widehat{P}_{n/2}(\ell(g_1(\mu(x)))) = \widehat{P}_{n/2}\left(\frac{\alpha\beta^{-t/2}D_t(\mu(x), \beta) + 2\alpha}{\lambda_1^2}\right). \tag{3.11}$$

Now from Proposition 2.2 we infer that

$$-\frac{2\beta^{t/2}\alpha\beta^{-t/2}}{\lambda_1^2} = -\frac{2\alpha}{\lambda_1^2}$$

is one of the two extrema of the polynomial $\alpha\beta^{-t/2}D_t(\mu(x), \beta)/(\lambda_1^2)$ and it is of P-type $(2, 2, \ldots, 2)$, as t is even. Therefore we have

$$\frac{\alpha\beta^{-t/2}D_t(\mu(x), \beta) + 2\alpha}{\lambda_1^2} = q(x)^2$$

for some $q(x) \in \mathbb{Q}[x]$. Thus $g(x)$ is of the form (iii), which completes the proof.

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