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OPTIMAL CONTINUED FRACTIONS AND THE MOVING AVERAGE ERGODIC THEOREM

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Abstract

We use the moving average ergodic theorem of A. Bellow, R. Jones and J. Rosenblatt to derive various results in metric number theory primarily concerning moving averages of various sequences attached to the optimal continued fraction expansion of a real number.

1. Introduction

We begin by introducing some notation. Let Z be a collection of points in $\mathbf{Z} \times \mathbf{N}$ and let

$$Z^h = \{(n,k) : (n,k) \in Z \text{ and } k \geq h\},$$
$$Z^h_\alpha = \{(z,s) \in \mathbf{Z}^2 : |z-y| < \alpha(s-r) \text{ for some } (y,r) \in Z^h\}$$

and

$$Z^h_\alpha(\lambda) = \{n : (n,\lambda) \in Z^h_\alpha\}. \qquad (\lambda \in \mathbf{N})$$

Geometrically we can think of Z^1_α as the lattice points contained in the union of all solid cones with aperture α and vertex contained in $Z^1 = Z$. We say a sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^\infty$ is *Stoltz* if there exists a collection of points Z in $\mathbf{Z} \times \mathbf{N}$, and a function $h = h(t)$ tending to infinity with t such that $(n_l, k_l)_{l=t}^\infty \in Z^{h(t)}$ and there exist h_0, α_0 and $A > 0$ such that for all integers $\lambda > 0$ we have $|Z^{h_0}_{\alpha_0}(\lambda)| \leq A\lambda$. This technical condition is interesting because of the following theorem [BJR].

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THEOREM 1. *Let (X,β,μ,T) denote a dynamical system, with set X, a σ-algebra of its subsets β, a measure μ defined on the measurable space (X,β) such that $\mu(X)=1$ and a measurable, measure preserving map T from X to itself. Suppose f is in $L^1(X,\beta,\mu)$ and that the sequence of pairs on natural numbers $(n_l,k_l)_{l=1}^{\infty}$ is Stoltz then if (X,β,μ,T) is ergodic,*

$$m_f(x)=\lim_{l\to\infty}\frac{1}{k_l}\sum_{i=1}^{k_l}f(T^{n_l+i}x),$$

exists almost everywhere with respect to Lebesgue measure.

Note that if $m_{l,f}(x)=\frac{1}{k_l}\sum_{i=1}^{k_l}f(T^{n_l+i}x)$ then

$$m_{l,f}(Tx)-m_{l,f}(x)=k_l^{-1}(f(T^{n_l+k_l+1})-f(T^{n_l+1}x)).$$

This means that $m_f(Tx)=m_f(x)$ μ-almost everywhere. A dynamical system (X,β,μ,T) is called ergodic if given any $A\in\beta$ the relation $T^{-1}A:=\{x\in X : Tx\in A\}=A$ implies that A has either full or null measure. A standard fact in ergodic theory is that if (X,β,μ,T) is ergodic and $m_f(Tx)=m_f(x)$ almost everywhere, then $m_f(x)=\int_X fd\mu$ μ almost everywhere [CFS]. The term Stoltz is used here because the condition on $(k_l,n_l)_{l=1}^{\infty}$ is analogous to the condition required in the classical non-radial limit theorem for harmonic functions also called a Stoltz condition. See [BJR]. Averages where $k_l=1$ for all l will be called non-moving. Moving averages satisfying the above hypothesis can be constructed by taking for instance $n_l=2^{2^l}$ and $k_l=2^{2^{l-1}}$.

We next introduce the notion of a semi-regular continued fraction expansion. For a continued fraction expansion of a real number x there are two standard notations which we can write in the form

$$x=[c_0;\epsilon_1c_1,\epsilon_2c_2,\cdots]=c_0+\cfrac{\epsilon_1}{c_1+\cfrac{\epsilon_2}{c_2+\ddots}},$$

where $(c_i)_{i=1}^{\infty}$ is a sequence of integers and $\epsilon_i\in\{-1,1\}$. The numbers c_i $(i=1,2,\ldots)$ are called the partial quotients of the expansion and for each natural number k the truncates

$$\frac{p_k}{q_k}=[c_0;\epsilon_1c_1,\cdots,\epsilon_kc_k]=c_0+\cfrac{\epsilon_1}{c_1+\ddots+\cfrac{\epsilon_k}{c_k}},$$

are called the convergents of the expansion. The expansion is called semi-regular if (i) c_i is a natural number, for positive i, (ii) $\epsilon_{i+1}+c_{i+1}\geq 1$ and (iii) $\epsilon_{i+1}+c_{i+1}\geq 2$ for infinitely many i if the expansion is itself infinite. Central to the class of semi-regular continued fraction expansions is the regular continued fraction expansion which is also the most familiar and obtained when c_i is a natural number and $\epsilon_i = 1$ for all i. Here and henceforth for a real number y let $\lfloor y \rfloor$ denote the greatest integer less than y and let $\{y\}$ denote its fractional part, that is $y - \lfloor y \rfloor$. Notice that for the regular continued fraction expansion $c_0 = \lfloor x \rfloor$. It is thus convenient and no real restriction to assume x is in $[0, 1)$. If this is done we define the Gauss map

$$Tx = \left\{ \frac{1}{x} \right\},\ x \neq 0;\ T0 = 0$$

on $[0, 1)$. We see that $c_i(x) = c_1(T^{i-1}x)$ $(i = 1, 2, \ldots)$.

Each regular convergent $\frac{P_n}{Q_n}$ $(n \geq 1)$ to x is always a best approximation to x in the sense that there do not exist better approximations with smaller denominators. That is, for all integers r and s such that $0 < s \leq Q_n$, if for some rational $\frac{r}{s}$ we have

$$\left| x - \frac{r}{s} \right| \leq \left| x - \frac{P_n}{Q_n} \right|$$

then $\frac{r}{s} = \frac{P_n}{Q_n}$. The converse does not hold [Pe §16]. It is nonetheless possible to find best approximants to x by looking at convergents arising from other continued fraction expansions in the semi-regular class, while at the same time requiring that these be as sparse as possible and at the same time remain members of the sequence $(\frac{P_n}{Q_n})_{n=1}^{\infty}$. We now explain how this is done.

As a form of Dirchlet's theorem on diophantine approximation [HW] recall the inequality

$$\left| x - \frac{P_n}{Q_n} \right| \leq \frac{1}{Q_n^2},$$

satisfied by the convergents of the regular continued fraction expansion. Clearly if for each natural number n we set

$$\Psi_n(x) = Q_n^2 \left| x - \frac{P_n}{Q_n} \right|, \tag{2.1}$$

then for each x the sequence $(\Psi_n(x))_{n=1}^{\infty}$ lies in the interval $[0, 1]$. Analogously, if $(\frac{p_k}{q_k})_{k=1}^{\infty}$ are the convergents to x for a semi-regular continued fraction expansion and $(k \geq 1)$, we can set

$$\psi_k(x) = q_k^2 \left| x - \frac{p_k}{q_k} \right|.$$

It was observed by H. Minkowski [Mink], [Pe] that the regular convergents $(\frac{P_{n(k)}}{Q_{n(k)}})_{k=1}^{\infty}$ for which $\Psi_{n(k)}(x) < 1/2$ are the convergents $(\frac{p_k}{q_k})_{k=1}^{\infty}$ of a semi-regular continued fraction expansion. That is $\frac{P_{n(k)}}{Q_{n(k)}} = \frac{p_k}{q_k}$ $(k \geq 1)$. In addition, a theorem of Legendre tells us that if $Q|Qx - P| < 1/2$ then P/Q is a regular convergent [HW]. We will therefore confine attention from now on to expansions for which $\psi_k(x) < 1/2$ holds for all natural numbers k. More particularly, among semi-regular continuous fractions such that $\psi_k(x) < 1/2$ $(k \geq 1)$ we are interested in the ones with convergents $(\frac{p_k}{q_k})_{k=1}^{\infty}$ that are as sparse as possible in $(\frac{P_n}{Q_n})_{n=1}^{\infty}$. There is a restriction on how sparse the sequence $(\frac{p_k}{q_k})_{k=1}^{\infty}$ can be however. This is because to remain the convergents of a semi-regular expansion, one of any two consequtive terms of $(\frac{p_k}{q_k})_{k=1}^{\infty}$ must remain in $(\frac{P_n}{Q_n})_{n=1}^{\infty}$.

A semi-regular continued fraction expansion is called *closest* if the first requirement, namely that $\psi_k(x) < 1/2$, is true for all natural numbers k and is called *fastest* if $(\frac{p_k}{q_k})_{k=1}^{\infty}$ skips the maximal number of regular convergents $(\frac{P_n}{Q_n})_{n=1}^{\infty}$. In particular you can seek to find best approximants $(\frac{p_k}{q_k})_{k=1}^{\infty}$ to x that are also closest and fastest. The optimal continued fraction expansion introduced in [Bo] is the unique semi-regular continued fraction expansion that is both closest and fastest. The purpose of this paper is to study the metrical theory of the optimal continued fraction expansion using Theorem 1. An analogous study of the regular continued fraction transformation appears in [KN]. Results analogous to ours in the case of non-moving averages appear in [BK]. In the next section we introduce and describe the optimal continued fraction expansion. We then obtain new results on the distribution of the sequence $(\psi_n(x))_{n=1}^{\infty}$ for almost all x with respect to Lebesgue measure in the case of the optimal continued fraction expansion.

2. Applying the ergodic theorem to the optimal continued fraction expansion

Let x be an irrational real number and suppose it lies in the interval $(c_0 - \frac{1}{2}, c_0 + \frac{1}{2})$ for some integer c_0 and put $t_0 = x - c_0$, $\epsilon_1(x) = \operatorname{sgn}(t_0)$ and

$$p_1 = 1, \quad p_0 = c_0, \quad q_1 = 0, \quad q_0 = 1, \tag{2.2}$$

and $v_0 = 0$. Suppose t_i, p_i, q_i, c_i, v_i and ϵ_{i+1} have been defined for $i \leq k$ and some positive integer k. Then define $t_{k+1}, p_{k+1}, q_{k+1}, c_{k+1}, v_{k+1}$ and ϵ_{k+2} are defined inductively as follows. Let

$$c_{k+1} = \Big[|t_k|^{-1} + \frac{[|t_k|^{-1}] + \epsilon_{k+1} v_k}{2([|t_k|^{-1}] + \epsilon_{k+1} v_{k+1}) + 1}\Big],$$

$$t_{k+1} = |t_k|^{-1} - c_{k+1}, \qquad f\epsilon_{k+2} = \operatorname{sgn}(t_{k+1}),$$

for $k \geq 1$ let

$$p_{k+1} = c_{k+1} p_k + \epsilon_{k+1} p_{k-1}\ ;\ q_{k+1} = c_{k+1} q_k + \epsilon_{k+1} q_{k-1} \tag{2.3}$$

and let $v_{k+1} = \frac{q_k}{q_{k+1}}$. Now the optimal continued fraction expansion of x is

$$x = [c_0; \epsilon_1 c_1, \epsilon_2 c_2, \ldots].$$

One straightforwardly verifies that

$$t_k = [0; \epsilon_{k+1}c_{k+1}, \epsilon_{k+2}c_{k+2}, \ldots], \tag{2.4}$$

and

$$v_k = [0; c_k, \epsilon_k c_{k-1}, \ldots, \epsilon_2 c_1]. \tag{2.5}$$

The sequence $(\frac{p_k}{q_k})_{k=1}^{\infty}$ are the convergents and as we said in the introduction are a subsequence of the sequence of regular convergents $(\frac{P_n}{Q_n})_{n=1}^{\infty}$ and if we define the function $n: \mathbf{N} \to \mathbf{N}$ by $\frac{p_k}{q_k} = \frac{P_{n(k)}}{Q_{n(k)}}$ then $n(k+1) = n(k)+1$ if and only if $\epsilon_{k+2} = 1$ and $n(k+1) = n(k)+2$ otherwise, once we have set $n(0) = 0$ for $x > 0$ and $n(0) = 1$ otherwise. Define $\Gamma \subset \Omega = ([0,1) \setminus \mathbf{Q}) \times [0,1]$ by

$$\Gamma = \Big\{ (T,V) \in \Omega \,:\, V < \min\Big(T, \frac{2T-1}{1-T}\Big)\Big\}$$

and put $H = \Omega \setminus \Gamma$. We have the following lemma [BK]. Also on Ω define the map

$$\mathcal{T}(x,y) = \Big(Tx, \frac{1}{[\frac{1}{x}] + y}\Big).$$

.

Lemma 2.1. *If x is irrational and n is a natural number, the following are equivalent:*

(i) *the regular continued fraction convergent $\frac{P_n}{Q_n}$ is not an optimal continued fraction convergent;*

(ii) $c_{n+1} = 1$, *$\psi_{n-1} < \psi_n$ and $\psi_n > \psi_{n+1}$;*

(iii) $\mathcal{T}^n(x,0) = (T_n, V_n)$ *is in* Γ.

We now define the map $U: H \to H$, by

$$U(T,V) = \begin{cases} \mathcal{T}(T,V) & \text{if } \mathcal{T}(T,V) \in H; \\ \mathcal{T}^2(T,V) & \text{if } \mathcal{T}(T,V) \notin H. \end{cases}$$

It is convenient to write $g = (1-\sqrt{5})/2$ and $G = \frac{1}{2}(1+\sqrt{5})/2$ henceforth. Let β_H denote the σ-algebra of Borel subsets of H and μ_H the probability measure on H with density $(\log G)^{-1}(1+xy)^{-2}$. The dynamical system (H, β_H, μ_H, U), which is in fact the system induced on H by $\mathcal{T}$, is ergodic, because ergodicity is preserved by inducing on positive measure subsets [CSF]. It is possible to describe a dynamical system explicitly which is isomorphic to (H, β_H, μ_H, U) and which is not described indirectly as an induced system. We do this as follows. Let $\Delta \subset (-1,1) \times (-1,1)$ be defined by

$$\Delta = \Big\{ (t,v) \in (-1,1) \times (-1,1) \,:\, v \le \min\Big(\frac{2t+1}{t+1}, \frac{t+1}{t+2}\Big); v \ge \max\Big(0, \frac{2t-1}{1-t}\Big)\Big\}.$$

Define a map W from Δ to itself by

$$W(t,v) = \Big(|t|^{-1} - \beta(t,v), \frac{1}{\beta(t,v) \ + \ \mathrm{sgn}(t)v}\Big),$$

where

$$\beta(t,v) = \Big[|t|^{-1} + \frac{[|t_k|^{-1}] + \mathrm{sgn}(t)v}{2([|t_k|^{-1}] + \mathrm{sgn}(t)v) + 1}\Big].$$

Also define a measure μ_Δ on Δ by setting its Radon–Nikodym derivative relative to two-dimensional Lebesgue measure to be $(\log G)^{-1}(1+xy)^{-2}$. Finally note that if x is in $(-1/2, 1/2)$ then $W^k(x,0) = (t_k, v_k)$ for all positive integers k. The dynamical system $(\Delta, \beta_\Delta, \mu_\Delta, W)$, where β_Δ is the σ-algebra of Borel sets on Δ, is exact [Kr] and hence ergodic.

We have the following theorem from which all the other results of this paper may be derived.

Theorem 2.2. *Suppose* $(t_k, v_k)_{k=1}^\infty$ *is as defined by* (2.4) *and* (2.5)*. Then if the sequence of pairs of natural numbers* $(n_l, k_l)_{l=1}^\infty$ *is Stoltz, for each element* A *of* β_H *we have*

$$\lim_{l\to\infty} \frac{1}{k_l} \sum_{i=1}^{k_l} \chi_A(t_{n_l+i}, v_{n_l+i}) = \frac{1}{\log G} \int_A \frac{dtdv}{(1+tv)^2},$$

almost everywhere with respect to Lebesgue measure.

Proof. Note that for all y such that (x,y) is in Δ we have

$$\lim_{n\to\infty} (W^n(x,y) - (W^n(x,0)) = 0,$$

and that $W^n(x,0) = (t_n, v_n)$. Then Theorem 2.2 is an immediate consequence of Theorem 1.

□

We now consider applications of this theorem. Let

$$\Pi = \{(w,z) \in \mathbf{R}\times\mathbf{R} : \ w>0, \ z>0, \ 4w^2+z^2<1, \ w^2+4z^2<1\}.$$

Theorem 2.3. *If* A *is a Borel subset of the set* Π *and the sequence of pairs of natural numbers* $(n_l, k_l)_{l=1}^\infty$ *is Stoltz, then we have*

$$\lim_{l\to\infty} \frac{1}{k_l} \sum_{i=1}^{k_l} \chi_A(\psi_{n_l+i-1}(x), \psi_{n_l+i}(x)) = \int_{A\cap\Pi} \Big(\frac{1}{\sqrt{1-4wt}} + \frac{1}{\sqrt{1+4wt}}\Big) dwdz$$

almost everywhere with respect to Lebesgue measure.

PROOF. Let ψ denote the two-to-one map from Δ to Π defined by

$$\Psi(t,v) = \Big(\frac{v}{1+tv}, \frac{\epsilon(t)t}{1+tv}\Big),$$

where $\epsilon(t)$ denotes the sign of t. We note that $\Psi(t_k, v_k) = (\psi_{k-1}, \psi_k)$ for each natural number k. To see this note that from a standard fact from the elementary theory of continued fractions we have

$$x = \frac{p_k + t_k p_{k-1}}{q_k + t_k q_{k-1}} \tag{2.6}$$

and so

$$\psi_k = \frac{\epsilon_{k-1} t_k}{1 + t_k v_k}. \tag{2.7}$$

Set

$$\Delta_{-1} = \{(t,v) \in \Delta \ : \ \epsilon(t) = -1\} \ \text{ and } \ \Delta_1 = \{(t,v) \in \Delta \ : \ \epsilon(t) = 1\}.$$

Also let $\Psi_{-1} = \Psi_{|\Delta_{-1}}$ and $\Psi_1 = \Psi_{|\Delta_1}$. These maps are then continuously differentiable bijective maps from Δ_{-1} (resp. Δ_1) to Π. Using the coordinate change formula for measures, the image measure corresponding to

$$\mu(A) = \frac{1}{\log G} \iint_{A\cap\Pi} \frac{dtdw}{(1+tv)^2}$$

under both maps Ψ_{-1} and Ψ_{-1} is given by

$$(\Psi_{-1}\mu)(B) = (\Psi_1\mu)(B) = \frac{1}{\log G} \iint_{B\cap\Pi} \Big(\frac{1+xy}{1-xy}\Big) dxdy.$$

Now by (2.6) and (2.7) if $\epsilon(t_k) = \epsilon_{k+1} = 1$ then

$$\Big(\frac{1 - t_k v_k}{1 + t_k v_k}\Big)^2 = 1 - 4\psi_{k-1}\psi_k$$

and if $\epsilon(t_k) = \epsilon_{k+1} = -1$ then

$$\Big(\frac{1 - t_k v_k}{1 + t_k v_k}\Big)^2 = 1 + 4\psi_{k-1}\psi_k.$$

Hence the image of μ under ψ is given by

$$(\Psi\mu)(A) = \iint_{A\cap\Pi} \Big(\frac{1}{\sqrt{1-4wt}} + \frac{1}{\sqrt{1+4wt}}\Big) dwdt.$$

The result now follows from Theorem 2.2.

□

In [BK] it is shown that for each irrational x we have $0 < \psi_{k-1} + \psi_k < 2/\sqrt{5}$. Let

$$h(z) = \begin{cases} \frac{1}{\log G}(\log\sqrt{1+z} - \log\sqrt{1-z} + \arctan z), & \text{if } z \in [0, 1/2], \\ \frac{1}{2\log G}\Big(\log\Big(\frac{5\sqrt{5-4z^2}-5z}{\sqrt{5-4z^2}+z}\Big) + 2\arctan\Big(\frac{2\sqrt{5-4z^2}-3z}{5\sqrt{1+z^2}}\Big)\Big), & \text{if } z \in [1/2, 2/\sqrt{5}]. \end{cases}$$

Theorem 2.4. *Let h be as above. Then if the sequence of pairs of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz, then*

$$\lim_{l\to\infty} \frac{1}{k_l}|\{1 \le i \le k_l \,:\, \psi_{n_l+i-1}(x) + \psi_{n_l+i}(x) < a\}| = \int_0^a h(t)dt$$

almost everywhere with respect to Lebesgue measure.

Proof. The result follows immediately by applying Theorem 2.3 to the function $w + t =$ const.

$\square$

In [BK] it is shown that for each irrational x we have $0 \le |\psi_{n-1} - \psi_n| \le 1/2$ for each natural number k. Let

$$j(z) = \frac{1}{\log G}\Big(\log\Big(\frac{5\sqrt{5-4z^2}-5z}{1+z}\Big) - \arctan z + \arcsin\Big(\frac{2\sqrt{5-4z^2}-3z}{\sqrt{1+z^2}}\Big)\Big).$$

We have the following theorem.

Theorem 2.5. *Let j be as defined above. Then if the sequence of natural numbers $(n_l, k_l)_{l=1}^{\infty}$ is Stoltz and a is in $[0, 1/2)$, then we have*

$$\lim_{l\to\infty} \frac{1}{k_l}|\{1 \le i \le k_l : |\psi_{n_l+i-1}(x) - \psi_{n_l+i}(x)| < a\}| = \int_0^a j(t)dt$$

almost everywhere with respect to Lebesgue measure.

Proof. The proof of this result is an immediate consequence of Theorem 2.3 and the appropriate choice of A.

$\square$

In [BK] it is shown that for irrational x, $\psi_k(x)$ is in $(0, \frac{1}{2})$. Let

$$k(z) = \begin{cases} \frac{1}{\log G}, & \text{if } z \in (0, 1/\sqrt{5}), \\ \frac{1}{\log G}\frac{\sqrt{1-4z^2}}{z}, & \text{if } z \in [1/\sqrt{5}, 1/2). \end{cases}$$

We have the following result:

Theorem 2.6. *Suppose k is defined as just above. Then if the sequence of pairs of natural numbers $(n_l, k_n)_{l=1}^{\infty}$ is Stoltz and a is in $[0, 1/2)$, then we have*

$$\lim_{l\to\infty} \frac{1}{k_l}\sum_{i=1}^{N} \chi_A(\psi_{n_l+i}(x)) = \int_{A\cap(0,\frac{1}{2})} k(z)dz$$

almost everywhere with respect to Lebesgue measure.

PROOF. Apply Theorem 2.4 with $w < z$. $\square$

Also calculating the first moment of k we have

THEOREM 2.7. *If the sequence of pairs natural numbers* $(n_l, k_l)_{l=1}^{\infty}$ *is Stoltz then*

$$\lim_{l\to\infty} \frac{1}{k_l} \sum_{l=1}^{k_l} \psi_{n_l+i}(x) = \frac{1}{4\log G} \arctan \frac{1}{2}$$

almost everywhere with respect to Lebesgue measure.

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