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CHARACTERIZATION OF ASYMPTOTIC DISTRIBUTION FUNCTIONS OF RATIO BLOCK SEQUENCES

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Abstract

In this paper we give necessary and sufficient conditions for the block sequence of the set $X = \{x_1 < x_2 < \cdots < x_n < \cdots\} \subset \mathbb{N}$ to have an asymptotic distribution function in the form x^λ.

1. Introduction

Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively. In the whole paper we will assume that X is an infinite set of positive integers. Denote by $R(X) = \{\frac{x}{y}; x \in X,\ y \in X\}$ the *ratio set of* X and say that a set X is (R)-dense if $R(X)$ is (topologically) dense in the set $\mathbb{R}^+$. Let us note that the concept of (R)-density was defined and first studied in the papers [12] and [13].

Now let $X = \{x_1, x_2, \dots\}$ where $x_n < x_{n+1}$ are positive integers. The (R)-density of the set X is equivalent to the everywhere density in $[0,1]$ of the sequence

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1}$$

It is also called the *ratio block sequence* of the set X and we see that it is composed by blocks

$$X_n = \Big(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\Big), \quad n = 1, 2, \dots \tag{2}$$

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which can be studied individually. The density of the sequence of induvidual blocks (2) implies the (R)-density of the set X. Also, if the distribution functions of (1) or (2) are increasing, then again the set X is (R)-dense. This is a motivation for the study of sets $G(x_m/x_n)$ and $G(X_n)$ of distribution functions of (1) and (2), respectively (defined in Par. 2), cf. [17], [18] and [7].

The second motivation for the study of block sequence (2) is that also other kinds of block sequences were studied by several autors, see [3], [8],[9], [14], [20], etc.

2. Definitions and basic results

In the following we use standard notations and definitions from [2], [9] and [16].

By a *distribution function* we mean any function $f\colon \langle 0,1\rangle \to \langle 0,1\rangle$ such that $f(0)=0,\ f(1)=1$ and f is nondecreasing in $\langle 0,1\rangle$.

For the block sequence (1), for $n\in\mathbb{N}$ and $x\in\langle 0,1\rangle$ denote

$$A(X_n,x)=\#\Big\{i : i\le n,\ \frac{x_i}{x_n}\le x\Big\} \quad \text{and} \quad \overline{A}(X_n,x)=\sum_{j=1}^{n}A(X_j,x).$$

Then we can attach to the sequence of blocks (X_n) and to the block sequence (1) the following distribution functions:

$$\begin{aligned} F(X_n,x)&=\frac{A(X_n,x)}{n}\,,\\ F_N(x_m/x_n,x)&=\frac{\#\{[i,j] : 1\le i\le j\le k,\frac{x_i}{x_j}\le x\}+\#\{i : i\le l,\frac{x_i}{x_{k+1}}\le x\}}{N}\\ &=\frac{\overline{A}(X_k,x)+O(k)}{N}=\frac{\overline{A}(X_k,x)}{N}+O\Big(\frac{1}{\sqrt{N}}\Big), \end{aligned}$$

where $x\in\langle 0,1\rangle$ and $N=\frac{k(k+1)}{2}+l$ with $0\le l<k+1$. Consequently

$$\lim_{N\to\infty}\Big(F_N\Big(\frac{x_m}{x_n},x\Big)-\frac{\overline{A}(X_k,x)}{k(k+1)/2}\Big)=0.$$

Denote by $G(X_n)$ the set of all distribution functions $g(x)$ for which there exists an increasing sequence of indices $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty}F(X_{n_k},x)=g(x)$$

almost everywhere (abbreviated as a.e.) in $\langle 0,1\rangle$.

Similarly $G(x_m/x_n)$ denotes the set of all distribution functions $g(x)$ of the block sequence (1) for which there exists an increasing sequence of indices $\{N_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} F_{N_k}\left(\frac{x_m}{x_n}, x\right) = g(x)$$

a.e. in $\langle 0, 1\rangle$.

If the set $G(X_n)$ is a singleton: $G(X_n) = \{g(x)\}$, then we say that the sequence X_n has the *asymptotic distribution function* $g(x)$ (abbreviated as a.d.f.). Similarly if $G(x_m/x_n) = \{g'(x)\}$, then we say that the block sequence (1) of the set X has a.d.f. $g'(x)$. In these cases

$$\lim_{n\to\infty} F(X_n, x) = g(x) \quad \text{and} \quad \lim_{N\to\infty} F_N\left(\frac{x_m}{x_n}, x\right) = g'(x)$$

holds for almost all $x \in \langle 0, 1\rangle$.

Especially, if $G(X_n) = \{g(x) = x\}$, resp. $G(x_m/x_n) = \{g'(x) = x\}$, then we say that the sequence X_n is uniformly distributed (abbreviated as u.d.), resp. the block sequence (1) of the set X is uniformly distributed.

Distribution functions of the sequence X_n and the block sequence (1) of the set X were first investigated in the paper [17], where the next statement is proved ([17], Theorem 8.1, Theorem 8.2, Theorem 8.4).

(A1) If $G(X_n) = \{g(x)\}$, then $G(x_m/x_n) = \{g(x)\}$.

(A2) Let $G(X_n) = \{g(x)\}$. Then one of the following equalities holds:
(i) $g(x) = c_0(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \in (0, 1\rangle \end{cases}$, or
(ii) $g(x) = x^{\lambda}$ for some $0 < \lambda \le 1$.

(A3) $G(X_n) = \{c_0(x)\}$ iff one of the following equalities holds:
(i)

$$\lim_{n\to\infty} \frac{1}{nx_n} \sum_{i=1}^{n} x_i = 0,$$

(ii)

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1}^{m}\sum_{j=1}^{n} \left|\frac{x_i}{x_m} - \frac{x_j}{x_n}\right| = 0.$$

We will use the following notation.

Let $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{R}$, and $E \subset \langle 0, 1\rangle$. We define

$$A(E, X_n) = \#\left\{i : i \le n, \frac{x_i}{x_n} \in E\right\}$$

and

$$\overline{A}(E, X_n) = \#\left\{[i, j] : 1 \le i \le j \le n, \frac{x_i}{x_j} \in E\right\} = \sum_{j=1}^{n} A(E, X_j).$$

Suppose that $G(x_m/x_n) = \{g(x)\}$ and that g is continuous on $\langle 0,1\rangle$. Let $0 \le a < b \le 1$ be real numbers. If $E = (a,b), \langle a,b), (a,b\rangle$ or $\langle a,b\rangle$, then it is obvious that

$$\lim_{n\to\infty} \frac{\overline{A}(E,X_n)}{n(n+1)/2} = g(b) - g(a)\,.$$

3. Results

In this paper we give necessary and sufficient conditions for the block sequence of the set X to have an a.d.f. of the form x^λ. The main result of this paper is the following theorem.

THEOREM 1. *Let $\lambda > 0$ be a real number and $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$. Then $G(x_m/x_n) = \{x^\lambda\}$ iff for every $k \in \mathbb{N}$*

$$\lim_{n\to\infty} \frac{x_{kn}}{x_n} = k^{\frac{1}{\lambda}}\,. \tag{3}$$

We give a sufficient condition for block sequences not to have an asymptotic distribution function in the form x^λ.

LEMMA 1. *Let $\lambda > 0$ be a real number and $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$. Suppose that there exist numbers α and β with $1 < \alpha < \beta$ such that for infinitely many $n, m \in \mathbb{N}, m < n$*

$$\frac{x_n}{x_m} > \beta \qquad \textit{and} \qquad \frac{n}{m} < \alpha^\lambda\,, \tag{4}$$

or

$$\frac{x_n}{x_m} < \alpha \qquad \textit{and} \qquad \frac{n}{m} > \beta^\lambda\,. \tag{5}$$

Then $G(x_m/x_n) \ne \{x^\lambda\}$.

PROOF. We prove the first part of Lemma 1 by contradiction.

Let $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$ be such set that (4) and $G(x_m/x_n) = \{x^\lambda\}$ hold.

By (4) there exists a sequence $\{[m_i,n_i]\}_{i=1}^\infty$, such that for every $i \in \mathbb{N}$ $m_i < n_i$, $n_i < n_{i+1}$ and

$$\frac{x_{n_i}}{x_{m_i}} > \beta \qquad \text{and} \qquad \frac{n_i}{m_i} < \alpha^\lambda\,. \tag{6}$$

Choose $d \in \mathbb{R}$ so that

$$1 < d < 2\Big(\frac{\beta}{\alpha}\Big)^\lambda - 1\,. \tag{7}$$

We now investigate the value of

$$\overline{A}\Big(\Big(\frac{1}{\beta},1\Big\rangle,X_{[dn_i]}\Big)\,.$$

It is obvious that

$$\overline{A}\Big(\Big(\frac{1}{\beta},1\Big\rangle,X_{[dn_i]}\Big)=\overline{A}\Big(\Big(\frac{1}{\beta},1\Big\rangle,X_{n_i}\Big)+\sum_{j=n_i+1}^{[dn_i]}A\Big(\Big(\frac{1}{\beta},1\Big\rangle,X_j\Big)\,. \tag{8}$$

From (6) we obtain

$$A\Big(\Big(\frac{1}{\beta},1\Big\rangle,X_{n_i}\Big)<n_i\Big(1-\frac{1}{\alpha^\lambda}\Big)\,. \tag{9}$$

Then

$$A\Big(\Big(\frac{1}{\beta},1\Big\rangle,X_{n_i+j}\Big)\le A\Big(\Big(\frac{1}{\beta},1\Big\rangle,X_{n_i}\Big)+j, \tag{10}$$

since if $\frac{x_k}{x_{n_i+j}}>\frac{1}{\beta}$, then $\frac{x_k}{x_{n_i}}>\frac{1}{\beta}$ or $k>n_i$. Inequalities (8) and (10) imply that

$$\begin{aligned}\overline{A}\Big(\Big(\frac{1}{\beta},1\Big\rangle,X_{[dn_i]}\Big)\le\overline{A}\Big(\Big(\frac{1}{\beta},1\Big\rangle,X_{n_i}\Big)&+([dn_i]-n_i)n_i\Big(1-\frac{1}{\alpha^\lambda}\Big)+\\&+\frac{([dn_i]-n_i)([dn_i]\quad-n_i+1)}{2}\,.\end{aligned} \tag{11}$$

Then $G(x_m/x_n)=\{x^\lambda\}$ yields

$$\lim_{i\to\infty}\frac{\overline{A}((\frac{1}{\beta},1\rangle,X_{[dn_i]})}{([dn_i])([dn_i]+1)/2}=1-\frac{1}{\beta^\lambda} \tag{12}$$

and

$$\lim_{i\to\infty}\frac{\overline{A}((\frac{1}{\beta},1\rangle,X_{n_i})}{n_i(n_i+1)/2}=1-\frac{1}{\beta^\lambda}\,. \tag{13}$$

On the other hand, using (13) we obtain

$$\begin{aligned}&\lim_{i\to\infty}\frac{\overline{A}((\frac{1}{\beta},1\rangle,X_{n_i})+([dn_i]-n_i)n_i(1-\frac{1}{\alpha^\lambda})+\frac{([dn_i]-n_i)([dn_i]-n_i+1)}{2}}{[dn_i]([dn_i]+1)/2}\\&\quad=\lim_{i\to\infty}\frac{\frac{\overline{A}((\frac{1}{\beta},1\rangle,X_{n_i})}{n_i(n_i+1)/2}\frac{n_i(n_i+1)}{2}+(d-1)n_i^2(1-\frac{1}{\alpha^\lambda})+\frac{n_i^2(d-1)^2}{2}}{n_i^2d^2/2}\\&\quad=\frac{(1-\frac{1}{\beta^\lambda})+2(d-1)(1-\frac{1}{\alpha^\lambda})+(d-1)^2}{d^2}\,.\end{aligned}$$

From this, (11) and (12) we derive that

$$1-\frac{1}{\beta^\lambda}\le\frac{(1-\frac{1}{\beta^\lambda})+2(d-1)(1-\frac{1}{\alpha^\lambda})+(d-1)^2}{d^2}\,.$$

Elementary calculations give

$$\begin{aligned}\Big(1-\frac{1}{\beta^\lambda}\Big)(d^2-1) &\le 2(d-1)\Big(1-\frac{1}{\alpha^\lambda}\Big)+(d-1)^2\,,\\ \Big(1-\frac{1}{\beta^\lambda}\Big)(d+1) &\le 2\Big(1-\frac{1}{\alpha^\lambda}\Big)+(d-1)\,,\\ d-\frac{d}{\beta^\lambda}+1-\frac{1}{\beta^\lambda} &\le 1-\frac{2}{\alpha^\lambda}+d\,,\\ d &\ge 2\Big(\frac{\beta}{\alpha}\Big)^\lambda-1\,,\end{aligned}$$

contradicting (7).

Now we prove by a contradiction the second part of Lemma 1. Let $X = \{x_1 < x_2 < \cdots\}$ be such that (5) and $G(x_m/x_n) = \{x^\lambda\}$ hold. By (5) there exists a sequence $\{[m_i, n_i]\}_{i=1}^\infty$ such that $m_i < n_i < n_{i+1}$ and

$$\frac{x_{n_i}}{x_{m_i}} < \alpha \qquad \text{and} \qquad \frac{n_i}{m_i} > \beta^\lambda \tag{14}$$

for every $i \in \mathbb{N}$.

Let $\gamma \in (\alpha, \beta)$ be arbitrary.

First we show that for every sufficiently large n_i

$$\frac{x_{[(\gamma/\alpha)^\lambda n_i]}}{x_{n_i}} \le \frac{\beta}{\alpha}\,. \tag{15}$$

For a contradiction assume that for infinitely many n_i

$$\frac{x_{[(\gamma/\alpha)^\lambda n_i]}}{x_{n_i}} > \frac{\beta}{\alpha}\,. \tag{16}$$

Let $\gamma < \gamma' < \beta$. It is obvious that

$$\lim_{i\to\infty}\frac{[(\gamma/\alpha)^\lambda n_i]}{n_i} = \Big(\frac{\gamma}{\alpha}\Big)^\lambda < \Big(\frac{\gamma'}{\alpha}\Big)^\lambda\,.$$

Thus there exists $i_0 \in \mathbb{N}$, such that for every $i \ge i_0$

$$\frac{[(\gamma/\alpha)^\lambda n_i]}{n_i} < \Big(\frac{\gamma'}{\alpha}\Big)^\lambda\,.$$

From this, (16) and the fact that $\frac{\gamma'}{\alpha} < \frac{\beta}{\alpha}$, using the first part of the proof one deduces that $G(x_m/x_n) \ne \{x^\lambda\}$. Hence (15) holds for every sufficiently large n_i.

Let n_i be sufficiently large. We will make estimations for

$$\overline{A}\Big(\Big(\frac{1}{\beta}, 1\Big\rangle, X_{[(\gamma/\alpha)^\lambda n_i]}\Big)\,.$$

It is obvious that

$$\overline{A}\Big(\Big(\frac{1}{\beta},1\Big\rangle, X_{[(\frac{\gamma}{\alpha})^\lambda n_i]}\Big) = \overline{A}\Big(\Big(\frac{1}{\beta},1\Big\rangle, X_{n_i}\Big) + \sum_{j=1}^{[(\frac{\gamma}{\alpha})^\lambda n_i]-n_i} A\Big(\Big(\frac{1}{\beta},1\Big\rangle, X_{n_i+j}\Big). \quad (17)$$

Moreover, for every $n_i < k \le [(\gamma/\alpha)^\lambda n_i]$ inequalities (14) and (15) imply that

$$\frac{x_{m_i}}{x_k} \ge \frac{x_{m_i}}{x_{[(\gamma/\alpha)^\lambda n_i]}} = \frac{x_{m_i}}{x_{n_i}}\frac{x_{n_i}}{x_{[(\gamma/\alpha)^\lambda n_i]}} > \frac{1}{\alpha}\frac{\alpha}{\beta} = \frac{1}{\beta}.$$

From this using (14) we obtain for every $1 \le j \le [(\gamma/\alpha)^\lambda n_i] - n_i$

$$A\Big(\Big(\frac{1}{\beta},1\Big\rangle, X_{n_i+j}\Big) \ge n_i - m_i + j \ge n_i\Big(1 - \frac{1}{\beta^\lambda}\Big) + j. \quad (18)$$

Putting (18) into (17) we obtain

$$\begin{aligned}\overline{A}\Big(\Big(\frac{1}{\beta},1\Big\rangle, X_{[(\frac{\gamma}{\alpha})^\lambda n_i]}\Big) &\ge \overline{A}\Big(\Big(\frac{1}{\beta},1\Big\rangle, X_{n_i}\Big) + \sum_{j=1}^{[(\gamma/\alpha)^\lambda n_i]-n_i} n_i\Big(1-\frac{1}{\beta^\lambda}\Big) + j \\ &\ge \frac{\overline{A}\Big(\Big(\frac{1}{\beta},1\Big\rangle, X_{n_i}\Big)}{n_i(n_i+1)/2}\frac{n_i(n_i+1)}{2} + \\ &\quad + \Big(\Big[\Big(\frac{\gamma}{\alpha}\Big)^\lambda n_i\Big] - n_i\Big) n_i\Big(1-\frac{1}{\beta^\lambda}\Big) + \\ &\quad + \frac{([(\frac{\gamma}{\alpha})^\lambda n_i] - n_i)([(\gamma/\alpha)^\lambda n_i] - n_i + 1)}{2}.\end{aligned} \quad (19)$$

On the other hand, the assumption $G(x_m/x_n) = \{x^\lambda\}$ implies that

$$\lim_{i\to\infty} \frac{\overline{A}((\frac{1}{\beta},1\rangle, X_{[(\gamma/\alpha)^\lambda n_i]})}{[(\gamma/\alpha)^\lambda n_i]([(\gamma/\alpha)^\lambda n_i]+1)/2} = 1 - \frac{1}{\beta^\lambda} \quad (20)$$

and

$$\lim_{i\to\infty} \frac{\overline{A}((\frac{1}{\beta},1\rangle, X_{n_i})}{n_i(n_i+1)/2} = 1 - \frac{1}{\beta^\lambda}. \quad (21)$$

From (19), (20) and (21) we can derive

$$1 - \frac{1}{\beta^\lambda} \ge \frac{(1-\frac{1}{\beta^\lambda}) + 2((\gamma/\alpha)^\lambda - 1)(1-\frac{1}{\beta^\lambda}) + ((\gamma/\alpha)^\lambda - 1)^2}{(\gamma/\alpha)^{2\lambda}}.$$

Using elementary computations we obtain

$$\Big(1-\frac{1}{\beta^\lambda}\Big)\Big(\Big(\frac{\gamma}{\alpha}\Big)^{2\lambda} - 2\Big(\frac{\gamma}{\alpha}\Big)^\lambda + 1\Big) \ge \Big(\Big(\frac{\gamma}{\alpha}\Big)^\lambda - 1\Big)^2,$$

hence

$$1 - \frac{1}{\beta^\lambda} \ge 1,$$

a contradiction.

□

The following lemma is required for the proof of Theorem 1.

Lemma 2. *Let* $\varepsilon > 0$ *and* $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$. *Suppose that for every* $k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{x_{kn}}{x_n} = k^{\varepsilon}\,. \tag{22}$$

Then for every real number $\alpha > 0$

$$\lim_{n\to\infty} \frac{x_{[\alpha n]}}{x_n} = \alpha^{\varepsilon}\,.$$

Proof. We will prove the lemma in two steps. First we prove the following claim. If $\lim\limits_{n\to\infty} \frac{x_{kn}}{x_n} = k^{\varepsilon}$ for some $\varepsilon > 0$ and every natural number k, then

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1\,. \tag{23}$$

Let k be a fixed positive integer and n a sufficiently large positive integer. Then there exists a positive integer m such that

$$km \le n < (k+1)m\,.$$

Then obviously $km \le n+1 \le (k+1)m$. From this we obtain

$$1 \le \frac{x_{n+1}}{x_n} \le \frac{x_{(k+1)m}}{x_{km}} = \frac{x_{(k+1)m}/x_m}{x_{km}/x_m} = V(m)\,.$$

If $n \to \infty$, then $m \to \infty$ and obviously $V(m) \to (\frac{k+1}{k})^{\varepsilon}$. Hence

$$1 \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n} \le \Big(\frac{k+1}{k}\Big)^{\varepsilon}\,.$$

This inequality holds for every $k \in \mathbb{N}$. For $k \to \infty$ we obtain from this inequality that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1\,.$$

Now we prove the lemma for the case $\alpha \in \mathbb{Q}^+$.

Let $\alpha = p/q$ where $p, q \in \mathbb{N}$. Let $n \ge q$ be an integer. Let l be the integer given by

$$ql \le n < q(l+1)\,.$$

Then obviously

$$\frac{x_{pl}/x_l}{x_{q(l+1)}/x_{l+1}} \frac{x_l}{x_{l+1}} = \frac{x_{pl}}{x_{q(l+1)}} \le \frac{x_{[\alpha n]}}{x_n} \le \frac{x_{p(l+1)}}{x_{ql}} = \frac{x_{p(l+1)}/x_{l+1}}{x_{ql}/x_l} \frac{x_{l+1}}{x_l}\,.$$

From this, (22), (23) we obtain

$$\lim_{n\to\infty} \frac{x_{[\alpha n]}}{x_n} = \frac{p^{\varepsilon}}{q^{\varepsilon}} = \alpha^{\varepsilon}\,.$$

It remains to prove that the lemma is valid also for irrational $\alpha > 0$.

Let $\delta > 0$. From the facts that the set $\mathbb{Q}^+$ is dense in $\mathbb{R}^+$ and that the function $f(x) = x^\varepsilon$ is increasing and continuous it follows that there are numbers $\alpha_1, \alpha_2 \in \mathbb{Q}$ such that $0 < \alpha_1 < \alpha < \alpha_2$ and

$$\alpha^\varepsilon - \alpha_1^\varepsilon < \delta \quad \text{and} \quad \alpha_2^\varepsilon - \alpha^\varepsilon < \delta\,.$$

Then from the inequalities

$$\alpha^\varepsilon - \delta < \alpha_1^\varepsilon = \lim_{n\to\infty} \frac{x_{[\alpha_1 n]}}{x_n} \le \liminf_{n\to\infty} \frac{x_{[\alpha n]}}{x_n} \le \limsup_{n\to\infty} \frac{x_{[\alpha n]}}{x_n} \le \lim_{n\to\infty} \frac{x_{[\alpha_2 n]}}{x_n} = \alpha_2^\varepsilon < \alpha^\varepsilon + \delta$$

it follows that

$$\lim_{n\to\infty} \frac{x_{[\alpha n]}}{x_n} = \alpha^\varepsilon\,.$$

□

Using Lemma 1 and Lemma 2 we can prove Theorem 1.

Proof of Theorem 1. We will prove the theorem in two steps.

I) The condition (3) is necessary for block sequences to have asymptotic distribution function in the form x^λ.

Suppose that there exists a set $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$ and $k \in \mathbb{N}$ such that $G(x_m/x_n) = \{x^\lambda\}$, but

$$\lim_{n\to\infty} \frac{x_{kn}}{x_n} \ne k^{1/\lambda}\,.$$

(It is obvious that $k \ge 2$.) Then there are two possibilities:

1) $\liminf_{n\to\infty}(x_{kn}/x_n) < k^{1/\lambda}$,

2) $\limsup_{n\to\infty}(x_{kn}/x_n) > k^{1/\lambda}$.

In the first case put

$$\alpha' = \liminf_{n\to\infty} \frac{x_{kn}}{x_n} \quad \text{and} \quad \beta' = k^{1/\lambda}\,.$$

Since $1 \le \alpha' < \beta'$, we can take numbers α, β such that $\alpha' < \alpha < \beta < \beta'$. Then we have for infinitely many $n \in \mathbb{N}$

$$\frac{x_{kn}}{x_n} < \alpha \quad \text{and} \quad \frac{kn}{n} = k = (\beta')^\lambda > \beta^\lambda\,.$$

Using Lemma 1 we obtain $G(x_m/x_n) \ne \{x^\lambda\}$, a contradiction.

In the second case put

$$\beta' = \limsup_{n\to\infty} \frac{x_{kn}}{x_n} \quad \text{and} \quad \alpha' = k^{1/\lambda}\,.$$

(It is possible that $\beta' = +\infty$.) Since $1 \le \alpha' < \beta'$, we can take numbers α, β, such that $\alpha' < \alpha < \beta < \beta'$. Then we have for infinitely many $n \in \mathbb{N}$

$$\frac{x_{kn}}{x_n} > \beta \quad \text{and} \quad \frac{kn}{n} = k = (\alpha')^\lambda < \alpha^\lambda .$$

Using Lemma 1 we obtain $G(x_m/x_n) \ne \{x^\lambda\}$.

II) The condition (3) is sufficient for block sequences to have asymptotic distribution function in the form x^λ.

From (A1) it follows that it is sufficient to prove that $G(X_n) = \{x^\lambda\}$. Let $\alpha \in (0,1)$. We will show that

$$\lim_{n\to\infty} F(X_n, \alpha) = \lim_{n\to\infty} \frac{\#\{i : i \le n, \frac{x_i}{x_n} < \alpha\}}{n} = \alpha^\lambda .$$

Let $0 < \alpha_1 < \alpha$. Lemma 2 implies that

$$\lim_{n\to\infty} \frac{x_{[\alpha_1^\lambda n]}}{x_n} = (\alpha_1^\lambda)^{1/\lambda} = \alpha_1 < \alpha .$$

Then there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$

$$\frac{x_{[\alpha_1^\lambda n]}}{x_n} < \alpha .$$

Hence, if $i \le [\alpha_1^\lambda n]$, then $x_i/x_n < \alpha$. This means that

$$\#\Big\{ i : i \le n,\ \frac{x_i}{x_n} < \alpha \Big\} \ge [\alpha_1^\lambda n] .$$

Then

$$\liminf_{n\to\infty} \frac{\#\{i : i \le n,\ x_i/x_n < \alpha\}}{n} \ge \lim_{n\to\infty} \frac{[\alpha_1^\lambda n]}{n} = \alpha_1^\lambda . \tag{24}$$

Since (24) holds for every $0 < \alpha_1 < \alpha$, we have

$$\liminf_{n\to\infty} \frac{\#\{i : i \le n,\ x_i/x_n < \alpha\}}{n} \ge \alpha^\lambda. \tag{25}$$

Similarly

$$\limsup_{n\to\infty} \frac{\#\{i : i \le n,\ x_i/x_n < \alpha\}}{n} \le \alpha^\lambda. \tag{26}$$

Inequalities (25) and (26) imply that

$$\lim_{n\to\infty} \frac{\#\{i : i \le n,\ x_i/x_n < \alpha\}}{n} = \alpha^\lambda .$$

This completes the proof of Theorem 1.

□

COROLLARY 1. *The block sequence* (1) *is uniformly distributed iff*

$$\lim_{n\to\infty} \frac{x_{kn}}{x_n} = k$$

for all $k \in \mathbb{N}$.

PROOF. The statement is an immediate consequence of Theorem 1. □

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References

[1] H. G. Barone, Limit points of sequences and their transforms by methods of summability, *Duke Math. J.*, **5** (1939), 740–752.

[2] M. Drmota and R. F. Tichy, *Sequences, Discrepancies and Applications*, Lecture Notes in Mathematics 1651, Springer-Verlag, Berlin, 1997.

[3] E. Hlawka, *The Theory of Uniform Distribution*, AB Academic publishers, London, 1984.

[4] F. Filip, L. Mišík and J. T. Tóth, Dispersion of ratio block sequences and asymptotic density, *Acta Arith.*, **131** (2008), 183–191.

[5] F. Filip, L. Mišík and J. T. Tóth, On distribution functions of certain block sequences, *Uniform Distribution Theory*, **2** (2007), 115–126.

[6] F. Filip and J. T. Tóth, On estimations of dispersions of certain dense block sequences, *Tatra Mt. Math. Publ.*, **31** (2005), 65–74.

[7] G. Grekos and O. Strauch, Distribution functions of ratio sequences, II, *Uniform Distribution Theory*, **2** (2007), 53–77.

[8] S. Knapowski, Über ein Problem der Gleichverteilung, *Colloq. Math.*, **5** (1958), 8–10.

[9] L Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, John Wiley & Sons, New York, 1974, reprint: Dover Publications, Inc. Mineola, New York, 2006.

[10] G. Myerson, A sampler of recent developments in the distribution of sequences, *Number theory with an emphasis on the Markoff spectrum* (Provo, UT, 1991), Lecture Notes in Pure and App.Math. Vol. 147, Marcel Dekker, Basel, 1993, 163–190.

[11] Š. Porubský, T. Šalát and O. Strauch, On a class of uniform distributed sequences, *Math. Slovaca*, **40** (1990), 143–170.

[12] T. Šalát, On ratio sets of sets of natural numbers, *Acta Arith.*, **XV** (1969), 273–278.

[13] T. Šalát, Quotientbasen und (R)-dichte Mengen, *Acta Arith.*, **19** (1971), 63–78.

[14] I. J. Schoenberg, Über die asymptotische Verteilung reeller Zahlen mod 1, *Math. Z.*, **28** (1928), 171–199.

[15] O. Strauch, *Unsolved Problem 1.9.2.*, Unsolved Problems Section on the homepage of Uniform Distribution Theory, http://udt.mat.savba.sk/udt_unsolv.htm.

[16] O. Strauch and Š. Porubský, *Distribution of Sequences: A Sampler*, Peter Lang, Frankfurt am Main, 2005.

[17] O. Strauch, J. T. Tóth, Distribution functions of ratio sequences, *Publ. Math. Debrecen*, **58** (2001), 751–778.

[18] O. Strauch and J. T. Tóth, Corrigendum to Theorem 5 of the paper "Asymptotic density of $A \subset \mathbb{N}$ and density of ratio set $R(A)$" (Acta Arith. 87 (1998), 67–78), *Acta Arith.*, **103** (2002), 191–200.

[19] J. T. Tóth, L. Mišík and F. Filip, On some properties of dispersion of block sequences of positive integers, *Math. Slovaca*, **54** (2004), 453–464.

[20] R. F. Tichy, Three examples of triangular arrays with optimal discrepancy and linear recurrences, *Applications of Fibonacci numbers*, **7** (1998), 415–423.

[21] R. Winkler, On the distribution behaviour of sequences, *Math. Nachr.*, **186** (1997), 303–312.