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TRANSITIVE PARTIAL ACTIONS OF GROUPS

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Abstract

J. Kellendonk and M. V. Lawson established that each partial action of a group G on a set Y can be extended to a global action of G on a set Y_G containing a copy of Y. In this paper we classify transitive partial group actions. When G is a topological group acting on a topological space Y partially and transitively we give a condition for having a Hausdorff topology on Y_G such that the global group action of G on Y_G is continuous and the injection Y into Y_G is an open dense equivariant embedding.

1. Introduction

Throughout this paper we shall always assume that G is a group with multiplication gh $(g, h \in G)$. 1 denotes the identity of G and g^{-1} denotes the inverse of g in G. Recall that an *action* of G on the set X is a function $G \times X \to X$, $(g, x) \mapsto g \cdot x$ such that $1 \cdot x = x$, $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in G$, and that it can also be defined by means of a homomorphism from G to the symmetric group on X. Two G-actions on X and X' are said to be *equivalent* if there is a bijection $f\colon X \to X'$ such that $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$. Such a map f is called an *isomorphism* between two G-actions.

An *inverse semigroup* S is a semigroup in which for every $s \in S$ there exists a unique element s^{-1}, called the inverse of s, satisfying $ss^{-1}s = s$, $s^{-1}ss^{-1} = s^{-1}$. The Wagner–Preston representation theorem ([6], [9]) states that every inverse monoid can be embedded in a symmetric inverse monoid $I(X)$ on a set X: this

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consists of all partial bijections on the set X under the usual operation of composition of partial functions. Let $\leq$ be the natural partial order on the inverse monoid $I(X)$. Then for $s, t \in I(X)$, $s \leq t$ if and only if the domain of t contains the domain of s and the two partial maps s and t agree on the domain of s.

A *partial action* (this terminology was introduced by R. Exel, [7]) of G on a non-empty set Y is a function $\theta\colon G \to I(Y)$ satisfying the following three conditions:
(P1) $\theta(g^{-1}) = \theta(g)^{-1}$ for all $g \in G$;
(P2) $\theta(g)\theta(h) \leq \theta(gh)$ for all $g, h \in G$;
(P3) $\theta(1) = 1_Y$, the identity on Y.

Notice that the essential difference between partial group actions and group actions lies in condition (P2): $\theta(gh)$ is an extension of $\theta(g)\theta(h)$ for all $g, h \in G$.

For an action of G on a set X, $G \times X \to X, (g, x) \mapsto g \cdot x$, and a non-empty subset $Y \subset X$, each element of $g \in G$ induces a partial bijection of Y whose domain is given by $\{y \in Y \,:\, g \cdot y \in Y\}$, and hence there is a natural partial action θ of G on Y defined by

$$\theta(g)\colon \mathrm{dom}(g) := \{y \in Y \,:\, g \cdot y \in Y\} \to Y, \quad \theta(g)(x) = g \cdot x.$$

We say that this partial action arises by *restricting* the global action.

Two partial actions $\theta\colon G \to I(Y)$, $\theta'\colon G \to I(Y')$ are said to be *equivalent* if there exists a bijection $f\colon Y \to Y'$ such that for $x \in Y$, $\theta(g)(x)$ is defined if and only if $\theta'(g)(f(x))$ is defined, and in this case $\theta'(g)(f(x)) = f(\theta(g)(x))$. Such a map f is called an *isomorphism* from θ to θ'.

In [10] J. Kellendonk and M. V. Lawson established that each partial group action is the restriction of a global group action. In this paper, we are interested in transitive partial group actions which are common in projective geometry (e.g., pseudogroups of conformal transformations). We first describe the partial action of Möbius transformations on $\mathbb{R}$ and present some of its key properties in order to illustrate our approach.

Let $I(\mathbb{R})$ be the inverse monoid of all partial bijections on $\mathbb{R}$. Let $G := \mathrm{GL}(2, \mathbb{R})$. Then the mapping defined by

$$\theta\colon G \to I(\mathbb{R}),\ \theta(g)(x) := \frac{ax+b}{cx+d},\ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R})$$

is a partial action of G on $\mathbb{R}$ ([10]). This partial action is *transitive* in the sense that for each pair of points x, y in $\mathbb{R}$ there is an element $g \in G$ such that $\theta(g)(x) = y$. Note that the translation $\mathbb{R} \ni z \mapsto z + (y - x) \in \mathbb{R}$ is an element of $\theta(G)$. The set G_0 of elements of G fixing 0 the origin ($\theta(g)(0)$ is defined and $\theta(g)(0) = 0$) can be represented by lower-triangular matrices and it forms a subgroup of G. Thus there is a natural group action of G on the quotient set G/G_0. Furthermore, the set G^0 of all elements of G such that $\theta(g)(0)$ is defined can be represented by elements of the form $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R}), d \neq 0$. Then the partial action θ of G on $\mathbb{R}$ is equivalent to the partial action arising by restricting the coset action to the set $\pi(G^0)$, where $\pi: G \to G/G_0$ denotes the natural projection.

However, the group G is a Lie group and G_0 is a *closed subgroup* of G. It is well-known that the coset space G/G_0 is diffeomorphic to the Riemann sphere $\mathbb{R}^\infty$. The evaluation mapping $\mathrm{ev}_0: G^0 \to \mathbb{R}, g \mapsto \theta(g)(0)$ is *continuous.* Furthermore, the set G^0 is an *open dense* subset of G leading to the open dense equivariant embedding of $\mathbb{R}$ into the coset space G/G_0.

In Section 2, we prove that each transitive partial group action is the restriction of a group coset action. In Section 3, we shall define a group bounded inverse submonoid of a symmetric inverse monoid $I(Y)$ and shall show that there is a global action of G on a set X such that each non-zero element of the semigroup can be uniquely extended to a symmetry on X induced by the action. In Section 4, we restrict our attention to transitive partial actions of topological groups. Mainly, we study Hausdorff globalisation conditions with open dense equivalent embedding ([4]), and revisit the partial actions of conformal transformations and show how these examples fit within our framework.

2. Transitive partial group actions

Let $\theta: G \to I(Y)$ be a partial action of G on Y. If $\theta(g)(x)$ is defined, then we shall write $\theta(g)(x) = g \cdot x$. For convenience, we shall write $\exists g \cdot x$ to mean that $g \cdot x$ is defined. The partial action θ of G on Y defines a partial function from $G \times Y$ to Y which satisfies the following conditions:

(PA1) $\exists g \cdot x$ implies that $\exists g^{-1} \cdot (g \cdot x)$ and $g^{-1} \cdot (g \cdot x) = x$;

(PA2) $\exists g \cdot (h \cdot x)$ implies that $\exists (gh) \cdot x$ and $g \cdot (h \cdot x) = (gh) \cdot x$;

(PA3) $\exists 1 \cdot x$ for all $x \in Y$, and $1 \cdot x = x$.

Conversely, a partial function

$$G \times Y \to Y, \ (g, x) \mapsto g \cdot x$$

which satisfies (PA1), (PA2), and (PA3) induces a partial action of G on Y, [10].

A *globalisation* of a partial action $\theta: G \to I(Y)$ is an action (G, X) together with an injection $\imath: Y \to X$ such that the partial action θ and the induced partial action of G on $\imath(Y)$ from the action (G, X) are equivalent. Let $\theta': G \to I(\imath(Y))$ be the partial action induced by the action $\alpha: G \times X \to X$, $\alpha(g, x) := g \cdot x$. Then $\exists\, \theta(g)(x)$ if and only if $\exists\, \theta'(g)(\imath(x))$ and $\theta'(g)(\imath(x)) = g \cdot \imath(x)$ for $g \in G$ and $x \in Y$.

A globalisation $\{(G, X), \imath\}$ of the partial action θ is said to be *minimal* if for any globalisation $\{(G, X'), \imath'\}$ of θ, there exists an injection $\lambda: X \to X'$ such that $\lambda(g \cdot x) = g \cdot \lambda(x)$ for all $g \in G$ and $x \in X$.

In [10] J. Kellendonk and M. V. Lawson has proved that every partial action has a unique (up to equivalence) minimal globalisation: Let $\theta: G \to I(Y)$ be a partial action of the group G on Y. Define the relation $\sim$ on the set $G \times Y$ by $(g, x) \sim (h, y)$ if and only if $\exists (h^{-1}g) \cdot x$ and $(h^{-1}g) \cdot x = y$. The relation $\sim$ is an equivalence relation on $G \times Y$. If we denote the set of $\sim$-equivalence classes on $G \times Y$ by Y_G, and denote the $\sim$-equivalence class containing the element (g, x) by $[g, x]$, then the function

$$G \times Y_G \to Y_G,\ (h, [g, x]) \mapsto h \cdot [g, x] = [hg, x]$$

is an action of G on Y_G and the function

$$\imath: Y \to Y_G,\ \imath(x) = [1, x]$$

is injective, [10].

Lemma 2.1. *Let $g, h \in G$ and let $x \in Y$. If $\exists h \cdot x$, then $[g, h \cdot x] = [gh, x]$. Furthermore, $\exists g \cdot x$ if and only if $\exists g \cdot [1, x]$.*

Proof. By (PA1), $(h^{-1}g^{-1}g)\cdot(h\cdot x) = h^{-1}\cdot(h\cdot x) = x$. Thus $[g, h\cdot x] = [gh, x]$. Suppose that $g \cdot x = y \in Y$. Then $g \cdot [1, x] = [g, x] = [1, g \cdot x] = [1, y] \in \imath(Y)$. Conversely, suppose that $g \cdot [1, x] = [g, x] \in \imath(Y)$. Then $[g, x] = [1, y]$ for some $y \in Y$. By definition, $\exists g \cdot x$ and $g \cdot x = y$.

$\square$

Theorem 2.2. (Kellendonk and Lawson) *Let $\theta: G \to I(Y)$ be a partial action. Then $\{(G, Y_G), \imath\}$ is a unique minimal globalisation of θ up to equivalence.*

Proof. Lemma 2.1 shows that $\{(G, Y_G), \imath\}$ is a globalisation of θ. Let $\{(G, X), \jmath\}$ be any globalisation of θ. Define $\alpha: Y_G \to X$ by $\alpha([g, x]) = g \cdot \jmath(x)$.

To show that α is well-defined, suppose that $[g, x] = [h, y]$. Then $\exists\, (h^{-1}g) \cdot x$ and $(h^{-1}g) \cdot x = y$. Thus we have that $\jmath(y) = \jmath((h^{-1}g) \cdot x) = (h^{-1}g) \cdot \jmath(x)$, and hence $h \cdot \jmath(y) = g \cdot \jmath(x)$. It follows that $\alpha([g, x]) = g \cdot \jmath(x) = h \cdot \jmath(y) = \alpha([h, y])$.

We now show that α is injective. Suppose that $\alpha([g, x]) = \alpha([h, y])$. Then $g \cdot \jmath(x) = h \cdot \jmath(y)$, and hence we have $\jmath((h^{-1}g) \cdot x) = (h^{-1}g) \cdot \jmath(x) = \jmath(y)$. Thus $(h^{-1}g) \cdot x = y$, and we have that $[g, x] = [h, y]$. Clearly, $\alpha(g \cdot x) = g \cdot \alpha(x)$ for all $g \in G$ and $x \in Y_G$. Thus we have that $\{(G, Y_G), \imath\}$ is a minimal globalisation of θ.

Suppose that $\{(G, Z), \jmath\}$ is a minimal globalisation of θ. Then there are injections $\lambda: Y_G \to Z,\ \beta: Z \to Y_G$ such that $\lambda(g \cdot x) = g \cdot \lambda(x)$, $\beta(h \cdot z) = h \cdot \beta(z)$ for all $x \in Y_G,\ z \in Z,\ g, h \in G$. Notice that

$$\begin{aligned}\beta \circ \lambda([g, y]) &= \beta \circ \lambda(g \cdot [1, y]) \\ &= \beta(g \cdot \lambda([1, y])) = g \cdot \beta(\lambda([1, y])) \\ &= g \cdot \beta(\lambda(\imath(y))) = g \cdot \beta(\jmath(y)) = g \cdot [1, y] = [g, y].\end{aligned}$$

Thus we have that $\beta \circ \lambda = 1_{Y_G}$. Since β and λ are injective we get that the actions (G, Y_G) and (G, Z) are equivalent.

$\square$

Proposition 2.3. *Each isomorphism between two partial G-actions $\theta: G \to I(Y)$ and $\theta': G \to I(Y')$ extends an isomorphism between G-actions $(G, Y_G, \imath_Y)$ and $(G, Y'_G, \imath_{Y'})$ sending $\imath_Y(Y)$ onto $\imath_{Y'}(Y')$ and vice versa.*

Proof. Let $f: Y \to Y'$ be an isomorphism from $\theta: G \to I(Y)$ to $\theta': G \to I(Y')$. Define $\overline{f}: Y_G \to Y'_G$ by $\overline{f}([g,x]) = [g, f(x)]$. We will first prove that $\overline{f}$ is well-defined and is injective. Suppose that $[g,x] = [h,y]$ in Y_G. Then $\exists h^{-1}g \cdot x$ and $h^{-1}g \cdot x = y$. Since f is an isomorphism, $\exists h^{-1}g \cdot f(x)$ and $f(y) = f(h^{-1}g \cdot x) = h^{-1}g \cdot f(x)$. This implies that $[g, f(x)] = [h, f(y)]$. Thus $\overline{f}$ is well-defined. By the same argument, we have that $\overline{f}$ is injective. Obviously, $\overline{f}$ is surjective. Since

$$\overline{f}(h \cdot [g,x]) = \overline{f}([hg,x]) = [hg, f(x)] = h \cdot [g, f(x)] = h \cdot \overline{f}([g,x]),$$

$\overline{f}$ is an isomorphism from $(G, Y_G, \imath_Y)$ to $(G, Y'_G, \imath_{Y'})$. Clearly, $\overline{f}(\imath_Y(Y)) = \imath_{Y'}(Y')$ and $\overline{f}$ is an extension of f.

Conversely, suppose that $\overline{f}: Y_G \to Y'_G$ is an isomorphism such that $\overline{f}(\imath_Y(Y)) = \imath_{Y'}(Y')$. Let $f := \imath_{Y'}^{-1} \circ \overline{f} \circ \imath_Y : Y \to Y'$. Then f is a bijection. It is easy to show that $\overline{f}([1,x]) = [1, f(x)]$ for all $x \in Y$. Suppose that $\exists g \cdot x$. Then since $g \cdot [1,x] = [1, g \cdot x]$, we have

$$g \cdot [1, f(x)] = g \cdot [1, x'] = g \cdot \overline{f}([1,x]) = \overline{f}(g \cdot [1,x]) = \overline{f}([1, g \cdot x]) \in \imath_{Y'}(Y'),$$

and in this case $f(g \cdot x) = g \cdot f(x)$. Since $\overline{f}^{-1}$ is also an isomorphism from $(G, Y'_G, \imath_{Y'})$ to $(G, Y_G, \imath_Y)$, by the same argument, we conclude that $\exists g \cdot x$ if and only if $\exists g \cdot f(x)$, and in this case $f(g \cdot x) = g \cdot f(x)$. Therefore f is an isomorphism from θ to θ'. □

A partial action $\theta: G \to I(Y)$ is said to be *transitive* if for any $x, y \in Y$, there is an element $g \in G$ such that $\exists g \cdot x$ and $y = g \cdot x$.

Proposition 2.4. *Let $\theta: G \to I(Y)$ be a partial action. Then θ is transitive if and only if the action of G on Y_G is transitive.*

Proof. Suppose that θ is transitive. Let $[g,x], [h,y] \in Y_G$. Since the partial action is transitive, $x = k \cdot y$ for some $k \in G$. Then by Lemma 2.1, $(hk^{-1}g^{-1}) \cdot [g,x] = [hk^{-1}, k \cdot y] = [h,y]$.

Conversely, suppose that G acts transitively on Y_G. Let $x, y \in Y$. Then there exists $g \in G$ such that $g \cdot [1,x] = [1,y]$. Since $g \cdot [1,x] = [g,x]$, we have that $\exists g \cdot x$ and $g \cdot x = y$. Therefore the partial action of G on Y is transitive. □

Proposition 2.5. *Let $\theta: G \to I(Y)$ be a partial action. Then for each $x \in Y$,*

$$G_x := \{g \in G : \exists g \cdot x \text{ and } g \cdot x = x\}$$

is a subgroup of G. In particular, if θ is transitive then G_x and G_y are conjugate for all $x, y \in Y$.

Proof. Obviously, the identity $1 \in G$ is an element of G_x. Suppose that $g \in G_x$. Then $\exists g \cdot x$ and $g \cdot x = x$. By (PA1), $g^{-1} \in G_x$. Suppose that $g, h \in G_x$. Then $\exists g \cdot (h \cdot x)$ and hence by (PA2), $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$. Thus $gh \in G_x$. Therefore G_x is a subgroup of G.

Suppose that θ is transitive. Let $x, y \in Y$. Choose $h \in G$ such that $h \cdot x = y$. Then $g \cdot y = y$ if and only if $g \cdot (h \cdot x) = (gh) \cdot x = h \cdot x$ (by (PA2)) if and only if $h^{-1} \cdot ((gh) \cdot x) = x$ (by (PA1)) if and only if $(h^{-1}gh) \cdot x = x$ if and only if $h^{-1}gh \in G_x$. Therefore, G_x and G_y are conjugate.

□

Theorem 2.6. *Suppose that the partial action of G on Y is transitive. Then the action (G, Y_G) is equivalent to the left coset action $(G, G/G_x)$ for any $x \in Y$.*

Proof. Let x_0 be an element of Y. Define $\phi: G/G_{x_0} \to Y_G,\ gG_{x_0} \mapsto [g, x_0]$. We show first that ϕ is a well-defined bijection. Suppose that $gG_{x_0} = hG_{x_0}$. Then $h = gk$ for some $k \in G_{x_0}$. Since $[g, x_0] = [g, k \cdot x_0] = [gk, x_0] = [h, x_0]$ by Lemma 2.1, $\phi(gG_{x_0}) = \phi(hG_{x_0})$. Thus ϕ is a well-defined map. Suppose that $\phi(gG_0) = \phi(hG_0)$. Then $[g, x_0] = [h, x_0]$. By the definition of the equivalence relation, $\exists (h^{-1}g) \cdot x_0$ and $(h^{-1}g) \cdot x_0 = x_0$. Thus $h^{-1}g \in G_{x_0}$ and $gG_{x_0} = hG_{x_0}$. Let $[g, x] \in Y_G$ be any element. Since G acts transitively on Y_G, there is $h \in G$ such that $[g, x] = h \cdot [1, x_0]$. Since $h \cdot [1, x_0] = [h, x_0]$, we conclude that $[g, x] = \phi(hG_{x_0})$. Therefore the map ϕ is a bijection.

Finally, we show that $(G, G/G_{x_0})$ and (G, Y_G) are equivalent via the map ϕ. It follows that $\phi(ghG_{x_0}) = [gh, x_0] = g \cdot [h, x_0] = g \cdot \phi(hG_{x_0})$.

□

Let $\theta: G \to I(Y)$ be a partial and transitive action. Fix $x_0 \in Y$. We let

$$G^{x_0} := \{g \in G : \exists g \cdot x_0\}.$$

Let $\phi: G/G_{x_0} \to Y_G,\ gG_{x_0} \mapsto [g, x_0]$, and let $\pi: G \to G/G_{x_0},\ \pi(g) = gG_{x_0}$. Then it is easy to see that $\phi^{-1}(\imath(Y)) = \pi(G^{x_0})$. Note that the map

$$\imath_\pi: Y \ni y \mapsto gG_{x_0} \in G/G_{x_0},\ g \cdot x_0 = y$$

is a well-defined injection. By Theorem 2.2 and Theorem 2.6, we have

Corollary 2.7. *For a transitive partial action $\theta: G \to I(Y)$ and $x_0 \in Y$,*

$$\{(G, G/G_{x_0}), \imath_\pi\}$$

is the unique (up to equivalence) minimal globalisation of the partial action θ.

By the following decomposition of G

$$G = G_{x_0} \cup (G^{x_0} \setminus G_{x_0}) \cup (G \setminus G^{x_0}),$$

the next result is immediate.

Proposition 2.8. *Suppose that the partial action of G on Y is transitive. Let $x_0 \in Y$. Then*

$$Y_G = \{[1, x_0]\} \cup \{[g, x_0] : g \in G^{x_0} \setminus G_{x_0}\} \cup \{[g, x_0] : g \in G \setminus G^{x_0}\}.$$

Furthermore, we have

(i) $[1, x_0] \neq [g, x_0]$ *for all* $g \in G \setminus G_{x_0}$*;*
(ii) $[g, x_0] \neq [h, x_0]$ *for all* $g \in G^{x_0}, h \in G \setminus G^{x_0}$*;*
(iii) *for* $g, h \in G$, $[g, x_0] = [h, x_0]$ *if and only if* $h^{-1}g \in G_{x_0}$*;*
(iv) $\{[1, y] : y \in Y \setminus \{x_0\}\} = \{[g, x_0] : g \in G^{x_0} \setminus G_{x_0}\}$*, and hence*

$$\imath(Y) = \{[1, x_0]\} \cup \{[g, x_0] : g \in G^{x_0} \setminus G_{x_0}\}.$$

Proof. (i), (ii), and (iii): Straightforward.

(iv) Let $y \in Y \setminus \{x_0\}$. By the transitivity of the partial action of G on Y, there exists $g \in G$ such that $\exists g \cdot x_0$ and $g \cdot x_0 = y (\neq x_0)$. It follows that $g \in G^{x_0} \setminus G_{x_0}$. By Lemma 2.1, $[g, x_0] = [1, g \cdot x_0] = [1, y]$. Conversely, if $g \in G^{x_0} \setminus G_{x_0}$, then $\exists g \cdot x_0$ and $g \cdot x_0 \neq x_0$. Thus $g \cdot x_0 \in Y \setminus \{x_0\}$ and $[g, x_0] = [1, g \cdot x_0]$.

□

Example. Let $G = \mathrm{SL}(2, \mathbb{R}), Y = \mathbb{R}$. Then G acts partially on Y via linear fractional (Möbius) transformations

$$g \cdot x = \frac{ax + b}{cx + d}, \qquad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G.$$

Since $t_x := \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] \in G$ for each $x \in \mathbb{R}$ and since $t_{y-x} \cdot x = y$, the partial action is transitive.

One computes that

$$\begin{aligned} G_0 &= \left\{ \begin{bmatrix} a & 0 \\ c & 1/a \end{bmatrix} \in G : a \neq 0 \right\}, \\ G^0 &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G : d \neq 0 \right\} = \{t_x : x \in \mathbb{R}\} G_0, \\ G \setminus G^0 &= \left\{ \begin{bmatrix} a & b \\ -1/b & 0 \end{bmatrix} \in G : b \neq 0 \right\}. \end{aligned}$$

A particular property of this action is that if $g, h \in G \setminus G^0$ then $h^{-1}g \in G_0$. Thus the set $\{[g, 0] : g \in G \setminus G^0\}$ consists of a single element. Then by Proposition 2.8 we can identify $\mathbb{R}_G$ with $\mathbb{R} \cup \{\infty\}$, the set obtained from $\mathbb{R}$ by the adjunction of the extra point ∞ (cf. [10]).

3. Group bounded inverse monoids

Let G be a group acting effectively on a Hausdorff space X, $G \times X \to X, (g, x) \mapsto g \cdot x$. Suppose that X is a G-space. Then the map $x \mapsto g \cdot x$ is a homeomorphism from X to itself for all $g \in G$. Let Y be an open dense subspace of X, and let $\theta: G \to I(Y)$ be the partial action of G on Y induced by the action of G on X. Since Y is open and dense, each element of the inverse submonoid S generated by $\{\theta(g) : g \in G\}$ is non-zero and has an open dense domain. Let σ be the minimum group congruence [9] on the inverse monoid S. It turns out ([4], [5]) that each σ-class contains a unique maximal element of the form $\theta(g)$, and hence we have a natural semigroup homomorphism $f: S \to S/\sigma \to G$.

Let S be an inverse monoid with product $s \circ t$. Let 1_S be the identity element of S. We denote by 0 the zero element of S if it has one. Let G be a group with identity 1_G. A *partial homomorphism* from S to G is a function $f: S \setminus \{0\} \to G$ such that $f(s \circ t) = f(s)f(t)$ whenever $s \circ t \neq 0$. It is easy to see that if f is a partial homomorphism then f maps all non-zero idempotents to the identity of G. We say that an inverse monoid S is *group bounded* if there is a partial homomorphism f from $S \setminus \{0\}$ to a group G (called a *bounding group*) such that for each $g \in G$, there exists a unique element $s_g \in f^{-1}(g)$ satisfying for $g, h \in G$, $s_g \circ s_h \neq 0$ and $t \leq s_g$ for all $t \in f^{-1}(g)$. We remark that f is always surjective.

Proposition 3.1. *Let S be a group bounded inverse monoid. Then*

(i) $1_S = s_{1_G}$ *and* $f^{-1}(1_G)$ *is equal to the set of all non-zero idempotents of S.*

(ii) *If $s \leq t$ and $s \neq 0$ then $f(s) = f(t)$. In particular,*

$$f^{-1}(g) = \{s \in S \setminus \{0\} : s \leq s_g\}.$$

(iii) *The set $\{s_g : g \in G\}$ is equal to the set of all maximal elements of S.*

(iv) $s_g^{-1} = s_{g^{-1}}$, *for any $g \in G$.*

(v) $s_g \circ s_h \leq s_{gh}$, *for any $g, h \in G$.*

Proof. (i) Clearly 1_S is a maximal element of S and $f(1_S) = 1_G$. By the definition of f, $1_S \leq s_{1_G}$ and hence $1_S = s_{1_G}$. Let s be an element of $f^{-1}(1_G)$. Then $s \leq s_{1_G} = 1_S$ and hence s must be an idempotent. Therefore, $f^{-1}(1_G)$ consists of all non-zero idempotents of S.

(ii) Since $s = t \circ e$ for some non-zero idempotent e, $f(s) = f(t \circ e) = f(t)f(e) = f(t)1_G = f(t)$.

(iii) Suppose that $s_g \leq t$ for some $t \in S$. Then by (ii), $g = f(s_g) = f(t)$ and hence $t \leq s_g$. Therefore s_g is a maximal element of S. Conversely, suppose that s is a maximal element of S. Then $s \neq 0$ since $0 \leq t$ for all $t \in S$. Thus $s \leq s_{f(s)}$ and hence $s = s_{f(s)}$ by maximality.

(iv) Since s_g is maximal, s_g^{-1} is a maximal element of S. Hence $s_g^{-1} = s_h$ for some $h \in G$. Since $s_g \circ s_h \neq 0$, we have $1_G = f(s_g \circ s_h) = f(s_g)f(s_h) = gh$ and hence $h = g^{-1}$.

(v) Since $s_g \circ s_h \neq 0$, $f(s_g \circ s_h) = gh$ and hence $s_g \circ s_h \leq s_{gh}$.

□

REMARK 3.2. (i) By Proposition 3.1(i) and (ii), we have that every group bounded inverse monoid is 0-E-unitary. Here, an inverse semigroup with zero is said to be *0-E-unitary* if $0 \neq e \leq s$, where e is an idempotent, implies that s is an idempotent.

(ii) Let S be a group bounded inverse monoid with a bounding group G. Define a map $\theta: G \to S$ by $\theta(g) = s_g$ for all $g \in G$. Then by Proposition 3.1(i), (iv), (v) and Theorem 4 in [10], there exists a unique monoid homomorphism $\theta^*: \tilde{G}^{\mathcal{R}} \to S$ such that $\theta^* \circ i = \theta$, where $\tilde{G}^{\mathcal{R}}$ is the Birget–Rhodes expansion of G ([5], [10]) and $i: G \to \tilde{G}^{\mathcal{R}},\ i(g) = (1_G, \{1_G, g\})$.

COROLLARY 3.3. *The bounding group of a group bounded inverse monoid is unique up to isomorphism.*

PROOF. Let $f_1: S\backslash\{0\} \to G_1$, $f_2: S\backslash\{0\} \to G_2$ be two group bounding partial homomorphisms of S. Let $g \in G_1$. Then s_g is a maximal element of S and hence $s_g = s_{f_2(s_g)}$ (by Proposition 3.1(iii)). Define $F: G_1 \to G_2$ by $F(g) = f_2(s_g)$. One may show that F is bijective. Since $s_g \circ s_h \leq s_{gh}$, $F(g)F(h) = f_2(s_g)f_2(s_h) = f_2(s_g \circ s_h) = f_2(s_{gh}) = F(gh)$, F is a homomorphism.

□

We remark that for a group bounded inverse monoid S, if the inverse submonoid S_M generated by the maximal elements of S does not contain the zero element then the bounding group is isomorphic to the group S_M/σ, where σ is the minimum group congruence on S_M.

An inverse submonoid S of a symmetric inverse monoid $I(Y)$ is said to be *transitive* if for every pair of points x and y in Y, there exists an element $s \in S$ such that $s(x)$ is defined and $s(x) = y$.

Let G be a group acting on a set X, $G \times X \to X$, $(g, x) \mapsto g \cdot x$. Then the map $x \mapsto g \cdot x$ is a bijection from X to itself for all $g \in G$, which is called a *symmetry* of X induced by the action.

THEOREM 3.4. *Let G be a bounding group of an (respectively, transitive) inverse submonoid $S \subset I(Y)$ with $1_S = 1_Y$. Then there exists a set X such that the group G acts on X (respectively, transitively) and each non-zero element of S can be uniquely extended to a symmetry of X induced by the group action.*

Proof. By Proposition 3.1(iv) and (v), the map $\theta\colon G \to S \subset I(Y)$ defined by $\theta(g) = s_g$ is a partial action. By Theorem 2.2, we have a global action of G on $X := Y_G$. Note that for each $g \in G$ the partial bijection $\theta(g) = s_g$ on Y can be extended to the symmetry on X induced by g. Let $s \in S \setminus \{0\}$. Then $s \leq s_g$ for some $g \in G$, and hence s can be extended to a symmetry on X by g. If s can be extended to an another symmetry induced by $h \in G$, then the partial symmetries s_g and s_h bound the element s hence $g = h$ by Proposition 3.1(ii).

Finally, let $x, y \in Y$. Then since S is transitive, $s(x) = y$ for some non-zero element $s \in S$. Let s_g be the maximal element of S bounding s. Then $\theta(g)(x) = s_g(x) = y$. Therefore the partial action of G on Y is transitive and hence the group action on $X = Y_G$ is transitive by Proposition 2.4.

□

4. Partial actions of topological groups

Throughout this section we assume that G is a topological group acting on a set Y partially and transitively. Let X be a topological space. An action $G \times X \to X$ is called continuous if the map is continuous. Here we consider $G \times X$ as a topological space with the product topology.

Proposition 4.1. *The following statements are equivalent.*

(i) *There exists a Hausdorff topology on Y_G such that the action of G on Y_G is continuous.*

(ii) *G_{x_0} is closed for some $x_0 \in Y$.*

(iii) *G_x is closed for all $x \in Y$.*

Proof. Since G is a topological group, each translation by an element of G is a homeomorphism. Thus by Proposition 2.5, (ii) and (iii) are equivalent.

(i) $\Longrightarrow$ (ii): Suppose that there exists a Hausdorff topology on Y_G such that the action α of G on Y_G is continuous. For fixed $x_0 \in Y$, if we consider the following maps,

$$\begin{array}{ccccc} G & \xrightarrow{f} & G \times Y_G & \xrightarrow{\alpha} & Y_G \\ g & \mapsto & (g, [1, x_0]) & \mapsto & [g, x_0] \end{array}$$

then clearly $\phi \circ \pi = \alpha \circ f$, where $\phi\colon G/G_{x_0} \to Y_G$, $gG_{x_0} \mapsto [g, x_0]$, $\pi\colon G \to G/G_{x_0}$ is the natural projection. Since f and α are continuous maps, the map $\phi \circ \pi$ is continuous, and so ϕ is continuous from the fact π is quotient. Thus ϕ is bijective (by Theorem 2.6) and continuous, and hence G/G_{x_0} is a Hausdorff space. Therefore G_{x_0} must be closed.

(ii) $\Longrightarrow$ (i): Suppose that G_{x_0} is a closed subgroup of G. Then the coset space G/G_{x_0} is a Hausdorff space and the natural action $G \times G/G_{x_0} \to G/G_{x_0}$, $g\cdot(hG_{x_0}) = (gh)G_{x_0}$ is continuous. Then the bijection map $\phi\colon G/G_{x_0} \to Y_G$, $gG_{x_0} \mapsto [g, x_0]$

gives a topology on Y_G which is homeomorphic to G/G_{x_0}. Then the action (G, Y_G) is continuous.

□

The abstract globalisation problem of [4] in general fails to have a Hausdorff solution (see Example 4.12 of [13]). Here, for a Hausdorff globalisation, we consider the space Y_G with the topology induced by the space G/G_{x_0}.

We remark that if G_{x_0} is a closed subgroup of G and if G and Y_G are locally compact and second countable then the space Y_G must be homeomorphic to the quotient space G/G_{x_0} (see Lemma 2.9.1 of [14]).

THEOREM 4.2. *Suppose that there exists x_0 such that G_{x_0} is closed in G. Consider the space Y_G with the topology induced by the space G/G_{x_0}. Then*

(a) *$\imath(Y)$ is open (respectively, dense) in Y_G if and only if G^{x_0} is open (respectively, dense) in G.*

(b) *If the set Y has a topology and G^{x_0} is open in G then the inclusion $\imath\colon Y \to Y_G$ is an open mapping if and only if the following evaluation mapping*

$$\mathrm{ev}_{x_0}\colon G^{x_0} \ni g \mapsto g \cdot x_0 \in Y$$

is continuous.

PROOF. Let us consider the following maps:

$$\begin{array}{ccccccc} G & \xrightarrow{\pi} & G/G_{x_0} & \xrightarrow{\phi} & Y_G & \xleftarrow{\imath} & Y \\ g & \mapsto & gG_{x_0} & \mapsto & [g, x_0] & & \end{array}.$$

One computes easily that $\pi^{-1}(\phi^{-1}(\imath(Y))) = G^{x_0}$.

(a) Since π is a quotient mapping, G^{x_0} is open if and only if $\phi^{-1}(\imath(Y))$ is open in G/G_{x_0} if and only if $\imath(Y)$ is an open set of Y_G. Since π is an open and continuous mapping, we have that G^{x_0} is dense in G if and only if $\phi^{-1}(\imath(Y))$ is dense in G/G_{x_0} if and only if $\imath(Y)$ is dense in Y_G.

(b) Suppose that Y is a topological space. Note that for a subset U of Y, $\pi^{-1}(\phi^{-1}(\imath(U)) = \mathrm{ev}_{x_0}^{-1}(U)$. Therefore, since G^{x_0} is open in G, $\mathrm{ev}_{x_0}^{-1}(U)$ is open in G^{x_0} if and only if $\mathrm{ev}_{x_0}^{-1}(U)$ is open in G. Thus the injection $\imath\colon Y \to Y_G$ is an open mapping if and only if $\mathrm{ev}_{x_0}^{-1}(U)$ is open for all open set U in Y if and only if ev_{x_0} is continuous.

□

Let G be a finite-dimensional Lie group and let H be a closed subgroup of G. Then there exists exactly one analytic structure on G/H (the quotient manifold) which converts it into an analytic manifold such that the natural action of G on G/H is analytic [14]. Therefore we have

Corollary 4.3. *Let G be a finite-dimensional Lie group acting partially on a manifold Y. Then there is an analytic structure on Y_G such that the action of G on Y_G is analytic, transitive and the injection $\imath: Y \to Y_G$ is an open dense embedding if and only if there exists $x \in Y$ such that G_x is closed, G^x is open, dense, and the evaluation map at x is continuous.*

Example. Let us consider the Möbius partial action of $G = \mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}$. Then it is a Lie group. Set

$$N^+ := \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\},$$
$$N^- := \left\{ \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} : x \in \mathbb{R} \right\},$$
$$H := \left\{ \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} : a \in \mathbb{R} \setminus \{0\} \right\}.$$

Then $N^{\pm}$ and H are closed subgroups of G. One may see that $G_0 = HN^-, G^0 = N^+HN^-$. Using the continuous map $G \to \mathbb{R}$, $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto d$, the set N^+HN^- is an open dense subset of G (one may show that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in N^+HN^-$ if and only if $d \neq 0$), and the triple decomposition is uniquely determined. The evaluation map $\mathrm{ev}_0: N^+HN^- \to \mathbb{R}$ is just the N^+-projection, and hence it is continuous. Thus the inclusion $\imath: \mathbb{R} \to \mathbb{R}_G = G/G_0$ is an open dense embedding.

Example. (Pseudogroups of conformal transformations) We refer to the book of S. Kobayashi [11] (respectively, [8], [1], [2], [3]) for the definition of a pseudogroup of transformations on a differential manifold (respectively, for a basic theory of Jordan algebras and conformal transformations on Jordan algebras).

Let V be a finite-dimensional semi-simple Jordan algebra having no ideals isomorphic to $\mathbb{R}$ or $\mathbb{C}$, and let H be the structure group of V. A *conformal transformation* of V is a locally defined diffeomorphism $\varphi: V \supset U \to W \subset V$ such that for all $x \in U$, the differential $D\varphi(x)$ of φ at x belongs to the linear group H. We assume that the mapping defined on the empty set is a conformal transformation. Then the set of all conformal transformations forms an inverse submonoid S with zero element of the symmetric inverse monoid $I(V)$. For $x \in V, t_x: V \to V, t_x(y) = x + y$ is a conformal transformation. Set $V^{-1} := \{x \in V : x \text{ is invertible}\}$. Then V^{-1} is an open dense subset of V, and the Jordan inverse $j: V^{-1} \to V^{-1}$ is a conformal transformation [8]. Furthermore, each element of H is a conformal mapping on V.

Let $\mathrm{Co}(V)$ be the subgroup of the group of all birational maps [12] on V generated by the following birational maps on V; $t_x, x \in V, j$ and elements of H. The group $G := \mathrm{Co}(V)$ is called a *conformal* or *Kantor–Koecher–Tits group* of V. A remarkable fact about $\mathrm{Co}(V)$ is that it is a Lie group ([12], [1], [2]). Let $\theta: G \to I(V)$ be the specialization of the group of birationals to partial bijections on

V. Then θ is a partial action of G on V. We note that $\theta(G)$ does not form a group under the usual composition of partial mappings. By [1], it turns out that each non-zero conformal transformation on V can be uniquely extended to a member of $\theta(G)$. Therefore $\theta(g)$ is a maximal element of S for $g \in G$. Furthermore the domain of $\theta(g)$ is an open dense subset of V for all $g \in G$. Hence we have a map $f\colon S \setminus \{0\} \to G$ that makes S a group bounded inverse monoid. Clearly, the partial action θ is transitive because of the group N^+ of translations. It is known that if x_0 is the zero vector of V then $G_{x_0} = HN^-$, where $N^- = j \circ N^+ \circ j$ is a closed subgroup. Furthermore, $G^{x_0} = N^+HN^-$ is an open dense subset of $\mathrm{Co}(V)$. The triple decomposition N^+HN^- is uniquely determined. Hence the evaluation map $G^{x_0} \ni g \to g(x_0) \in V$ is continuous because it is the composition of the N^+-projection and the natural identification map $N^+ \ni t_x \to x$. Hence we have an open dense embedding of V into G/G_{x_0}, which is known as the conformal compactification of V.

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