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# THE ILLUMINATION CONJECTURE AND ITS EXTENSIONS

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**Abstract**

The Illumination Conjecture was raised independently by Boltyanski and Hadwiger in 1960. According to this conjecture any $d$-dimensional convex body can be illuminated by at most $2^d$ light sources. This is an important fundamental problem. The paper surveys the state of the art of the Illumination Conjecture.

## 1. The Illumination Conjecture

Let $\mathbf{K}$ be a convex body (i.e. a compact convex set with nonempty interior) in the $d$-dimensional Euclidean space $\mathbb{E}^d, d \geq 2$. According to Hadwiger [22] an exterior point $\mathbf{p} \in \mathbb{E}^d \setminus \mathbf{K}$ of $\mathbf{K}$ illuminates the boundary point $\mathbf{q}$ of $\mathbf{K}$ if the half line emanating from $\mathbf{p}$ passing through $\mathbf{q}$ intersects the interior of $\mathbf{K}$ (at a point not between $\mathbf{p}$ and $\mathbf{q}$). Furthermore, a family of exterior points of $\mathbf{K}$ say, $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ illuminates $\mathbf{K}$ if each boundary point of $\mathbf{K}$ is illuminated by at least one of the point sources $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$. Finally, the smallest $n$ for which there exist $n$ exterior points of $\mathbf{K}$ that illuminate $\mathbf{K}$ is called the *illumination number* of $\mathbf{K}$

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denoted by $I(\mathbf{K})$. In 1960, Hadwiger [22] raised the following amazingly elementary but, very fundamental question. An equivalent but somewhat different looking concept of illumination was introduced by Boltyanski in [13]. There he proposed to use directions (i.e. unit vectors) instead of point sources for the illumination of convex bodies. Based on these circumstances the following conjecture we call the *Boltyanski–Hadwiger Illumination Conjecture.*

Conjecture 1.1. *The illumination number $I(\mathbf{K})$ of any convex body $\mathbf{K}$ in $\mathbb{E}^d$, $d \geq 3$ is at most $2^d$ and $I(\mathbf{K}) = 2^d$ if and only if $\mathbf{K}$ is an affine d-cube.*

It is quite easy to prove the Illumination Conjecture in the plane (see for example [6]). Also, it has been noticed by several people that the illumination number of any smooth convex body in $\mathbb{E}^d$ is exactly $d+1$ ([6]). (A convex body of $\mathbb{E}^d$ is called smooth if through each of its boundary points there exists a uniquelly defined supporting hyperplane of $\mathbb{E}^d$.) However, the illumination conjecture is widely open for convex d-polytopes as well as for non-smooth convex bodies in $\mathbb{E}^d$ for all $d \geq 3$. In fact, a proof of the Illumination Conjecture for polytopes alone would not immediately imply its correctness for convex bodies in general mainly, because of the so-called upper semicontinuity of the illumination numbers of convex bodies. More exactly, here we refer to the following statement ([18]).

Theorem 1.2. *Let $\mathbf{K}$ be a convex body in $\mathbb{E}^d$. Then for any convex body $\mathbf{K}'$ sufficiently close to $\mathbf{K}$ in the Hausdorff metric of the convex bodies in $\mathbb{E}^d$ the inequality $I(\mathbf{K}') \leq I(\mathbf{K})$ holds (often with strict inequality).*

In what follows we survey the major results known about the Illumination Conjecture. For earlier and by now less updated accounts on the status of this problem we refer the reader to the survey papers [6] and [28].

## 2. Equivalent formulations

There are two equivalent formulations of the Illumination Conjecture that are often used in the literature (for more details see [28]). The first of these, was raised by Gohberg and Markus [21]. (In fact, they came up with their problem independently from Boltyanski and Hadwiger by studying some geometric properties of normed spaces.) It is called the *Gohberg–Markus Covering Conjecture.*

Conjecture 2.1. *Let $\mathbf{K}$ be an arbitrary convex body in $\mathbb{E}^d, d \geq 3$. Then $\mathbf{K}$ can be covered by $2^d$ smaller positively homothetic copies and $2^d$ copies are needed only if $\mathbf{K}$ is an affine d-cube.*

Another equivalent formulation was found independently by P. Soltan and V. Soltan [32] (who formulated it for the centrally symmetric case only) and by

K. Bezdek [2] (see also [3]). In the formulation below of the *K. Bezdek – P. Soltan – V. Soltan Separation Conjecture* a face of a convex body means the intersection of the convex body with a supporting hyperplane.

CONJECTURE 2.2. *Let* $\mathbf{K}$ *be an arbitrary convex body in* $\mathbb{E}^d, d \geq 3$ *and* $\mathbf{o}$ *an arbitrary interior point of* $\mathbf{K}$. *Then there exist* $2^d$ *hyperplanes of* $\mathbb{E}^d$ *such that each face of* $\mathbf{K}$ *can be strictly separated from* $\mathbf{o}$ *by at least one of the* $2^d$ *hyperplanes. Furthermore,* $2^d$ *hyperplanes are needed only if* $\mathbf{K}$ *is the convex hull of* $d$ *linearly independent line segments which intersect at the common relative interior point* $\mathbf{o}$.

## 3. Estimates and the status of the Illumination Conjecture in dimension three

The following theorem collects the best upper bounds for the illumination numbers of convex bodies in dimensions greater than 3. The first upper bound follows from the results of Erdős and Rogers [20] and Rogers and Shephard [30], the second is due to Lassak [25] (see also [36]).

THEOREM 3.1. *If* $\mathbf{K}$ *is an arbitrary convex body in the* $d$*-dimensional Euclidean space* $\mathbb{E}^d$, $d \geq 2$, *then*

$$I(\mathbf{K}) \leq \min\left\{\binom{2d}{d}(d\ln d + d\ln\ln d + 5d), (d+1)d^{d-1} - (d-1)(d-2)^{d-1}\right\}.$$

The best upper bound known on the illumination numbers of convex bodies in $\mathbb{E}^3$ is due to I. Papadoperakis [29].

THEOREM 3.2. *The illumination number of any convex body in* $\mathbb{E}^3$ *is at most* 16.

It is quite encouraging that the Illumination Conjecture is known to hold for some "relatively large" classes of convex bodies in $\mathbb{E}^3$ as well as in $\mathbb{E}^d$, $d \geq 4$. In what follows, first we survey the 3-dimensional results.

K. Bezdek [2] succeded to prove the following theorem.

THEOREM 3.3. *If* $\mathbf{P}$ *is a convex polyhedron of* $\mathbb{E}^3$ *with affine symmetry, i.e., if the affine symmetry group of* $\mathbf{P}$ *consists of the identity and at least one other affinity of* $\mathbb{E}^3$, *then the illumination number of* $\mathbf{P}$ *is at most* 8.

On the other hand, also the following theorem holds. The first part of that was proved by Lassak [24] (in fact, this paper was published before the publication of Theorem 3.3) and the second part by Dekster [19] extending the above theorem of K. Bezdek on polyhedra to convex bodies with center or plane symmetry.

Theorem 3.4.

(i) *If* $\mathbf{K}$ *is a centrally symmetric convex body in* $\mathbb{E}^3$, *then* $I(\mathbf{K}) \leq 8$.

(ii) *If* $\mathbf{K}$ *is a convex body symmetric about a plane in* $\mathbb{E}^3$, *then* $I(\mathbf{K}) \leq 8$.

Lassak [26] and later also Weissbach [35] gave a proof of the following.

Theorem 3.5. *The illumination number of any convex body of constant width in* $\mathbb{E}^3$ *is at most* 6.

It is tempting to conjecture the following even stronger result. If true, then it would give a new proof and insight of the well-known theorem, conjectured by Borsuk long ago (see for example [1]), that any set of diameter 1 in $\mathbb{E}^3$ can be partitioned into (at most) four subsets of diameter smaller than 1. Before stating that conjecture we quote the following theorem from [12], which is a slightly stronger version of the previous theorem.

Theorem 3.6. *Let* $X \subset \mathbb{E}^3$ *be an arbitrary set of diameter at most* 1 *and let* $\mathbf{B}[X]$ *be the intersection of the closed* 3*-dimensional unit balls centered at the points of* $X$. *Then* $\mathbf{B}[X]$ *can be illuminated by* 6 *directions (i.e. unit vectors) forming the vertices of a regular octahedron one pair of opposite vertices (i.e. one pair of opposite generating unit vectors) of which one can choose in an arbitrary direction.*

Perhaps, the following even stronger result holds.

Conjecture 3.7. *The illumination number of* $\mathbf{B}[X]$ *is exactly* 4.

As a last remark we feel we have to mention the following. In [15] Boltyanski announced a solution of the Illumination Conjecture in dimension 3. Unfortunately, even today the proposed proof of this result remains incomplete. In other words, one has to regard the Illumination Conjecture as a still open problem in dimension 3.

## 4. Proving the Illumination Conjecture for special convex bodies in high dimensions

Schramm [31] has proved the Illumination Conjecture for any convex body of constant width of dimension at least 16. In fact, his theorem can be extended in a straighforward way to a somewhat larger class of convex bodies as follows.

Theorem 4.1. *Let* $X \subset \mathbb{E}^d, d \geq 3$ *be an arbitrary set of diameter at most* 1 *and let* $\mathbf{B}[X]$ *be the intersection of the closed d-dimensional unit balls centered at the points of* $X$. *Then*

$$I(\mathbf{B}[X]) < 5d\sqrt{d}(4+\ln d)\left(\frac{3}{2}\right)^{\frac{d}{2}}.$$

It seems that the Illumination Conjecture has not yet been proved for $d$-dimensional convex bodies of constant width for any $4 \le d \le 15$.

Recall that a convex polytope is called a belt polytope if to each side of any of its 2-faces there exists a parallel (opposite) side on the same 2-face. This class of polytopes is wider than the class of zonotopes moreover, it is easy to see that any convex body of $\mathbb{E}^d$ can be represented as a limit of a covergent sequence of belt polytopes with respect to the Hausdorff metric in $\mathbb{E}^d$. The following theorem on belt polytopes was proved by Martini in [27]. The result that it extends to the class of convex bodies called belt bodies (including zonoids) is due to Boltyanski [14]. (See also [17] for a somewhat sharper result on the illumination numbers of belt bodies.)

Theorem 4.2. *Let* $\mathbf{P}$ *be an arbitrary d-dimensional belt polytope (resp., belt body) different from a parallelotope in* $\mathbb{E}^d$, $d \ge 2$. *Then*

$$I(\mathbf{P}) \le 3 \cdot 2^{d-2}.$$

Now, let $\mathbf{K}$ be an arbitrary convex body in $\mathbb{E}^d$ and let $T(\mathbf{K})$ be the family of all translates of $\mathbf{K}$ in $\mathbb{E}^d$. The Helly dimension $\mathrm{him}(\mathbf{K})$ of $\mathbf{K}$ is the smallest integer $h$ such that for any finite family $\mathcal{F} \subset T(\mathbf{K})$ with $\mathrm{card}\mathcal{F} > h+1$ the following assertion holds: if every $h+1$ members of $\mathcal{F}$ have a point in common, then all the members of $\mathcal{F}$ have a point in common. As it is well-known $1 \le \mathrm{him}(\mathbf{K}) \le d$. Using this notion Boltyanski [16] gave a proof of the following theorem.

Theorem 4.3. *Let* $\mathbf{K}$ *be a convex body with* $\mathrm{him}(\mathbf{K}) = 2$ *in* $\mathbb{E}^d$, $d \ge 3$. *Then*

$$I(\mathbf{K}) \le 2^d - 2^{d-2}.$$

In fact, in [16] Boltyanski concejtures the following more general inequality.

Conjecture 4.4. *Let* $\mathbf{K}$ *be a convex body with* $\mathrm{him}(\mathbf{K}) = h > 2$ *in* $\mathbb{E}^d, d \ge 3$. *Then*

$$I(\mathbf{K}) \le 2^d - 2^{d-h}.$$

K. Bezdek and Bisztriczky gave a proof of the Illumination Conjecture for the class of dual cyclic polytopes in [10]. Their upper bound for the illumination numbers of dual cyclic polytopes has been improved by Talata in [34]. So, we have the following statement.

Theorem 4.5. *The illumination number of any d-dimensional dual cyclic polytope is at most* $\frac{(d+1)^2}{2}$ *for all* $d \ge 2$.

In connection with the results of this section quite a number of questions remain open including the following ones.

PROBLEM 4.6.

(i) What are the illumination numbers of cyclic polytopes?

(ii) Can one give a proof of the Separation Conjecture for zonotopes (resp., belt polytopes)?

(iii) Is there a way to prove the Separation Conjecture for $0/1-$polytopes?

## 5. The Generalized Illumination Conjecture on the successive illumination numbers of convex bodies

Let $\mathbf{K}$ be a convex body in $\mathbb{E}^d$, $d \geq 2$. The following definitions were introduced by K. Bezdek in [9] (see also [2] that introduced the concept of the first definition below).

Let $L \subset \mathbb{E}^d \setminus \mathbf{K}$ be an affine subspace of dimension $l$, $0 \leq l \leq d-1$. Then $L$ illuminates the boundary point $\mathbf{q}$ of $\mathbf{K}$ if there exists a point $\mathbf{p}$ of $L$ that illuminates $\mathbf{q}$ on the boundary of $\mathbf{K}$. Moreover, we say that the affine subspaces $L_1, L_2, \ldots, L_n$ of dimension $l$ with $L_i \subset \mathbb{E}^d \setminus \mathbf{K}, 1 \leq i \leq n$ illuminate $\mathbf{K}$ if every boundary point of $\mathbf{K}$ is illuminated by at least one of the affine subspaces $L_1, L_2, \ldots, L_n$. Finally, let $I_l(\mathbf{K})$ be the smallest positive integer $n$ for which there exist $n$ affine subspaces of dimension $l$ say, $L_1, L_2, \ldots, L_n$ such that $L_i \subset \mathbb{E}^d \setminus \mathbf{K}$ for all $1 \leq i \leq n$ and $L_1, L_2, \ldots, L_n$ illuminate $\mathbf{K}$. $I_l(\mathbf{K})$ is called the *$l$-dimensional illumination number* of $\mathbf{K}$ and the sequence $I_0(\mathbf{K}), I_1(\mathbf{K}), \ldots, I_{d-2}(\mathbf{K}), I_{d-1}(\mathbf{K})$ is called the *successive illumination numbers* of $\mathbf{K}$. Obviously, $I_0(\mathbf{K}) \geq I_1(\mathbf{K}) \geq \cdots \geq I_{d-2}(\mathbf{K}) \geq I_{d-1}(\mathbf{K}) = 2$.

Let $\mathbb{S}^{d-1}$ be the unit sphere centered at the origin of $\mathbb{E}^d$. Let $HS^l \subset \mathbb{S}^{d-1}$ be an $l$-dimensional open great-hemisphere of $\mathbb{S}^{d-1}$, where $0 \leq l \leq d-1$. Then $HS^l$ illuminates the boundary point $\mathbf{q}$ of $\mathbf{K}$ if there exists a unit vector $\mathbf{v} \in HS^l$ that illuminates $\mathbf{q}$ in other words, for which it is true that the half line emanating from $\mathbf{q}$ and having direction vector $\mathbf{v}$ intersects the interior of $\mathbf{K}$. Moreover, we say that the $l$-dimensional open great-hemispheres $HS_1^l, HS_2^l, \ldots, HS_n^l$ of $\mathbb{S}^{d-1}$ illuminate $\mathbf{K}$ if each boundary point of $\mathbf{K}$ is illuminated by at least one of the open great-hemispheres $HS_1^l, HS_2^l, \ldots, HS_n^l$. Finally, let $I'_l(\mathbf{K})$ be the smallest number of $l$-dimensional open great-hemispheres of $\mathbb{S}^{d-1}$ that illuminate $\mathbf{K}$. Obviously, $I'_0(\mathbf{K}) \geq I'_1(\mathbf{K}) \geq \cdots \geq I'_{d-2}(\mathbf{K}) \geq I'_{d-1}(\mathbf{K}) = 2$.

Let $L \subset \mathbb{E}^d$ be a linear subspace of dimension $l$, $0 \leq l \leq d-1$ in $\mathbb{E}^d$. The $l$-th order circumscribed cylinder of $\mathbf{K}$ generated by $L$ is the union of translates of $L$ that have a nonempty intersection with $\mathbf{K}$. Then let $C_l(\mathbf{K})$ be the smallest number of translates of the interiors of some $l$-th order circumscribed cylinders of $\mathbf{K}$ the union of which contains $\mathbf{K}$. Obviously, $C_0(\mathbf{K}) \geq C_1(\mathbf{K}) \geq \cdots \geq C_{d-2}(\mathbf{K}) \geq C_{d-1}(\mathbf{K}) = 2$.

The following theorem, which was proved in [9], collects the basic information known about the quantities just introduced. (The inequality (ii) was in fact, first proved in [4] and reproved in a different way in [7].)

THEOREM 5.1. *Let $\mathbf{K}$ be an arbitrary convex body of $\mathbb{E}^d$.*

(i) *Then $I_l(\mathbf{K}) = I'_l(\mathbf{K}) = C_l(\mathbf{K})$ for all $0 \leq l \leq d-1$;*

(ii) $\left\lceil \frac{d+1}{l+1} \right\rceil \leq I_l(\mathbf{K})$ *for all* $0 \leq l \leq d-1$ *with equality for any smooth* $\mathbf{K}$*;*
(iii) $I_{d-2}(\mathbf{K}) = 2$ *for all* $d \geq 3$*.*

The *Generalized Illumination Conjecture* was phraised by K. Bezdek in [9] as follows.

CONJECTURE 5.2. *Let* $\mathbf{K}$ *be an arbitrary convex body and* $\mathbf{C}$ *be a* $d$*-dimensional affine cube in* $\mathbb{E}^d$*. Then*

$$I_l(\mathbf{K}) \leq I_l(\mathbf{C})$$

*holds for all* $0 \leq l \leq d-1$*.*

The above conjecture was proved for zonotopes and zonoids in [9]. The results of part (i) and (ii) of the next theorem are taken from [9], where they were proved for zonotopes (resp., zonoids). However, in the light of the more recent works in [14] and [17] these results extend to the class of belt polytopes (resp., belt bodies) in a rather straightforward way so, we present them in that form. The lower bound of part (iii) was proved in [9] and the upper bound of part (iii) is the major result of [23]. Finally, part (iv) was proved in [8].

THEOREM 5.3. *Let* $\mathbf{K}'$ *be a belt polytope (resp., belt body) and* $\mathbf{C}$ *be a* $d$*-dimensional affine cube in* $\mathbb{E}^d$*. Then*

(i) $I_l(\mathbf{K}') \leq I_l(\mathbf{C})$ *holds for all* $0 \leq l \leq d-1$*;*
(ii) $I_{\lfloor \frac{d}{2} \rfloor}(\mathbf{K}') = \cdots = I_{d-1}(\mathbf{K}') = 2$*;*
(iii) $\frac{2^d}{\sum_{i=0}^{l} \binom{d}{i}} \leq I_l(\mathbf{C}) \leq K(d,l)$*, where* $K(d,l)$ *denotes the minimum cardinality of binary codes of length* $d$ *with covering radius* $l$*,* $0 \leq l \leq d-1$*.*
(iv) $I_1(\mathbf{C}) = \frac{2^d}{d+1}$ *provided that* $d+1 = 2^m$*.*

We close this section with a conjecture of Kiss ([23]) on the illumination numbers of affine cubes and call the attention of the reader to the problem of proving the Generalized Illumination Conjecture in a stronger form for convex bodies of constant width in $\mathbb{E}^4$.

CONJECTURE 5.4. *Let* $\mathbf{C}$ *be a* $d$*-dimensional affine cube in* $\mathbb{E}^d$*. Then* $I_l(\mathbf{C}) = K(d,l)$*, where* $K(d,l)$ *denotes the minimum cardinality of binary codes of length* $d$ *with covering radius* $l$*,* $0 \leq l \leq d-1$*.*

CONJECTURE 5.5. *Let* $\mathbf{K}''$ *be a convex body of constant width in* $\mathbb{E}^4$*. Then* $I_0(\mathbf{K}'') \leq 8$ *(and so,* $I_1(\mathbf{K}'') \leq 4$*).*

## 6. Estimating the illumination parameter as well as the vertex index of convex bodies

Let $\mathbf{K_o}$ be a convex body in $\mathbb{E}^d$, $d \geq 2$ symmetric about the origin $\mathbf{o}$ of $\mathbb{E}^d$. Then $\mathbf{K_o}$ defines the norm

$$\|\mathbf{x}\|_{\mathbf{K_o}} = \inf\{\lambda \mid \mathbf{x} \in \lambda\mathbf{K_o}\}$$

of any $\mathbf{x} \in \mathbb{E}^d$ (with respect to $\mathbf{K_o}$).

The *illumination parameter* $\mathrm{ill}(\mathbf{K_o})$ of $\mathbf{K_o}$ was introduced by K. Bezdek in [5] as follows.

$$\mathrm{ill}(\mathbf{K_o}) = \inf\left\{\sum_i \|\mathbf{p_i}\|_{\mathbf{K_o}} \;\middle|\; \{\mathbf{p_i}\} \text{ illuminates } \mathbf{K_o}\right\}.$$

Clearly this insures that far-away light sources are penalised. The following theorem was proved in [5]. In the same paper the problem of finding the higher dimensional analogue of that claim was raised as well.

Theorem 6.1. *If $\mathbf{K_o}$ is a centrally symmetric convex domain of $\mathbb{E}^2$, then* $\mathrm{ill}(\mathbf{K_o}) \leq 6$ *with equality for any affine regular convex hexagon.*

Motivated by the notion of the illumination parameter Swanepoel [33] introduced the *covering parameter* $\mathrm{cov}(\mathbf{K_o})$ of $\mathbf{K_o}$ in the following way.

$$\mathrm{cov}(\mathbf{K_o}) = \inf\left\{\sum_i (1-\lambda_i)^{-1} \;\middle|\; \mathbf{K_o} \subset \bigcup_i (\lambda_i\mathbf{K_o} + \mathbf{t_i}),\ 0 < \lambda_i < 1,\ \mathbf{t_i} \in \mathbb{E}^d\right\}.$$

In this way homothets almost as large as $\mathbf{K_o}$ are penalised. Swanepoel [33] proved the following fundamental inequalities.

Theorem 6.2. *For any $\mathbf{o}$-symmetric convex body $\mathbf{K_o}$ in $\mathbb{E}^d, d \geq 2$ we have that*

(i) $\mathrm{ill}(\mathbf{K_o}) \leq 2\mathrm{cov}(\mathbf{K_o}) \leq O\left(2^d d^2 \ln d\right)$;

(ii) $v(\mathbf{K_o}) \leq \mathrm{ill}(\mathbf{K_o})$, *where $v(\mathbf{K_o})$ is the maximum possible degree of a vertex in a $\mathbf{K_o}$-Steiner minimal tree.*

Based on the above theorems, it is natural to study the following rather basic question (the part of which on the upper bound was first asked by Swanepoel [33]). This problem one can regard as the quantitative analogue of the Illumination Conjecture.

Problem 6.3. Prove or disprove that the inequalities $2d \leq \mathrm{ill}(\mathbf{K_o}) \leq O\left(2^d\right)$ hold for all $\mathbf{o}$-symmetric convex bodies $\mathbf{K_o}$ of $\mathbb{E}^d$.

The following very recent related concept was introduced by K. Bezdek and Litvak in [11]. Let $\mathbf{K_o}$ be a convex body in $\mathbb{E}^d, d \geq 2$ symmetric about the origin $\mathbf{o}$ of $\mathbb{E}^d$. Now, we place $\mathbf{K_o}$ in a convex polytope, say $\mathbf{P}$, with vertices $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$, where $n \geq d+1$. Then it is natural to measure the closeness of the vertex set of $\mathbf{P}$ to the origin $\mathbf{o}$ by computing $\sum_{1 \leq i \leq n} \|\mathbf{p}_i\|_{\mathbf{K_o}}$, where $\|\mathbf{x}\|_{\mathbf{K_o}} = \inf\{\lambda > 0 \mid \mathbf{x} \in \lambda \mathbf{K_o}\}$ denotes the norm of $\mathbf{x} \in \mathbb{E}^d$ with respect to $\mathbf{K_o}$. Finally, we look for the convex polytope that contains $\mathbf{K_o}$ and whose vertex set has the smallest possible closeness to $\mathbf{o}$ and introduce the *vertex index*, $\mathrm{vein}(\mathbf{K_o})$, of $\mathbf{K_o}$ as follows:

$$\mathrm{vein}(\mathbf{K_o}) = \inf\left\{ \sum_i \|\mathbf{p}_i\|_{\mathbf{K_o}} \;\middle|\; \mathbf{K_o} \subset \mathrm{conv}\{\mathbf{p}_i\} \right\}.$$

We note that $\mathrm{vein}(\mathbf{K_o})$ is an affine invariant quantity assigned to $\mathbf{K_o}$, i.e. if $A : \mathbb{E}^d \to \mathbb{E}^d$ is an (invertible) linear map, then $\mathrm{vein}(\mathbf{K_o}) = \mathrm{vein}(A(\mathbf{K_o}))$. Moreover, it is also clear that

$$\mathrm{vein}(\mathbf{K_o}) \leq \mathrm{ill}(\mathbf{K_o})$$

holds for any $\mathbf{o}$-symmetric convex body $\mathbf{K_o}$ in $\mathbb{E}^d$ with equality for smooth convex bodies.

The main results of [11] are lower and upper estimates on $\mathrm{vein}(\mathbf{K_o})$. This question seems to raise a new fundamental problem that is connected to some important problems of analysis and geometry including the problem of estimating the illumination parameters of convex bodies, the problem of covering a convex body by another one and the problem of estimating the Banach–Mazur distances between convex bodies. Next we summarize the major results of [11].

THEOREM 6.4. *For every $d \geq 2$ one has*

$$\frac{d^{3/2}}{\sqrt{2\pi e}} \leq \mathrm{vein}(\mathbf{B}_2^d) \leq 2d^{3/2},$$

*where $\mathbf{B}_2^d$ denotes the Euclidean unit ball in $\mathbb{E}^d$. Moreover, if $d = 2, 3$ then $\mathrm{vein}(\mathbf{B}_2^d) = 2d^{3/2}$.*

In connection with this result K. Bezdek and Litvak [11] conjecture the following.

CONJECTURE 6.5. *For every $d \geq 2$ one has*

$$\mathrm{vein}\left(\mathbf{B}_2^d\right) = 2d^{3/2}.$$

It is proved in [11] that the above conjecture implies the inequality $2d \leq \mathrm{vein}(\mathbf{K_o})$ for any $\mathbf{o}$-symmetric convex body $\mathbf{K_o}$ in $\mathbb{E}^d$ with equality for $d$-dimensional crosspolytopes.

We finish our survey paper with the following general result of K. Bezdek and Litvak [11].

THEOREM 6.6. *There are absolute constants $c > 0$, $C > 0$ such that for every $d \geq 2$ and every 0-symmetric convex body $\mathbf{K_o}$ in $\mathbb{E}^d$ one has*

$$\frac{d^{3/2}}{\sqrt{2\pi e}\ \mathrm{ovr}(\mathbf{K_o})} \leq \mathrm{vein}(\mathbf{K_o}) \leq C\ d^{3/2}\ \ln(2d),$$

*where* $\mathrm{ovr}(\mathbf{K_o}) = \inf\left(\mathrm{vol}(\mathcal{E})/\,\mathrm{vol}(\mathbf{K_o})\right)^{1/d}$ *is the outer volume ratio of* $\mathbf{K_o}$ *with the infimum taken over all ellipsoids* $\mathcal{E} \supset \mathbf{K_o}$ *and with* $\mathrm{vol}(\cdot)$ *denoting the volume.*

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