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ON A STRONGER FORM OF SALEM–ZYGMUND'S INEQUALITY FOR RANDOM TRIGONOMETRIC SUMS WITH EXAMPLES

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Abstract

By applying the majorizing measure method, we obtain a new estimate of the supremum of random trigonometric sums. We show that this estimate is strictly stronger than the well-known Salem–Zygmund's estimate, as well as recent general formulations of it obtained by the author. This improvement is obtained by considering the case when the characters are indexed on sub-exponentially growing sequences of integers. Several remarkable examples are studied.

1. Introduction

Let $\mathbf{T} = [0,1[= \mathbb{R}/\mathbb{Z}$ be the torus endowed with the normalized Lebesgue measure m. Let $\underline{p} = (p_k)_{k\ge 1}$, $\underline{\theta} = (\theta_k)_{k\ge 1}$ be two sequences of reals; and denote by $\tilde{p}_N = \max\{[2+|p_k|], 1 \le k \le N\}$, where $[x]$ stands for the integer part of x. Let also $\mathcal{X} = \{X_1, X_2, \dots\}$ and $\mathcal{Y} = \{Y_1, Y_2, \dots\}$ be two sequences of real random variables defined on a common probability space $(\Omega, \mathcal{A}, \mathbf{P})$. We will be mainly interested in the cases when $\mathcal{X}$ and $\mathcal{Y}$ are sequences of centered, independent random variables.

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Consider for $N = 1, 2\ldots$ the sequence of random trigonometric sums

$$
\begin{aligned}
Z_N(\omega, t) &= \sum_{k=1}^{N} \theta_k \Big\{ X_k(\omega) \cos 2\pi p_k t + Y_k(\omega) \sin 2\pi p_k t \Big\}, \\
Q_N &:= \sup_{0 \le t \le 1} |Z_N(t)| .
\end{aligned} \tag{1}
$$

A fundamental problem consists in the search of good estimates for the supremum Q_N. These sort of estimates are moreover particularly relevant in ergodic theory, where by means of the spectral lemma, problems of evaluating norms are reduced to Fourier analysis questions. The purpose of this paper is to investigate this problem by developing an approach based on the majorizing measure method. This method initiated by Garsia–Rodemich–Rumsey [GRR] has been since extensively studied and developed under the main impulse of Talagrand. We refer for this work to his fundamental paper [T]. As it will be seen, the application of this method to these questions turns to be elementary, allowing to obtain bounds for Q_N that are proved to be as good as the previous ones known, and strictly sharper when $\underline{p}$ grows faster than polynomially. This case is a critical case, since (see Remark 3.-2)) the classical estimate of Salem–Zygmund as well as the general form of it showed in [We] is trivial when $\underline{p}$ grows geometrically. This improvement obtained in Theorem 4 (Section 3) is the main contribution of the paper. To illustrate the strength of the result we obtain, consider the following example in which $\underline{p}$ grows sub-exponentially. Let $\mathcal{X}$ and $\mathcal{Y}$ be two mutually independent Rademacher sequences; let $\beta \ge 1$ and $0 \le \alpha < 1$ and put

$$
\zeta_N(\omega, t) = \sum_{k=1}^{N} \frac{1}{k^{\beta/2}} \Big\{ X_k(\omega) \cos 2\pi e^{k^{1-\alpha}} t + Y_k(\omega) \sin 2\pi e^{k^{1-\alpha}} t \Big\}.
$$

We consider in what follows the Orlicz space $L^G(\mathbf{P})$ with Orlicz norm $\|.\|_G$ (see Section 2) associated to the Young function $G(t) = \exp(t^2) - 1$. If $\beta > 1$, applying Salem–Zygmund's inequality (or the general form of it given in Theorem 1) provides

$$
\Big\| \sup_{t \in \mathbf{T}} |\zeta_N(t)| \Big\|_G \le C(\alpha, \beta) N^{\frac{1-\alpha}{2}},
$$

whereas by Theorem 4 (Section 3), we get

$$
\Big\| \sup_{t \in \mathbf{T}} |\zeta_N(t)| \Big\|_G = \begin{cases} \mathcal{O}\left(N^{1 - \left(\frac{\beta+\alpha}{2}\right)} \right) & \text{if } \beta + \alpha < 2, \\ \mathcal{O}\left(\log N \right) & \text{if } \beta + \alpha = 2, \\ \mathcal{O}\left(1 \right) & \text{if } \beta + \alpha > 2. \end{cases}
$$

If $\beta = 1$, Theorem 4 still provides better estimates (see Example 1 in Section 4). Several important classes of examples are studied. Finally, an application to uniform convergence of Rademacher random Fourier series in given. In view of further discussions, comparing results, and also in order to introduce the necessary data to make these comparizons possible, we begin by recalling in the next section some results in [We] and give a brief idea of proof.

2. Applying the metric entropy method

An approach based on the metric entropy method has been proposed in [We]. It is showed that the study amounts to applying the metric entropy method in the simplest case: the real line provided with the usual distance. We obtained a general estimate ([We], Theorem 1), which not only applies to the case where $\mathcal{X}$, $\mathcal{Y}$ are iid sequences, but also when these are bounded martingales differences or Gaussian stationary sequences with finite decoupling coefficients (see Examples 1, 2, 3 therein). As a particular case, we recovered the well-known estimate of Salem–Zygmund [SZ : Theorem 7]. Also our proof presents an important difference: we do not use Bernstein's inequality for polynomials unlike in [SZ] or [K]. Although stated here in a slightly more general form than initially, they are proved through exactly the same line of reasoning than in the quoted paper. We will therefore only briefly sketch the idea of proof. Put for $s, t \in [0, 1]$

$$d_N(s,t) = 2\left(\sum_{k=1}^N \theta_k^2 \sin^2 \pi p_k(s-t)\right)^{1/2}. \tag{2}$$

When $\mathcal{X}$ and $\mathcal{Y}$ are independent random variables with $\mathbf{E}X_k = \mathbf{E}Y_k = 0$ and $\mathbf{E}X_k^2 = \mathbf{E}Y_k^2 = 1$, then it is easily seen by means of elementary computation that $d_N(s,t) = \|Z_N(s) - Z_N(t)\|_2$. Introduce now an assumption about the increments of the process Z_N. Let $\Phi : \mathbb{R} \to \mathbb{R}^+$ be a Young function (convex, even and $\Phi(0) = 0$, $\lim_{x\to\infty} \Phi(x) = \infty$) with associated Orlicz's norm $\|f\|_\Phi = \inf\{\alpha > 0 : \mathbf{E}\Phi(|f|/\alpha) \le 1\}$, $f \in L^0(\mathbf{P})$. Let L^Φ be the subspace of $L^0(\mathbf{P})$ consisting with elements f verifying $\|f\|_\Phi < \infty$; then L^Φ endowed with norm $\|f\|_\Phi$ is a Banach space. If $\Phi(t) = |t|^p$, L^Φ is the usual L^p space.

We assume that for some constant B

$$\forall N \ge 1,\ \forall 0 \le s, t \le 1, \qquad \begin{cases} \|Z_N(s) - Z_N(t)\|_G \le B d_N(s,t) \\ \|Z_N(s)\|_G \le B\left(\sum\limits_{k=1}^N \theta_k^2\right)^{1/2}. \end{cases} \tag{3}$$

These assumptions are satisfied when $\mathcal{X}$ and $\mathcal{Y}$ are independent Rademacher or Gaussian random variables; but also in other interesting cases (see Examples 1-3 in [We]).

A standard but important result (see for instance [T] Theorem 1.2 p.2) from the metric entropy method can be stated as follows: let (E, d) be a pseudo-metric space; by pseudo-metric we mean that d satisfies the properties of a metric, except for the implication: $d(s,t) = 0 \Rightarrow s = t$. For any real $u > 0$, the entropy number $N(E, d, u)$ is the smallest (possibly infinite) covering number of E by open d-balls of radius u.

Auxiliary result. *Let E be a countable set provided with a pseudo-metric d, and let $X = \{X(\omega,t),\ \omega \in \Omega,\ t \in E\}$ be a family of random variables indexed on E and satisfying the following increment condition:*

$$\forall s,t \in E, \qquad \big\|X_s - X_t\big\|_\Phi \le d(s,t). \tag{4}$$

Assume that the integral

$$\mathcal{I}_\Phi(E,d) = \int_0^{\operatorname{diam}(E,d)} \Phi^{-1}\big(N(E,d,u)\big)du \tag{5}$$

is convergent. Then X is almost surely d-continuous, and there exists a universal constant C such that:

$$\Big\| \sup_{s,t\in E} |X_s - X_t\,| \Big\|_\Phi \le C\ \mathcal{I}_\Phi(E,d). \tag{6}$$

Only (6) will be used – note that Z_N is continuous – as follows: let S be a subdivision of $\mathbf{T}$ (later we will choose $S = \big\{\frac{j}{4\tilde p_n}, j = 0,1,\dots,4\tilde p_n\big\}$). One can compare $\sup_{t\in\mathbf{T}}|Z_N(t)|$ with $\sup_{t\in S}|Z_N(t)|$, next control the error by estimating for each $s \in S$, $\sup_{t\in V(s)}|Z_N(t) - Z_N(s)|$ where $V(s)$ is some suitable neighbourhood of s. This last point is carried out by means of the above mentionned result. This is exactly the idea of proof of Theorem 1 in [We], which we slightly refine below.

Theorem 1. *Under assumption (3), there exists a constant C (which is a function of the constant B from (3) only) such that for any integer $N \ge 1$,*

$$\|Q_N\|_G \le C\,(\log \tilde p_N)^{1/2}\left(\sum_{k=1}^N \theta_k^2\right)^{1/2}.$$

The following random exponential sums will be also considered in this work. Let $\mathcal{U} = (U_k)_{k=1}^\infty$ be a sequence of independent, centered real random variables. Put

$$Z'_N(\omega,t) = \sum_{k=1}^N U_k(\omega)e^{2i\pi p_k t}, \qquad Q'_N := \sup_{0\le t\le 1}|Z'_N(t)|\,. \tag{1'}$$

By using a Gaussian randomization (see for instance Lemma 2.3 p. 269 in [PSW]), we have

$$\mathbf{E}Q'_N \le \sqrt{8\pi}\,\mathbf{E}Q^*_N, \tag{7}$$

where $Q^*_N = \sup_{0\le t\le 1}|Z^*_N(t)|$ and Z^*_N is defined by (1) with $\mathcal{X}$, $\mathcal{Y}$ iid $\mathcal{N}(0,1)$ sequences, $\underline{p} = \mathcal{U}$, and $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{U}$ mutually independent. The following corollary is then easily deduced from Theorem 1.

Corollary 2. *Let $\mathcal{U} = (U_k)_{k=1}^{\infty}$ be a sequence of independent, centered real random variables. Then*

$$\mathbf{E} \sup_{0\le t\le 1} \left|\sum_{k=1}^{N} U_k e^{2i\pi p_k t}\right| \le C \min\left\{ (\log \tilde{p}_N)^{1/2} \left(\sum_{k=1}^{N} \mathbf{E} U_k^2\right)^{1/2}, \sum_{k=1}^{N} \mathbf{E}|U_k| \right\},$$

where C is a universal constant.

Remarks 3. Some comments are necessary, the two first are already mentionned in [We], but are of interest in what follows.

1) First of all, the estimate of Theorem 1 is optimal. Indeed, assume that $X_n = \xi_{2n}$, $Y_n = \xi_{2n+1}$ where $(\xi_n)_{n\ge 0}$ is a sequence of independent Rademacher random variables. Assume also that $\theta_k = 1$ and $p_k = k$ $(k \ge 1)$. It follows from (50) and Fatou lemma that we have

$$\forall N \ge 1, \qquad \mathbf{E}\, Q_N \ge C\, (N \log N)^{1/2},$$

where C is a universal constant.

2) Next, both estimates are only interesting when $(p_m)_{m\ge 1}$ grows at most geometrically. Consider for instance Corollary 2 and observe indeed by means of Cauchy–Schwarz's inequality, that

$$\frac{\left|\sum_{k=N+1}^{M} U_k e^{2i\pi p_k t}\right|}{\left(\log \bar{p}_M \sum_{k=N+1}^{M} U_k^2\right)^{1/2}} \le \frac{\sum_{k=N+1}^{M} |U_k|}{\left(\log \bar{p}_M \sum_{k=N+1}^{M} U_k^2\right)^{1/2}} \le \frac{(M-N)^{1/2}}{(\log \bar{p}_M)^{1/2}}.$$

So that if $\underline{p}$ is λ-lacunary $(\lambda > 1)$, that is $p_{m+1} \ge \lambda p_m$ for all $m \ge 1$, the estimate is trivial. This naturally raises the following question: when the sequence $\underline{p}$ grows faster than polynomially, what is the correct order of $\|Q_N\|_\infty$? We will see in this paper that the rate of growth of the sequence $\underline{p}$ indeed plays a role and explain how a better estimate can be obtained in that case.

3) One might think that the bound $(\log \tilde{p}_N)^{1/2} \left(\sum_{k=1}^{N} \mathbf{E} U_k^2\right)^{1/2}$ in Corollary 2 is always better than the trivial bound $\sum_{k=1}^{N} \mathbf{E}|U_k|$. This is however not the case. Consider the following example. We assume that each random variable U_k takes only two values as follows:

$$U_k = \begin{cases} 1/k & \text{with probability } 1-\varepsilon_k, \\ -(1-\varepsilon_k)/(k\varepsilon_k) & \text{with probability } \varepsilon_k, \end{cases}$$

where $0 < \varepsilon_k < 1$ and ε_k decreases to 0. Then $\mathbf{E} U_k = 0$, $\mathbf{E} U_k^2 = (1-\varepsilon_k)/k^2 + (1-\varepsilon_k)^2/(k^2\varepsilon_k)$. Assume that $\lim_{k\to\infty} k^2 \varepsilon_k = 1$. Then $\mathbf{E} U_k^2 \sim 1$ as k tends to infinity. And so $\left(\log \tilde{p}_N \sum_{k=1}^{N} \mathbf{E} U_k^2\right)^{1/2} \sim (N \log \tilde{p}_N)^{1/2}$, as N tends to infinity. But $\mathbf{E}|U_k| = 2(1-\varepsilon_k)/k$, so that $\sum_{k=1}^{N} \mathbf{E}|U_k| \sim C \log N$, which provides a much better bound.

4) The same proof combined with a simple form of the Borell–Sudakov–Tsirelson inequality (operating the same way as in [We] p. 451) also serves to establish a multidimensional version of Theorem 1. Let τ be some positive integer. Let $(\mathbf{p}_k)_{k\geq 1}$ be a sequence of elements of $\mathbb{R}^\tau$, and denote by $\tilde{\mathbf{p}}_N = \max\big\{ \big[2+\mathbf{p}_k^i\big], 1\leq k\leq N, 1\leq i\leq \tau\big\}$; here we have denoted $\mathbf{p}_k = (\mathbf{p}_k^1,\dots,\mathbf{p}_k^\tau)$. For $\mathbf{t}\in\mathbf{T}^\tau$, define analogously to (1)

$$Z_N^\tau(\omega,\mathbf{t}) = \sum_{k=1}^N \theta_k \Big\{ X_k(\omega)\cos 2\pi\langle \mathbf{p}_k,\mathbf{t}\rangle + Y_k(\omega)\sin 2\pi\langle \mathbf{p}_k,\mathbf{t}\rangle\Big\},$$

$$Q_N^\tau = \sup_{\mathbf{t}\in\mathbf{T}^\tau} \left|Z_N^\tau(\mathbf{t})\right|.$$

The corresponding pseudo-metric to (3) is defined for $\mathbf{s},\mathbf{t}\in\mathbf{T}^\tau$ by

$$d_{N,\tau}(\mathbf{s},\mathbf{t}) = 2\left(\sum_{k=1}^N \theta_k^2 \sin^2\pi\langle \mathbf{p}_k, \mathbf{t}-\mathbf{s}\rangle\right)^{1/2}, \tag{2'}$$

When $\mathcal{X}$ and $\mathcal{Y}$ are independent random variables with $\mathbf{E}X_k = \mathbf{E}Y_k = 0$ and $\mathbf{E}X_k^2 = \mathbf{E}Y_k^2 = 1$, then $\mathbf{E}\big(Z_N^\tau(\mathbf{s}) - Z_N^\tau(\mathbf{t})\big)^2 = d_{N,\tau}^2(\mathbf{s},\mathbf{t})$. Analogously, we assume that for some constant B

$$\forall N\geq 1,\ \forall \mathbf{s},\mathbf{t}\in\mathbf{T}^\tau,\quad \begin{cases} \|Z_N(\mathbf{s}) - Z_N(\mathbf{t})\|_G \leq B d_{N,\tau}(\mathbf{s},\mathbf{t}) \\ \|Z_N(\mathbf{s})\|_G \leq B\left(\sum\limits_{k=1}^N \theta_k^2\right)^{1/2}. \end{cases} \tag{3'}$$

Then, under assumption (3') there exists a constant C (which is a function of τ and the constant B from (3') only) such that for any integer $N\geq 1$,

$$\|Q_N^\tau\|_G \leq C\,(\tau\log\tilde{\mathbf{p}}_N)^{1/2}\left(\sum_{k=1}^N \theta_k^2\right)^{1/2}$$

and also,

$$\begin{aligned} &\mathbf{E}\sup_{\mathbf{t}\in\mathbf{T}^\tau}\left|\sum_{k=1}^N U_k e^{2i\pi\langle \mathbf{p}_k,\mathbf{t}\rangle}\right| \\ &\qquad \leq C\min\left\{ (\tau\log\tilde{\mathbf{p}}_N)^{1/2}\left(\sum_{k=1}^N \mathbf{E}U_k^2\right)^{1/2},\ \sum_{k=1}^N \mathbf{E}|U_k|\right\}, \end{aligned} \tag{8}$$

where C is a universal constant.

5) The fact that the sequence $\underline{p}$ takes values in $\mathbb{R}$ rather than in $\mathbb{Z}$ does not represent at this stage such a substantial gain since periodicity is destroyed for noninteger valued sequences $\underline{p}$, and an estimation of the supremum over $\mathbb{R}$ would be more suitable than over $\mathbf{T}$ in that case. However, the same argument of proof allows to estimate the supremum over $[-A, A]$, A arbitrary at the price of an extra factor A in the right hand side of Theorem 1 and Corollary 2: e.g. $\left\|\sup_{-A\le t\le A}|Z_n(t)|\right\|_G \le C\left(\log(A\tilde p_N).\sum_{k=1}^N\theta_k^2\right)^{1/2}$. See also Example 5 in Section 4 where a problem of evaluating the sup-norm over $\mathbb{R}$ is reduced to a problem of evaluating the sup-norm over $\mathbf{T}^\nu$ for suitable ν; and moreover an estimation of the sup-norm over $\mathbb{R}$ is obtained under a diophantine type condition regarding the sequence $\underline{p}$.

Sketch of proof. From the trivial inequality $|\sin x| \le (|x|\wedge 1)$, we get $d_N^2(s,t) \le 4\pi^2|s-t|^2\sum_{k=1}^N\theta_k^2\left(p_k^2\wedge\frac{1}{\pi^2|s-t|^2}\right)$. Now divide $\mathbf{T}$ in sub-intervals: $I_{N,j} = \left[\frac{j-1}{4\tilde p_N},\frac{j}{4\tilde p_N}\right[$, $j=1,2,\dots,4\tilde p_N$, and observe for $s,t\in I_{N,j}$ that $d_N(s,t)\le 2\pi|s-t|\left(\sum_{k=1}^N\theta_k^2p_k^2\right)^{1/2}$. Put for $j=1,2,\dots,4\tilde p_N$ and $t\in I_{N,j}$,

$$\mathcal{Y}_N(t) = \left[Z_N(t) - Z_N\left(\frac{j-1}{4\tilde p_N}\right)\right]\Big/\, 2\pi\left(\sum_{k=1}^N\theta_k^2p_k^2\right)^{1/2}.$$

Then

$$Q_N \le \sup_{1\le j\le 4\tilde p_N}\left|Z_N\left(\frac{j-1}{4\tilde p_N}\right)\right| + 2\pi\left(\sum_{k=1}^N\theta_k^2p_k^2\right)^{1/2}\sup_{1\le j\le 4\tilde p_N}\ \sup_{t\in I_{N,j}}|\mathcal{Y}_N(t)|. \tag{9}$$

Using the classical inequality (see for instance [GPW], inequality (3.5) p.62)):

$$\left\|\sup_{1\le j\le n}|f_j|\right\|_G \le ([2/\log 2]\log n)^{1/2}\sup_{1\le j\le n}\left\||f_j|\right\|_G,$$

we get

$$\begin{aligned}
&\frac{\|Q_N\|_G}{([2/\log 2]\log 4\tilde p_N)^{\frac12}}\\
&\quad\le \sup_{1\le j\le 4\tilde p_N}\left\|Z_N\left(\frac{j-1}{4\tilde p_N}\right)\right\|_G + 2\pi\left(\sum_{k=1}^N\theta_k^2p_k^2\right)^{1/2}.\sup_{1\le j\le 4\tilde p_N}\left\|\sup_{t\in I_{N,j}}|\mathcal{Y}_N(t)|\right\|_G\\
&\quad\le B\left(\sum_{k=1}^N\theta_k^2\right)^{1/2} + 2\pi\left(\sum_{k=1}^N\theta_k^2p_k^2\right)^{1/2}.\sup_{1\le j\le 4\tilde p_N}\left\|\sup_{t\in I_{N,j}}|\mathcal{Y}_N(t)|\right\|_G
\end{aligned}$$

As for $0<u\le 1/4\tilde p_N$, $N(I_{N,j},|\,.\,|,u)\le 1+\left[\frac{1/4\tilde p_N}{2u}\right]\le 1+\frac{1/4\tilde p_N}{2u}\le\frac{1}{2u\tilde p_N}$, we also have

$$\mathcal{I}(I_{N,j},|\,.\,|) \le \int_0^{1/4\tilde p_N}\sqrt{\log(1/2u\tilde p_N)}\,du \le C/\tilde p_N. \tag{10}$$

In view of the auxiliary result and since $\mathcal{Y}\Big(\frac{j-1}{4\tilde{p}_N}\Big) = 0$, it follows that for any countable subset E of $I_{N,j}$

$$\Big\| \sup_{t\in E} |\mathcal{Y}_N(t)| \Big\|_G \le \Big\| \sup_{s,t\in E} |\mathcal{Y}_N(s) - \mathcal{Y}_N(t)| \Big\|_G \le C/\tilde{p}_N, \tag{11}$$

where C depends on B only. By continuity of $Z_N(t,.)$, we have in fact $\Big\| \sup_{t\in I_{N,j}} |\mathcal{Y}_N(t)| \Big\|_G \le C/\tilde{p}_N$. We thus obtain

$$\begin{aligned} \|Q_N\|_G &\le C\left(\log 4\tilde{p}_N\right)^{\frac{1}{2}} \left\{ \left(\sum_{k=1}^N \theta_k^2\right)^{1/2} + \frac{1}{\tilde{p}_N}\left(\sum_{k=1}^N \theta_k^2 p_k^2\right)^{1/2} \right\} \\ &\le C\left(\log \tilde{p}_N \sum_{k=1}^N \theta_k^2\right)^{1/2}, \end{aligned}$$

as required. □

It follows from the above proof that $\|Q_N\|_G$ is controlled by two different quantities:

$$a_N = \left(\sum_{k=1}^N \theta_k^2\right)^{1/2}, \qquad b_N = \frac{1}{\tilde{p}_N}\left(\sum_{k=1}^N \theta_k^2 p_k^2\right)^{1/2}.$$

Obviously $b_N \le a_N$. But b_N is not necessary of some order as a_N; we may have $b_N = o(a_N)$. Indeed, if $\underline{p}$ increases very fast, say exponentially, and $\underline{\theta}$ no more than polynomially, then the right order of b_N can be $\sup_{1\le k\le N} |\theta_k|$, which is quite different from a_N. This is important to observe, in fact this was the starting point of this work. So, the natural question to be drawn from it is: which from a_N and b_N really reflects the right size's order of $\|Q_N\|_G$? The answer turns to be a bit subtle. This will be clarified in section 4.

3. Applying the majorizing measure method

From now on, we assume for simplicity that the sequence $\underline{p}$ is an increasing sequence of positive reals greater than 1. By using the majorizing measure method, another estimate for the supremum of Z_N can be obtained. Put for $r = 1, \dots, N$

$$\varepsilon_r^2 = p_r^{-2} \sum_{k=1}^r \theta_k^2 p_k^2 + \sum_{k=r+1}^N \theta_k^2, \tag{12}$$

and observe first that the sequence $\varepsilon_r, r = 1, \dots N$ is *decreasing*. Indeed,

$$\varepsilon_r^2 = p_r^{-2} \sum_{k=1}^r \theta_k^2 p_k^2 + \sum_{k=r+1}^N \theta_k^2 > p_{r+1}^{-2} \sum_{k=1}^r \theta_k^2 p_k^2 + \frac{\theta_{r+1}^2 p_{r+1}^2}{p_{r+1}^2} + \sum_{k=r+2}^N \theta_k^2 = \varepsilon_{r+1}^2.$$

Moreover $\varepsilon_1^2 = \sum_{k=1}^N \theta_k^2 = a_N^2$, whereas $\varepsilon_N^2 = p_N^{-2}\sum_{k=1}^N \theta_k^2 p_k^2 = \left(\frac{[2+p_N]}{p_N}\right)^2 b_N^2$.

THEOREM 4. *Under assumption (3), there exist constants C_i, $i = 0, 1, 2$ (which are functions of the constant B from (3) only) such that for any integer $N \geq 1$,*

$$\left\|\sup_{s,t\in\mathbf{T}} |Z_N(s) - Z_N(t)|\right\|_G \leq C_0 \left\{\varepsilon_N\sqrt{\log p_N} + \sum_{r=2}^N (\varepsilon_{r-1} - \varepsilon_r)\sqrt{\log p_r}\right\},$$

and

$$\left\|\sup_{t\in\mathbf{T}} |Z_N(t)|\right\|_G \leq C_1\varepsilon_1 + C_2\left\{\varepsilon_N\sqrt{\log p_N} + \sum_{r=2}^N (\varepsilon_{r-1} - \varepsilon_r)\sqrt{\log p_r}\right\}.$$

The last inequality follows from the first and from assumption (3), by the triangle inequality. The right-hand side being clearly bounded above by $\max(C_1, C_2)\cdot \varepsilon_1\sqrt{\log p_N}$, it follows that Theorem 4 contains Theorem 1. Before giving the proof, we are first going to establish a lemma. Let $\psi(x) = \sqrt{\log(x+1)}$, $x \geq 0$.

LEMMA 5. *For any positive integer N,*

$$\sup_{\alpha\in\mathbb{R}} \int_0^{2\varepsilon_1} \psi\left(\frac{1}{m\left(B_{d_N}(\alpha,\varepsilon)\right)}\right) d\varepsilon \leq C\varepsilon_N\psi(\pi p_N) + 2\sum_{r=2}^N \big(\varepsilon_{r-1} - \varepsilon_r\big)\psi(\pi p_r),$$

where $B_{d_N}(\alpha,\varepsilon)$ is the d_N-ball of radius ε centered at point α, and C is an absolute constant.

PROOF. Let $1 \leq r < N$ and let $\alpha, \beta \in \mathbb{R}$ be such that $\frac{1}{\pi p_{r+1}} \leq |\alpha - \beta| < \frac{1}{\pi p_r}$. Then,

$$\begin{aligned} d_N^2(\alpha,\beta) &\leq 4\sum_{k=1}^N \theta_k^2\big((\pi p_k|\alpha-\beta|)^2 \wedge 1\big) \\ &= 4\left(\sum_{k=1}^r \pi^2\theta_k^2 p_k^2\right)|\alpha-\beta|^2 + 4\sum_{k=r+1}^N \theta_k^2 \qquad (13) \\ &\leq 4p_r^{-2}\sum_{k=1}^r \theta_k^2 p_k^2 + 4\sum_{k=r+1}^N \theta_k^2 = 4\varepsilon_r^2. \end{aligned}$$

For $r=N$, $\varepsilon_r^2 = p_N^{-2}\sum_{k=1}^N \theta_k^2 p_k^2$. Now, if $|\alpha-\beta| < \frac{1}{\pi p_N}$ then

$$\begin{aligned} d_N^2(\alpha,\beta) &\le 4\sum_{k=1}^N \theta_k^2\big((\pi p_k|\alpha-\beta|)^2\wedge 1\big) \\ &= 4\left(\sum_{k=1}^N \pi^2\theta_k^2 p_k^2\right)|\alpha-\beta|^2 \le 4\varepsilon_N^2. \end{aligned} \tag{14}$$

Let $1\le r_0 < N$; then the ball $B_{d_N}(\alpha, 2\varepsilon_{r_0})$ contains the interval $\big]\alpha - \frac{1}{\pi p_{r_0}}, \alpha + \frac{1}{\pi p_{r_0}}\big[$. Hence, $m\big(B_{d_N}(\alpha,\varepsilon_{r_0})\big) \ge \frac{1}{\pi p_{r_0}}$. Therefore

$$\begin{aligned} \int_{2\varepsilon_N}^{2\varepsilon_1} \psi\left(\frac{1}{m\big(B_{d_N}(\alpha,\varepsilon)\big)}\right)d\varepsilon &= \sum_{r_0=2}^N \int_{2\varepsilon_{r_0}}^{2\varepsilon_{r_0-1}} \psi\left(\frac{1}{m\big(B_{d_N}(\alpha,\varepsilon)\big)}\right)d\varepsilon \\ &\le 2\sum_{r_0=2}^N \big(\varepsilon_{r_0-1}-\varepsilon_{r_0}\big)\psi(\pi p_{r_0}). \end{aligned} \tag{15}$$

Let now $0<\varepsilon\le 2\varepsilon_N$ and $0<\tau\le 1$. Let $|\alpha-\beta| < \tau/\pi p_N$. Then, $d_N(\alpha,\beta) < 2\tau\varepsilon_N$. The ball $B_{d_N}(\alpha,\tau\varepsilon_N)$ contains the interval $\big]\alpha - \frac{\tau}{\pi p_N}, \alpha + \frac{\tau}{\pi p_N}\big[$. And,

$$\begin{aligned} \int_0^{2\varepsilon_N} \psi\left(\frac{1}{m\big(B_{d_N}(\alpha,\varepsilon)\big)}\right)d\varepsilon &= 2\varepsilon_N\int_0^1 \psi\left(\frac{1}{m\big(B_{d_N}(\alpha,\tau\varepsilon_N)\big)}\right)d\tau \\ &\le 2\varepsilon_N\int_0^1 \psi\left(\frac{\pi p_N}{\tau}\right)d\tau \le C\varepsilon_N\psi(\pi p_N), \end{aligned} \tag{16}$$

since $\sqrt{(\log(1+\pi p_N/\tau))} \le \sqrt{(\log[(1+\pi p_N)(1+1/\tau)]} \le \sqrt{(\log(1+\pi p_N)} + 1/\sqrt{\tau}$. Thus,

$$\int_0^{2\varepsilon_1} \psi\left(\frac{1}{m\big(B_{d_N}(\alpha,\varepsilon)\big)}\right)d\varepsilon \le C\varepsilon_N\psi(\pi p_N) + 2\sum_{r=2}^N \big(\varepsilon_{r-1}-\varepsilon_r\big)\psi(\pi p_r). \tag{17}$$

Since the bound in (17) is independent from $\alpha\in\mathbb{R}$, we have thus proved the Lemma. □

Note that if ψ is another non decreasing function such that for $u,v\ge 1$, $\psi(uv)\le K\psi(u)\psi(v)$ and $\int_0^1\psi(u^{-1})du < \infty$, we have also

$$\sup_{\alpha\in\mathbb{R}}\int_0^{2\varepsilon_1} \psi\left(\frac{1}{m\big(B_{d_N}(\alpha,\varepsilon)\big)}\right)d\varepsilon \le C_\psi\varepsilon_N\psi(\pi p_N) + 2\sum_{r=2}^N \big(\varepsilon_{r-1}-\varepsilon_r\big)\psi(\pi p_r),$$

where C_ψ depends on ψ only.

Now, recall some facts from majorizing measures. Let (T,d) be a compact metric space and denote by D the diameter of T. For $x\in T$ and $\varepsilon>0$, let $B(x,\varepsilon)$ denote the open d-ball of T with center x and radius ε. Let $X=\{X_t, t\in T\}$ be a stochastic process (namely a collection of random variables indexed by T) defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Let also $\Phi(x) = \int_0^x \phi(t)dt$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing continuous, $\phi(0) = 0$, be a Young function. Let Ψ be the conjugate Young function of Φ: $\Psi(x) = \int_0^x \phi^{-1}(t)dt$. We say that Ψ satisfies the Δ_2-condition if for some constant C, and all $x \geq 1$, we have $\Psi(2x) \leq C\Psi(x)$. Consider the increment condition

$$\|X_s - X_t\|_\Phi \leq d(s,t) \qquad (s,t \in T) \tag{18}$$

Assume Ψ satisfies the Δ_2-condition and that there exists a probability measure (a majorizing measure) μ on T such that:

$$\sup_{x\in T} \int_0^D \Psi\left(\frac{1}{\mu(B(x,\varepsilon))}\right) d\varepsilon = M. \tag{19}$$

It follows from Theorem 4.6 p. 27 in [T], that each separable process satisfying the increment condition (18), also satisfies

$$\left\| \sup_{s,t\in T} (X_s - X_t) \right\|_\Phi \leq K_\Phi M, \tag{20}$$

where K_Φ depends on Φ only. A stochastic process is separable (with respect to the metric d), if there exists a countable d-dense subset T_0 of T such that for each t in T, $X_t \overset{a.s.}{=} \lim_{s\to t, s\in T_0} X_s$. In our case this last condition is trivially satisfied since Z_N is continuous everywhere, thus separable.

Proof of Theorem 4. By Lemma 5, m is a majorizing measure for $(\mathbf{T}, d)$ and $\Phi = G$, and so condition (19) is realized. Theorem 4 just follows from estimates (20) and(3). □

Remark. If $\underline{p}'$ is an increasing sequence of positive reals such that $p_k \leq p'_k$ for all k, then

$$\begin{aligned} d_N^2(\alpha,\beta) &\leq 4\sum_{k=1}^N \theta_k^2\big((\pi p_k|\alpha-\beta|)^2 \wedge 1\big) \leq 4\sum_{k=1}^N \theta_k^2\big((\pi p'_k|\alpha-\beta|)^2 \wedge 1\big), \\ &= 4\left(\sum_{k=1}^r \pi^2\theta_k^2(p'_k)^2\right)|\alpha-\beta|^2 + 4\sum_{k=r+1}^N \theta_k^2 \\ &\leq 4(p'_r)^{-2}\sum_{k=1}^r \theta_k^2(p'_k)^2 + 4\sum_{k=r+1}^N \theta_k^2 := 4(\varepsilon'_r)^2. \end{aligned}$$

Consequently the bound in $\underline{p}$, $\underline{\theta}$ given in Theorem 4 is less than the same bound expressed with $\underline{p}'$, $\underline{\theta}$. We will use this trivial observation in the next Section as follows: if $p_k = [p'_k]$, where $[x]$ stands for the integer part of x; in order to apply Theorem 4, it is enough to calculate quantities related to $\underline{\theta}$ and $\underline{p}'$.

4. Some examples and discussion

We begin with studying two examples, the first will show that Theorem 4 is strictly better than Theorem 1. On the second example, both Theorems provide the same estimate. However this example will give a hint for another reading of estimate in Theorem 1, leading to exhibit large classes of sequences $\underline{p}$, $\underline{\theta}$ for which more handable uniform estimates of the sup-norm are possible to obtain.

Example 1 (*sub-exponential case*). Consider two increasing differentiable functions $\psi, \varphi : \mathbb{R}^+ \to [1, \infty[$. We define $\underline{p}$ and $\underline{\theta}$ as follows: $p_k = [\exp\{k/2\psi(k)\}]$, $\theta_k^2 = 1/\varphi(k)$. We assume that

$$\frac{x\psi'(x)}{\psi(x)} \sim c \in [0,1[, \quad \frac{\psi(x)\varphi'(x)}{\varphi(x)} = o(1), \quad \psi'(x) = o(1) \quad (x \to \infty). \tag{21}$$

Note that $(p_r/p_{r-1}) \sim 1$ if $\psi(x) \uparrow \infty$ as x tends to infinity, and that in any case $(p_r/p_{r-1}) \leq C < \infty$, C independent of r if the values of $\psi(x)$ are bounded below by some strictly positive constant. The Lemma below in which we put $\Phi(y) = \int_1^y \frac{du}{\varphi(u)}$ is elementary.

Lemma 6. *The following estimates in which C is an absolute constant, are valid when $N, r \to \infty$,*

$$\begin{cases} 1) \quad p_r^{-2} \sum_{k=1}^{r} \theta_k^2 p_k^2 \leq C \dfrac{\psi(r)}{\varphi(r)}, \\ 2) \quad \varepsilon_{r-1}^2 - \varepsilon_r^2 = \left[p_{r-1}^{-2} - p_r^{-2}\right] \sum_{k=1}^{r-1} \theta_k^2 p_k^2 \leq C \dfrac{1}{\varphi(r)}, \\ 3) \quad \varepsilon_r^2 \geq \Phi(N) - \Phi(r+1), \\ 4) \quad \sum_{k=2}^{N} (\varepsilon_{k-1} - \varepsilon_k) \sqrt{\log p_k} \leq C \sum_{k=2}^{N} \left(\dfrac{k}{\varphi(k)^2 \psi(k) [\Phi(N) - \Phi(k)]} \right)^{1/2}, \\ 5) \quad \sum_{k=2}^{N-2} \left(\dfrac{k}{\varphi(k)^2 \psi(k) [\Phi(N) - \Phi(k)]} \right)^{1/2} \\ \qquad \leq C \displaystyle\int_2^{N-1} \left(\dfrac{x}{\varphi(x)^2 \psi(x) [\Phi(N) - \Phi(x)]} \right)^{1/2} dx, \\ 6) \quad \varepsilon_N \sqrt{\log p_N} \leq C \left[\dfrac{N}{\varphi(N)} \right]^{1/2}, \\ \qquad \varepsilon_1 \sqrt{\log p_N} \leq C \left[\left(\dfrac{1}{\varphi(1)} + \Phi(N) \right) \dfrac{r}{\psi(r)} \right]^{1/2}. \end{cases}$$

Proof. This follows from the asymptotics:

$$(\exp\{x/\psi(x)\})' \sim (1-c)\exp\{x/\psi(x)\}/\psi(x)$$

and

$$\begin{aligned}\left(\frac{\psi(x)}{\varphi(x)}\exp\{x/\psi(x)\}\right)' &\sim \frac{1}{\varphi(x)}\exp\{x/\psi(x)\}\left[(1-c)+\psi'(x)-\frac{\psi(x)\varphi'(x)}{\varphi(x)}\right]\\ &\sim \frac{(1-c)}{\varphi(x)}\exp\{x/\psi(x)\},\end{aligned}$$

as $x \to \infty$. □

Note that $\varepsilon_N^2 \sim \frac{\psi(N)}{\varphi(N)}$ whereas $\varepsilon_1^2 \sim \frac{1}{\varphi(1)} + \Phi(N)$, and therefore all the balls $B_{d_N}(t, \varepsilon_r)$ have a contribution in estimates (15) and (16). For the discussion, we choose $\psi(x) = x^\alpha$, $\varphi(x) = x^\beta$ with $\beta \geq 1$, $0 \leq \alpha < 1$. The set of conditions (21) is fulfilled if $0 < \alpha < 1$ as well as in the limit case $\alpha = 0$, corresponding to the exponential case. First consider the case $\beta > 1$. Then

$$\begin{aligned}\int_2^{N-1}\left(\frac{x}{\varphi(x)^2\psi(x)[\Phi(N)-\Phi(x)]}\right)^{1/2}dx &= \int_2^{N-1}\left(\frac{x^{1-2\beta-\alpha}}{[x^{1-\beta}-N^{1-\beta}]}\right)^{1/2}\\ (x = Nu)\qquad &= N^{1-(\frac{\beta+\alpha}{2})}\int_{2/N}^{1-1/N}\left[\frac{u^{1-2\beta-\alpha}}{|u^{1-\beta}-1|}\right]^{1/2} du.\end{aligned}$$

But

$$\begin{cases}\displaystyle\int_0^1\left[\frac{u^{1-2\beta-\alpha}}{|u^{1-\beta}-1|}\right]^{1/2} du < \infty & \text{if } \beta+\alpha < 2,\\ \displaystyle\int_{2/N}^{1-1/N}\left[\frac{u^{1-2\beta-\alpha}}{|u^{1-\beta}-1|}\right]^{1/2} du = \mathcal{O}(\log N) & \text{if } \beta+\alpha = 2,\\ \displaystyle\int_{2/N}^{1-1/N}\left[\frac{u^{1-2\beta-\alpha}}{|u^{1-\beta}-1|}\right]^{1/2} du = \mathcal{O}(N^{-1+(\frac{\beta+\alpha}{2})}) & \text{if } \beta+\alpha > 2.\end{cases}$$

The residual terms in Lemma 6, inequality (4): $(\varepsilon_{N-1}-\varepsilon_N)\sqrt{\log p_N}$ and $(\varepsilon_{N-2}-\varepsilon_{N-1})\sqrt{\log p_{N-1}}$, have a contribution which is at most $N^{(1-\beta)/2} \leq N^{1-(\frac{\beta+\alpha}{2})}$. It follows that

$$\sum_{r=2}^N(\varepsilon_{r-1}-\varepsilon_r)\sqrt{\log p_r} = \begin{cases}\mathcal{O}\left(N^{1-(\frac{\beta+\alpha}{2})}\right) & \text{if } \beta+\alpha < 2,\\ \mathcal{O}(\log N) & \text{if } \beta+\alpha = 2,\\ \mathcal{O}(1) & \text{if } \beta+\alpha > 2.\end{cases}$$

From Lemma 6 we also have that

$$\varepsilon_N\sqrt{\log p_N} = \mathcal{O}\left(N^{\frac{1-\beta}{2}}\right), \qquad \varepsilon_1\sqrt{\log p_N} = \mathcal{O}\left(N^{\frac{1-\alpha}{2}}\right).$$

Consider the case $\beta+\alpha<2$. By Theorem 4,

$$\Big\| \sup_{t\in\mathbf{T}} |Z_N(t)| \Big\|_G \le C(\alpha,\beta) N^{1-(\frac{\beta+\alpha}{2})}. \tag{22a}$$

whereas by Theorem 1

$$\| \sup_{t\in\mathbf{T}} |Z_N(t)| \|_G \le C(\alpha,\beta) N^{\frac{1-\alpha}{2}}. \tag{22b}$$

As we assumed $\beta>1$, it follows that $1-\left(\frac{\beta+\alpha}{2}\right)<\frac{1-\alpha}{2}$, therefore implying that Theorem 4 is strictly stronger than Theorem 1. In the case $\beta+\alpha\ge 2$, this fact is evident. Now to recover the estimates given in the Introduction, it suffices to change the choice of ψ into $\psi_1=2\psi$. Then ψ_1 with φ still satisfy condition (21), and as multiplying ψ by a constant does not affect the previous calculations, the claimed estimates are thus deduced by these ones.

Now if $\beta=1$, we find with Theorem 4 an estimate which is $\mathcal{O}\big(N^{\frac{1-\alpha}{2}}\big)$, whereas with Theorem 1 we get $\mathcal{O}\big((N^{1-\alpha}\log N)^{1/2}\big)$. In particular, in the exponential case $\alpha=0$, we find an order of type $\mathcal{O}\big(N^{1/2}\big)$ again strictly better than $\mathcal{O}\big((N\log N)^{1/2}\big)$.

Finally, consider for $M>N$ the increment

$$Q_{N,M} := \sup_{t\in\mathbf{T}} |Z_M(t)-Z_N(t)|. \tag{23}$$

This case is a bit more delicate and the corresponding sequence (ε_r) is given by

$$\varepsilon_r^2 = p_r^{-2}\sum_{k=N+1}^{r}\theta_k^2p_k^2 + \sum_{k=r+1}^{M}\theta_k^2, \qquad r=N+1,\dots,M \tag{24}$$

and $\varepsilon_{N+1}^2=\sum_{k=N+1}^M\theta_k^2$, $\varepsilon_M^2=p_M^{-2}\sum_{k=N+1}^M\theta_k^2p_k^2$. The previous calculations and the use of the trivial bound $\sum_{k=N+1}^r\theta_k^2p_k^2\le\sum_{k=1}^r\theta_k^2p_k^2$ show here that

$$\begin{aligned}\sum_{k=N+2}^{M-1}(\varepsilon_{k-1}-\varepsilon_k)\sqrt{\log p_k} &\le C\int_{N+2}^{M-1}\left(\frac{x}{\varphi(x)^2\psi(x)[\Phi(M)-\Phi(x)]}\right)^{1/2}dx,\\ \varepsilon_{N+1}\sqrt{\log p_M} &\le C\left([\Phi(M)-\Phi(N)]\frac{M}{\varphi(M)}\right)^{1/2}\\ \varepsilon_M\sqrt{\log p_M} &\le C\left(\int_N^M\frac{x}{\psi(x)\varphi(x)}\,dx\right)^{1/2}.\end{aligned} \tag{25}$$

For the last estimate, we used the fact that

$$\begin{aligned}\varepsilon_M\sqrt{\log p_M} &= \left(p_M^{-2}\log p_M\sum_{k=N+1}^{M}\theta_k^2p_k^2\right)^{1/2}\\ &\le \left(\sum_{k=N+1}^{M}\theta_k^2\log p_k\right)^{1/2}\\ &= \left(\sum_{k=N+1}^{M}\frac{k}{\psi(k)\varphi(k)}\right)^{1/2}.\end{aligned}$$

Choose again for the discussion $\psi(x)=x^\alpha$, $\varphi(x)=x^\beta$ with $\beta\ge 1$, $0\le\alpha<1$. Assume first that $\beta>1$, $\alpha+\beta<2$ and for technical reason $M\ge N+6$. We shall distinguish when $\eta:=\frac{M-N}{M}$ is small or not as M,N tend to infinity. With the change of variables $x=Mu$, the integral in (25) is rewritten as

$$M^{1-(\frac{\alpha+\beta}{2})}\int_{(N+2)/M}^{1-1/M}\left[\frac{u^{1-2\beta-\alpha}}{|u^{1-\beta}-1|}\right]^{1/2}du.$$

Since $\alpha+\beta<2$, the integral converges. The order is thus at most $M^{1-(\frac{\alpha+\beta}{2})}$. But if η is small, since $(N+2)/M=1-\eta+2/M$, we see a contribution of the integration near 1. Operating the change of variables $u=1-h$, we get

$$\int_{1-\eta+2/M}^{1-1/M}\left[\frac{u^{1-2\beta-\alpha}}{|u^{1-\beta}-1|}\right]^{1/2}du\le C_{\alpha,\beta}\int_{1/M}^{\eta-2/M}\frac{dh}{\sqrt{h}}\le C_{\alpha,\beta}\left(\frac{M-N}{M}\right)^{1/2},$$

where we used the fact that $\eta-3/M\le\eta/2$, since $\eta>6/M$. Consequently, we get

$$\sum_{k=N+2}^{M-1}(\varepsilon_{k-1}-\varepsilon_k)\sqrt{\log p_k}\le C_{\alpha,\beta}M^{1-(\frac{\alpha+\beta}{2})}\left(\frac{M-N}{M}\right)^{1/2}. \tag{26}$$

By (25) we have

$$\varepsilon_M\sqrt{\log p_M}\le\begin{cases}C_{\alpha,\beta}\left(\frac{M-N}{N^{\alpha+\beta-1}}\right)^{1/2} & \text{if } M-N\le N,\\ C_{\alpha,\beta}\left(\frac{1}{N^{\alpha+\beta-2}}\right)^{1/2} & \text{if } M-N\ge N.\end{cases}$$

Thus we get by Theorem 4

$$\begin{aligned}&\|Q_{N,M}\|_G\\ &\le\begin{cases}C_{\alpha,\beta}\left\{\left(\frac{M-N}{N^{\alpha+\beta-1}}\right)^{1/2}+M^{1-(\frac{\alpha+\beta}{2})}\left(\frac{M-N}{M}\right)^{1/2}\right\} & \text{if } M-N\le N,\\ C_{\alpha,\beta}\left\{\left(\frac{1}{N^{\alpha+\beta-2}}\right)^{1/2}+M^{1-(\frac{\alpha+\beta}{2})}\left(\frac{M-N}{M}\right)^{1/2}\right\} & \text{if } M-N\ge N.\end{cases}\end{aligned} \tag{27a}$$

Now using again (25), we deduce from Theorem 1

$$\|Q_{N,M}\|_G \le C_{\alpha,\beta}\left(\left[N^{1-\beta}-M^{1-\beta}\right]M^{1-\alpha}\right)^{1/2}$$

$$\le \begin{cases} C_{\alpha,\beta}\left(\frac{(M-N)M^{1-\alpha}}{N^{\beta}}\right)^{1/2}, & M-N\le N,\\ C_{\alpha,\beta}\left(N^{1-\beta}M^{1-\alpha}\right)^{1/2}, & M-N\ge N.\end{cases} \tag{27b}$$

Thus here again Theorem 4 provides better bounds than Theorem 1. If $\alpha+\beta=2$, we find by Theorem 4

$$\|Q_{N,M}\|_G \le \begin{cases} C_{\alpha,\beta}\log\left(e\frac{M}{N}\right), & M-N\le N,\\ C_{\alpha,\beta}\log\left(e\frac{M}{N}\right)\left(\frac{M-N}{M}\right)^{1/2}, & M-N\ge N.\end{cases}$$

whereas if $\alpha+\beta>2$,

$$\|Q_{N,M}\|_G \le \begin{cases} C_{\alpha,\beta}\log\left(e\frac{M}{N}\right), & M-N\le N,\\ C_{\alpha,\beta}\log\left(e\frac{M}{N}\right)\left(\frac{M-N}{M}\right)^{1/2}, & M-N\ge N.\end{cases}$$

again better than those obtained via Theorem 1.

Example 2 (*polynomial case*). Consider another case: $p_k=\left[k^{s/2}\right]$, $\theta_k^2=\frac{1}{\log k}$. This corresponds to the choice $\psi(x)=x/(s\log x)$ and $\varphi(x)=1/\log x$. In that case, we will see that $\varepsilon_r \asymp \varepsilon_1$. This means that there is only one big ball at the origin. Theorems 1 and 4 will produce similar estimates. As said before, this example is also very instructive for the sequel. At first,

$$p_r^{-2}\sum_{k=1}^{r}\theta_k^2p_k^2 \sim \frac{r}{(2s+1)\log r}, \qquad \sum_{k=1}^{r}\theta_k^2\sim\frac{r}{\log r}, \qquad (r\to\infty).$$

And $\varepsilon_r^2=p_r^{-2}\sum_{k=1}^r\theta_k^2p_k^2+\sum_{k=r+1}^N\theta_k^2\sim\frac{r}{(2s+1)\log r}+\sum_{k=r+1}^N\frac{1}{\log r}$. By distinguishing the cases $r\le N/2$ and $r\ge N/2$, we easily see that for N large

$$C_1\frac{N}{\log N}\le\varepsilon_r^2\le C_2\frac{N}{\log N} \qquad 1\le r\le N$$

$C_1, C_2,\dots$ being absolute constants, therefore showing that $\varepsilon_r\asymp\varepsilon_1$ [we don't forget that these numbers are defined once the value of N has been fixed].

Now as $p_{r-1}^{-2}-p_r^{-2}\sim 2s/r^{2s+1}$, we get $\varepsilon_{r-1}^2-\varepsilon_r^2\sim 2s/\log r$, and combining these estimates

$$\varepsilon_{r-1}-\varepsilon_r \asymp sC_3\sqrt{\frac{\log N}{N}}\frac{1}{\log r} \qquad (r\to\infty).$$

Consequently

$$\sum_{r=2}^{N}(\varepsilon_{r-1}-\varepsilon_r)\sqrt{\log p_r}\sim s^{3/2}C_4\sqrt{\frac{\log N}{N}}\sum_{r=2}^{N}\frac{1}{\sqrt{\log r}}\sim s^{3/2}C_5\sqrt{N}$$

and $\varepsilon_N\sqrt{\log p_N} \sim \sqrt{N}$, $\varepsilon_1\sqrt{\log p_N} \sim \sqrt{N}$. Then

$$\varepsilon_N\sqrt{\log p_N} + \sum_{r=2}^{N}(\varepsilon_{r-1} - \varepsilon_r)\sqrt{\log p_r} \sim \sqrt{N}$$

when N tends to infinity. Hence by Theorems 1 or 4,

$$\Big\|\sup_{t\in\mathbf{T}}|Z_N(t)|\Big\|_G \le C(s)\sqrt{N}. \tag{28}$$

It is interesting to observe in this example that

$$\sum_{r=2}^{N}(\varepsilon_{r-1} - \varepsilon_r)\sqrt{\log p_r} \asymp \frac{\sum_{r=1}^{N}\theta_r^2\sqrt{\log p_r}}{\big(\sum_{r=1}^{N}\theta_r^2\big)^{1/2}}, \tag{29a}$$

and by Cauchy–Schwarz inequality this is less than $\big(\sum_{r=1}^{N}\theta_r^2\log p_r\big)^{1/2}$, which has same order in $\sqrt{N}$. As one also always has

$$\varepsilon_N\sqrt{\log p_N} = \left(p_N^{-2}\log p_N\sum_{k=1}^{N}\theta_k^2p_k^2\right)^{1/2} \le \left(\sum_{k=1}^{N}\theta_k^2\log p_k\right)^{1/2}, \tag{29b}$$

we have by Theorem 4 the following bound

$$\Big\|\sup_{t\in\mathbf{T}}|Z_N(t)|\Big\|_G \le C\left(\sum_{k=1}^{N}\theta_k^2\log p_k\right)^{1/2}. \tag{29c}$$

That expression is of course much more handable than $\sqrt{\log p_N}\big(\sum_{r=1}^{N}\theta_r^2\big)^{1/2}$. It is therefore interesting to search whether a set of conditions on $\underline{p}$ and $\underline{\theta}$ guaranteeing the validity of (29c) is possible to define. This goes as follows.

We assume that there exists a sequence $\underline{c} = \{c_k, k \ge 1\}$ of reals and a real $0 < \Delta \le 1$ such that

$$(\mathcal{C})\quad \begin{cases} 1)\quad \limsup_{r\to\infty}\sum_{k=1}^{2r}\theta_k^2/\sum_{k=r}^{2r}\theta_k^2 < \infty, \\ 2)\quad \limsup_{r\to\infty}\left[p_r^{-2} - p_{r+1}^{-2}\right]c_r^{-2}\sum_{k=1}^{r}\theta_k^2p_k^2 < \infty, \\ 3)\quad \limsup_{r\to\infty}\sum_{k=1}^{r}c_k^2/\sum_{k=1}^{r}\theta_k^2 < \infty, \\ 4)\quad p_{[r/2]} \ge \Delta p_r. \end{cases}$$

Observe at first that $\left[p_r^{-2} - p_{r+1}^{-2}\right]$ behaves like $\left[\sum_{k=1}^{r}p_k^2\right]^{-1}$ if $p_k = k^s$, $(s > 0)$ or if $p_k = 2^k$, in which case it is also like p_r^{-2}. Practically $(\mathcal{C}2)$ reads as follows:

$$\limsup_{r\to\infty}\frac{\sum_{k=1}^{r}\theta_k^2p_k^2}{c_r^2\big(\sum_{k=1}^{r}p_k^2\big)} < \infty, \tag{$\mathcal{C}$2'}$$

which is satisfied in many cases. Condition $(\mathcal{C}1)$ is satisfied once we have that $\sum_{k=1}^{r}\theta_k^2 \asymp \kappa(r)$, where κ is some regularly varying function near infinity. The requirement also implies that the series $\sum_{k=1}^{\infty}\theta_k^2$ diverges.

Condition $(\mathcal{C}3)$ complements $(\mathcal{C}2)$ on comparing the growth of $\underline{\theta}$ and $\underline{c}$. Finally, condition $(\mathcal{C}4)$ means that the sequence $\underline{p}$ grows at most polynomially.

Proposition 7. *Under assumption $(\mathcal{C})$, there exists a constant C such that for all N large enough*

$$\Big\|\sup_{t\in\mathbf{T}}|Z_N(t)|\Big\|_G \le C\left(\sum_{r=1}^{N}\theta_r^2\log p_r\right)^{1/2}.$$

Proof. By assumption, for some suitable real $0<c<1$ we have for all r large enough

$$\begin{cases} 1) \quad \displaystyle\sum_{r}^{2r}\theta_k^2 \ge c\sum_{1}^{2r}\theta_k^2, \\ 2) \quad c\left[p_r^{-2}-p_{r+1}^{-2}\right]\displaystyle\sum_{k=1}^{r}\theta_k^2p_k^2 \le c_r^2, \\ 3) \quad \displaystyle\sum_{k=1}^{r}\theta_k^2 \ge c\sum_{k=1}^{r}c_k^2. \end{cases}$$

Using (1) and $(\mathcal{C}3)$ we get

$$\begin{aligned}\varepsilon_r^2 \ge \varepsilon_N^2 = p_N^{-2}\sum_{k=1}^{N}\theta_k^2p_k^2 &\ge p_N^{-2}p_{[N/2]}^2\sum_{N/2\le k\le N}\theta_k^2 \\ &\ge c\Delta^2\sum_{k=1}^{N}\theta_k^2 = c\Delta^2\varepsilon_1^2.\end{aligned}$$

Now by $(\mathcal{C}2)$ and estimate 3) above

$$\varepsilon_{r-1}-\varepsilon_r = \frac{\varepsilon_{r-1}^2-\varepsilon_r^2}{\varepsilon_{r-1}+\varepsilon_r} \le \frac{\left[p_{r-1}^{-2}-p_r^{-2}\right]\sum_{k=1}^{r-1}p_k^2\theta_k^2}{\Delta\left[c\sum_{k=1}^{N}\theta_k^2\right]^{1/2}} \le \frac{c_r^2}{\Delta c^2\left[\sum_{k=1}^{N}c_k^2\right]^{1/2}}.$$

Therefore, by applying Cauchy-Schwarz's inequality

$$\sum_{r=2}^{N}(\varepsilon_{r-1}-\varepsilon_r)\sqrt{\log p_r} \le \sum_{r=2}^{N}\frac{c_r^2\sqrt{\log p_r}}{\Delta c^2\left[\sum_{k=1}^{N}c_k^2\right]^{1/2}} \le \frac{1}{\Delta c^2}\left(\sum_{r=2}^{N}c_r^2\log p_r\right)^{1/2}.$$

One concludes by applying Theorem 4. □

There is an interesting case where Proposition 7 applies. We assume that $\mathcal{X}$ and $\mathcal{Y}$ are either independent iid Rademacher sequences or independent iid $\mathcal{N}(0,1)$ sequences. Let $\mathcal{U} = \{U_k, k \geq 1\}$ be a sequence of independent random variables defined on a joint probability space $(\Upsilon, \mathcal{F}, \mathbf{\Theta})$.

Consider also a sequence $\underline{c} = \{c_k, k \geq 1\}$ of reals and choose in (1)

$$\theta_k = c_k U_k \qquad k = 1, 2, \ldots \tag{30}$$

It is clear with the choice made for $\mathcal{X}$ and $\mathcal{Y}$ that condition (3) is satisfied, conditionally to $\mathcal{U}$ (one can take $B = 18\sqrt{2}$, or $B = 18\sqrt{\pi}$ in the Gaussian or Rademacher case, see [We] p. 445, Example 1). We now impose on $\mathcal{U}$ to satisfy the two following weighted strong laws of large numbers:

$$\lim_{N\to\infty} \frac{\sum_{k=1}^N c_k^2 U_k^2}{\sum_{k=1}^N c_k^2} \overset{a.s.}{=} a_1, \qquad \lim_{N\to\infty} \frac{\sum_{k=1}^N p_k^2 c_k^2 U_k^2}{\sum_{k=1}^N p_k^2 c_k^2} \overset{a.s.}{=} a_2, \tag{31}$$

where $0 < a_1, a_2 < \infty$. When the random variables U_k are moreover *identically distributed* and $a = \mathbf{E}U_1^2 < \infty$, then according to Theorem 3, p. 42 of Jamison–Orey–Pruitt in [JOP], the strong laws in (31) are respectively verified as soon as

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t} \sharp \left\{ r : \frac{\sum_{k=1}^r c_k^2}{c_r^2} \leq t \right\} < \infty, \\ &\limsup_{t\to\infty} \frac{1}{t} \sharp \left\{ r : \frac{\sum_{k=1}^r p_k^2 c_k^2}{p_r^2 c_r^2} \leq t \right\} < \infty, \end{aligned} \tag{32}$$

in which case $a_1 = a_2 = a$.

Condition (32) allows to catch a wide range of examples, for instance $p_k = k^s$ and $c_k = k^\beta$ with $s \geq 1$ and β real are suitable. Put $H(r) = \sum_{k=1}^r c_k^2$, $r \geq 1$. We do assume that the sequence $\underline{p}$ is polynomially growing and that the extra assumption linking both $\underline{p}$ and $\underline{c}$ holds as well: *there exists* $C > 1$ *such that for any* r *large enough*

$$\begin{aligned} &\text{a)} \qquad H(2r) \geq CH(r) \\ &\text{b)} \qquad [p_r^{-2} - p_{r+1}^{-2}] \sum_{k=1}^r c_k^2 p_k^2 \leq C c_r^2. \end{aligned} \tag{33}$$

The requirement (33a), implying the divergence of the series $\sum_{k=1}^\infty c_k^2$, is satisfied for instance if $H(r) \asymp \kappa(r)$ where κ is a regularly varying function with positive Karamata index, but not if κ is slowly varying. Let us look at the effect of assumptions (31), (33) on the control of the quantities appearing in conditions ($\mathcal{C}1$), ($\mathcal{C}2$) and ($\mathcal{C}3$). In the one hand, for any $C > C' > 1$, by using (31) and (33a)

$$\frac{\sum_{k=1}^{2r} c_k^2 U_k^2}{\sum_{k=1}^r c_k^2 U_k^2} = \left(\frac{\sum_{k=1}^{2r} c_k^2 U_k^2}{H(2r)} \right) \Big/ \left(\frac{\sum_{k=1}^r c_k^2 U_k^2}{H(r)} \right) \left(\frac{H(2r)}{H(r)} \right) \geq C',$$

almost surely, for r large. So that $\sum_{k=r+1}^{2r} c_k^2 U_k^2 \geq (C'-1) \sum_{k=1}^{r} c_k^2 U_k^2$, r large, thus implying that condition ($\mathcal{C}1$) is checked. In the other hand, by (31) and (33b)

$$\begin{aligned}\left[p_r^{-2} - p_{r+1}^{-2}\right] \sum_{k=1}^{r} c_k^2 U_k^2 p_k^2 &= \left[p_r^{-2} - p_{r+1}^{-2}\right] \left(\sum_{k=1}^{r} c_k^2 p_k^2\right) \frac{\sum_{k=1}^{r} c_k^2 U_k^2 p_k^2}{\sum_{k=1}^{r} c_k^2 p_k^2} \\ &\leq 2\left[p_r^{-2} - p_{r+1}^{-2}\right] \sum_{k=1}^{r} c_k^2 p_k^2 \leq 2C c_r^2,\end{aligned}$$

almost surely, for r large. This implies that condition ($\mathcal{C}2$) is satisfied. Finally, concerning condition ($\mathcal{C}3$), we observe by assumption (31) that $\lim_{r\to\infty} \frac{\sum_{k=1}^{r} c_k^2}{\sum_{k=1}^{r} c_k^2 U_k^2} = (a_1)^{-1}$, so that it is trivially satisfied. Consequently we can state:

Corollary 8. *The sequences $\mathcal{X}$ and $\mathcal{Y}$ being fixed as before, let $\underline{p}$ be polynomially growing. Let also $\mathcal{U}$ be a sequence of independent random variables defined on a joint probability space $(\Upsilon, \mathcal{F}, \boldsymbol{\Theta})$. Let $\underline{c}$ be a sequence of reals. We assume that $\mathcal{U}$, $\underline{p}$ and $\underline{c}$ satisfy conditions (31) and (33). If $\underline{\theta}$ is defined by (30), for almost all υ in Υ there exists $C_\upsilon < \infty$ such that for all N*

$$\left\| \sup_{t\in\mathbf{T}} |Z_N(t)| \right\|_G \leq C_\upsilon \left(\sum_{r=1}^{N} c_r^2 \log p_r\right)^{1/2}.$$

And specifying this for iid square integrable sequences, we get:

Corollary 9. *The sequences $\mathcal{X}$ and $\mathcal{Y}$ being fixed as before, let $\underline{p}$ be polynomially growing. Now let $\mathcal{U}$ be a sequence of iid square integrable random variables defined on a joint probability space$(\Upsilon, \mathcal{F}, \boldsymbol{\Theta})$. Let $\underline{p}$ and $\underline{c}$ be satisfying (32), (33). With $\underline{\theta}$ defined by (30), for almost all υ in Υ there exists $C_\upsilon < \infty$ such that for all N*

$$\left\| \sup_{t\in\mathbf{T}} |Z_N(t)| \right\|_G \leq C_\upsilon \left(\sum_{r=1}^{N} c_r^2 \log p_r\right)^{1/2}.$$

Example 3 (*a median case*). In the two preceding examples, both sequences $\underline{\theta}$, $\underline{p}$ exhibited different type of growth in the sense that: $\theta_k^{-1} = o(p_k)$. Here, to the contrary θ_k^{-1} and p_k will be of comparable order.

Assume first that $\theta_k = p_k^{-1}$. Plainly, $\varepsilon_{r-1}^2 - \varepsilon_r^2 = \left(\frac{p_r^2 - p_{r-1}^2}{p_r^2 p_{r-1}^2}\right)(r-1)$ and $\varepsilon_r^2 = p_r^{-2} \sum_{k=1}^{r} \theta_k^2 p_k^2 + \sum_{k=r+1}^{N} \theta_k^2 \geq p_r^{-2} r$. It follows that $\varepsilon_{r-1} - \varepsilon_r \leq \left(\frac{p_r^2 - p_{r-1}^2}{p_r p_{r-1}^2}\right) r^{1/2}$.

Moreover $\varepsilon_1 = \left(\sum_{k=1}^N 1/p_k^2\right)^{1/2}$ $\varepsilon_N = \sqrt{N}/p_N$. Thus,

$$\varepsilon_N(\log p_N)^{1/2} + \sum_{r=2}^N \big(\varepsilon_{r-1} - \varepsilon_r\big)(\log p_r)^{1/2} \le \frac{(N\log p_N)^{1/2}}{p_N} + \sum_{r=2}^N \left(\frac{p_r^2 - p_{r-1}^2}{p_r p_{r-1}^2}\right)(r\log p_r)^{1/2}.$$

If for instance $p_k = k^R$, $R \ge 1$, this is uniformly bounded in N; so that by Theorem 4 we get $\left\|\sup_{t\in\mathbf{T}}|Z_N(t)|\right\|_G = \mathcal{O}(1)$, whereas by Theorem 1 we only get $\left\|\sup_{t\in\mathbf{T}}|Z_N(t)|\right\|_G = \mathcal{O}\left(\sqrt{\log N}\right)$.

Now assume that $\theta_k = p_k^{-1/2}$ and $p_k = k^\sigma$ with $\sigma > 1$. Then, $\varepsilon_{r-1}^2 - \varepsilon_r^2 \asymp r^{-\sigma}$ and $\varepsilon_r^2 \asymp r^{1-\sigma}$. Thus

$$\sum_{r=2}^N(\varepsilon_{r-1} - \varepsilon_r)\sqrt{\log p_r} \asymp \sum_{r=2}^N r^{-(1+\sigma)/2}\sqrt{\log r} = \mathcal{O}(1).$$

As

$$\varepsilon_N\sqrt{\log p_N} \asymp N^{(1-\sigma)/2}\sqrt{\log N} \quad \text{and} \quad \varepsilon_1\sqrt{\log p_N} \asymp \sqrt{\log N},$$

it follows from Theorem 4 that $\left\|\sup_{t\in\mathbf{T}}|Z_N(t)|\right\|_G = \mathcal{O}(1)$, whereas by Theorem 1 we only obtain a bound of order $\mathcal{O}\left(\sqrt{\log N}\right)$.

If $\sigma = 1$, then $\varepsilon_N \asymp 1$, $\varepsilon_1 \asymp \sqrt{\log N}$. Further $\varepsilon_{r-1}^2 - \varepsilon_r^2 \asymp r^{-1}$ and $\varepsilon_r^2 \asymp \max(1, \log(N/r))$, so that $\varepsilon_N\sqrt{\log p_N} \asymp \sqrt{\log N}$ and $\varepsilon_1\sqrt{\log p_N} \asymp \log N$. Now

$$\sum_{r=2}^{N-2}(\varepsilon_{r-1} - \varepsilon_r)\sqrt{\log p_r} \asymp \sum_{r=2}^{N-2} r^{-1}\left(\frac{\log r}{\max(1, \log(N/r))}\right)^{1/2} \asymp \int_2^{N-2}\left(\frac{\log x}{\max(1, \log(N/x))}\right)^{1/2}\frac{dx}{x}.$$

But $\int_{N/2}^{N-2}\left(\frac{\log x}{\max(1,\log(N/x))}\right)^{1/2}\frac{dx}{x} \asymp \sqrt{\log N}$ and

$$\int_2^{N/2}\left(\frac{\log x}{\log(N/x)}\right)^{1/2}\frac{dx}{x} \asymp \int_{\log 2}^{\log(N/2)}\left(\frac{u}{\log N - u}\right)^{1/2}du \asymp (\log N)\int_{\frac{\log 2}{\log N}}^{\frac{\log(N/2)}{\log N}}\left(\frac{t}{1-t}\right)^{1/2}dt \asymp \log N,$$

so that Theorems 1 and 4 coincide on this case.

Example 4 (*arithmetical weights*). So far we have been concerned with regular (decreasing) weights, except for Corollaries 8 and 9, in which we considered random independent weights. In this example we study one symptomatic case of weights arising from arithmetic number theory. Let $d(n) = \sharp\{d : d \mid n\}$ be the divisor function and consider the case $p_k = \left[k^{s/2}\right]$, $\theta_k = d(k)$. In this case the weights are very irregular, but their sums behave regularly. According to Eq. 18.2.1 p. 263 of [HW] and Eq. (B) p. 81 of [R] (see [Wi] for a proof) we recall, in effect, that

$$\sum_{n=1}^{N} d(n) \sim N \log N, \qquad \sum_{n=1}^{N} d^2(n) \sim \left(\frac{N}{\pi^2}\right) \log^3 N.$$

as N tends to infinity. It follows from Theorem 1 or 4 that

$$\|Q_N\|_G \le C(s) N^{1/2} (\log N)^2.$$

This case is also one example where the sums of the weights grows to infinity. It is natural to also compare Theorems 1 and 4 when the weights are growing. We shall perform this on the limit case: $p_k^2 = M^k$, where $M > 1$ is fixed. We assume that there exists a non decreasing differentiable function ℓ such that $\ell(r) = \sum_{k=1}^{r} \theta_k^2 / r$, and $x\ell'(x) \le c_0 \ell(x)$. Recall Abel summation: $\sum_{k=1}^{r} u_k y_k = \sum_{j=1}^{r-1} D_j (y_j - y_{j+1}) + D_r y_r$, where $D_j = \sum_{k=1}^{j} u_k$. Applying it with $u_k = 1$, $y_k = M^k$ gives the relation $\frac{M^{r+1}-1}{M-1} = M^r r - \sum_{j=1}^{r-1} j M^j (M-1)$. Applying now with $u_k = \theta_k^2$ arbitrary and using the latter relation gives

$$\begin{aligned}
\sum_{k=1}^{r} \theta_k^2 p_k^2 &= \ell(r) r M^r - \sum_{j=1}^{r-1} \ell(j) j M^j (M-1) \\
&\ge \ell(r) \left(r M^r - \sum_{j=1}^{r-1} j M^j (M-1) \right) \\
&= \ell(r) \frac{M^{r+1} - 1}{M - 1}.
\end{aligned}$$

Conversely as $rM^r = \frac{M^{r+1}-1}{M-1} + \sum_{j=1}^{r-1} \ell(j) j M^j (M-1)$,

$$\begin{aligned}
\sum_{k=1}^{r} \theta_k^2 p_k^2 &= \ell(r) r M^r - \sum_{j=1}^{r-1} \ell(j) j M^j (M-1) \\
&= \ell(r) \frac{M^{r+1} - 1}{M - 1} + \sum_{j=1}^{r-1} j M^j (M-1) [\ell(r) - \ell(j)].
\end{aligned}$$

But, as $\ell(r) - \ell(j) \le (r-j)\ell'(j)$ and

$$\begin{aligned}
\sum_{j=1}^{r-1} j M^j (M-1)(r-j)\ell'(j) &\le C \sum_{j=1}^{r-1} M^j (M-1)(r-j)\ell(j) \\
&\le C \ell(r) M^r \sum_{k=1}^{r-1} M^{-k} (M-1) k,
\end{aligned}$$

we get $\sum_{k=1}^r \theta_k^2 p_k^2 \le \ell(r)M^r \left\{\frac{M}{M-1} + C\sum_{k=1}^\infty M^{-k}(M-1)k\right\}$. Consequently, for some constants C_1, C_2 depending on M and ℓ only, one has $C_1\ell(r)M^r \le \sum_{k=1}^r \theta_k^2 p_k^2 \le C_2\ell(r)M^r$. And this now implies that

$$C_1' \sum_{r=2}^N \frac{\ell(r)\sqrt{r}}{\sqrt{D(N)-D(r)}} \le \sum_{r=2}^N (\varepsilon_{r-1}-\varepsilon_r)\sqrt{\log p_r} \le C_2' \sum_{r=2}^N \frac{\ell(r)\sqrt{r}}{\sqrt{D(N)-D(r)}}.$$

Fix some $\alpha > 1$ such that $c_0 \log(1/\alpha) < 1$. Since $\ell(x) \le \ell(x\alpha) + \int_{x\alpha}^x \ell'(u)du \le \ell(x\alpha) + c_0 \int_{x\alpha}^x (\ell(u)/u)du \le \ell(x\alpha) + [c_0 \log(1/\alpha)]\ell(x)$, it follows that $\ell(x) \le c_\alpha \ell(x\alpha)$. Thus

$$\sum_{r=2}^N (\varepsilon_{r-1}-\varepsilon_r)\sqrt{\log p_r} \ge C_1 \sum_{r=2}^N \frac{\ell(r)\sqrt{r}}{\sqrt{N\ell(N)}} \ge C_1' \frac{\ell(N\alpha)}{\sqrt{N\ell(N)}} \sum_{N\ge r\ge N\alpha} \sqrt{r} \ge C_\alpha N\ell(N)^{1/2}.$$

But in view of Theorem 1, $\|Q_N\|_G \le CN\ell(N)^{1/2}$, so that in this case both theorems produce equivalent estimates.

Example 5 (*Dirichlet polynomials*). On this interesting example, a problem of evaluating the sup-norm over $\mathbb{R}$ is reduced to a problem of evaluating the sup-norm over $\mathbf{T}^\nu$ for suitable ν, in the case where $\underline{p}$, which is no longer integer valued, enjoys some diophantine properties (linear independence over $\mathbb{Q}$). Let $\mathcal{U} = (U_k)_{k=1}^\infty$ be a sequence of independent, centered real random variables. Let $s = \sigma + it$. Consider the Dirichlet polynomial $P_N(s) = \sum_{n=1}^N U_n n^{-is}$. Let $\rho_1 < \rho_2 < \dots$ denote the prime numbers. We denote $\tau = \pi(N)$ the number of prime numbers less than or equal to N. For $n \le N$, $n = \rho_1^{a_1(n)} \dots \rho_\tau^{a_\tau(n)}$, denote $\mathbf{p}_n = (a_1(n), \dots, a_\tau(n))$. Consider also on $\mathbf{T}^\tau$ the polynomial

$$\Pi_N(\mathbf{t}) = \sum_{n=1}^N U_n n^{-\sigma} e^{2i\pi\langle \mathbf{p}_n, \mathbf{t}\rangle}, \qquad \mathbf{t} = (t_1, \dots, t_\tau) \in \mathbf{T}^\tau.$$

According to Bohr's observation (see [Q1], [Q2])

$$\|P_N\|_\infty := \sup_{t\in\mathbb{R}} |P_N(t)| = \|\Pi_N\|_\infty = \sup_{\mathbf{t}\in\mathbf{T}^\tau} |\Pi_N(\mathbf{t})|.$$

This follows from Kronecker's Theorem (see [HW], Theorem 442 p. 382). In view of this Theorem, indeed, there are infinitely many values of j such that $(j\log 2, \dots, j\log\rho_\tau)$ is arbitrary close to any given element $\alpha = (\alpha_1, \dots, \alpha_\tau)$ of $\mathbf{T}^\tau$. With this reduction, we are led to estimate the supremum of a polynomial of type given in

Part 4) of Remark 3. From the elementary estimates for $n \le N$, $\sum_{k=1}^{\tau} a_k(n) \le \log N/\log 2$ and $\tau = \mathcal{O}(N/\log N)$, follows from (7) and (8) that

$$\mathbf{E}\ \|P_N\|_\infty \le C \min\left\{ \left(\frac{N\log\log N}{\log N}\right)^{1/2} \left(\sum_{n=1}^{N} \frac{\mathbf{E}U_n^2}{n^{2\sigma}}\right)^{1/2},\quad \sum_{n=1}^{N} \frac{\mathbf{E}|U_n|}{n^{\sigma}} \right\},$$

where C is a universal constant. This was observed by Queffélec ([Q1] p. 535) when $\mathcal{U}$ is a Rademacher sequence and $\sigma = 0$, in which case one gets the bound $CN\sqrt{\frac{\log\log N}{\log N}}$. By another result of Bohr (see [Q2], Theorem 2.1 p. 46), also follows that $\mathbf{E}\ \|P_N\|_\infty \ge C \sum_\rho' \rho^{-\sigma}\mathbf{E}|U_\rho|$ where the summation $\sum_\rho'$ is taken over the primes ρ less than or equal to N. In the Rademacher case this produces a lower bound of type $N/\log N$. In ([Q2] Theorem 4.1 p. 51) Queffélec gave a probabilistic proof of Halász's two-sided estimate $c_\sigma\left(\frac{N^{1-\sigma}}{\log N}\right) \le \mathbf{E}\|P_N\|_\infty \le C_\sigma\left(\frac{N^{1-\sigma}}{\log N}\right)$, valid for $0 < \sigma < 1/2$ and $N \ge 2$. If the summation in the definition of P_N is not taken over the interval of integers $[2, N]$, but over an arbitrary set of integers E, one may wonder what could remain from the previous estimates in that case.

It is natural to ask whether Theorem 4 admits a version with sup-norm over $\mathbb{R}$ for related polynomials. For, we introduce a condition on the sequence $\underline{p}$, which is in fact not exactly related to that one of Kronecker's Theorem. Define

$$J_N(\underline{p}, \rho) = \big\{ j \in \mathbb{Z} : \forall 1 \le k \le N,\ \exists \nu_k \in \mathbb{Z} : |jp_k - \nu_k| < \rho \big\}$$

$$\delta_{N,T}(\underline{p}, \rho) = \frac{1}{T}\sharp\big\{ J_N(\underline{p}, \rho) \cap [-T/2, T/2] \big\}, \qquad \delta_N(\underline{p}) = \liminf_{T\to\infty} \delta_{N,T}(\underline{p}, p_1/p_N).$$

Thus $\delta_N(\underline{p})$ relates to the distribution of integers j for which there exist integers $\nu_1, \ldots, \nu_k$ such that $|p_k - \frac{\nu_k}{j}| < \frac{p_1}{jp_N}$, $1 \le k \le N$.

Proposition 10. *Assume that $\delta_N(\underline{p}) > 0$. Then, there exists a constant C which depends on the constant B of (3) only such that:*

$$\left\| \sup_{s,t\in\mathbb{R}} |Z_N(s) - Z_N(t)| \right\|_G \le C\varepsilon_N \psi\left(\frac{2\pi p_N}{\delta_N(\underline{p})}\right) + 4\sum_{r=2}^{N} (\varepsilon_{r-1} - \varepsilon_r)\psi\left(\frac{2\pi p_r}{\delta_N(\underline{p})}\right).$$

The study of the condition $\delta_N(\underline{p}) > 0$ requires some substantial extra work, and this will be made elsewhere. A thorough study of the supremum of Dirichlet polynomials indexed on arithmetical sets is undertaken in the joint work [LW].

Proof. Consider for $T > 0$ the interval $[-T/2, T/2]$ equipped with the normalized Haar measure $m_T(dt) = \frac{1}{T}\chi_{[-T/2,T/2]}(t)m(dt)$. Let $j \in J_N(\underline{p}, p_1/p_N)$

and $\beta = a + \tau + j$ with $1/(\pi p_{r+1}) < \tau \leq 1/(\pi p_r)$. Using (13) gives

$$\begin{aligned} d_N^2(\alpha, \beta) &= 4 \sum_{k=1}^{N} \theta_k^2 \sin^2 \pi p_k(\tau + j) = 4 \sum_{k=1}^{N} \theta_k^2 \sin^2 \pi(p_k \tau + (p_k j - \nu_k)) \\ &\leq 4 \sum_{k=1}^{N} \theta_k^2 \left(\pi^2 \left(p_k \tau + (p_k j - \nu_k) \right)^2 \wedge 1 \right) \\ &\leq 16 \left(\sum_{k=1}^{r} \theta_k^2 \frac{p_k^2}{p_r^2} \right) + 4 \sum_{k=r+1}^{N} \theta_k^2 \leq 16 \varepsilon_r^2. \end{aligned}$$

And if $|\tau| < \frac{1}{\pi p_N}$, then $d_N^2(\alpha, \beta) \leq 16 \varepsilon_N^2$. For $1 \leq r_0 < N$, the ball $B_{d_N}(\alpha, 2\varepsilon_{r_0})$ thus contains $\sum_{j \in J_N(\underline{p}, p_1/p_N)} \left] \alpha - \frac{1}{\pi p_{r_0}}, \alpha + \frac{1}{\pi p_{r_0}} \right[$. Hence, $m_T\big(B_{d_N}(\alpha, \varepsilon_{r_0})\big) \geq \frac{1}{2\pi p_{r_0}} \delta_{N,T}(\underline{p}, p_1/p_N)$. Arguing now as in the proof of Lemma 5, gives

$$\begin{aligned} &\int_0^{4\varepsilon_1} \psi \left(\frac{1}{m_T \left(B_{d_N}(\alpha, \varepsilon) \right)} \right) d\varepsilon \\ &\qquad \leq C \varepsilon_N \psi \left(\frac{2\pi p_N}{\delta_{N,T} \left(\underline{p}, p_1/p_N \right)} \right) + 4 \sum_{r=2}^{N} \left(\varepsilon_{r-1} - \varepsilon_r \right) \psi \left(\frac{2\pi p_r}{\delta_{N,T} \left(\underline{p}, p_1/p_N \right)} \right), \end{aligned}$$

where the bound obtained is independent of α in $[-T/2, T/2]$. It follows from (20) that

$$\begin{aligned} &\left\| \sup_{s,t \in [-T/2, T/2]} |Z_N(s) - Z_N(t)| \right\|_G \\ &\qquad \leq C \varepsilon_N \psi \left(\frac{2\pi p_N}{\delta_{N,T} \left(\underline{p}, p_1/p_N \right)} \right) + 4 \sum_{r=2}^{N} \left(\varepsilon_{r-1} - \varepsilon_r \right) \psi \left(\frac{2\pi p_r}{\delta_{N,T} \left(\underline{p}, p_1/p_N \right)} \right). \end{aligned}$$

Consequently, by taking the limsup as T tends to infinity in both sides, gives

$$\begin{aligned} &\left\| \sup_{s,t \in \mathbb{R}} |Z_N(s) - Z_N(t)| \right\|_G \\ &\qquad \leq C \varepsilon_N \psi \left(\frac{2\pi p_N}{\delta_N \left(\underline{p} \right)} \right) + 4 \sum_{r=2}^{N} \left(\varepsilon_{r-1} - \varepsilon_r \right) \psi \left(\frac{2\pi p_r}{\delta_N \left(\underline{p} \right)} \right), \end{aligned} \tag{34}$$

which proves the result. □

5. Uniform convergence of random Fourier series

Let $\mathcal{C}$ be the space of complex valued continuous functions on $\mathbf{T}$ equipped with the sup-norm $\|f\| = \sup_{0\le t\le 1}|f(t)|$, $f \in \mathcal{C}$. Let $\mathcal{U} = (U_k)_{k=1}^{\infty}$ be a sequence of independent symmetric real random variables, and let $\underline{p}$ be a non-decreasing sequence of positive integers. In [We] (Theorem 10) we showed that the following condition:

there exist integers $0 := n_0 < n_1 < n_2 < \dots$ such that the series

$$\sum_{i=0}^{\infty}\sqrt{\log(p_{n_{i+1}})}\mathbf{E}\left[\sum_{k=n_i+1}^{n_{i+1}}|U_k|^2\right]^{1/2} \quad \textit{converges.} \tag{35}$$

is enough to ensure the uniform convergence of the random Fourier series (1') for almost all ω.

This result is deduced from a uniform estimate of the sup-norm of the increments of (Z_N) defined in (1) by combining Theorem 1 with the Borell-Sudakov-Tsirelson isoperimetric inequality for Gaussian processes. In the light of the previous Section, however, it is clear that this condition is only efficient for polynomially growing sequences $\underline{p}$. In concrete cases, it is often enough to choose $n_k = 2^k$ to obtain a sharp sufficient condition on $\mathcal{U}$ and $\underline{p}$. But there are examples (for instance Rademacher Fourier series with $\underline{p}$ and $\underline{\theta}$ defined by (39)) for which the correct choice is $n_k = 2^{2^k}$, which show that the appearance of the sequence $(n_k)_k$ in the above condition is meaningful.

In what follows, we would like to use the results from the previous Section to investigate this question more specifically. We will restrict the scope of the study to Rademacher random Fourier series. Let $\underline{\varepsilon} = \{\varepsilon_k, k \ge 1\}$, $\underline{\varepsilon}' = \{\varepsilon'_k, k \ge 1\}$ be two independent Rademacher sequences. We assume in (1) that $\mathcal{X} = \underline{\varepsilon}$, $\mathcal{Y} = \underline{\varepsilon}'$ and define for integers $M \ge N$:

$$\begin{aligned} Z_{N,M}(\omega,t) &= Z_M(\omega,t) - Z_N(\omega,t) \\ &= \sum_{k=N+1}^{M}\theta_k\Big\{\varepsilon_k(\omega)\cos 2\pi p_k t + \varepsilon'_k(\omega)\sin 2\pi p_k t\Big\}. \end{aligned} \tag{36}$$

We investigate the uniform convergence of the series $\sum_{k=1}^{\infty}\theta_k\big\{\varepsilon_k(\omega)\cos 2\pi p_k t + \varepsilon'_k(\omega)\sin 2\pi p_k t\big\}$. Consider first the polynomial case. We establish another type of sufficient condition for uniform convergence in which we get rid of the sequence (n_k). We consider sequences $\underline{p}$ and $\underline{\theta}$ linked by the conditions:

$$\begin{aligned} &(i) \qquad \forall N \ge 1, \qquad \frac{1}{p_m^2}\sum_{k\le m}\theta_k^2 p_k^2 = o(1), \\ &(ii) \qquad \exists \Gamma < \infty \ : \quad \big[p_{m-1}^{-2} - p_m^{-2}\big]\sum_{k\le m}\theta_k^2 p_k^2 \le \Gamma\theta_m^2. \end{aligned} \tag{37}$$

The examples studied in the previous section justify the introduction of the following set.

$$\mathcal{D} = \big\{(\underline{p}, \underline{\theta}) : \textit{condition } (37) \textit{ is fulfilled}\big\}. \tag{38}$$

The pairs $(\underline{p}, \underline{\theta})$ studied in Examples 1 and 2 belong to $\mathcal{D}$, as well as for instance the pair defined by

$$p_k^2 = e^{\log^\theta k}, \qquad \theta_k^2 = \frac{1}{k \log^\mu k}, \tag{39}$$

where $\mu > 1$ and $\theta > 0$.

THEOREM 11. *Let* $(\underline{p}, \underline{\theta}) \in \mathcal{D}$. *Assume that*

$$a) \quad \sum_{r=1}^{\infty} \theta_r^2 \log p_r < \infty,$$

$$b) \quad \lim_{N\to\infty} \limsup_{M\to\infty} \sum_{N\le r\le M} \frac{\theta_{r-1}^2 \sqrt{\log p_r}}{\left(\sum_{r<k\le M} \theta_k^2\right)^{1/2}} = 0.$$

Then the random Fourier series $\sum_{k=1}^\infty \theta_k \big\{\varepsilon_k(\omega)\cos 2\pi p_k t + \varepsilon_k'(\omega)\sin 2\pi p_k t\big\}$ *converges in* $\mathcal{C}(\mathbf{T})$ *for almost all* ω.

PROOF OF THEOREM 11. Let $0 < \gamma < 1$ be fixed. Using (37-i), we define recursively the following sequence of integers

$$N_1 = 1,$$
$$N_j = \sup\left\{ m > N_{j-1} : \frac{1}{p_m^2} \sum_{N_{j-1}\le k\le m} \theta_k^2 p_k^2 \ge \gamma \sum_{N_{j-1}\le k\le m} \theta_k^2 \right\}. \tag{40}$$

For $N_{j-1} < r \le N_j$, we denote $\varepsilon_r^2 = \frac{1}{p_r^2}\sum_{N_{j-1}<k\le r} \theta_k^2 p_k^2 + \sum_{r+1<k\le N_j} \theta_k^2$. It follows that

$$\varepsilon^2_{N_{j-1}+1} \ge \varepsilon^2_{N_j} \ge \gamma\, \varepsilon^2_{N_{j-1}+1}. \tag{41}$$

Now, using (37-ii) and (41), we get

$$\begin{aligned} \varepsilon_{r-1} - \varepsilon_r &= \frac{\varepsilon_{r-1}^2 - \varepsilon_r^2}{\varepsilon_{r-1} + \varepsilon_r} \\ &\le \left(\frac{\Gamma}{\sqrt{\gamma}}\right) \frac{\theta_{r-1}^2}{\left(\sum_{N_{j-1}<k\le N_j} \theta_k^2\right)^{1/2}}. \end{aligned} \tag{42}$$

It follows that

$$
\begin{aligned}
&\sum_{r=N_{j-1}+2}^{N_j} (\varepsilon_{r-1}-\varepsilon_r)\sqrt{\log p_r} \\
&\qquad \le \Big(\frac{\Gamma}{\sqrt{\gamma}}\Big) \sum_{r=N_{j-1}+2}^{N_j} \frac{\theta_{r-1}^2\sqrt{\log p_r}}{\Big(\sum_{N_{j-1}<k\le N_j}\theta_k^2\Big)^{1/2}} \\
&\qquad \le \Big(\frac{\Gamma}{\sqrt{\gamma}}\Big) \frac{\Big(\sum_{r=N_{j-1}+2}^{N_j}\theta_{r-1}^2\Big)^{1/2}\Big(\sum_{r=N_{j-1}+2}^{N_j}\theta_{r-1}^2\log p_r\Big)^{1/2}}{\Big(\sum_{N_{j-1}<k\le N_j}\theta_k^2\Big)^{1/2}} \qquad (43)\\
&\qquad = \Big(\frac{\Gamma}{\sqrt{\gamma}}\Big)\Bigg(\sum_{r=N_{j-1}+2}^{N_j}\theta_{r-1}^2\log p_r\Bigg)^{1/2}.
\end{aligned}
$$

Applying now Theorem 4, we get

$$
\Big\|\sup_{t\in\mathbf{T}}\big|Z_{N_{j-1},N_j}(t)\big|\Big\|_G \le C_{\Gamma,\gamma}\Bigg(\sum_{r=N_{j-1}+2}^{N_j}\theta_{r-1}^2\log p_r\Bigg)^{1/2}. \qquad (44)
$$

And by means of Levy's inequality, which we recall for reader's convenience (see [LT] for a proof):

> *Let $\{\xi_k, k\ge 1\}$ be a sequence of independent and symmetric random variables with values in a separable Banach space $(B,\|.\|)$. Denote $S_n=\sum_{k=1}^n\xi_k$ for all $n\ge 1$. Then we have:*
>
> $$\mathbf{E}\sup_{1\le j\le n}\|S_j\|^p \le 2\mathbf{E}\|S_n\|^p,$$
>
> *for all $0<p<\infty$ and all $n\ge 1$,*

we get

$$
\mathbf{E}\sup_{N_{j-1}<R\le N_j}\sup_{t\in\mathbf{T}}\big|Z_{N_{j-1},R}(t)\big|^2 \le 2C_{\Gamma,\gamma}^2\Bigg(\sum_{r=N_{j-1}+2}^{N_j}\theta_{r-1}^2\log p_r\Bigg). \qquad (45)
$$

In view of (45) and assumption a) of the Theorem, we deduce that the sequence (Z_n) converges in $\mathcal{C}(\mathbf{T})$ almost surely, if and only if, the subsequence (Z_{N_j}) converges in $\mathcal{C}(\mathbf{T})$ almost surely. Let $L<J$ be fixed. By Levy's inequality

$$
\mathbf{E}\sup_{L\le l\le j\le J}\sup_{t\in\mathbf{T}}\big|Z_{N_l,N_j}(t)\big|^2 \le 2\,\mathbf{E}\sup_{t\in\mathbf{T}}\big|Z_{N_L,N_J}(t)\big|^2.
$$

For $N_L < r \le N_J$, we denote this time $\varepsilon_r^2 = \frac{1}{p_r^2}\sum_{N_L<k\le r}\theta_k^2 p_k^2 + \sum_{r+1<k\le N_J}\theta_k^2$. Plainly $\varepsilon_r^2 \ge \sum_{r+1<k\le N_J}\theta_k^2$, and by (37)

$$\varepsilon_{r-1} - \varepsilon_r = \frac{\varepsilon_{r-1}^2 - \varepsilon_r^2}{\varepsilon_{r-1} + \varepsilon_r} \le \Gamma \frac{\theta_{r-1}^2}{\Big(\sum_{r<k\le N_J}\theta_k^2\Big)^{1/2}}.$$

So that

$$\sum_{N_L<r\le N_J} (\varepsilon_{r-1} - \varepsilon_r)\sqrt{\log p_r} \le \Gamma \sum_{N_L<r\le N_J} \frac{\theta_{r-1}^2\sqrt{\log p_r}}{\Big(\sum_{r<k\le N_J}\theta_k^2\Big)^{1/2}}. \tag{46}$$

We deduce from assumption b) of the Theorem that

$$\lim_{L\to\infty}\limsup_{J\to\infty}\mathbf{E}\sup_{L\le l\le j\le J}\sup_{t\in\mathbf{T}}\big|Z_{N_l,N_j}(t)\big|^2 = 0, \tag{47}$$

which clearly implies that the subsequence (Z_{N_j}) converges in $\mathcal{C}(\mathbf{T})$ almost surely. □

Now consider for the sub-exponential case again Example 1. Using estimates (27a) with $\alpha+\beta>2$, one can prove that the random Fourier series arising from (1) converges uniformly almost surely; which cannot be obtained from existing results nor Theorem 11. Let indeed $N_k = k^R$ where R is chosen so that $R(\alpha+\beta-2)>1$. Then, one has for $j\ge k$

$$\begin{aligned}\big\|Q_{N_k,N_{k+1}}\big\|_G &\le C_{\alpha,\beta}k^{-1/2-R(2-\alpha-\beta)/2}\\ \big\|Q_{N_k,N_l}\big\|_G &\le C_{\alpha,\beta}k^{-R[(\alpha+\beta)/2-1]}.\end{aligned} \tag{48}$$

Therefore by Levy's inequality

$$\begin{aligned}\mathbf{E}\sup_{N_{k-1}<R\le N_k}\sup_{t\in\mathbf{T}}\big|Z_{N_{k-1},R}(t)\big|^2 &\le C_{\alpha,\beta}k^{-1/2-R(2-\alpha-\beta)/2}\\ \mathbf{E}\sup_{L\le l\le j\le J}\sup_{t\in\mathbf{T}}\big|Z_{N_l,N_j}(t)\big|^2 &\le 2\mathbf{E}\sup_{t\in\mathbf{T}}\big|Z_{N_L,N_J}(t)\big|^2\\ &\le C_{\alpha,\beta}k^{-R[(\alpha+\beta)/2-1]}.\end{aligned} \tag{49}$$

Remark. One can show on example (39) that Theorem 4 is still stronger than Theorem 1. We omit details of calculation. By Theorem 1, we indeed get an order $\mathcal{O}\big((\log N)^{\theta/2}\big)$ because $\mu>1$. But, calculating the sum $\sum_{r=2}^N(\varepsilon_{r-1}-\varepsilon_r)\sqrt{\log p_r}$ in Theorem 4 leads to the integral

$$\begin{aligned}\int_1^N \frac{(\log x)^{\theta/2-\mu}dx}{x\big[(\log x)^{-\mu+1} - (\log N)^{-\mu+1}\big]^{1/2}} &\overset{(x=e^u)}{=} \int_0^{\log N}\frac{u^{\theta/2-\mu}du}{\big[(u^{-\mu+1} - (\log N)^{-\mu+1}\big]^{1/2}}\\ &\overset{(u=(\log N)v)}{=} (\log N)^{\frac{1+\theta-\mu}{2}}\int_0^1 \frac{v^{\theta/2-\mu}du}{\big[v^{-\mu+1}-1\big]^{1/2}}.\end{aligned}$$

As

$$\int_1^N \frac{(\log x)^{\theta/2-\mu}dx}{x\big[(\log x)^{-\mu+1}-(\log N)^{-\mu+1}\big]^{1/2}} = \begin{cases} \mathcal{O}\left((\log N)^{\frac{1+\theta-\mu}{2}}\right) & \text{if } \theta+1>\mu,\\ \mathcal{O}(\log\log N) & \text{if } \theta+1=\mu,\\ \mathcal{O}(1) & \text{if } \theta+1<\mu,\end{cases}$$

we find the better order $\mathcal{O}\left((\log N)^{\frac{1+\theta-\mu}{2}}\right)$, since the contribution of the residual term $\varepsilon_N\sqrt{\log p_N}$ is $\mathcal{O}\left((\log N)^{-\theta/2-\mu+1}\right)$.

Concluding remarks

1) Let $\underline{\varepsilon}=\{\varepsilon_k, k\geq 1\}$ be an independent Rademacher sequence and consider the special case $f_n(t)=\sum_{k=1}^n \varepsilon_k\cos kt$. In [SZ], it is showed that with probability one

$$\frac{1}{2\sqrt{6}} \leq \liminf_{n\to\infty} \frac{\sup_{0\leq t\leq 2\pi}|f_n(t)|}{\sqrt{n\log n}} \leq \limsup_{n\to\infty} \frac{\sup_{0\leq t\leq 2\pi}|f_n(t)|}{\sqrt{n\log n}} \leq 1.$$

Hayman in [H2] (problem 4.17) asked whether there is a limit with probability one. In [H1] (communicated to us by István Berkes), Halász answered the question in the affirmative, and gave in addition the second order term in the approximation of $\sup_{0\leq t\leq 2\pi}|f_n(t)|$:

$$\sqrt{n\log n}-4\sqrt{\frac{n}{\log n}}\log\log n \leq \sup_{0\leq t\leq 2\pi}|f_n(t)| \leq \sqrt{n\log n}+3\sqrt{\frac{n}{\log n}}\log\log n.$$

It is natural to ask whether such a degree of precision can be reached when replacing the sequence of naturals in the definition of $f_n(t)$ by any increasing sequence of integers. In the light of Theorem 4 and Example 1, the case of subexponentially growing sequences seems to be of particular interest.

2) If in $(1')$, $\mathcal{U}$ is a sequence of independent $\mathcal{N}(0,1)$ random variables, then in that case a two-sided estimate of $\|Q'_N\|_G$ exists and is given by

$$C_1\mathcal{I}_G(\mathbf{T},d_N)\leq \|Q'_N\|_G \leq C_2\mathcal{I}_G(\mathbf{T},d_N), \tag{50}$$

C_1, C_2 being absolute positive constants. Although theoretically difficult to apply on concrete examples, more effort should be given in order to also derive lower bounds for $\|Q'_N\|_G$.

3) One may naturally wonder what kind of results the majorizing measure method could produce when applied to non random polynomials. Such a question might be surprising at first glance, since there is no random structure there. Not quite in fact; consider indeed the polynomials $X^N(t) = \sum_{k=1}^N \theta_k e^{2i\pi p_k t}$, as well as the *shifted* polynomials

$$X_\alpha^N(t) = \sum_{k=1}^N \theta_k e^{2i\pi p_k(\alpha+t)}, \qquad (\alpha \in \mathbf{T}) \tag{51}$$

The parameter t is then used as a random parameter, and our probability space will be $(\mathbf{T}, m)$. Letting $t_- = -t \; mod(1)$, we note that $\sup_{s\in\mathbf{T}} \left|X^N(s) - X^N(0)\right| = \sup_{\alpha\in\mathbf{T}} \left|X_\alpha^N(t) - X_{t_-}^N(t)\right|$. Thus

$$\begin{aligned} \sup_{s\in\mathbf{T}} \left|X^N(s) - X^N(0)\right| &\leq \sup_{\alpha,\beta\in\mathbf{T}} \left|X_\alpha^N(t) - X_\beta^N(t)\right| \\ &\leq 2 \sup_{s\in\mathbf{T}} \left|X^N(s) - X^N(0)\right|. \end{aligned} \tag{52}$$

But

$$\left\|X_\alpha^N - X_\beta^N\right\|_2 = d_N(\alpha, \beta), \tag{53}$$

because we assumed $\underline{p}$ to be an increasing sequence of integers. And we get from the proof of Lemma 5 and the remark made right after that for any positive integer N,

$$\sup_{\alpha\in\mathbb{R}} \int_0^{2\varepsilon_1} \frac{1}{m(B_{d_N}(\alpha,\varepsilon))^{1/2}}\, d\varepsilon \leq C\varepsilon_N \sqrt{p}_N + 2\sum_{r=2}^N \big(\varepsilon_{r-1} - \varepsilon_r\big)\sqrt{p}_r, \tag{54}$$

C being some absolute constant. Then by (53), (54) conditions (18) and (19) are fulfilled with the choice $\Phi(t) = t^2$. Applying (20) gives, in view of (52)

$$\sup_{s\in\mathbf{T}} \left|X^N(s) - X^N(0)\right| \leq C\left(\varepsilon_N \sqrt{p}_N + 2\sum_{r=2}^N (\varepsilon_{r-1} - \varepsilon_r)\sqrt{p}_r\right). \tag{55}$$

C being some absolute constant. However when applied on examples, this does not provide more than the trivial bound $|X_\alpha^N(t) - \sum_{k=1}^N \theta_k| \leq 2\sum_{k=1}^N |\theta_k|$. The fact is that the supremum of non random polynomials relies rather upon their L^1-norms than L^2-norms, unlike the class of random polynomials studied here, which is governed by assumption (3). Maybe possible refinements of the approach would permit some improvements.

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