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Inverse subsumption for complete explanatory induction

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Abstract Modern explanatory inductive logic programming methods like Progol, Residue procedure, CF-induction, HAIL and Imparo use the principle of inverse entailment (IE). Those IE-based methods commonly compute a hypothesis in two steps: by first constructing an intermediate theory and next by generalizing its negation into the hypothesis with the inverse of the entailment relation. Inverse entailment ensures the completeness of generalization. On the other hand, it imposes many non-deterministic generalization operators that cause the search space to be very large. For this reason, most of those methods use the inverse relation of subsumption, instead of entailment. However, it is not clear how this logical reduction affects the completeness of generalization. In this paper, we investigate whether or not inverse subsumption can be embedded in a complete induction procedure; and if it can, how it is to be realized. Our main result is a new form of inverse subsumption that ensures the completeness of generalization. Consequently, inverse entailment can be reduced to inverse subsumption without losing the completeness for finding hypotheses in explanatory induction.

Keywords Inverse entailment · Inverse subsumption · Learning from entailment · Explanatory induction · Inductive logic programming

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Fig. 1 Hypothesis finding based on inverse entailment

$$B \wedge \neg E \models F_1 \models \cdots \models F_i \quad \text{(entailment)}$$

$$\text{(anti-entailment)} \quad \neg F_i \;\dashv\; \cdots \;\dashv\; \neg F_n \dashv H$$

1 Introduction

Learning from entailment (Muggleton 1995; Flach 1996; De Raedt 1997; Inoue 2004) is one of the most widely studied frameworks for the paradigm of inductive machine learning. Given a background theory B and examples E, the task of learning from entailment is to find a hypothesis H such that $B \wedge H \models E$ where $B \wedge H$ is consistent. This style of inductive learning is alternatively called *explanatory induction* (Flach 1996) and is used as a standard setting in *inductive logic programming* (ILP) (Muggleton and De Raedt 1994; Nienhuys-Cheng and De Wolf 1997). By the principle of *inverse entailment* (IE) (Muggleton 1995), the above task is logically equivalent to finding a consistent hypothesis H such that $B \wedge \neg E \models \neg H$. This equivalence means that the inductive hypothesis H can be computed by deriving its negation $\neg H$ from B and $\neg E$. We can represent this derivation process as:

$$B \wedge \neg E \models F_1 \models \cdots \models F_i \models \cdots \models F_n \models \neg H \tag{1}$$

where each F_i $(1 \le i \le n)$ denotes a clausal theory.

Modern explanatory ILP methods like Progol (Muggleton 1995; Tamaddoni-Nezhad and Muggleton 2009), Residue procedure (Yamamoto 2003), CF-induction (Inoue 2004; Yamamoto et al. 2008), HAIL (Ray et al. 2003; Ray and Inoue 2008; Ray 2009) and Imparo (Kimber et al. 2009) are based on IE. These IE-based methods compute a hypothesis H in two steps: by first constructing an intermediate theory F_i in Relation (1) and next generalizing its negation $\neg F_i$ into the hypothesis H. The relation between $\neg F_i$ and H can be obtained from the contrapositive of Relation (1) as:

$$\neg(B \wedge \neg E) \dashv \neg F_1 \dashv \cdots \dashv \neg F_i \dashv \cdots \dashv \neg F_n \dashv H \tag{2}$$

where $\dashv$ denotes the inverse relation of entailment, simply called *anti-entailment*.[1] In other words, every IE-based method first uses the entailment relation to construct F_i in Relation (1), and then switches to anti-entailment to generate the hypothesis H in Relation (2). (See Fig. 1.)

Inverse entailment ensures the completeness of generalization in the sense of generating *any* hypothesis H such that $F_i \models \neg H$ for an intermediate theory F_i in Fig. 1. On the other hand, it needs a variety of different operators such as *inverse resolution* (Muggleton and Buntine 1988) which applies the inverse of the resolution principle. There are several such operators each of which can be applied in many different ways. This fact leads to a large number of choice points that cause the huge search space of IE-based methods. For this reason, some methods use the inverse relation of subsumption, simply called *anti-subsumption*, due to computational efficiency. However, it was not clear whether or not their generalization becomes incomplete by reducing anti-entailment to anti-subsumption, and thus they may fail to find a relevant hypothesis worth considering. To distinguish their specific approach using anti-subsumption from IE, we term it *inverse subsumption* (IS).

[1] We distinguish two terms: anti-entailment and inverse entailment. Note that inverse entailment generally indicates the *approach* to find hypotheses using anti-entailment in ILP.

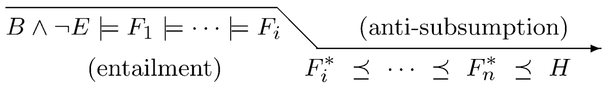

Fig. 2 Hypothesis finding based on inverse subsumption

For this open problem, the paper investigates whether or not inverse subsumption can be embedded in a complete inductive procedure; and if it can, how it is to be realized. Consequently, the paper shows a new form of inverse subsumption that can ensure the completeness of generalization. Our result is applicable to every previously proposed method. Firstly, it enables to logically characterize the possible hypotheses obtained by each IS-based method (Muggleton 1995; Yamamoto 2003; Ray et al. 2003; Ray and Inoue 2008; Ray 2009; Tamaddoni-Nezhad and Muggleton 2009; Kimber et al. 2009). Secondly, it enables to logically simplify the generalization procedure in each IE-based method (Inoue 2004; Yamamoto et al. 2008) using the new form of inverse subsumption without losing the completeness for finding hypotheses.

The key idea lies in the logical relation $F_i \models \neg H$ for a ground intermediate theory F_i and a ground hypothesis H in Relation (1). We show that there is a certain clausal theory F_i^* such that F_i^* is logically equivalent to $\neg F_i$ and $F_i^* \preceq H$. Note here that $\preceq$ denotes anti-subsumption, that is, F_i^* is subsumed by H.

Example 1 We give the intuition of our idea by drawing a passage from Hamlet. Given $E_1 = \mathit{defeat}(\mathit{claudius})$, suppose the following hypothesis:

$$H_1 = \mathit{risk\_life}(\mathit{hamlet}) \wedge (\mathit{risk\_life}(\mathit{hamlet}) \supset \mathit{defeat}(\mathit{claudius})).$$

H_1 may describe Hamlet's decision to risk his life to defeat Claudius. Let an intermediate theory F_i be $\neg E_1$. The complement $\overline{F_i} = \mathit{defeat}(\mathit{claudius})$ is entailed by H_1, but is not subsumed by it. We then consider another intermediate theory F_i' obtained by adding the tautology $\mathit{risk\_life}(\mathit{hamlet}) \vee \neg \mathit{risk\_life}(\mathit{hamlet})$ to F_i. Since F_i' is logically equivalent to F_i, the following complement $\overline{F_i'}$:

$$(\mathit{risk\_life}(\mathit{hamlet}) \vee \mathit{defeat}(\mathit{claudius})) \wedge (\mathit{risk\_life}(\mathit{hamlet}) \supset \mathit{defeat}(\mathit{claudius}))$$

is also entailed by H_1. Besides, $\overline{F_i'}$ is subsumed by H_1, unlike the case of $\overline{F_i}$. Hence, adding the tautology (i.e. Hamlet risks his life or not) plays a role of reducing anti-entailment to anti-subsumption. Note that the above F_i^* can be regarded as $\overline{F_i'}$.

This feature enables us to logically reduce the derivation process based on inverse entailment in Fig. 1 to the one based on inverse subsumption described in Fig. 2.

This paper uses two kinds of clausal theories regarded as F_i^*, called *residue* and *minimal* complements, respectively. Together with these two complements, we show how inverse entailment can be reduced to inverse subsumption. In both cases, inverse subsumption is sufficient to ensure the completeness of generalization.

The rest of this paper is organized as follows. After Sect. 2 describes the theoretical background, we show inverse subsumption with residue and minimal complements ensure the completeness of generalization in Sects. 3 and 4, respectively. In Sect. 5, we clarify some commonness between two approaches with residue and minimal complements and apply them to each IE-based method. In Sect. 6, we conclude.

2 Background

2.1 Preliminaries

We assume the reader to be familiar with the basic concepts in first-order logic and inductive logic programming (Nienhuys-Cheng and De Wolf 1997). A *clause* is a finite disjunction of literals which is often identified with the set of its literals. A clause of the form $\{\neg B_1, \dots, \neg B_n, A_1, \dots, A_m\}$, where each A_i, B_j is an atom, is also written as $B_1 \wedge \cdots \wedge B_n \supset A_1 \vee \cdots \vee A_m$. Every variable of a clause is assumed to be universally quantified at the front. A *Horn clause* is a clause which contains at most one positive literal; otherwise it is a *non-Horn clause*. It is known that a clause is a *tautology* if it has two complementary literals A and $\neg A$. We denote by $\bot$ the *empty clause* which contains no literal. Note that $\bot$ is inconsistent.

A *clausal theory* is a finite set of clauses, which represents the conjunction of the clauses in it. A clausal theory is *full* if it contains at least one non-Horn clause. A clausal theory is *ground* if it contains no variable. A (*universal*) *conjunctive normal form* (CNF) formula is a conjunction of clauses, and a *disjunctive normal form* (DNF) formula is a disjunction of conjunctions of literals. A clausal theory is identified with the CNF formula that is the conjunction of clauses in it. We denote by $\models$ the classical logical entailment relation and by $=\!|$ the inverse relation of entailment, called *anti-entailment*. Let S and T be two clausal theories. S and T are (*logically*) *equivalent*, denoted by $S \equiv T$, if $S \models T$ and $T \models S$. A clause C is a *consequence* of S if $S \models C$.

A (ground) *substitution* θ replaces variables $x_1, \dots, x_k$ occurring in a clause C to (ground) terms $t_1, \dots, t_k$ in $C\theta$. Note that $C\theta$ is called an *instance* of C. Let L_1 and L_2 be two literals. A substitution θ is called a *unifier* for L_1 and L_2 if $L_1\theta = L_2\theta$. A unifier θ is a *most general unifier* (mgu) if there is no other unifier σ for which the unified literal $L_1\sigma$ is more general than $L_1\theta$. Let C and D be two clauses. C *subsumes* D, denoted by $C \succeq D$, if there is a substitution θ such that $C\theta \subseteq D$. C *properly subsumes* D if $C \succeq D$ but $D \not\succeq C$.

Definition 1 (Theory-subsumption) Let S and T be two clausal theories. Then, S (*theory-*) *subsumes* T, denoted by $S \succeq T$, if for any clause $D \in T$, there is a clause $C \in S$ such that $C \succeq D$. We denote by $\preceq$ the inverse relation of the (theory-) subsumption, called *anti-subsumption*.

Now, we have two concepts: entailment and subsumption to characterize the logical relation between two clausal theories S and T. It is known that $S \models T$ holds if $S \succeq T$, though $S \succeq T$ does not necessarily hold even if $S \models T$. Suppose two clausal theories $S = \{p(X) \supset p(f(X))\}$ and $T = \{p(Y) \supset p(f(f(Y)))\}$ are given. Indeed, $S \not\succeq T$, but $S \models T$ holds, since T is derivable from S using the resolution principle. The two concepts can be logically connected in the context of the resolution principle by Lee's theorem (Lee 1967), alternatively called the *Subsumption theorem* (Nienhuys-Cheng and De Wolf 1997).

Let C and D (called *parent clauses*) be two clauses, and L_c and L_d two literals in C and D, respectively. If there is a most general unifier θ for L_c and $\neg L_d$, then the clause

$$(C\theta - \{L_c\theta\}) \cup (D\theta - \{L_d\theta\})$$

is called a *resolvent* of C and D. For example, recall the above S and T. The clause in T is a resolvent of two copies of the clause in S: $R_1 = p(X_1) \supset p(f(X_1))$ and $R_2 = p(X_2) \supset p(f(X_2))$, since two literals $p(f(X_1))$ and $p(X_2)$ has a mgu θ that replaces X_1 and X_2 to

Y and $f(Y)$, respectively. Then, the resolvent $(R_1\theta - \{p(f(Y))\}) \cup (R_2\theta - \{\neg p(f(Y))\})$ corresponds to the clause in T.

A *derivation* of a clause C from S is a finite sequence of clauses $R_1, \dots, R_k = C$ such that each R_i is either in S, or is a resolvent of two clauses in $\{R_1, \dots, R_{i-1}\}$. Then, the Subsumption theorem states that $S \models T$ if and only if for each clause $C \in T$, C is a tautology or there is a derivation of a clause D from S such that D *subsumes* C.

2.2 Hypothesis finding based on inverse entailment

We give the definition of a hypothesis H in the setting of learning from entailment:

Definition 2 (Hypothesis) Let B and E be clausal theories, representing a background theory and positive examples, respectively. A clausal theory H is a *hypothesis* wrt B and E if H satisfies that $B \wedge H \models E$ and $B \wedge H$ is consistent.

We refer to a hypothesis instead of a hypothesis wrt B and E if no confusion arises.

Example 2 Suppose that

$$B_2 = \{buy(john, diaper) \vee buy(john, beer)\}, \qquad E_2 = \{shopping(john, at\_night)\}$$

are given. Then,

$$H_2 = \{buy(X, diaper) \supset buy(X, beer), \tag{3}$$

$$buy(Y, beer) \supset shopping(Y, at\_night)\} \tag{4}$$

is a hypothesis, since $B_2 \wedge H_2 \models E_2$ and $B_2 \wedge H_2$ is consistent. Note here that the clause (3) means that customers who buy diapers also tend to buy beer,[2] and the clause (4) means that customers who buy beer tend to go shopping at night.

Hypothesis finding in Definition 2 is logically equivalent to seeking a consistent hypothesis H such that $B \wedge \neg E \models \neg H$. Using this alternative condition, IE-based methods (Muggleton 1995; Yamamoto 2003; Ray et al. 2003; Ray and Inoue 2008; Ray 2009; Tamaddoni-Nezhad and Muggleton 2009; Kimber et al. 2009) compute a hypothesis H in two steps. First, they construct an intermediate theory F such that F is ground and $B \wedge \neg E \models F$. Hereafter, we call F a *bridge theory* wrt B and E as follows.

Definition 3 (Bridge theory) Let B and E be a background theory and examples, respectively. Let F be a ground clausal theory. Then F is a *bridge theory* wrt B and E if $B \wedge \neg E \models F$ holds. If no confusion arises, a bridge theory wrt B and E will simply be called a bridge theory.

After constructing a bridge theory F, they next generalize its negation $\neg F$ to a hypothesis H such that $H \models \neg F$.

[2] This rule is often used to introduce the market basket analysis that detects cross-selling opportunities from the custom behavior.

Example 3 Recall Example 2. Let a ground clausal theory F_2 be as follows:

$$\{buy(john, diaper) \vee buy(john, beer),\ \neg shopping(john, at\_night)\}.$$

Since $F_2 = B_2 \cup \neg E_2$, F_2 is a bridge theory wrt B_2 and E_2. We easily have the DNF formula of $\neg F_2$ using De Morgan's laws. By translating this DNF formula into CNF with the standard equivalent operations, we get $\neg F_2$ as the following clausal theory:

$$\neg F_2 = \{buy(john, diaper) \supset shopping(john, at\_night), \quad (5)$$

$$buy(john, beer) \supset shopping(john, at\_night)\}. \quad (6)$$

The clause (5) is subsumed by the resolvent of two parent clauses (3) and (4) in H_2. The other clause (6) is also subsumed by the clause (4) in H_2. Hence, $H_2 \models \neg F_2$ holds, though H_2 does not subsume $\neg F_2$.

Every IE-based method generalizes the negation of a constructed bridge theory to a hypothesis in its own way. On the one hand, CF-induction (Inoue 2004) generalizes it based on anti-entailment. There are several well-known operators to realize this generalization, such as *inverse resolution* (Muggleton and Buntine 1988) which applies the inverse of resolution, *anti-weakening* which adds some clauses, *anti-instantiation* which replaces ground terms with variables and *dropping* which drops some literals from a clause. These generalization operators are soundly applied, and can jointly generate any hypothesis H such that $H \models \neg F$. For example, H_2 is generated from $\neg F_2$ in such a way that we first replace the term $john$ in the clause (5) with a variable using anti-instantiation, and next derive the two parent clauses (3) and (4) by applying inverse resolution to the clause (5).

Note here that there are many ways to apply inverse resolution to a clause, because inverse resolution can generate whatever two parent clauses of it. In turn, the other generalization operators are also applicable in many ways. Moreover, any combination of them can be applied as another operator. This fact makes generalization with anti-entailment highly non-deterministic and causes the search space to be very large.

Because of this situation, most IE-based methods (Muggleton 1995; Yamamoto 2003; Ray et al. 2003; Ray and Inoue 2008; Ray 2009; Tamaddoni-Nezhad and Muggleton 2009; Kimber et al. 2009), except for CF-induction, are based on inverse subsumption that generalizes the negation of a bridge theory using anti-subsumption, instead of anti-entailment. Generalization with anti-subsumption has been actively studied in the context of *refinement operators* (Nienhuys-Cheng and De Wolf 1997; Badea and Stanciu 1999; Bratko 1999; Riguzzi 2005; Tamaddoni-Nezhad and Muggleton 2009). They systematically explore the hypothesis space structured by a bounded subsumption lattice. However, it was not yet clarified how their logical reduction from inverse entailment to inverse subsumption affects the completeness of generalization. For example, though H_2 can be generated from $\neg F_2$ by inverse entailment, inverse subsumption cannot do as H_2 does not subsume $\neg F_2$. For this problem, the following two sections show that given a bridge theory F and a hypothesis H such that $H \models \neg F$, there is a certain clausal theory F^* such that $F^* \equiv \neg F$ and $H \succeq F^*$.

2.3 Residue and minimal complements

We define two kinds of clausal theories regarded as the above F^*. A clausal theory S is *irredundant* if there is no clause $C \in S$ such that $S - \{C\} \equiv S$; otherwise it is *redundant*. Note that S becomes redundant if S contains either tautologies or clauses that are properly subsumed by others. $\tau(S)$ denotes the clausal theory obtained by removing all the tautologies from S. $\mu(S)$ denotes the clausal theory obtained by removing from S all clauses that are properly subsumed by clauses in S. We say S is *subsume-minimal* if $S = \mu(S)$ holds.

Let S be a ground clausal theory $\{C_1, C_2, \ldots, C_n\}$ where each clause C_i $(1 \le i \le n)$ $= l_{i,1} \vee l_{i,2} \vee \cdots \vee l_{i,m_i}$. The *complement* of S, denoted by $\overline{S}$, is defined as follows:

$$\overline{S} = \left\{ \neg l_{1,k_1} \vee \neg l_{2,k_2} \vee \cdots \vee \neg l_{n,k_n} \;\middle|\; \begin{array}{c} 1 \le k_1 \le m_1,\ 1 \le k_2 \le m_2, \\ \ldots,\ 1 \le k_n \le m_n \end{array} \right\}.$$

In case that S is empty, $\overline{S}$ is defined as the set $\{\bot\}$ where $\bot$ is the empty clause. Note that $\overline{S}$ is a CNF formula such that $\overline{S} \equiv \neg S$. Accordingly, $\tau(\overline{S})$ and $\mu(\overline{S})$ are also CNF formulas logically equivalent to $\neg S$. In the following, we denote $\tau(\overline{S})$ and $\mu(\overline{S})$ as the functions $R(S)$ and $M(S)$, called the *residue* and *minimal* complement of S, respectively. For sake of simplicity, we often denote $R(R(S))$ and $M(M(S))$ by $R^2(S)$ and $M^2(S)$, respectively. Note that $R(S) \equiv M(S) \equiv \neg S$ and $R^2(S) \equiv M^2(S) \equiv S$.

Example 4 Let S be the clausal theory $\{a \vee b,\ b \vee c,\ \neg c\}$. Then, $\overline{S}$, $R(S)$ and $M(S)$ are as below. $M(S)$ is obtained by removing two clauses $\neg a \vee \neg b \vee c$ and $\neg b \vee \neg c \vee c$, which are subsumed by the clause $\neg b \vee c$. Note that $M(S)$ contains a tautology. On the other hand, $R(S)$ is obtained by all of the tautologies in $\overline{S}$, though $R(S)$ contains a redundant clause subsumed by another.

$$\overline{S} = \{\neg a \vee \neg b \vee c,\ \neg a \vee \neg c \vee c,\ \neg b \vee c,\ \neg b \vee \neg c \vee c\},$$

$$M(S) = \{\neg a \vee \neg c \vee c,\ \neg b \vee c\}, \qquad R(S) = \{\neg a \vee \neg b \vee c,\ \neg b \vee c\}.$$

In next two sections, we use the residue and minimal complements as the two kinds of clausal theories representing the above F^* such that $H \succeq F^*$ holds, and show that inverse subsumption with each of them ensures the completeness for finding hypotheses.

3 Inverse subsumption with residue complements

Let S and T be two ground clausal theories such that $S \models T$. Our approach is based on the fact that the logical relation between the two CNF formulas translated from $\neg S$ and $\neg T$ is represented by anti-subsumption. We intend to apply this feature to Relation (1). Since $\neg S$ and $\neg T$ are DNF formulas after applying De Morgan's laws, there are several ways to represent $\neg S$ and $\neg T$ into CNF. In this section, we use the residue complement and consider the logical relation between $R(S)$ and $R(T)$, which is represented primarily by the following theorem[3].

Theorem 1 (Yamamoto 2003) *Let S and T be two clausal theories such that T is ground and both S and T do not include any tautologies. If $S \models T$, there is a finite subset S' of ground instances from S such that $R(T) \succeq R(S')$.*

By Theorem 1, the following holds, when S is ground.

[3] Theorem 1 and Lemma 1 have been proved in the literature (Yamamoto 2003). We give their alternative proofs in the appendix.

Corollary 1 *Let S and T be two ground clausal theories such that S and T do not include any tautologies. If $S \models T$, then $R(T) \succeq R(S)$.*

We first recall the following lemma[4] to prove Corollary 1.

Lemma 1 (Yamamoto 2003) *For ground clausal theories S and T that do not include tautologies, $T \subseteq S$ implies $R(T) \succeq R(S)$.*

Using Lemma 1 and Theorem 1, Corollary 1 is proved as follows:

Proof of Corollary 1 By Theorem 1, there is a ground theory S' such that $S' \subseteq S$ such that $R(T) \succeq R(S')$. By Lemma 1, $R(S') \succeq R(S)$ holds. Hence, $R(T) \succeq R(S)$ holds. □

We apply Corollary 1 to the logical relation $F \models \neg H$ where F is a bridge theory and H is a ground hypothesis. We represent $\neg H$ using the residue complement $R(H)$. Suppose that F does not include any tautologies. Then, by Corollary 1, $R^2(H) \succeq R(F)$ holds. In other words, $R^2(H)$, which is logically equivalent to H, can be obtained from $R(F)$ using anti-subsumption.

Theorem 2 *Let F be a bridge theory such that F does not include tautologies, and H be a hypothesis such that $F \models \neg H$. Then, there is a hypothesis H^* such that $H^* \equiv H$ and $H^* \succeq R(F)$.*

Proof of Theorem 2 By Herbrand's theorem,[5] there is a ground clausal theory H_g such that $H \succeq H_g$ and $F \models \neg H_g$. Since $\neg H_g \equiv R(H_g)$ holds, $R^2(H_g) \succeq R(F)$ holds by Corollary 1. Assume the clausal theory $H^* = H \cup R^2(H_g)$. Since $H \succeq H_g$ and $H_g \equiv R^2(H_g)$, $H \models R^2(H_g)$ holds. Accordingly, $H \models H^*$ holds. Hence, $H^* \equiv H$ holds. Since $R^2(H_g) \succeq R(F)$ and $H^* \supseteq R^2(H_g)$, $H^* \succeq R(F)$ holds. □

Example 5 Let B_3, E_3 and H_3 be a background theory, examples and a target hypothesis as follows:

$$B_3 = \{p \supset q\}, \qquad E_3 = \{p \supset r\},$$
$$H_3 = \{q \supset r\}.$$

Suppose the clausal theory $F_3 = \{p \supset q,\ p,\ \neg r\}$. Since $\neg E_3 = p \wedge \neg r$, F_3 corresponds to $B_3 \wedge \neg E_3$. Then, F_3 is a bridge theory wrt B_3 and E_3 such that $F_3 \models \neg H_3$. The residue complement $R(F_3)$ is $\{\neg q \vee p \vee r\}$. We notice that $R(F_3)$ is subsumed by H_3. Hence, the hypothesis H_3 can be obtained from $R(F_3)$ using anti-subsumption.

Theorem 2 means that for every hypothesis H, its equivalent hypothesis H^* can be derived from the residue complement $R(F)$ using anti-subsumption. In this sense, inverse subsumption with residue complements ensures the completeness of generalization.

[4] In case that T is empty, $\overline{T} = \{\bot\}$ holds. Since $R(T) = \tau(\overline{T})$, $R(T)$ contains the empty clause $\bot$ which subsumes any clause. Hence, for any clausal theory S, $R(T) \succeq R(S)$ holds.

[5] A set of clauses Σ is unsatisfiable if and only if a finite set of ground instances of clauses of Σ is unsatisfiable (Chang and Lee 1973).

However, every target hypothesis itself is not necessarily obtained from the residue complement by anti-subsumption. The below example describes such a case.

Example 6 Let B_4, E_4 and H_4 be a background theory, examples and a target hypothesis as follows:

$$B_4 = \{p(a)\}, \qquad E_4 = \{p(f(f(a))\},$$
$$H_4 = \{p(a) \supset p(f(a)),\ p(f(a)) \supset p(f(f(a)))\}.$$

Let F_4 be the clausal theory $\{p(a), \neg p(f(f(a)))\}$. Since F_4 corresponds to $B_4 \wedge \neg E_4$, F_4 is a bridge theory wrt B_4 and E_4 such that $F_4 \models \neg H_4$. $R(F_4)$ is $\{p(a) \supset p(f(f(a)))\}$. Then we notice that $R(F_4)$ is not subsumed by H_4. Indeed, $R(F_4)$ is the resolvent of two clauses in H_4. Hence, we need to apply an inverse resolution operator to $R(F_4)$ for obtaining the target hypothesis H_4. Note that $R(H_4)$ and $R^2(H_4)$ are as follows:

$$R(H_4) = \{p(a) \vee p(f(a)),\ p(a) \vee \neg p(f(f(a))),\ \neg p(f(a)) \vee \neg p(f(f(a)))\},$$
$$R^2(H_4) = \{\neg p(a) \vee p(f(a)),\ \neg p(a) \vee p(f(f(a))),\ \neg p(f(a)) \vee p(f(f(a))),$$
$$\neg p(a) \vee p(f(f(a))) \vee p(f(a)),\ \neg p(a) \vee p(f(f(a)) \vee \neg p(f(a))\}.$$

We notice that $R^2(H_4)$ contains the unique clause $\neg p(a) \vee p(f(f(a)))$ in $R(F_4)$. Hence, $R^2(H_4)$ subsumes $R(F_4)$. Since $R^2(H_4) \equiv H_4$, Theorem 2 holds by regarding the equivalent hypothesis H_4^* as $R^2(H_4)$.

The problem described in the above example is caused by the fact that $R^2(H) = H$ cannot necessarily hold. Indeed, the key idea in Theorem 2 lies in the logical relation $R^2(H) \succeq R(F)$. If $R^2(H) = H$ should not hold, H cannot be obtained from $R(F)$ using anti-subsumption. We thus need another CNF formula $F(H)$ for representing the negation of a hypothesis H such that $F(F(H)) = H$.

4 Inverse subsumption with minimal complements

4.1 Properties of minimal complements

We here investigate minimal complements. Firstly, the following theorem holds.

Theorem 3 *Let S be a ground clausal theory. Then, $M^2(S) = \mu(S)$ holds.*

Proof The proof of Theorem 3 is given in the appendix. □

This theorem can be regarded as a fixpoint theorem on the function M computing the minimal complement of $\mu(S)$. Unlike residue complements, $M^2(S)$ corresponds to S itself in case that S is subsume-minimal. Thus, minimal complements may not cause the problem of residue complements that they cannot necessarily obtain a target hypothesis using anti-subsumption, as described in Sect. 3.

Example 7 We recall Example 4. Then, $\overline{S}$, $R(S)$, $R^2(S)$, $M(S)$ and $M^2(S)$ are as follows. In fact, $M^2(S) = S$ holds, whereas $R^2(S)$ does not.

$$\overline{S} = \{\neg a \vee \neg b \vee c,\ \neg a \vee \neg c \vee c,\ \neg b \vee c,\ \neg b \vee \neg c \vee c\},$$

$$R(S) = \{\neg a \vee \neg b \vee c,\ \neg b \vee c\}, \qquad R^2(S) = \{a \vee b,\ \neg c \vee a,\ b,\ \neg c \vee b,\ \neg c\},$$

$$M(S) = \{\neg a \vee \neg c \vee c,\ \neg b \vee c\}, \qquad M^2(S) = \{a \vee b,\ b \vee c,\ \neg c\}.$$

On the other hand, unlike residue complements, the logical relation $M(T) \succeq M(S)$ does not necessarily hold whenever $S \models T$ holds for ground clausal theories S and T.

Example 8 We recall Example 6. $M(H_4)$ is as follows:

$$\{p(a) \vee p(f(a)),\ p(a) \vee \neg p(f(f(a))),\ \neg p(f(a)) \vee \neg p(f(f(a))),\ \neg p(f(a)) \vee p(f(a))\}.$$

Suppose the same bridge theory $F_4 = \{p(a),\ \neg p(f(f(a)))\}$. Then, $F_4 \models M(H_4)$ holds. But, $M^2(H_4) \succeq M(F_4)$ does not hold. Because $M^2(H_4)$ corresponds to H_4 by Theorem 3, and H_4 does not subsume $M(F_4) = \{\neg p(a) \vee p(f(f(a)))\}$.

This is because minimal complements can include tautologies that residue complements never have. Indeed, Corollary 1, which shows the logical relation between $R(T)$ and $R(S)$, does not allow any tautologies to be included in S and T. We then extend Corollary 1 so as to deal with tautologies as follows:

Theorem 4 *Let S and T be ground clausal theories such that $S \models T$ and for every tautology $D \in T$, there is a clause $C \in S$ such that $C \succeq D$. Then,*

$$\tau(M(T)) \succeq \tau(M(S)).$$

Proof The proof of Theorem 4 is given in the Appendix B. □

Example 9 Recall Example 6. We have $F_4 \models M(H_4)$, but $M(H_4)$ contains one tautology: $\neg p(f(a)) \vee p(f(a))$, which is not subsumed by any clause in F_4. Suppose that this tautology is added to F_4. We denote by F_4' the added clausal theory. Since $F_4' = \{p(a),\ \neg p(f(f(a))),\ \neg p(f(a)) \vee p(f(a))\}$, $\tau(M(F_4'))$ is as follows:

$$\{\neg p(a) \vee \boxed{p(f(f(a))) \vee \neg p(f(a)),}$$
$$\boxed{\neg p(a)} \vee p(f(f(a)) \vee \boxed{p(f(a))}\}.$$

We then notice that H_4 subsumes $\tau(M(F_4'))$ (See the dotted surrounding parts). This subsumption relation can be derived using Theorem 4. Since $F_4' \equiv F_4$ and $F_4 \models M(H_4)$, it holds that $F_4' \models M(H_4)$. Since the tautology in $M(H_4)$ is also contained in F_4', we can use Theorem 4, and then have $\tau(M^2(H_4)) \succeq \tau(M(F_4'))$. By Theorem 3, $\tau(M^2(H_4)) = \tau(\mu(H_4))$ holds. Since H_4 is subsume-minimal and it does not contain any tautologies, it holds that $\tau(\mu(H_4)) = H_4$. Hence, we obtain $H_4 \succeq \tau(M(F_4'))$.

4.2 Generalization with minimal complements

Theorems 3 and 4 enable us to construct an alternative generalization procedure using minimal complements. To describe the hypotheses that can be found by this, we first introduce the following language bias, called an *induction field*:

Definition 4 (Induction field) An *induction field*, denoted by $\mathcal{I}_H = \langle \mathbf{L} \rangle$, where $\mathbf{L}$ is a finite set of literals to appear in ground hypotheses. A ground hypothesis H_g *belongs to* $\mathcal{I}_H$ if every literal in H_g is included in $\mathbf{L}$. Given an induction field $\mathcal{I}_H = \langle \mathbf{L} \rangle$, $Taut(\mathcal{I}_H)$ is defined as the set of tautologies $\{\neg A \vee A \mid A \in \mathbf{L} \text{ and } \neg A \in \mathbf{L}\}$.

We next define the target hypotheses using the notion of an induction field $\mathcal{I}_H$, together with a bridge theory F as follows:

Definition 5 (Hypothesis wrt $\mathcal{I}_H$ and F) Let H be a hypothesis. H is a *hypothesis* wrt $\mathcal{I}_H$ and F if there is a ground hypothesis H_g such that H_g consists of instances from H, $F \models \neg H_g$ and H_g belongs to $\mathcal{I}_H$.

Now, the generalization procedure based on inverse subsumption with minimal complements is as follows:

Definition 6 Let B, E and $\mathcal{I}_H = \langle \mathbf{L} \rangle$ be a background theory, examples and an induction field, respectively. Let F be a bridge theory wrt B and E. A clausal theory H is derived by *inverse subsumption with minimal complements* from F wrt $\mathcal{I}_H$ if H is constructed as follows.

Step 1. Compute $Taut(\mathcal{I}_H)$;
Step 2. Compute $\tau(M(F \cup Taut(\mathcal{I}_H)))$;
Step 3. Construct a clausal theory H satisfying the condition:

$$H \succeq \tau(M(F \cup Taut(\mathcal{I}_H))). \tag{7}$$

Inverse subsumption with minimal complements ensures the completeness for finding hypotheses wrt $\mathcal{I}_H$ and F, by way of (7).

Main Theorem *Let B, E and $\mathcal{I}_H$ be a background theory, examples and an induction field, respectively. Let F be a bridge theory* wrt *B and E. For every hypothesis H* wrt *$\mathcal{I}_H$ and F, H is derived by inverse subsumption with minimal complements from F* wrt *$\mathcal{I}_H$.*

Proof of Main Theorem It is sufficient to prove the following lemma. □

Lemma 2 *Let B, E and $\mathcal{I}_H$ be a background theory, examples and an induction field, respectively. Let F be a bridge theory* wrt *B and E. For every hypothesis H* wrt *$\mathcal{I}_H$ and F, H satisfies the following condition*:

$$H \succeq \tau(M(F \cup Taut(\mathcal{I}_H))).$$

Proof of Lemma 2 Since H is a hypothesis wrt $\mathcal{I}_H$ and F, there is a ground hypothesis H_g such that $H \succeq H_g$, $F \models \neg H_g$ and H_g belongs to $\mathcal{I}_H$. Since $\neg H_g \equiv M(H_g)$, $F \models M(H_g)$

holds. Accordingly, $F \cup Taut(\mathcal{I}_H) \models M(H_g)$ holds. Since H_g belongs to $\mathcal{I}_H$, every literal in H_g is included in $\mathcal{I}_H$. Then, for every tautological clause $D \in M(H_g)$, there is a clause $C \in Taut(\mathcal{I}_H)$ such that $C \succeq D$. By Theorem 4, $\tau(M^2(H_g)) \succeq \tau(M(F \cup Taut(\mathcal{I}_H)))$ holds. Since $\mu(H_g) = M^2(H_g)$ by Theorem 3, $\tau(\mu(H_g)) \succeq \tau(M(F \cup Taut(\mathcal{I}_H)))$ holds. Since $H_g \supseteq \tau(\mu(H_g))$, $H_g \succeq \tau(\mu(H_g))$ holds. Hence, $H \succeq \tau(M(F \cup Taut(\mathcal{I}_H)))$ holds. □

4.3 Examples

We show how a target hypothesis is derived by inverse subsumption with minimal complements using the below examples.

Example 10 Recall Example 6 that could not be solved using residue complements. Let an induction field $\mathcal{I}_{H_4}$ be as follows:

$$\mathcal{I}_{H_4} = \langle \{\neg p(a),\ p(f(a)),\ \neg p(f(a)),\ p(f(f(a)))\} \rangle.$$

H_4 belongs to $\mathcal{I}_{H_4}$ and $H_4 \models \neg F_4$ holds. Then, H_4 is a hypothesis wrt $\mathcal{I}_{H_4}$ and F_4. $Taut(\mathcal{I}_{H_4})$ is the set $\{p(f(a)) \vee \neg p(f(a))\}$. Note that $F_4 \cup Taut(\mathcal{I}_{H_4})$ corresponds to F_4' in Example 9. Then, H_4 subsumes $\tau(M(F_4 \cup Taut(\mathcal{I}_{H_4})))$ as shown in Example 9. Hence, H_4 is derivable by inverse subsumption with minimal complements.

Example 11 Recall H_2 and F_2 in Example 3. $R(F_2)$ corresponds to $\neg F_2$ consisting of two clauses (5) and (6). Then, H_2 does not subsume $R(F_2)$, and cannot be generated from $R(F_2)$ by inverse subsumption. In contrast, it is derivable by inverse subsumption with minimal complement. Let an induction field $\mathcal{I}_{H_2}$ be as follows:

$$\langle \{\neg buy(john, diaper),\ buy(john, beer),\ \neg buy(john, beer),\ shopping(john, at\_night)\} \rangle.$$

Consider the following ground hypothesis H_{g_2} consisting of instances from H_2:

$$H_{g_2} = \{buy(john, diaper) \supset buy(john, beer),$$
$$buy(john, beer) \supset shopping(john, at\_night)\}.$$

H_{g_2} belongs to $\mathcal{I}_{H_2}$ and $F_2 \models \neg H_{g_2}$ holds. Then, H_2 is a hypothesis wrt $\mathcal{I}_{H_2}$ and F_2. $Taut(\mathcal{I}_{H_2})$ contains one tautology: $buy(john, beer) \vee \neg buy(john, beer)$. After adding this tautology to F_2, we compute $\tau(M(F_2 \cup Taut(\mathcal{I}_{H_2})))$ represented as follows.

$$\{\boxed{\neg buy(john, diaper) \vee buy(john, beer)} \vee shopping(john, at\_night),$$
$$\boxed{\neg buy(john, beer) \vee shopping(john, at\_night)}\}.$$

Then, H_2 indeed subsumes $\tau(M(F_2 \cup Taut(\mathcal{I}_{H_2})))$ (See the dotted surrounding parts). Hence, H_2 can be generated by inverse subsumption with this minimal complement.

Example 12 We next consider the following example on pathway completion:

$$B_5 = \{arc(a, b),\ arc(X, Y) \wedge path(Y, Z) \supset path(X, Z)\}, \qquad E_5 = \{path(a, c)\},$$
$$\mathcal{I}_{H_5} = \{arc(b, c),\ \neg arc(b, c),\ path(b, c),\ \neg path(b, c)\},$$
$$H_5 = \{arc(b, c),\ arc(X, Y) \supset path(X, Y)\}.$$

Note that $arc(X,Y)$ (*resp.* $path(X,Y)$) means there is an arc (*resp.* a path) from a node X to a node Y. B_5 contains one fact that there is an arc from a to b and one rule that, if there is an arc from X to Y and a path from Y to Z, then there is a path from X to Y. However, only B_5 cannot logically explain E_5 that there is a path from a to c. One possible cause is that one arc from b to c and another rule defining the concept of pathways are missing in the background theory. Then, we seek for the hypothesis H_5 that completes these missing fact and rule. To complete H_5, both abduction and induction must involve, but most current ILP systems cannot compute it. This advanced inference has a possibility to be effectively applied when we need to complete both facts and rules that are missing in a prior background theory. In fact, there is a recent work to use it for finding *master reactions* from incomplete biochemical networks in systems biology (Yamamoto et al. 2009b).

Let F_5 be the clausal theory $\{arc(a,b),\ arc(a,b) \wedge path(b,c) \supset path(a,c),\ \neg path(a,c)\}$. Since F_5 is the set of ground instances from $B_5 \wedge \neg E_5$, F_5 is a bridge theory wrt B_5 and E_5. Since there is a ground hypothesis $H_{g_5} = \{arc(b,c),\ arc(b,c) \supset path(b,c)\}$ such that H_{g_5} consists of instances from H_5, $F_5 \models \neg H_{g_5}$ and H_{g_5} belongs to $\mathcal{I}_{H_5}$, H_5 is a hypothesis wrt $\mathcal{I}_{H_5}$ and F_5. Then, H_5 could be derived by inverse subsumption with minimal complements. We first compute $Taut(\mathcal{I}_{H_5})$. Then, $Taut(\mathcal{I}_{H_5})$ is the set $\{\neg arc(b,c) \vee arc(b,c),\ \neg path(b,c) \vee path(b,c)\}$. After adding $Taut(\mathcal{I}_{H_5})$ to F_5, we compute $\tau(M(F \cup Taut(\mathcal{I}_{H_5})))$ represented as follows.

$$\{\neg arc(a,b) \vee path(b,c) \vee \boxed{arc(b,c)} \vee path(a,c),$$
$$\neg arc(a,b) \vee \boxed{\neg arc(b,c) \vee path(b,c)} \vee path(a,c)\}.$$

We then notice that H_5 subsumes $\tau(M(F \cup Taut(\mathcal{I}_{H_5})))$ (See the dotted surrounding parts). Therefore, H_5 can be derived by inverse subsumption with minimal complements.

In contrast, Since $R(F_5)$ is $\{\neg arc(a,b) \vee path(b,c) \vee path(a,c)\}$, H_5 does not subsume the residue $R(F_5)$. Hence, H_5 cannot be obtained from the residue complement, whereas the minimal complement can do with inverse subsumption.

Example 13 We lastly consider a biological example to find cellular regulations.

$$B_6 = \{glucose\_ext \supset induced(hxt) \vee active(snf3)\},$$

$$E_6 = \{glucose\_ext \supset glycolysis\_on\},$$

$$H_6 = \{active(snf3) \supset induced(hxt),\ induced(hxt) \supset glycolysis\_on\}.$$

Most eukaryotic cells, including yeasts and humans, can sense the availability of carbon sources in their surroundings and, in the presence of their favorite sugar (often glucose), they transport glucose into the cell and use it through the glycolysis pathway to produce energy (Westergaard et al. 2006). The example E_6 describes this causality between glucose and glycolysis. Now, we know that if glucose is available, the hexose transporter Hxt can be induced or the sensing protein Snf3 can be active. This prior background theory B_6 cannot logically explain E_6, and thus there are some missing links between B_6 and E_6. In recent work (Westergaard et al. 2006), it has been made known that a signal triggered by Snf3 leads to induce Hxt, and then glucose can be moved into the cell via the transporter Hxt. Then, H_6, which describes these cellular regulations, is a considerable hypothesis. However, it is not straightforward for most current IE-based methods to generate the target hypothesis. In the following, we show how our proposal can solve this example. Let a bridge theory F_6 be $B_6 \wedge \neg E_6$. We give the induction field $\mathcal{I}_{H_6}$ as follows:

$$\langle\{\neg active(snf3),\ induced(hxt),\ \neg induced(hxt),\ glycolysis\_on\}\rangle.$$

Since H_6 belongs to $\mathcal{I}_{H_6}$ and $F_6 \models \neg H_6$ holds, H_6 is a hypothesis wrt $\mathcal{I}_{H_6}$ and F_6. $Taut(\mathcal{I}_{H_6})$ contains one tautology: $induced(hxt) \vee \neg induced(hxt)$. After adding the tautology to F_6, we compute $\tau(M(F_6 \cup Taut(\mathcal{I}_{H_6})))$ as follows:

$$\{\neg glucose\_ext \vee \boxed{\neg active(snf3) \vee induced(hxt)} \vee glycolysis\_on,$$
$$\neg glucose\_ext \vee \boxed{\neg induced(hxt) \vee glycolysis\_on}\}.$$

H_6 subsumes $\tau(M(F_6 \cup Taut(\mathcal{I}_{H_6})))$ (See the dotted surrounding parts). Then, H_6 is derivable by inverse subsumption with minimal complements. Note that the residue complement $R(F_6)$ is as follows:

$$\{\neg glucose\_ext \vee \neg active(snf3) \vee glycolysis\_on,$$
$$\neg glucose\_ext \vee \neg induced(hxt) \vee glycolysis\_on\}.$$

Then, H_6 does not subsume $R(F_6)$, and then cannot be generated by inverse subsumption with residue complements.

5 Further topics and related work

5.1 The commutative property of residue and minimal complements

We have proposed two approaches with residue and minimal complements for inverse subsumption. The derivable hypotheses in two approaches are characterized as Theorem 2 and Main Theorem, respectively. In this section, we clarify some commonness between these approaches and investigate what aspect causes their crucial difference.

Lemma 3 *Let S be a clausal theory. Then,* $\tau(\mu(S)) = \mu(\tau(S))$.

Proof of Lemma 3 We show that, for any clause $C \in S$, $C \notin \mu(\tau(S))$ if and only if $C \notin \tau(\mu(S))$. ($\Rightarrow$) Suppose $C \notin \mu(\tau(S))$. There is a clause $D \in \tau(S)$ such that $D \succeq C$. Since $D \in S$ holds, we have $C \notin \mu(S)$. Then, $C \notin \tau(\mu(S))$ holds. ($\Leftarrow$) Suppose $C \notin \tau(\mu(S))$. C is a tautology or there is a clause $D \in S$ such that $D \succeq C$. If C is a tautology, $C \notin \mu(\tau(S))$ holds. In the other case, since $D \succeq C$ and C is not a tautology, D is also not, that is, $D \in \tau(S)$ holds. Then, $C \notin \mu(\tau(S))$ holds. □

By Lemma 3, $\tau(M(S)) = \mu(R(S))$ holds, since $\tau(M(S)) = \tau(\mu(\overline{S}))$ and $\mu(R(S)) = \mu(\tau(\overline{S}))$. Hence, residue and minimal complements satisfy the commutative property. Using this property, we obtain a new variation by replacing $\tau(M(S))$ with $\mu(R(S))$ in Lemma 2 as follows:

Corollary 2 *Let H be a hypothesis* wrt $\mathcal{I}_H$ *and F. Then,* $H \succeq R(F \cup Taut(\mathcal{I}_H))$.

Proof of Corollary 2 By Lemma 2, $H \succeq \tau(M(F \cup Taut(\mathcal{I}_H)))$ holds. Since $\mu(R(S)) = \tau(M(S))$, $H \succeq \mu(R(F \cup Taut(\mathcal{I}_H)))$, and thus $H \succeq R(F \cup Taut(\mathcal{I}_H))$ holds. □

Every hypothesis wrt $\mathcal{I}_H$ and F is also derivable by inverse subsumption with residue complements by adding $Taut(\mathcal{I}_H)$ to the original bridge theory. In contrast, even if the tautologies are not added, inverse subsumption with minimal complements ensures the equivalent completeness in the case of residue complements.

Corollary 3 *Let F be a bridge theory without tautologies, and H be a hypothesis such that $F \models \neg H$. Then, there is a hypothesis H^* such that $H^* \equiv H$ and $H^* \succeq \tau(M(F))$.*

Proof of Corollary 3 By Theorem 2, there is a hypothesis H^* such that $H^* \equiv H$ and $H^* \succeq R(F)$. Since $R(F) \succeq \mu(R(F))$, $H^* \succeq \mu(R(F))$ holds. Since $\mu(R(F)) = \tau(M(F))$, we get $H^* \succeq \tau(M(F))$. □

The completeness of inverse subsumption with either residue or minimal complements is varied by whether adding tautologies or not. In the case of adding tautologies, both approaches can derive every hypothesis H wrt $\mathcal{I}_H$ and F. In the other case, they can derive its equivalent hypothesis H^*, which can be characterized as follows:

Definition 7 (Maximal Hypothesis) Let H be a hypothesis. H is a *maximal hypothesis* if, for each consequence C of H, there is a clause D in H such that $D \succeq C$.

Example 14 Let H_7 and H_8 be two hypotheses as follows:

$$H_7 = \{p(X) \supset p(f(X))\}, \qquad H_8 = \{p(a),\ p(X) \supset q(X)\}.$$

H_7 is not a maximal hypothesis since a consequence $p(X) \supset p(f(f(X)))$ of H_7 is not contained in H_7. H_8 is also not, since the consequence $q(a)$ is not in H_8. In contrast, $H_8 \cup \{q(a)\}$ is a maximal hypothesis. Note that, like non-recursive rules, any hypothesis where no derivations exist is a maximal hypothesis.

Maximal hypotheses are derivable by inverse subsumption even if tautologies are not added.

Corollary 4 *Let H be a maximal hypothesis* wrt *$\mathcal{I}_H$ and F. Then, $H \succeq R(F)$.*

Proof of Corollary 4 There is a maximal hypothesis H_g such that H_g consists of ground instances from H and $F \models \neg H_g$. Since $\neg H_g \equiv R(H_g)$, $F \models R(H_g)$ holds. By Corollary 1, we have $R^2(H_g) \succeq R(F)$. Since H_g is a maximal hypothesis, for every consequence C of H_g, there is a clause $D \in H_g$ such that $D \succeq C$. Since $H_g \equiv R^2(H_g)$, every clause in $R^2(H_g)$ is regarded as a consequence of H_g. Then, $H_g \succeq R^2(H_g)$ holds. Hence, we get $H \succeq R(F)$ because of $H \succeq H_g$ and $R^2(H_g) \succeq R(F)$. □

Corollary 5 *Let H be a maximal hypothesis* wrt *$\mathcal{I}_H$ and F. Then, $H \succeq \tau(M(F))$.*

Proof of Corollary 5 By Corollary 4, $H \succeq R(F)$ holds. Since $\mu(R(F)) = \tau(M(F))$ and $R(F) \succeq \mu(R(F))$, we get $H \succeq \tau(M(F))$. □

5.2 Embedding inverse subsumption to IE-based methods

The results in the paper can be applied to previously proposed IE-based methods. Firstly, we review those methods in brief.

Progol (Muggleton 1995; Tamaddoni-Nezhad and Muggleton 2009), one of the state-of-the-art ILP systems in Horn clause learning, uses the technique of *Bottom Generalization*. Its bridge theory F corresponds to the conjunction of ground literals each of which is derived from B and $\neg E$. After constructing $\neg F$ called the *bottom clause* $\bot(B, E)$, Progol generalizes it with anti-subsumption, instead of anti-entailment.

HAIL (Ray et al. 2003; Ray and Inoue 2008) constructs so-called *Kernel Sets* to overcome some limitation on Bottom Generalization. Each ground clause C_i in a Kernel Set $KS = \{C_1, \ldots, C_n\}$ is given by the form of $B_1^i \wedge \cdots \wedge B_{m_i}^i \supset A^i$, where $B \cup \{A^1, \ldots, A^n\} \models E$ and $B \models \{B_1^1, \ldots, B_{m_n}^n\}$. After constructing a Kernel Set, HAIL also generalizes it using anti-subsumption like Progol. A Kernel Set KS is regarded as the negation of a certain bridge theory F. In other words, HAIL directly constructs the negation of F by separately computing head and body literals of each clause in the negation.

Example 15 We recall Example 6. The bottom clause $\bot(B_4, E_4) = p(a) \supset p(f(f(a)))$ and the Kernel set KS only contains the bottom clause. Then, H_4 does not subsume neither $\bot(B_4, E_4)$ nor KS. Hence, both Progol and HAIL cannot solve this example.

We remark there is an extension called X-HAIL (Ray 2009) which allows the body in a Kernel Set KS to contain such literals that are derived by B *with the head literals of* KS. There is a recent work to extend Kernel Sets into so-called *Connection Theories* with an iterative procedure, called Imparo (Kimber et al. 2009). Note that X-HAIL and Imparo can solve Example 6.

The residue procedure (Yamamoto 2003), which has been firstly proposed to find hypotheses in full clausal theories, constructs a bridge theory F consisting of ground instances from $B \wedge \overline{E\sigma}$, where σ is a ground substitution to skolemize E. It then computes the residue complement $R(F)$, and generalizes it with anti-subsumption. In contrast, CF-induction (Inoue 2004) is sound and *complete* for finding hypotheses in full clausal theories. It constructs a bridge theory F consisting of ground instances from so-called *characteristic clauses* of $B \wedge \overline{E\sigma}$. Each characteristic clause is a subsume-minimal consequence of $B \wedge \overline{E\sigma}$ that satisfies a given language bias. Then CF-induction translates the DNF formula $\neg F$ into a CNF formula and generalizes it with anti-entailment.

Every method in the above, except for CF-induction, is based on inverse subsumption. Hence, it reduces anti-entailment to anti-subsumption. Based on our result, we investigate the completeness of generalization in those IS-based methods.

Definition 8 (Completeness in generalization) Suppose an IS-based method Γ. Γ is (*resp. partially*) *complete in generalization* if for each bridge theory F_Γ of Γ and each induction field $\mathcal{I}_H$, Γ derives any (*resp. maximal*) hypothesis wrt $\mathcal{I}_H$ and F_Γ.

For simplicity, we denote by p, h, x, i and r Progol, HAIL, X-HAIL, Imparo and Residue procedure, respectively. Note that HAIL (X-HAIL) and Imparo directly compute the negations of certain bridge theories. Then, we regard their bridge theories F_h (F_x) and F_i as the minimal complements of a Kernel Set and a Connection theory, respectively.

Corollary 6 *For each* $\Gamma \in \{p,\ h,\ x,\ i,\ r\}$, Γ *is partially complete in generalization.*

Proof of Corollary 6 Let H be a maximal hypothesis wrt $\mathcal{I}_H$ and F_X where $X \in \{p,\ h,\ x,\ i\}$. By Corollary 5, $H \succeq \tau(M(F_X))$ holds. In case that $X = p$, $\tau(M(F_p))$ corresponds to the bottom clause. Then, H is derivable from the bottom clause by p. In case that $X \in \{h,\ x\}$, F_X corresponds to $M(KS)$ for some Kernel Set KS. Then, $\tau(M(F_X)) = \tau(M^2(KS))$ holds. By Theorem 3, $\tau(M(F_X)) = \tau(\mu(KS))$ holds. Since KS does not contain tautologies, $H \succeq \mu(KS)$ holds. Then, H is derivable from KS by $X \in \{h,\ x\}$. In case that $X = i$, F_i corresponds to $M(CT)$ for some Connection Theory CT. Since CT also does not contain tautologies, H is derivable from CT by i, just like in the above case. In case that $X = r$, by Corollary 4, $H \succeq R(F_r)$ holds. Hence, H is derivable from $R(F_r)$ by r. □

Every IS-based method at least ensures the partial completeness of generalization. Progol and HAIL are incomplete in generalization by Example 15. On the other hand, it is still an open question whether or not X-HAIL and Imparo are also incomplete. The residue procedure is incomplete in generalization by Example 6. However, by Corollary 2, it becomes complete by adding the tautologies to the original bridge theory.

Our results can be applied to CF-induction in order to logically simplify its generalization procedure. Previously, it generalizes the negation of F to a hypothesis H using anti-entailment. As shown in Sect. 2.1, this generalization involves many non-deterministic operators that are the cause of its huge search space. By Main Theorem, it is sufficient to generalize $\tau(M(F \cup Taut(\mathcal{I}_H)))$ to H using anti-subsumption. This simplification enables us to systematically search relevant hypotheses in the subsumption lattice bounded by $\tau(M(F \cup Taut(\mathcal{I}_H)))$. Indeed, the other IS-based methods developed refinement operators to efficiently explore with heuristics the lattice structure (Tamaddoni-Nezhad and Muggleton 2009). By our results, such previously proposed sophisticated techniques can be embedded in CF-induction, while it preserves the completeness for finding hypotheses.

6 Conclusion and future work

This paper has shown a new form of inverse subsumption that can be embedded in a complete induction procedure. Most IE-based methods use anti-subsumption, instead of anti-entailment, for their generalization. However, it has not yet been clarified whether or not this logical reduction affects the completeness of generalization. For this open problem, we have shown that inverse subsumption can ensure the completeness only by adding tautologies associated with a language bias to the original bridge theory.

We have investigated the possible hypotheses obtained by each previously proposed method like Progol, HAIL, X-HAIL, Imparo and the residue procedure. As a result, we have shown that they are at least partially complete in the sense that they can derive any maximal hypotheses. The residue procedure becomes complete by simply adding tautologies to its bridge theories. In contrast, it is an open question whether or not X-HAIL and Imparo preserve the completeness of generalization. It would be fruitful to consider this question in future: if they could construct the theory obtained by adding the tautologies as another bridge theory, then they should be complete.

We have also shown that CF-induction can be logically simplified using the new form of inverse subsumption. Inverse entailment needs many non-deterministic operators like inverse resolution which cause its huge search space. This simplification enables us to focus on the search space characterized as a bounded subsumption lattice. This search space never involve inverse resolution. We intend to investigate how the search space can be reduced by the simplification in future.

Efficient implementation of inverse subsumption is an important future work. There is an efficient algorithm for enumerating the minimal hitting sets (Satoh and Uno 2002; Uno 2002). This is applicable to computing the minimal complement, and is solvable in quasi-polynomial total time (Fredman and Khanchiyan 1996). However, if the induction field $\mathcal{I}_H$ contains many complementary literals, we need vast computational costs, since the number of tautologies in $Taut(\mathcal{I}_H)$ becomes large. To restrict them, one may consider a closed world assumption or does not allow new terms that do not appear in a prior knowledge base. This issue on how to provide relevant induction fields should be addressed in future work.

It is also necessary to develop an algorithm to systematically explore the subsumption lattice bounded by the minimal complement. This issue is related to refinement operators, which have been studied in ILP. They use heuristics for guiding like compression and the description length. We emphasis that inverse subsumption ensures the completeness of generalization. Hence, it can derive hypotheses which are beyond reach for incomplete methods. In this point of view, we believe the algorithm for our approach should keep its completeness in some way. For example, it would be beneficial to target an enumerating algorithm that produces hypotheses in an incremental way.

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Appendix A: Proof of Theorem 3

We first introduce the notion of minimal hitting sets to prove Theorem 3.[6]

Definition 9 ((Minimal) Hitting set) Let Π be a finite set and $\mathcal{F}$ be a subset family of Π. A finite set E is a *hitting set* of $\mathcal{F}$ if for every $F \in \mathcal{F}$, $E \cap F \neq \emptyset$. A finite set E is a *minimal hitting set* of $\mathcal{F}$ if E satisfies the following two conditions:

1. E is a hitting set of $\mathcal{F}$;
2. For every subset $E' \subseteq E$, if E' is a hitting set of $\mathcal{F}$, then $E' = E$.

It is known that minimal hitting sets satisfy the following lemma:

Lemma 4 (Uno 2002) *Let $\mathcal{F}$ be a family of sets and E be a hitting set of $\mathcal{F}$. E is a minimal hitting set of $\mathcal{F}$ if and only if for every element $e \in E$, there is a set $F \in \mathcal{F}$ such that*

$$E \cap F = \{e\}.$$

Proof of Lemma 4

($\Rightarrow$) Suppose that there is an element $e \in E$ such that $E \cap F \neq \{e\}$ for every set $F \in \mathcal{F}$. Let E' be the subset $E - \{e\}$ of E. Then, it holds that $E' \cap F \neq \emptyset$ for every set $F \in \mathcal{F}$, since E is a hitting set of $\mathcal{F}$ and $E \cap F \neq \{e\}$ for every $F \in \mathcal{F}$. By the second condition of Definition 9, E is not a minimal hitting set of $\mathcal{F}$.

($\Leftarrow$) Suppose that E is not a minimal hitting set of $\mathcal{F}$. There is a subset E' of E such that E' is a hitting set of $\mathcal{F}$. For every $F \in \mathcal{F}$, $E' \cap F \neq \emptyset$ holds. There is an element e in E such that $e \notin E'$. Since $E' \subseteq E - \{e\}$, for every $F \in \mathcal{F}$, $(E - \{e\}) \cap F \neq \emptyset$ also holds. Accordingly, $E \cap F \neq \{e\}$ holds for every $F \in \mathcal{F}$. Hence, there is an element $e \in E$ such that $E \cap F \neq \{e\}$ for every $F \in \mathcal{F}$.

[6] Note that the proof of Theorem 3 has been partially shown in Yamamoto et al. (2009a), which is not a journal paper. We then show the full proof in this paper.

□

Let S be a family of sets $\{C_1, \dots, C_n\}$ where each C_i $(1 \le i \le n)$ is a finite set of ground literals $\{l_i^1, \dots, l_i^{m_i}\}$. In the following, $\mathcal{F}(S)$ denotes the family of sets $\{C_1^*, C_2^*, \dots, C_n^*\}$ where each C_i^* is the set of literals $\{\neg l_i^1, \dots, \neg l_i^{m_i}\}$. Let S be a ground clausal theory.[7] Then, $\mathcal{F}(\mathcal{F}(S))$ corresponds to S.

Example 16 Let S be the clausal theory $\{a \vee \neg b,\ \neg b \vee \neg c,\ \neg b \vee \neg d\}$. Then, $\mathcal{F}(S)$ and $\mathcal{F}(\mathcal{F}(S))$ are represented as follows:

$$\mathcal{F}(S) = \{\{\neg a,\ b\},\ \{b,\ c\},\ \{b,\ d\}\},$$
$$\mathcal{F}(\mathcal{F}(S)) = \{\{a,\ \neg b\},\ \{\neg b,\ \neg c\},\ \{\neg b,\ \neg d\}\}.$$

Given a ground clausal theory S, the number of minimal hitting sets of the family $\mathcal{F}$(S) is finite. We denote by *MHS*(S) the finite set of all the minimal hitting sets of $\mathcal{F}(S)$. Then, *MHS*(S) corresponds to the minimal complement $M(S)$ as follows.

Example 17 Recall Example 16. Then, *MHS*(S), $\overline{S}$ and $M(S)$ are as follows:

$$MHS(S) = \{\{\neg a,\ c,\ d\},\ \{b\}\},$$
$$\overline{S} = \{\neg a \vee b,\ \neg a \vee b \vee d,\ \neg a \vee c \vee d,\ \neg a \vee c \vee d,\ b,\ b \vee d,\ b \vee c,\ b \vee c \vee d\},$$
$$M(S) = \{\neg a \vee c \vee d,\ b\}.$$

We notice that *MHS*(S) indeed corresponds to $M(S)$.

Lemma 5 *Let S be a ground clausal theory. Then $M(S) = MHS(S)$ holds.*

Proof of MHS$(S) \subseteq M(S)$ Let E be a minimal hitting set of $\mathcal{F}(S)$. We show $E \in \mu(\overline{S})$ since $\mu(\overline{S}) = M(S)$ by the definition of minimal complements. By Lemma 4, for each literal $e_i \in E$ $(1 \le i \le n)$, there is a set $F_i \in \mathcal{F}(S)$ such that $E \cap F_i = \{e_i\}$. We denote by $\mathcal{F}_E$ the subfamily $\{F_1, \dots, F_n\}$ of $\mathcal{F}(S)$ where each F_i is the above set for each literal $e_i \in E$. By the definition of complements, each clause in $\overline{S}$ is constructed by selecting one literal l from each set in $\mathcal{F}(S)$. Since E is a minimal hitting set of $\mathcal{F}(S)$, for each set F in $\mathcal{F}(S)$, $E \cap F \neq \emptyset$ holds. Then, E can be constructed by selecting the literal $e_i \in E$ from each set $F_i \in \mathcal{F}_E$ and by selecting any literal e in $E \cap F$ from another set $F \in \mathcal{F}(S) - \mathcal{F}_E$. Hence, $E \in \overline{S}$ holds. Suppose that $E \notin \mu(\overline{S})$. Then, there is a clause $D \in \overline{S}$ such that $D \subset E$. Since $D \in \overline{S}$, D is a hitting set of $\mathcal{F}(S)$. However, this contradicts that E is a minimal hitting set of $\mathcal{F}(S)$. Therefore, $E \in \mu(\overline{S})$ holds.

Proof of $M(S) \subseteq MHS(S)$ Suppose that

$$(*) \quad \text{there is a clause } D \in M(S) \text{ such that } D \notin MHS(S).$$

Since $D \in \mu(\overline{S})$ and $\mu(\overline{S}) \subseteq \overline{S}$, $D \in \overline{S}$ holds. By the definition of $\overline{S}$, D satisfies that $C \cap D \neq \emptyset$ for every $C \in \mathcal{F}(S)$. Hence, D is a hitting set of $\mathcal{F}(S)$. Accordingly, there is a clause D'

[7] Each clause in S is identified with the set of literals which it contains (See Sect. 2.1).

in $MHS(S)$ such that $D' \subseteq D$. Since we assume $D \notin MHS(S)$, $D \neq D'$ holds. Then, we get $D' \subset D$. Since $D' \in MHS(S)$ and $MHS(S) \subseteq M(S)$, we have $D' \in M(S)$. Hence, there is a clause $D' \in M(S)$ such that D' properly subsumes the clause D in $M(S)$. This contradicts the minimality of $M(S)$. Hence, the primary assumption (*) is false. Therefore for every clause $D \in M(S)$, $D \in MHS(S)$ holds. □

Based on Lemma 5, we consider Theorem 3 in the context of minimal hitting sets. Hereafter, we simply denote $MHS(MHS(S))$ by $MHS^2(S)$.

Lemma 6 *Let S be a ground clausal theory. Then $MHS^2(S) = \mu(S)$ holds.*

Proof of $\mu(S) \subseteq MHS^2(S)$ We show every clause in $\mu(S)$ is a minimal hitting set of $\mathcal{F}(MHS(S))$. By the definition of $MHS(S)$, for every clause $D \in MHS(S)$, D satisfies that $D \cap C \neq \emptyset$ for every set $C \in \mathcal{F}(S)$. In other words, for every set $C \in \mathcal{F}(S)$, C satisfies that $D \cap C \neq \emptyset$ for every clause $D \in MHS(S)$. Then, every set $C \in \mathcal{F}(S)$ is a hitting set of $MHS(S)$. Let C' be the set of negations of literals in C. Since $C \in \mathcal{F}(S)$, $C' \in \mathcal{F}(\mathcal{F}(S))$ holds. Since C is a hitting set of $MHS(S)$, C' is a hitting set of $\mathcal{F}(MHS(S))$. Accordingly, every set $C \in \mathcal{F}(\mathcal{F}(S))$ is a hitting set of $\mathcal{F}(MHS(S))$. Since the family $\mathcal{F}(\mathcal{F}(S))$ corresponds to S, it holds that every clause $C \in S$ is a hitting set of $\mathcal{F}(MHS(S))$. Since $\mu(S) \subseteq S$, every clause $C \in \mu(S)$ is a hitting set of $\mathcal{F}(MHS(S))$.

Suppose that there is a clause $C \in \mu(S)$ such that C is not a *minimal* hitting set of $\mathcal{F}(MHS(S))$. Then,

(∗) there is a literal $l \in C$ such that $C - \{l\}$ is a hitting set of $\mathcal{F}(MHS(S))$.

For every clause $C_i \in \mu(S)$, if $C_i \neq C$ then there is a literal $l_i \in C_i$ such that $l_i \notin C$. We then consider those literals $E = \{l_1, l_2, \ldots, l_n\}$ where each l_i is a literal of $C_i \in \mu(S) - \{C\}$ such that l_i is not included in C. Note that $E \cap C = \emptyset$ holds. On the other hand, for any literal $l \in C$, it holds that $(E \cup \{l\}) \cap D \neq \emptyset$ for every clause $D \in \mu(S)$. Hence, $E \cup \{l\}$ is a hitting set of $\mu(S)$. Accordingly, $E \cup \{l\}$ is also a hitting set of S. Then, there is a minimal hitting set E' of S such that $E' \subseteq E \cup \{l\}$. Since $MHS(S)$ is the set of minimal hitting sets of $\mathcal{F}(S)$, $\mathcal{F}(MHS(S))$ corresponds to the set of minimal hitting sets of $\mathcal{F}(\mathcal{F}(S))$. Since $\mathcal{F}(\mathcal{F}(S)) = S$, $\mathcal{F}(MHS(S))$ is the set of minimal hitting sets of S. Hence, we have $E' \in \mathcal{F}(MHS(S))$. Since $E' \subseteq E \cup \{l\}$, it holds that $E' \cap (C - \{l\}) \subseteq (E \cup \{l\}) \cap (C - \{l\})$. Since $E \cap C = \emptyset$, $(E \cup \{l\}) \cap (C - \{l\}) = \emptyset$ holds. Accordingly, we have $E' \cap (C - \{l\}) = \emptyset$. However, this contradicts that $C - \{l\}$ is a hitting set of $\mathcal{F}(MHS(S))$, since $E' \in \mathcal{F}(MHS(S))$. Then, the assumption (*) is false. Hence, every clause C in $\mu(S)$ is a minimal hitting set of $\mathcal{F}(MHS(S))$.

Proof of $MHS^2(S) \subseteq \mu(S)$ Let D be a clause in $MHS^2(S)$. Suppose that there is a clause $C \in \mu(S)$ such that $C \subset D$. Since $\mu(S) \subseteq MHS^2(S)$, $C \in MHS^2(S)$ holds. This contradicts with the minimality of $MHS^2(S)$. Hence, for every clause $C \in \mu(S)$, $C \not\subset D$. In other words, for ever clause $C \in \mu(S)$, $C = D$ or $C \nsubseteq D$. Suppose that

(∗) for any clause $C \in \mu(S)$, $C \neq D$ holds.

Then, $C \nsubseteq D$ holds. Accordingly, for every clause $C_i \in \mu(S)$, there is a literal $l_i \in C_i$ such that $l_i \notin D$. We consider the finite set $E = \{l_1, l_2, \ldots, l_n\}$ where each l_i is the above literal for each $C_i \in \mu(S)$. Note that $E \cap D = \emptyset$. On the other hand, the intersection of E and any clause in $\mu(S)$ is not empty. Hence, E is a hitting set of $\mu(S)$. Accordingly, E is a hitting set of S. Then, there is a minimal hitting set E' of S such that $E' \subseteq E$. Since $MHS(S)$ is the set of minimal hitting sets of $\mathcal{F}(S)$, $\mathcal{F}(MHS(S))$ corresponds to the set of minimal hitting sets of $\mathcal{F}(\mathcal{F}(S))$. Since $\mathcal{F}(\mathcal{F}(S)) = S$, $\mathcal{F}(MHS(S))$ is the set of minimal hitting sets of S. Hence, we have $E' \in \mathcal{F}(MHS(S))$. Since $E' \subseteq E$ and $E \cap D = \emptyset$, $E' \cap D = \emptyset$ holds. Note that since $D \in MHS^2(S)$, D is a minimal hitting set of $\mathcal{F}(MHS(S))$. However this contradicts that $E' \cap D = \emptyset$, since $E' \in \mathcal{F}(MHS(S))$. Then, the assumption (*) is false. Hence, there is a clause $C \in \mu(S)$ such that $C = D$. Therefore, $D \in \mu(S)$ holds. □

Using Lemma 5 and Lemma 6, Theorem 3 is proved as follows:

Proof of Theorem 3 By Lemma 5, $M^2(S) = MHS^2(S)$ holds. By Lemma 6, $MHS^2(S) = \mu(S)$ holds. Hence, $M^2(S) = \mu(S)$ holds. □

Appendix B: Proof of Theorem 4

We introduce the following deductive operators for proving this theorem:

Definition 10 (Deductive operators (Yamamoto 2003)) Let S and T be clausal theories. Then T is *directly-derivable* from S if T is obtained from S by one of the following three operators:

1. (resolution) $T = S \cup \{C\}$, where C is a resolvent of two clauses $D_1, D_2 \in S$.
2. (subsumption) $T = S \cup \{C\}$, where C is subsumed by some clause $D \in S$.
3. (weakening) $T = S - \{D\}$ for some clause $D \in S$.

We write $S \vdash_r T$, $S \vdash_s T$, $S \vdash_w T$ to denote that T is directly derivable from S by resolution, subsumption, weakening, respectively. $\vdash_X^*$ is the reflexive and transitive closure of $\vdash_X$, where X is one of the symbols r, s, w. Alternatively, $S \vdash_X^* T$ if T follows from S by application of zero or more $\vdash_X$.

Let S and T be clausal theories such that S and T contains no tautologies and $S \models T$. Then, T can be generated from S with a concatenation of those operators, represented by the following lemma:

Lemma 7 (Yamamoto et al. 2008) *Let S and T be clausal theories without any tautologies. If $S \models T$, then there are two clausal theories U and V such that*

$$S \vdash_r^* U \vdash_s^* V \vdash_w^* T.$$

Proof of Lemma 7 Let $T = \{C_1, \ldots, C_n\}$. Then $S \models C_i$ for each clause C_i in T. By the Subsumption Theorem there is a derivation $R_1^i, \ldots, R_{m_i}^i$ from S of a clause $R_{m_i}^i$ that subsumes C_i. Hence, it is sufficient to let $U = S \cup \{R_j^i : 1 \leq i \leq n, 1 \leq j \leq m_i\}$ and $V = U \cup T$. □

Using Lemma 7, we obtain the following lemma that allows tautologies to be included in S and T:

F_4 with the tautology		V		$M(H_4)$
$p(a)$	$\vdash_s^3$	$p(a)$	$\vdash_w^2$	$p(a) \vee p(f(a))$
$\neg p(f(f(a)))$		$p(a) \vee p(f(a))$		$\neg p(f(f(a))) \vee p(a)$
$\neg p(f(a)) \vee p(f(a))$		$\neg p(f(f(a)))$		$\neg p(f(f(a))) \vee \neg p(f(a))$
		$\neg p(f(f(a))) \vee p(a)$		$\neg p(f(a)) \vee p(f(a))$
		$\neg p(f(f(a))) \vee \neg p(f(a))$		
		$\neg p(f(a)) \vee p(f(a))$		

Fig. 3 A derivation from F_4 to $M(H_4)$ in Example 8

Lemma 8 *Let S and T be ground clausal theories such that $S \models T$ and for every tautology $D \in T$, there is a clause $C \in S$ such that $C \succeq D$. Then there are two ground clausal theories U and V such that*

$$S \vdash_r^* U \vdash_s^* V \vdash_w^* T.$$

Proof of Lemma 8 We denote the two sets of tautologies in S and T by $Taut_S$ and $Taut_T$, respectively. Since S theory-subsumes $Taut_T$, there is a ground clausal theory V_t such that $S \vdash_s^* V_t \vdash_w^* Taut_T$. By Lemma 7, there are ground clausal theories U' and V' such that $S - Taut_S \vdash_r^* U' \vdash_s^* V' \vdash_w^* T - Taut_T$. Hence, we get

$$S \vdash_r^* U' \cup Taut_S \vdash_s^* V' \cup V_t \vdash_w^* T. \qquad \square$$

Example 18 Firstly, we recall the bridge theory F_4 and the hypothesis H_4 in Example 8. Note that $M(H_4)$ contains the tautology $\neg p(f(a)) \vee p(f(a))$, though F_4 does not. Then, $M(H_4)$ is obtained from F_4 with the tautology (See the dotted surrounding parts) using the subsumption and weakening operators in Fig. 3.

Secondly, we recall Example 12. Let F_5 be the same bridge theory in Example 12 and H_{g_5} be the ground hypothesis as follows:

$$H_{g_5} = \{arc(b,c),\ arc(b,c) \supset path(b,c)\}.$$

Note that H_{g_5} consists of ground instances from the target hypothesis H_5 in Example 12 and $F_5 \models M(H_{g_5})$. $M(H_{g_5})$ is as follows:

$$M(H_{g_5}) = \{\neg arc(b,c) \vee \neg path(b,c),\ \neg arc(b,c) \vee arc(b,c)\}.$$

$M(H_{g_5})$ contains the tautology $\neg arc(b,c) \vee arc(b,c)$, though F_5 does not. Then, $M(H_{g_5})$ is obtained from F_5 with the tautology using the three: resolution, subsumption and weakening operators in Fig. 4.

Based on Lemma 8, we prove Theorem 4 by showing that $\tau(M(T)) \succeq \tau(M(S))$ when $S \vdash_X T$ holds for each symbol $X \in \{r,\ s,\ w\}$.

Lemma 9 *Let S and T be two ground clausal theories such that $S \vdash_r T$. Then,*

$$\tau(M(T)) \succeq \tau(M(S)).$$

Proof of Lemma 9 Since $S \vdash_r T$, T is written as $S \cup \{C\}$ where C is a resolvent of two clauses C_1 and C_2 in S. Since C_1 and C_2 are ground, the resolvent C is written as $(C_1 -$

F_5 with the tautology		U		V
$arc(a,b)$ $arc(a,b) \wedge path(b,c)$ $\supset path(a,c)$ $\neg path(a,c)$ $\neg arc(b,c) \vee arc(b,c)$	$\vdash_r^2$	$arc(a,b)$ $arc(a,b) \wedge path(b,c)$ $\supset path(a,c)$ $\neg path(a,c)$ $arc(b,c) \supset path(a,c)$ $\neg arc(b,c)$ $\neg arc(b,c) \vee arc(b,c)$	$\vdash_s^1$	$arc(a,b)$ $arc(a,b) \wedge path(b,c)$ $\supset path(a,c)$ $\neg path(a,c)$ $arc(b,c) \supset path(a,c)$ $\neg arc(b,c)$ $\neg arc(b,c) \vee \neg path(b,c)$ $\neg arc(b,c) \vee arc(b,c)$

V		$M(H_{g_5})$
$arc(a,b)$ $arc(a,b) \wedge arc(b,c)$ $\supset path(a,c)$ $\neg path(a,c)$ $arc(b,c) \supset path(a,c)$ $\neg arc(b,c)$ $\neg arc(b,c) \vee \neg path(b,c)$ $\neg arc(b,c) \vee arc(b,c)$	$\vdash_w^5$	$\neg arc(b,c) \vee \neg path(b,c)$ $\neg arc(b,c) \vee arc(b,c)$

Fig. 4 A derivation from F_5 to $M(H_{g_5})$ in Example 12

$\{l\}) \cup (C_2 - \{\neg l\})$ for some literal l in C_1. Let C', C_1' and C_2' be three sets such that C', C_1' and C_2' consist of the negations of literals in C, C_1 and C_2. Since $C_1, C_2 \in S$, C_1' and C_2' are included in $\mathcal{F}(S)$, and C' is in $\mathcal{F}(S \cup \{C\})$. Let D be a clause in $\tau(MHS(S))$. Since D is a minimal hitting set of $\mathcal{F}(S)$, $D \cap C_1' \neq \emptyset$ and $D \cap C_2' \neq \emptyset$ hold. Suppose that

$$(*) \quad D \text{ is not a hitting set of } \mathcal{F}(S \cup \{C\}).$$

Since $C' \in \mathcal{F}(S \cup \{C\})$, $D \cap C' = \emptyset$ should hold. Since $C' = (C_1' - \{\neg l\}) \cup (C_2' - \{l\})$, $D \cap C' = ((C_1' - \{\neg l\}) \cap D) \cup ((C_2' - \{l\}) \cap D)$ holds. Since $D \cap C' = \emptyset$, it holds that $(C_1' - \{\neg l\}) \cap D = \emptyset$ and $(C_2' - \{l\}) \cap D = \emptyset$. Since $D \cap C_1' \neq \emptyset$ and $D \cap C_2' \neq \emptyset$, we obtain that $D \cap \{\neg l\} \neq \emptyset$ and $D \cap \{l\} \neq \emptyset$. Then, D has complementary literals $\neg l$ and l. It contradicts that D is not a tautology, since $D \in \tau(MHS(S))$. Thus, the assumption (*) is false. Hence, D is a hitting set of $\mathcal{F}(S \cup \{C\})$. Accordingly, there is a clause $E \in MHS(S \cup \{C\})$ such that $E \succeq D$. Since D is not a tautology, E is also not. Then, $E \in \tau(MHS(S \cup \{C\}))$ holds. By Lemma 5, we have $\tau(MHS(S)) = \tau(M(S))$ and $\tau(MHS(S \cup \{C\})) = \tau(M(S \cup \{C\}))$. Accordingly, we have $D \in \tau(M(S))$ and $E \in \tau(M(S \cup \{C\}))$. Therefore, for each clause $D \in \tau(M(S))$, there is a clause $E \in \tau(M(S \cup \{C\}))$ such that $E \succeq D$. □

Lemma 10 *Let S and T be two ground clausal theories such that $S \vdash_s T$. Then,*

$$\tau(M(T)) \succeq \tau(M(S)).$$

Proof of Lemma 10 Since $S \vdash_s T$, T is written as $S \cup \{C\}$ where C is a clause such that $D \succeq C$ for some clause $D \in S$. Since D and C are ground, $D \subseteq C$ holds. Let C' and D' be two sets such that C' and D' consist of the negations of literals in C and D, respectively. Since $D \in S$, D' is included in $\mathcal{F}(S)$, and C' is in $\mathcal{F}(S \cup \{C\})$. Let E be a clause in $\tau(MHS(S))$. Since E is a minimal hitting set of $\mathcal{F}(S)$, $E \cap D' \neq \emptyset$ holds. Since $D \subseteq C$, $D' \subseteq C'$ holds.

Accordingly, $E \cap C' \neq \emptyset$ holds. Since $C' \in \mathcal{F}(S \cup \{C\})$, E is a hitting set of $\mathcal{F}(S \cup \{C\})$. Then, there is a clause $E' \in MHS(S \cup \{C\})$ such that $E' \succeq E$. Since $E \in \tau(MHS(S))$, E is not a tautology, and E' is also not. Hence, $E' \in \tau(MHS(S \cup \{C\}))$ holds. By Lemma 5, we have $\tau(MHS(S)) = \tau(M(S))$ and $\tau(MHS(S \cup \{C\})) = \tau(M(S \cup \{C\}))$. Therefore, for each clause $E \in \tau(M(S))$, there is a clause $E' \in \tau(M(S \cup \{C\}))$ such that $E' \succeq E$. □

Lemma 11 *Let S and T be two ground clausal theories such that $S \vdash_w T$. Then,*

$$\tau(M(T)) \succeq \tau(M(S)).$$

Proof or Lemma 11 Since $S \vdash_w T$, T is written as $S - \{C\}$ where C is a clause in S. Let E be a clause in $\tau(MHS(S))$. E is a minimal hitting set of $\mathcal{F}(S)$. Since $T \subset S$, $\mathcal{F}(T) \subset \mathcal{F}(S)$ holds. Then, E is a hitting set of $\mathcal{F}(T)$. Hence, there is a clause $E' \in MHS(T)$ such that $E' \succeq E$. Since E is not a tautology, E' is also not a tautology, that is, $E' \in \tau(MHS(T))$ holds. By Lemma 5, we have $\tau(MHS(S)) = \tau(M(S))$ and $\tau(MHS(T)) = \tau(M(S))$. Therefore, for each clause $E \in \tau(M(S))$, there is a clause $E' \in \tau(M(T))$ such that $E' \succeq E$. □

Using Lemmas 8, 9, 10 and 11, Theorem 4 is proved as follows:

Proof of Theorem 4 By Lemma 8, there are two ground clausal theories U and V such that

$$S \vdash_r^* U \vdash_s^* V \vdash_w^* T.$$

By Lemma 9, $\tau(M(U)) \succeq \tau(M(S))$ holds. By Lemma 10, $\tau(M(V)) \succeq \tau(M(U))$ holds. By Lemma 11, $\tau(M(T)) \succeq \tau(M(V))$ holds. Hence, the following formula holds:

$$\tau(M(T)) \succeq \tau(M(V)) \succeq \tau(M(U)) \succeq \tau(M(S)).$$ □

Appendix C: Proofs of Theorem 1 and Lemma 1

Using theoretical results in the paper, we prove Theorem 1 and Lemma 1.

Proof of Theorem 1 By Herbrand's Theorem, there is a finite subset S' of ground instances from S such that $S' \models T$. By Theorem 4, $\tau(M(T)) \succeq \tau(M(S'))$ holds. By Lemma 3, it holds that $\tau(M(T)) = \mu(R(T))$ and $\tau(M(S')) = \mu(R(S'))$. Then, $\mu(R(T)) \succeq \mu(R(S'))$ holds. Since $R(T) \succeq \mu(R(T))$ and $\mu(R(S')) \succeq R(S')$, $R(T) \succeq R(S')$ holds. □

Proof of Lemma 1 Since $T \subseteq S$, $S \vdash_w^* T$ holds. By Lemma 11, $\tau(M(T)) \succeq \tau(M(S))$ holds. By Lemma 3, $\mu(R(T)) \succeq \mu(R(S))$ holds. Hence, we have $R(T) \succeq R(S)$. □

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