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A Semantic Framework for the Impure Logic of Ground

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Abstract

There is a curious bifurcation in the literature on ground and its logic. On the one hand, there has been a great deal of work that presumes that logical complexity invariably yields grounding. So, for instance, it is widely presumed that any fact stated by a true conjunction is grounded in those stated by its conjuncts, that any fact stated by a true disjunction is grounded in that stated by any of its true disjuncts, and that any fact stated by a true double negation is grounded in that stated by the doubly-negated formula. This commitment is encapsulated in the system GG axiomatized and semantically characterized by [deRosset and Fine, 2023] (following [Fine, 2012]). On the other hand, there has been a great deal of important formal work on "flatter" theories of ground, yielding logics very different from GG [Correia, 2010] [Fine, 2016, 2017b]. For instance, these theories identify the fact stated by a self-conjunction $(\phi \wedge \phi)$ with that stated by its conjunct ϕ. Since, in these systems, no fact grounds itself, the "flatter" theories are inconsistent with the principles of GG. This bifurcation raises the question of whether there is a single notion of ground suited to fulfill the philosophical ambitions of grounding enthusiasts. There is, at present, no unified semantic framework employing a single conception of ground for simultaneously characterizing both GG and the "flatter" approaches. This paper fills this gap by specifying such a framework and demonstrating its adequacy.

Keywords Impure logic of ground · Truthmaker semantics · Logic of ground · Ground

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There is a curious bifurcation in the literature on ground and its logic. On the one hand, there has been a great deal of work that presumes that logical complexity invariably yields grounding. So, for instance, it is widely presumed that the fact stated by a true conjunction is grounded in those stated by its conjuncts, that the fact stated by a true disjunction is grounded in that stated by any of its true disjuncts, and that the fact stated by a true double negation is grounded in that stated by the doubly-negated formula.[1] These commitments are encapsulated in the system GG axiomatized and semantically characterized in [6] (following [8]).

On the other hand, there has been a great deal of important formal work on "flatter" theories of content, including concomitant explorations of corresponding logics of ground very different from GG. So, for instance, the semantic approaches of [3] and [9, 11] each yield a theory of content, AC, characterized axiomatically by [1]; see Section 3 below for a specification. AC requires, among other things, that we identify the fact stated by a self-conjunction $(\phi \wedge \phi)$ with that stated by its conjunct ϕ. Since, in these systems, no fact grounds itself, any "flatter" theory implying AC is inconsistent with the principles of GG. According to GG, conjoining ϕ with itself "raises" the content of ϕ, yielding something new; according to the "flatter" theories, by contrast, self-conjunction just gives us back the old fact.

So, we have two very different logics of ground, and also two very different views of the conditions under which sentences are *equivalent*, in the sense that they express the same fact.[2] As it turns out, the semantic approaches cited above and used to characterize these two different views are also very different, and neither seems readily adaptable to handle the logic of ground yielded by the other. So, we have no unified semantic framework suited to treat both the "raised" approach to content characteristic of GG (with one set of constraints on contents) and the various "flatter" approaches (with a different set of constraints on contents).

This bifurcation is regrettable. In particular, the "raised" approach differs from the extant "flatter" treatments in its interpretation of grounding claims. This motivates the idea that we do not have a single notion of ground treated by the disparate logics and challenges the claim that there is a single notion to treat.[3] It has even been suggested that the lack of a unified semantic framework for interpreting claims of ground provides a reason to be skeptical about the cogency or utility of the idea [12, p. 327].

[1] See [8], [14], and [15] for discussion and citations.

[2] This intuitive way of expressing the sort of equivalence at issue is potentially misleading, since the notion of *expressing a fact* may reasonably taken to be factive: a sentence expresses a fact only if the sentence is true. In the present context I intend the idea to be taken non-factively. So, for instance, the "flatter" theories imply that $(\phi \wedge \phi)$ and ϕ are equivalent in the relevant sense even when ϕ is false. So, in the idiom I indulge in the main text, this claim can be expressed by saying that $(\phi \wedge \phi)$ and ϕ express the same fact, even when ϕ is false. Correia [3, 4] calls this non-factive notion *factual equivalence*.

[3] In this connection, Correia [3, 4] explicitly distinguishes the notions of *representational grounding* and *worldly grounding*. In [3], he draws the distinction by appeal to the fact that worldly grounding has a more coarse-grained conception of the relata of the grounding relation. But, as indicated in the main text, the differences between extant "raised" and "flatter" treatments cannot be characterized merely by differences in how fine-grained the relata are supposed to be. The different treatments also involve different conceptions of the notion of ground itself.

To meet the challenges posed by these charges, we would like a single framework, with a single conception of ground, which, given different constraints on equivalence, yield the different logics of ground. A model for a unified framework of this sort is the now-standard relational possible worlds semantics for propositional modal logic. There we have a single conception of necessity as truth at all accessible worlds. Different constraints on accessibility then yield different modal logics. We are aiming for something similar for the logic of ground.

This paper offers reason for thinking that such a framework is ready to hand. For reasons that will become clear, there is no prospect of adapting the "raised" approach of [6] by differently constraining contents so that we get a "flatter" theory instead of GG. In particular, the conception of ground specified by deRosset and Fine [6, D2.1, p. 426] is unsuitable for a "flatter" treatment, because it encodes a characteristic commitment of the "raised" approach.[4] But a small variation on that semantics is more serviceable for the purpose. As we will see, the variant semantics, featuring a tweaked conception of ground, yields the logic GG on one set of constraints on contents, and AC and its associated logic of ground on another. The variant semantics thus provides a framework that unifies the hitherto bifurcated treatments of ground in the literature.

We will start (Section 1) by reviewing the semantics for GG specified by [6]. The variant semantics is then described, and GG's soundness and completeness are established (Section 2). This variant semantics appeals to a single constraint on contents, dubbed ($\leq$- MAXIMALITY). The soundness and completeness results show that, in the framework described, this constraint characterizes GG. Next, we state an alternative set of constraints, inconsistent with ($\leq$- MAXIMALITY) in the framework described, and establish the soundness (Section 3) and completeness (Section 4) of AC on the resulting semantics. Finally (Section 5), we show that the resulting interpretation of ground exactly corresponds to the definition of ground given by [3] and [8, 11]. Thus, we have a single semantic framework encapsulating a single conception of ground that is suitable for exactly characterizing the semantic assumptions of our disparate logics of ground.

1 Original Semantics for GG

Let's begin by describing the semantics for the impure logic of ground of [6, Section 2], which we will call the *selection space semantics*. We take as given the familiar idea of a space of *conditions*, which, intuitively, may either obtain or fail to obtain. We can pair these conditions into propositions, or candidate *contents* for sentences, which comprise both a truth-condition and a falsity-condition.

We are also given two ways in which conditions may be constructed out of contents. deRosset and Fine [6, pp. 421, 425-6] dub these two modes of construction *choice* and *combination*, respectively. They characterize them by appeal to what, following [10, p. 637ff.], they call *the theory of menus*. To appreciate the idea, consider a typical

[4] deRosset and Fine [6, p. 492] assert that their approach "can be modified and extended" to accommodate a "flatter" theory. But they do not say how, nor do they suggest a way of capturing both the "flatter" theory and the "raised" theory by employing a single conception of ground.

breakfast menu, offering a choice of either oatmeal with fruit or eggs with toast. Each of the two options is itself a combination of items, and the toast might itself comprise a choice between whole wheat and white toast. Thus, on typical menus, there is a hierarchical organization of choices and combinations, with the menu itself generally offering, at the highest level, a choice of options. Clearly, the character of choices on a menu is, intuitively, disjunctive, since any of the options on offer may be selected. Likewise, the character of combinations is, intuitively, conjunctive, since any selection includes all of the items together.

The choice and combination operations in selection space semantics are analogous operations on finite sequences of contents, with *choice* providing a semantic analogue of disjunction and *combination* a semantic analogue of conjunction. Thus, we may think of the choice of contents v and w (written $[v + w]$), as a condition comprising two ways in which it might obtain. So, if the condition in fact obtains, circumstances must somehow include a selection of one of those ways in which the condition obtains. Similarly, the combination of v and w (written $[v.w]$) may be thought of, intuitively, as a condition whose actual obtaining requires that circumstances include both v and w. Since there is, on this conception, no intuitive difference between a singleton combination and a singleton choice of a content v, deRosset and Fine [6] identify them, writing $[v]$ to denote such a choice/combination. Intuitively, one might think of $[v]$ as an “a la carte” item. Any non-empty set of conditions, together with choice and combination operations (denoted Σ and Π, respectively), is a member of the class of *selection spaces* that gives selection space semantics its name.

The selection space semantics interprets sentences of a propositional language with negation, conjunction, and disjunction compositionally, mapping each sentence to a content, *i.e.*, a pairing of truth- and falsity-conditions. We take as given an assignment of contents to atomic sentences. The truth-condition of $\neg\phi$ is the falsity condition of ϕ, and the falsity-condition of $\neg\phi$ is the “a la carte” choice/combination of ϕ’s content. The truth-condition of a disjunction is the choice of the contents of the disjuncts, and its falsity-condition is the combination of the contents of the disjuncts’ negations. Similarly, the truth-condition of a conjunction is the combination of the contents of the conjuncts, and its falsity-condition is the choice of the contents of the conjuncts’ negations. Formally, an interpretation is a function $\bar{\cdot}$ given by an assignment of contents to atomic sentences that is extended inductively to molecular sentences in the way just described:

1. $\overline{\neg\phi} = (\bar{\phi}_\ominus, [\bar{\phi}])$;
2. $\overline{(\phi \wedge \psi)} = ([\bar{\phi} \,.\, \bar{\psi}], [\overline{\neg\phi} + \overline{\neg\psi}])$; and
3. $\overline{(\phi \vee \psi)} = ([\bar{\phi} + \bar{\psi}], [\overline{\neg\phi} \,.\, \overline{\neg\psi}])$.

It remains to interpret claims of ground. Here deRosset and Fine [6] appeal to a distinction, standard in the literature on the logic of ground, between notions of *strict ground* ($<$) and *weak ground* ($\leq$). Strict ground is the more familiar idea, deployed by philosophers across a wide range of areas. Weak ground is indispensable for logical purposes, but is less familiar and less widely used. deRosset and Fine [6] note, however, that their target logic GG requires that weak ground be specifiable by appeal to strict ground: weak grounds for ϕ are exactly strict grounds for $\neg\neg\phi$.[5] But GG also requires

[5] This is their definition (W/S), [6, p. 423].

that strict ground be specifiable by appeal to weak ground: $\psi_1, \psi_2, \ldots$ strictly ground ϕ iff they *irreversibly* weakly ground ϕ, *i.e.*, they weakly ground ϕ, and there are no Γ such that ϕ, together with Γ, weakly grounds any of the ψ_i.[6]

Of course, at most one of these specifications can be designated as a formal *definition* of the relevant sort of grounding. deRosset and Fine [6] choose to elevate the specification of weak ground by appeal to strict ground to a definition. That is, they define weak grounds for ϕ, in effect, as strict grounds for $\neg\neg\phi$. They then define strict ground directly, by appeal to selection. The definition is inductive, starting with a notion of immediate selection. So, any content v is an immediate selection from a choice of contents that includes v, and the contents $v, w, \ldots$ are, collectively, an immediate selection from their combination. Immediate selections from the truth-condition of a content are strict grounds for that content: whenever G is an immediate selection from the truth condition of v (written $v_\oplus$), G is a strict ground of v. The definition of strict ground is then rounded out by closing under a series of natural chaining operations. It is worth stating the definition in full, since it will figure in what follows. In this definition, $\ll_{\mathfrak{F}}$ is used for immediate selection (relative to a given selection space $\mathfrak{F}$), $<_{\mathfrak{F}}$ for strict selection, and a *weak selection* claim of the form $G \leq_{\mathfrak{F}} v$ abbreviates $(\exists d)G <_{\mathfrak{F}} ([v], d)$:[7]

Definition 1.1

1. **Basis**: if $G \ll_{\mathfrak{F}} v_\oplus$, then $G <_{\mathfrak{F}} v$;
2. **Ascent**: if $G <_{\mathfrak{F}} w$ and $[w] = v_\oplus$, then $G <_{\mathfrak{F}} v$;
3. **Lower Cut**: if $(G^i \leq_{\mathfrak{F}} v^i)$, and $(v^i) <_{\mathfrak{F}} v$, then $(G^i) <_{\mathfrak{F}} v$; and
4. **Upper Cut**: if $(G^i <_{\mathfrak{F}} v^i)$, and $(v^i) \leq_{\mathfrak{F}} v$, then $(G^i) <_{\mathfrak{F}} v$. [6, D2.1, p. 426]

With this definition in hand, we can consider the resulting logic of grounding claims. deRosset and Fine show that two constraints on selection spaces yield a class of models for which GG is sound and complete. Thus, those constraints characterize the semantic presuppositions of GG, given the conception of ground captured by their definition Definition 1.1. The first constraint we have already encountered: it says, in effect, that strict ground is irreversible weak ground. Let's indicate a connection of *partial weak selection* in a selection space $\mathfrak{F}$ using $\preceq_{\mathfrak{F}}$: $v \preceq_{\mathfrak{F}} w$ iff there is a G such that $v, G \leq_{\mathfrak{F}} w$. This gives us a concise way to express irreversibility:

Irreversibility $G <_{\mathfrak{F}} v$ iff $G \leq_{\mathfrak{F}} v$ and $(\forall w \in G) v \not\preceq_{\mathfrak{F}} w$. [6, pp. 423-4,426]

The specification of strict ground as irreversible weak ground is a shared commitment of both GG and the "flatter" theories of ground mentioned above [3, 9, 11]. (IRREVERSIBILITY) is needed because it is not guaranteed by the definitions in [6] of selection spaces and ground on their own. Nothing in those definitions, for instance, prohibits there being a selection space $\mathfrak{F}$ in which the singleton choice/combination $[v]$ of a content v is the truth-condition of v. Then the content v will be an immediate selection from $v_\oplus = [v]$, and hence, by the definitions of both strict and weak selection, v is both a strict and a weak selection from itself. Since the weak selection $v \leq_{\mathfrak{F}} v$ is obviously reversible, $\mathfrak{F}$ witnesses a failure of (IRREVERSIBILITY).

6 This is their definition (S/W), [6, p. 424].

7 In what follows, we will refer to indexed sets using standard notation, writing $(x_i)_{i<n}$ for $\{x_i | i < n\}$. We will almost always omit the subscripted restriction '$i < n$'.

As already noted, the specification of strict ground as irreversible weak ground is shared with the "flatter" treatments. So (IRREVERSIBILITY) does not capture a *distinctive* commitment of GG. Thus, deRosset and Fine [6] must impose a further constraint to give a semantics for GG. One distinctive commitment of GG is already captured by the definition of ground offered above. Given that definition together with the assignment of contents to sentences, logical complexity of the sort treated invariably yields grounding connections. For instance, since the content of ϕ is always an immediate selection from its singleton choice/combination, and that singleton choice/combination is always the truth-condition of $\neg\neg\phi$, the claim that ϕ (strictly) grounds $\neg\neg\phi$ is valid. But GG is also committed to the claim that grounds for a logically complex sentence must "go through" the contents of its immediate constituents. Every (strict) ground for $\neg\neg\phi$, for instance, must somehow "go through" ϕ, in the sense of being a weak ground for ϕ. Thus, ϕ is a kind of maximal strict ground of $\neg\neg\phi$. Similar commitments govern grounds for conjunctions, disjunctions, and their DeMorgan equivalents. Say that $G_1, G_2, \ldots$ are a *covering of* G iff $G = G_1 \cup G_2 \cup \ldots$. Then this constraint can be encapsulated thus:

Maximality:

1. $G <_{\mathfrak{F}} ([v^0.v^1.\ldots], d)$ only if there is a covering $G_0, G_1, \ldots$ of G such that $G_i \leq_{\mathfrak{F}} v^i$, for each i; and
2. $G <_{\mathfrak{F}} ([v^0 + v^1 + \ldots], d)$ only if there is a non-empty subset $w^0, w^1, \ldots$ of $v^0, v^1, \ldots$ and a covering $G_0, G_1, \ldots$ of G such that $G_i \leq_{\mathfrak{F}} w^i$ for each i.[6, D2.2.2, p. 427]

deRosset and Fine show that GG is sound and complete for the class of models whose selection spaces satisfy (IRREVERSIBILITY) and (MAXIMALITY) [6, T3.1, T8.6, pp. 429, 489]. Please see [6, Section 2, pp. 425-7] for a formal specification of the semantics and [6, Section 3, pp. 427-9] for an explicit specification of the corresponding system of derivation GG.

2 The Variant Semantics

GG's characteristic commitments concerning the fineness of grain of contents are highly controversial. For instance, GG requires that we distinguish the content of ϕ from each of $\neg\neg\phi$, $\phi \wedge \phi$, and $\phi \vee \phi$, since $\phi \leq \phi$ is a theorem of GG, but each of $\neg\neg\phi \leq \phi$, $(\phi \wedge \phi) \leq \phi$, and $(\phi \vee \phi) \leq \phi$ is inconsistent in GG. Use $\approx$ to express *ground-theoretic equivalence* between formulae, so that $\phi \approx \psi$ iff $\phi \leq \psi$ and $\psi \leq \phi$. GG also requires counter-examples to the general ground-theoretic equivalence of $(\phi \vee (\psi \vee \chi))$ with $((\phi \vee \psi) \vee \chi)$ [6, Section 9.3]. For, as one might expect from inspection of (MAXIMALITY), GG requires that a strict ground for $(\phi \vee \psi)$ be either a weak ground for ϕ, a weak ground for ψ, or split (perhaps non-exclusively) into a weak ground for ϕ and a weak ground for ψ. Also, ϕ and ψ are each required in GG to be strict grounds for their disjunction. So, if $((\phi \vee \psi) \vee \psi) \approx (\phi \vee (\psi \vee \psi))$, then, according to GG, we have:

$$((\phi \vee \psi) \vee \psi) \leq (\phi \vee (\psi \vee \psi))$$
$$\vdash (\phi \vee \psi) < (\phi \vee (\psi \vee \psi))$$

$\vdash (\phi \vee \psi) \leq (\psi \vee \psi)$
$\vdash \phi < (\psi \vee \psi)$
$\vdash \phi \leq \psi$.
Since ϕ is arbitrary, we may substitute $\neg\neg\psi$ for ϕ in this derivation, yielding

$$((\neg\neg\psi \vee \psi) \vee \psi) \leq (\neg\neg\psi \vee (\psi \vee \psi)) \quad \vdash \quad \neg\neg\psi \leq \psi \quad \vdash \quad \psi < \psi.$$

In GG, however, nothing grounds itself, so $\psi < \psi$ is inconsistent. Thus, GG requires that disjunction not be associative. GG also turns out to require similar counter-examples to both the associativity of conjunction and the boolean distribution equivalences $(\phi \vee (\psi \wedge \chi)) \approx ((\phi \vee \psi) \wedge (\phi \vee \chi))$ and $(\phi \wedge (\psi \vee \chi)) \approx ((\phi \wedge \psi) \vee (\phi \wedge \chi))$. Moreover, GG allows us to distinguish all instances of $\phi \vee \psi$ from $\psi \vee \phi$, though it also allows their general ground-theoretic equivalence. Similar remarks apply to DeMorgan equivalences. These facts demonstrate the way in which GG imposes interesting constraints on the individuation of content. They thereby differentiate the theory of content required by GG from the "flatter" treatments we have already mentioned, which presuppose these ground-theoretic equivalences [3, 8, 11].

Our aim is to recover the impure logic of ground given by those "flatter" treatments by revising the constraint on selection spaces imposed by (IRREVERSIBILITY) and (MAXIMALITY). Since (IRREVERSIBILITY) is a commitment shared between the "raised" conception articulated by GG and the "flatter" treatments, one might hope that simply revising (MAXIMALITY) would do the trick. The hope is forlorn. The conception of ground encapsulated in deRosset and Fine's [6] definition of strict selection is itself unsuitable for the "flatter" treatments. As we saw in Section 1, on that definition, ϕ invariably strictly (and so irreversibly) grounds $\neg\neg\phi$, and weakly grounds itself. These two commitments are inconsistent with any of the "flatter" views, which identify the content of any sentence and its double negation, and maintain that strict ground is irreversible weak ground. So, as we saw above, the very conception of ground at issue incorporates a characteristic commitment of GG. Selection systems cannot provide the unified semantic framework we are seeking, on the conception of ground characterized by deRosset and Fine's Definition 1.1.

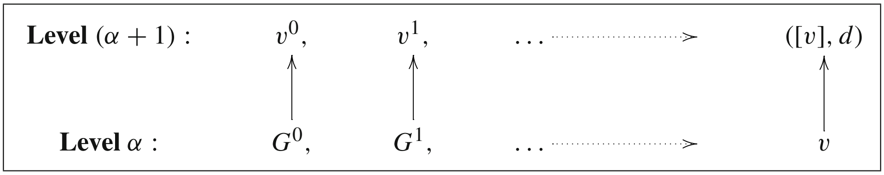
This problem can only be solved by starting with a different definition of selection. Fortunately, there is such a definition in the offing. Recall that weak and strict ground, in GG, can each be specified in terms of the other. As we saw, deRosset and Fine elect to take the specification of weak ground in terms of strict ground as a *definition* of weak ground: weak grounds for ϕ are defined, in effect, as strict grounds for $\neg\neg\phi$. Strict ground is then defined directly, as in Definition 1.1. We can do better by following the opposite procedure, defining weak ground directly, and then defining strict ground as irreversible weak ground.

Suppose we are given a selection space $\mathfrak{F}$.

Definition 2.1 The relation $\leq_1$ for $\mathfrak{F}$ is defined inductively:

1. BASIS: $G \ll v_\oplus$ or $G \ll [v] \Rightarrow G \leq_1 v$;
2. CUT: $(G_i \leq_1 v^i)$ and $(v^i) \leq_1 v \Rightarrow (G_i) \leq_1 v$; and
3. LEVEL: $(G_i \ll v^i_\oplus)$ and $(v^i) \leq_1 ([v], d) \Rightarrow (G_i) \leq_1 v$.

(BASIS) and (CUT) are familiar and straightforward. (LEVEL), by contrast, is more difficult. It is plausible to think that a weak grounding claim indicates that the weak grounds of some content v are *at or below* the explanatory level of v. What (LEVEL) says, on this way of thinking, is that if some contents $v^0, v^1, \ldots$ are at or below $[v]$'s level, we go down a level from $v^0, v^1, \ldots$ to get $G_0, G_1, \ldots$, and we go down a level from $[v]$ to get v, then the level of the G's will be at or below the level of v. A picture illustrates the idea:



Here the dotted arrow represents $\leq_1$ and the solid arrows represent relations of immediate selection connecting entities of a given level with entities one level up. In the context of GG's specification of weak selections from a content v as strict selections from the "a la carte" item $[v]$, the definition of $\leq_1$ can be seen as a way of adapting the original definition of strict selection in Definition 1.1 to yield a direct definition of weak selection. The immediate selection clause and ASCENT in Definition 1.1 get bundled into BASIS. The CUT clause in Definition 2.1 is a special case of LOWER CUT, where the major premise $(v^i) <_{\mathfrak{F}} v$ has the form $(v_i) <_{\mathfrak{F}} ([w], d)$, and so says, in the present context, that (v_i) are a weak selection from w. And LEVEL is a special case of the UPPER CUT clause of Definition 1.1 and the basis case. Whenever (G_i) are immediate, hence strict, selections, respectively, from (v^i) and (v^i) are, in turn, a weak selection from $([v], d)$, application of UPPER CUT implies that (G_i) are a strict selection from $([v], d)$, and hence a weak selection from v. In fact, we will show (Theorem 2.5) that deRosset and Fine's original definition of weak selection and the new definition Definition 1.1 are equivalent in any selection space. So, the new definition Definition 1.1 of weak selection simply presents the old relation in a somewhat unfamiliar guise.[8]

The framework of selection systems, together with the conception of ground corresponding to this definition of $\leq_1$, provides the unified approach we seek. What we will show is that the framework of selection spaces, together with this conception of ground, can be constrained one way to yield GG, and a different way to yield the logic of the "flatter" treatments.

We start by showing that there is a single, natural constraint on selection systems, obtained by strengthening deRosset and Fine's (MAXIMALITY) constraint, that yields the logic GG given the conception of ground corresponding to Definition 2.1. For this purpose, we will show that the class of selection systems meeting deRosset and Fine's two constraints is a subclass of the class meeting the strengthened (MAXIMALITY) constraint, and that, for each selection system in that class, the two pairs of definitions of strict and weak selection exactly coincide. This immediately implies, via deRosset and Fine's completeness theorem, that GG is complete for the class of

[8] The new definition of strict selection, however, is different from deRosset and Fine's notion. In particular, the new definition Definition 2.7 of strict selection, unlike Definition 1.1, does not imply without further constraints that ϕ strictly grounds each of $\neg\neg\phi$, $(\phi \vee \phi)$, and $(\phi \wedge \phi)$.

selection systems meeting the strengthened maximality constraint; the soundness of GG is straightforwardly established by a routine induction on derivations in GG.

The relation of strict selection $<$ is (directly) defined as before in Definition 1.1, and, to prevent confusion, we write $G \leq_2 v$ (instead of $G \leq v$) for $(\exists d)G < ([v], d)$. Lemma 2.2-Theorem 2.5 establish the somewhat surprising equivalence of $\leq_1$ and $\leq_2$ in any selection system. This vindicates the assertion above that the two definitions of weak selection present a single underlying phenomenon in two different ways.

The implication in one direction, from the new definition to deRosset and Fine's original definition, is straightforward.

Lemma 2.2 $G \leq_1 v \Rightarrow G \leq_2 v$.

Proof We prove the result by induction on Definition 2.1.

BASIS: Suppose $G \ll v_{\oplus}$. By Definition 1.1:

$$G \ll v_{\oplus} \stackrel{\text{BASIS}}{\Longrightarrow} G < v \stackrel{\text{ASCENT}}{\Longrightarrow} G < ([v], d) \Rightarrow G \leq_2 v.$$

Suppose instead that $G \ll [v]$. By Definition 1.1:

$$G \ll [v] \stackrel{\text{BASIS}}{\Longrightarrow} G < ([v], d) \Rightarrow G \leq_2 v.$$

CUT: Suppose $(G^i \leq_1 v^i)$ and $(v^i) \leq_1 v$. By IH, $(G^i \leq_2 v^i)$ and $(v^i) \leq_2 v$. By Definition 1.1, LOWER CUT, $(G^i) \leq_2 v$.

LEVEL: Suppose $(G^i \ll v^i_{\oplus})$ and $(v^i) \leq_1 ([v], d)$. By IH, $(v^i) \leq_2 ([v], d)$. By Definition 1.1,BASIS, $(G^i < v^i)$. So, by Definition 1.1,UPPER CUT, $(G^i) < ([v], d)$, i.e., $(G^i) \leq_2 v$.

To show the implication in the opposite direction, we prove a utility lemma that shows, in effect, that every strict selection in deRosset and Fine's sense can be represented in a convenient normal form. Write $G \leq_1 v^0, \ldots, v^i, \ldots$ when there is a covering $G^0, \ldots, G^i, \ldots$ of G such that $G^0 \leq_1 v^0, \ldots, G^i \leq_1 v^i, \ldots$. It is obvious by CUT that, if $G \leq_1 H$ and $H \leq_1 I$, then $G \leq_1 I$.

Write $(G^i) \ll (v^i)$ for $(G^i \ll v^i_{\oplus})$.

Lemma 2.3 *If $G < v$ then there are (u^i) and (H^i) such that*

$$G \leq_1 (H^i) \ll (u^i) \leq_1 v.$$

Proof We prove the result by induction on Definition 1.1.

BASIS: Suppose $G \ll v_{\oplus}$. Then $G \leq_1 G \ll v \leq_1 v$.

ASCENT: Suppose $G < w$ and $[w] = v_{\oplus}$. By IH, $G \leq_1 (H^i) \ll (u^i) \leq_1 w$ (for some $(H^i), (u^i)$). By Definition 2.1,BASIS and CUT, $G \leq_1 (H^i) \ll (u^i) \leq_1 w \leq_1 v$.

LOWER CUT: Suppose $(G^i \leq_2 v^i)$ and $(v^i) < v$. By IH, $(v^i) \leq_1 (H^j) \ll (u^j) \leq_1 v$ (for some $(H^j), (u^j)$). Also by IH, for each i, $G^i \leq_1 (I^k) \ll (x^k) \leq_1 ([v^i], d^i)$ (for some (I^k), (x^k)). By Definition 2.1,LEVEL, $(I^k) \leq_1 v^i$; and so, by Definition 2.1,CUT, $G^i \leq_1 v^i$. So,

$$(G^i) \leq_1 (v^i) \leq_1 (H^j) \ll (u^j) \leq_1 v.$$

By Definition 2.1,CUT, $(G^i) \leq_1 (H^j) \ll (u^j) \leq_1 v$.

UPPER CUT: Suppose $(G^i < v^i)$ and $(v^i) < ([v], d)$. By IH applied to $(G^i < v^i)$, for each i, $G_i \leq_1 (H^{ij})_j \ll (u^{ij})_j \leq_1 v^i$ (for some $(H^{ij})_j, (u^{ij})_j$). By IH applied to $(v^i) < ([v], d)$, $(v^i) \leq_1 (I^k) \ll (x^k) \leq_1 ([v], d)$ (for some (I^k), (x^k)). So, by Definition 2.1,LEVEL $(v^i) \leq_1 (I^k) \leq_1 v$. Putting all of this together:

$$(G_i) \leq_1 (H^{ij}) \ll (u^{ij}) \leq_1 (v^i) \leq_1 (I^k) \leq_1 v$$

So, by Definition 2.1,CUT, $(G^i) \leq_1 (H^{ij}) \ll (u^{ij}) \leq_1 v$.

Now we can establish a convenient CUT principle.

Lemma 2.4 *If* $(G^i < v^i)$ *and* $(v^i) \leq_1 ([v], d)$*, then* $(G^i) \leq_1 v$.

Proof By Lemma 2.3, for each i, $G_i \leq_1 (H^{ij})_j \ll (u^{ij})_j \leq_1 v^i$ (for some $(H^{ij})_j, (u^{ij})_j$). So, $(G^i) \leq_1 (H^{ij}) \ll (u^{ij}) \leq_1 (v^i) \leq_1 ([v], d)$. By Definition 2.1, CUT and LEVEL, $(H^{ij}) \leq_1 v$. So, $(G^i) \leq_1 (H^{ij}) \leq_1 v$.

This permits us to prove that weak selection in deRosset and Fine's original sense ($\leq_2$) implies weak selection in the new sense corresponding to Definition 2.1 ($\leq_1$), thereby establishing the equivalence of $\leq_1$ and $\leq_2$ in any selection system.

Theorem 2.5 $G \leq_1 v \Leftrightarrow G \leq_2 v$.

Proof $\Rightarrow$: Lemma 2.2

$\Leftarrow$: We prove the result by induction on Definition 1.1:

BASIS: Suppose $G \ll [v]$. By Definition 2.1,BASIS, $G \leq_1 v$.

ASCENT: Suppose $G < w$ and $[w] = [v]$. Then $w \ll [v]$. So, by Definition 2.1, BASIS, $w \leq_1 ([v], d)$. So, by Lemma 2.4, $G \leq_1 v$.

LOWER CUT: Suppose $(G^i \leq_2 v^i)$ and $(v^i) < ([v], d)$. By IH, $(G^i) \leq_1 (v^i) \leq_1 v$.

UPPER CUT: Suppose $(G^i < v^i)$ and $(v^i) \leq_2 ([v], d)$. By IH, $(v^i) \leq_1 ([v], d)$. By Lemma 2.4, $G \leq_1 v$.

Following [6, D2.2, p. 426], call a selection system a *<-frame* iff it meets both (MAXIMALITY) and (IRREVERSIBILITY). The next two lemmas use Lemma 2.5 to straightforwardly establish similar equivalences for the other grounding operators treated by [6] in every <-frame (not: in every selection system). Write $v \preceq_1 w$ for $(\exists H)v, H \leq_1 w$; $v \preceq_1 w$ says that v is a partial, weak selection from w, in the newly defined sense.

Lemma 2.6 *Suppose* $\mathfrak{F} = \langle F, \Sigma, \Pi \rangle$ *is a <-frame. Then* $G <_{\mathfrak{F}} v$ *iff* $G \leq_1 v$ *and* $(\forall w \in G) v \not\preceq_1 w$.

Proof $\mathfrak{F}$ is a <-frame and thus satisfies (IRREVERSIBILITY):

$$G <_{\mathfrak{F}} v \text{ iff } G \leq_2 v \text{ and } (\forall w \in G)(\forall H)v, H \not\leq_2 v.$$

The result is therefore immediate by Theorem 2.5.

Suppose $\mathfrak{F} = \langle F, \Sigma, \Pi \rangle$ is a <- frame. To prevent confusion as we establish a correspondence between the two definitions of strict selection on hand, we will now write $<_2$ for $<_{\mathfrak{F}}$, the original variety of strict selection defined by Definition 1.1. We offer a new definition of a variety of strict selection relation $<_1$, on which it is defined as irreversible weak selection:

Definition 2.7 $G <_1 v \Leftrightarrow G \leq_1 v$ and $v \not\leq_1 w$.

The following is then an immediate consequence of Lemma 2.6:

Lemma 2.8 *Suppose $\mathfrak{F} = \langle F, \Sigma, \Pi \rangle$ is a <-frame, and let $<_2$ and $<_1$ be defined as specified above. Then $G <_2 v$ iff $G <_1 v$.*

We can now specify our new semantics for GG, appealing to a strengthened maximality principle, defined by appeal to our new strict selection relation $<_1$.

Definition 2.9 A selection space $\mathfrak{F} = \langle F, \Sigma, \Pi \rangle$ is a $\leq$-*frame* iff it satisfies

$\leq$-maximality

1. $G <_1 ([v^0.v^1.\dots], d)$ iff $G \leq_1 (v^i)$; and
2. $G <_1 ([v^0 + v^1 + \dots], d)$ iff there is a subset (w^j) of (v^i) such that $G \leq_1 (w^j)$.

It is now straightforward to show that GG is sound and complete for the new semantics. We first prove a utility lemma, establishing that any <-frame satisfies a strengthened version of deRosset and Fine's [6] (MAXIMALITY) constraint.

Lemma 2.10 *If $\mathfrak{F}$ is a <-frame, then $\mathfrak{F}$ satisfies*

Strengthened Maximality

1. $G <_2 ([v^0.v^1.\dots], d)$ iff $G \leq_2 (v^i)$; and
2. $G <_2 ([v^0 + v^1 + \dots], d)$ iff there is a subset (w^j) of (v^i) such that $G \leq_2 (w^j)$.

Proof Suppose $\mathfrak{F}$ is a <-frame. It already satisfies (MAXIMALITY). So, we need only show (1.) and (2.) in the right-to-left direction.

(1.) Suppose $G \leq_2 (v^i)$. By Definition 1.1, BASIS, $(v^i) < ([v^0.v^1.\dots], d)$. So, by Definition 1.1, LOWER CUT, $(G^i) <_2 ([v^0.v^1.\dots], d)$.
(2.) Suppose there is a subset (u^j) of (v^i) such that $G \leq_2 (u^j)$. Then G has a covering (G^j) such that $(G^j \leq_2 u^j)$. Let $v = ([v^0 + v^1 + \dots])$. By Definition 1.1, BASIS, $(u^j <_2 v)$. So, by applications of Definition 1.1, LOWER CUT, $(G^j \leq_2 v)$. Also, by Definition 1.1, BASIS, $v \leq_2 v$. So, we have $(G^j \leq_2 v)$ and $v, v, \dots \leq_2 v$. By Definition 1.1, LOWER CUT, $(G^j) \leq_2 v$.

This lemma makes it easy to show that satisfaction of deRosset and Fine's constraints on selection spaces implies satisfaction of our new constraint $\leq$- MAXIMALITY.

Lemma 2.11 *If $\mathfrak{F}$ is a $<$-frame, then it satisfies* ($\leq$- MAXIMALITY).

Proof Lemma 2.10, Lemma 2.8, and Theorem 2.5.

Now we can define a notion of a model appropriate to our new definition of ground.

Definition 2.12 A $\leq$-frame is a selection space that satisfies ($\leq$- MAXIMALITY), and a $\leq$-*model* is a quadruple $\langle F, \Sigma, \Pi, \bar{\cdot} \rangle$, where $\bar{\cdot}$ is an interpretation, and $\mathfrak{F} = \langle F, \Sigma, \Pi \rangle$ is a $\leq$-frame.

Define truth in a $\leq$-model ($\vDash_1$) for grounding claims $\Delta < \phi$, $\Delta \leq \phi$, etc., in the obvious way, analogously to [6, D2.4, p. 427]. Write $\vDash_2$ for the notion of truth in a model in deRosset and Fine's semantics. The key relation between the two semantics is now easy to show:

Lemma 2.13

1. *Every model $\mathfrak{M} = \langle F, \Sigma, \Phi, \bar{\cdot} \rangle$ for a language $\mathscr{L}$ is also a $\leq$ $-$model for $\mathscr{L}$; and*
2. *for all models $\mathfrak{M}$ and grounding claims σ, $\mathfrak{M} \vDash_1 \sigma$ iff $\mathfrak{M} \vDash_2 \sigma$.*

Proof By Lemma 2.11, we need only check that $\mathfrak{M} \vDash_1 \sigma$ iff $\mathfrak{M} \vDash_2 \sigma$. We do this separately for the four kinds of grounding claims:[9]

$<$: Suppose $\sigma = \Delta < \phi$. Then

$$\mathfrak{M} \vDash_1 \Delta < \phi \Leftrightarrow \overline{\Delta} <_1 \overline{\phi} \overset{\text{Lemma 2.8}}{\Longleftrightarrow} \overline{\Delta} <_2 \overline{\phi} \Leftrightarrow \mathfrak{M} \vDash_2 \Delta < \phi$$

$\dot{\leq}$: Suppose $\sigma = \Delta \leq \phi$. Then

$$\mathfrak{M} \vDash_1 \Delta \leq \phi \Leftrightarrow \overline{\Delta} \leq_1 \overline{\phi} \overset{\text{Theorem 2.5}}{\Longleftrightarrow} \overline{\Delta} \leq_2 \overline{\phi} \Leftrightarrow \mathfrak{M} \vDash_2 \Delta \leq \phi$$

$\dot{\preceq}$: Suppose $\sigma = \delta \preceq \phi$. Then

$$\mathfrak{M} \vDash_1 \delta \preceq \phi \Leftrightarrow \overline{\delta} \preceq_1 \overline{\phi} \Leftrightarrow (\exists H)\overline{\delta}, H \leq_1 \overline{\phi} \overset{\text{Theorem 2.5}}{\Longleftrightarrow} (\exists H)\overline{\delta}, H \leq_2 \overline{\phi} \Leftrightarrow \mathfrak{M} \vDash_2 \delta \preceq \phi$$

$\dot{\prec}$: By Theorem 2.5,

($\bigstar$) $(\exists H)w, H \leq_1 v$ iff $(\exists H)w, H \leq_2 v$.

Suppose $\sigma = \delta \prec \phi$. Then

$$\begin{aligned} M \vDash_1 \sigma \; &\Leftrightarrow \; (\exists H)\overline{\delta}, H \leq_1 \overline{\phi} \text{ and } \neg(\exists I)\overline{\phi}, I \leq_1 \overline{\delta} \\ &\overset{(\bigstar)}{\Longleftrightarrow} (\exists H)\overline{\delta}, H \leq_2 \overline{\phi} \text{ and } \neg(\exists I)\overline{\phi}, I \leq_2 \overline{\delta} \Leftrightarrow \mathfrak{M} \vDash_2 \sigma. \end{aligned}$$

[9] *Strict partial ground* is indicated by $\prec$. ϕ is a strict partial ground of ψ iff $\phi \preceq \psi$, but $\psi \npreceq \phi$.

Recall that deRosset and Fine prove that GG is complete for their original semantics [6, T8.6, p. 489]. We can use that result, together with Lemma 2.13 to prove the completeness of GG for our variant semantics. Write $S \vDash_1 T$ to indicate that, for every $\leq$-model $\mathfrak{M}$, if $(\forall \sigma \in S)\mathfrak{M} \vDash_1 \sigma$, then $(\exists \tau \in T)\mathfrak{M} \vDash_1 \tau$.

Theorem 2.14 ***(Completeness)*** *If $S \vDash_1 T$, then $S \vdash T$.*

Proof Suppose $S \nvdash T$. By deRosset and Fine's completeness theorem [6, T8.6, p. 489], there is a model $\mathfrak{M}$ such that $\mathfrak{M} \vDash_2 S$, but, for each $\tau \in T$ $\mathfrak{M} \nvDash \tau$. By Lemma 2.13, $\mathfrak{M}$ witnesses that $S \nvDash_1 T$.

Soundness is easily proved by a routine induction on the length of derivations, omitted here. So, GG is sound and complete for our variant semantics.

One feature of the variant semantics bears mention. As we have seen, deRosset and Fine's [6] treatment distributes the characteristic commitments of GG between one of the two constraints on selection systems and the definition of ground. By contrast, the variant semantics encapsulates the characteristic commitments of GG into a single constraint, $\leq$- MAXIMALITY. It thus brings the characteristic commitments of GG into clearer view.

3 The System AC, Semantics for AC, and Soundness

The previous section showed that GG is sound and complete for the class of $\leq$-models. Semantically, the fineness of grain for contents required by $\leq$-models is enforced by the ($\leq$-MAXIMALITY) constraint. We now turn to the question of whether the "flatter" logics of ground that presuppose AC can be captured, semantically, by replacing that constraint with some alternative. If they can, then we have the semantic framework we seek.

We start by characterizing AC and a corresponding semantics. Previously, we used $\approx$ to express ground-theoretic equivalence. Abusing notation, let us now use $\approx$ to express the claim that formulae are equivalent, so that $\phi \approx \psi$ says, intuitively, that ϕ and ψ express the same fact, or, alternatively, that for it to be the case that ϕ is for it to be the case that ψ and vice versa [7].[10] Angell's theory AC can be characterized by the following axioms and rules:[11]

[10] Given that full, weak ground is reflexive, if ϕ and ψ are equivalent, then it will follow that they are mutual full, weak grounds of one another. A failure of ground-theoretic equivalence would show that the two sentences are not equivalent. So, the intuitive interpretation of $\approx$ as expressing equivalence entails its prior interpretation as expressing ground-theoretic equivalence. It turns out that the converse entailment also holds, on the interpretation of ground that accompanies Angell's theory of equivalence; see below.

[11] This axiomatization is stated in [9], following [1].

The System AC:

INVOL	$\vdash \phi \approx \neg\neg\phi$
IDEM($\wedge$)	$\vdash \phi \approx (\phi \wedge \phi)$
COMMUT($\wedge$)	$\vdash (\phi \wedge \psi) \approx (\psi \wedge \phi)$
ASSOC($\wedge$)	$\vdash (\phi \wedge (\psi \wedge \chi) \approx ((\phi \wedge \psi) \wedge \chi)$
IDEM($\vee$)	$\vdash \phi \approx (\phi \vee \phi)$
COMMUT($\vee$)	$\vdash (\phi \vee \psi) \approx (\psi \vee \phi)$
ASSOC($\vee$)	$\vdash (\phi \vee (\psi \vee \chi) \approx ((\phi \vee \psi) \vee \chi)$
DM($\neg\wedge$)	$\vdash \neg(\phi \wedge \psi) \approx (\neg\phi \vee \neg\psi)$
DM($\neg\vee$)	$\vdash \neg(\phi \vee \psi) \approx (\neg\phi \wedge \neg\psi)$
DISTRIB($\wedge/\vee$)	$\vdash (\phi \wedge (\psi \vee \chi)) \approx ((\phi \wedge \psi) \vee (\phi \wedge \chi))$
DISTRIB($\vee/\wedge$)	$\vdash (\phi \vee (\psi \wedge \chi)) \approx ((\phi \vee \psi) \wedge (\phi \vee \chi))$
SYMM	$\phi \approx \psi \vdash \psi \approx \phi$
TRANS	$\phi \approx \psi, \psi \approx \chi \vdash \psi \approx \chi$
SUB($\wedge$)	$\phi \approx \psi \vdash (\phi \wedge \chi) \approx (\psi \wedge \chi)$
SUB($\vee$)	$\phi \approx \psi \vdash (\phi \vee \chi) \approx (\psi \vee \chi)$

Intuitively, AC requires the identification of the contents of sentences that have the same disjunctive normal forms. This requires fewer identifications of content than, for instance, a boolean approach, which requires the identification of the contents of tautologically equivalent sentences. For example, AC allows us to deny $\phi \approx \phi \vee (\phi \wedge \psi)$. The boolean approach, by contrast, would require it, since $\phi \Leftrightarrow \phi \vee (\phi \wedge \psi)$ is a tautology. But AC also requires the identification of the contents of sentences in cases in which GG demands their distinctness. For instance, AC requires the equivalence of ϕ and $\neg\neg\phi$, while, as we have seen, GG demands that the second sentence express the result of "raising" the content of ϕ to yield something new. So, the conception of content characterized by AC is much "flatter" than that required by GG, though not nearly as "flat" as that required by a boolean approach.

We define the notion of an *Angellic frame* (an *A-frame*) in a way similar to the notion of a $\leq$-frame, but replacing STRENGTHENED MAXIMALITY with four constraints. The first is:

Commutativity + Unipolarity: $[(a_1, c_1).(a_2.c_2).\cdots] = [(b_1, c_1).(b_2.c_2).\cdots]$ if $\{a_1, a_2, \ldots\} = \{b_1, b_2, \ldots\}$; and $[(a_1, c_1)+(a_2.c_2)+\cdots] = [(b_1, c_1)+(b_2.c_2)+\cdots]$ if $\{a_1, a_2, \ldots\} = \{b_1, b_2, \ldots\}$.

Intuitively, (COMMUTATIVITY + UNIPOLARITY) (or (C+U) for short) says that the choice and combination operations on contents are blind to order, repetitions, and falsity conditions. (c+u) thus permits the definition of choice and combination operations on *conditions*, rather than contents, where $[a_1.a_2.\cdots]$ is the combination, for any (b_i), of $\langle(a_1, b_1), (a_2, b_2), \ldots\rangle$, and, similarly, for $[a_1+a_2+\cdots]$. The other three constraints on Angellic frames are then:[12]

(Involution): $[a] = a$;

(Associativity): $[a.[b.c]] = [[a.b].c]$ and $[a + [b + c]] = [[a + b] + c]$; and

(Distribution): $[a + [b.c]] = [[a + b].[a + c]]$ and $[a.[b + c]] = [[a.b] + [a.c]]$.

The notion of an interpretation $\bar{\cdot}$ for a language $\mathscr{L}$ has already been defined in Section 1. An *Angellic model* (*A-model*) is then a quadruple $\langle F, \Sigma, \Pi, \bar{\cdot}\rangle$, where $\bar{\cdot}$ is an interpretation, and $\mathfrak{F} = \langle F, \Sigma, \Pi\rangle$ is an A-frame. Sentences ϕ and ψ are equivalent when they have the same truth-condition: $\phi \approx \psi$ is true in an A-model iff $\overline{\phi}_\oplus = \overline{\psi}_\oplus$. $\phi \approx \psi$ is *valid* iff it is true in every A-model.

This semantics is sound and complete for AC. It is useful to note some basic facts concerning the interaction of the interpretation function and the choice and combination operations:

Lemma 3.1 *For any A-model $\langle F, \Sigma, \Pi, \bar{\cdot}\rangle$,*

1. $\overline{\neg\phi} \overset{\text{(INVOL)}}{=} (\bar{\phi}_\ominus, \bar{\phi}_\oplus)$.
2. *For* $\odot \in \{+, .\}$,

$$[v \odot w] \overset{\text{(C+U)}}{=} [w \odot v];$$

$$[v \odot \overline{(\phi \wedge \psi)}] \overset{\text{(C+U)}}{=} [v_\oplus \odot \overline{(\phi \wedge \psi)}_\oplus] = [v_\oplus \odot [\bar{\phi}.\bar{\psi}]] \overset{\text{(C+U)}}{=} [v_\oplus \odot [\bar{\phi}_\oplus.\bar{\psi}_\oplus]];\ \textit{and}$$

$$[v \odot \overline{(\phi \vee \psi)}] \overset{\text{(C+U)}}{=} [v_\oplus \odot \overline{(\phi \vee \psi)}_\oplus] = [v_\oplus \odot [\bar{\phi} + \bar{\psi}]] \overset{\text{(C+U)}}{=} [v_\oplus \odot [\bar{\phi}_\oplus + \bar{\psi}_\oplus]].$$

3. $\overline{(\phi \wedge \psi)}_\oplus = [\bar{\phi}.\bar{\psi}] \overset{\text{(C+U)}}{=} [\bar{\phi}_\oplus.\bar{\psi}_\oplus]$ *and* $\overline{(\phi \vee \psi)}_\oplus = [\bar{\phi} + \bar{\psi}] \overset{\text{(C+U)}}{=} [\bar{\phi}_\oplus + \bar{\psi}_\oplus]$.

Lemma 3.1 gives us structural information that makes it easy to prove the soundness of AC for the class of A-models by a straightforward induction on derivations in AC.

Lemma 3.2 *(**Soundness**) If $\vdash \phi \approx \psi$, then $\phi \approx \psi$ is valid.*

4 Completeness of AC

AC is also complete for the space of A-models in the sense that $\vdash \phi \approx \psi$ if $\phi \approx \psi$ is true in every A-model.[13] Completeness could be shown, as in [9] by directly constructing a canonical A-model in which $\phi \approx \psi$ is true iff it is a theorem of AC. Here, for the sake of brevity, we instead show how, given a model of the semantics of [9], to construct a corresponding A-model in which exactly the same equivalence claims are true. We then use the completeness of Fine's semantics for AC to establish the completeness of the selection space semantics described in Section 3. This procedure has the added benefit of showing how the models of Fine's semantics are systematically

12 (ASSOCIATIVITY) and (DISTRIBUTION) have infinitary analogues, but for present purposes, the weaker, finitary versions suffice.

13 This is a *weak* completeness result. See n. 14 below.

related to A-models. In particular, it enables us to see how Fine's models can be thought of as a special case of A-models, and thus of selection-spaces more generally.

Let's begin by recapitulating Fine's semantics. A *statespace* is a pair $\langle S, \sqsubseteq \rangle$, where S is nonempty and

1. $\sqsubseteq$ is a partial order on S, *i.e.*, $\sqsubseteq$ is reflexive, transitive, and anti-symmetric; and
2. Every subset of S has a $\sqsubseteq$-least upper bound.

Use lowercase letters r, s, and t (perhaps with superscripts or subscripts) for members of S, and a, b, and c for subsets of S. Intuitively, a state s is a part of a state t when part of what it is for t to obtain is that s obtains. Thus, if t is a state in which a certain house h in prehistoric Sumeria is a red house, part of what it is for t to obtain is for the state r in which h is red to obtain. By contrast, the obtaining of state r' in which either tea is expensive in China or it is not is presumably no part of what it is for t to obtain, since t's obtaining, intuitively, has nothing to do with the price of tea in China. The least upper bound of a set of states a is, intuitively, the state whose obtaining involves the obtaining of exactly the states in a and no more. Thus, we might think of the least upper bound of a as the state-theoretic analogue of conjunction. The requirement that every set of states has a least upper bound can then be understood as the apparently benign commitment that every set of states has such a conjunction. Fine [9, p. 205] calls this analogue of conjunction the *fusion* of the states. Unlike a conjunction of sentences, however, fusions are not generally uniquely decomposable. The fusion, for instance, of r, s, and t has a decomposition into r and the fusion of s and t, and another decomposition into t and the fusion of r and s.

We can now define analogues, in Fine's semantics, for the choice and combination operations in selection-space semantics. In fact, these operations will serve, in effect, as the choice and combination operations of an A-model we will define, corresponding to a given model of Fine's semantics. For any $a \subseteq S$ write $\bigsqcup a$ for the least upper bound of a, and, if $\{s^1, s^2, \ldots\} \subseteq S$, write $s^1 \sqcup s^2 \sqcup \ldots$ for $\bigsqcup\{s^1, s^2, \ldots\}$. Define two operations on sets $\{a^1, a^2, \ldots\}$ of subsets $a^1, a^2, \ldots$ of S, and a unary operation on subsets a of S:

Definition 4.1

1. $(a^1.a^2.\cdots) = \{s^1 \sqcup s^2 \sqcup \ldots | (s^i \in a^i)\}$;
2. $(a^1 + a^2 + \cdots) = a^1 \cup a^2 \cup \ldots$; and
3. The *complete, convex closure* $[a]'$ of a is $\{t | (\exists s \in a) s \sqsubseteq t \sqsubseteq \bigsqcup a\}$.

I often omit parentheses for $(a^1.a^2.\cdots)$ and $(a^1+a^2+\cdots)$ for readability. I also omit the superscript $'$ from $[a]'$ when it will not result in any confusion. So, for instance, in a convenient abuse of notation, we write $[a^1.a^2.\cdots]$ for the complete, convex closure of $(a^1.a^2.\cdots)$. Our strategy will be to show that the closure of $(a^1.a^2.\cdots)$, denoted either $[(a^1.a^2.\cdots)]'$ or $[a^1.a^2.\cdots]$, can serve as the combination of $a^1, a^2, \ldots$ in an appropriately defined A-model, and, similarly, for $+$ and choice; see Definition 4.12.

For the moment, however, we will focus on understanding Fine's semantics. We carry out that task by establishing results Lemma 4.2-Theorem 4.9. It is important to bear in mind in our discussion of these results that '$+$' and '.' are used to indicate the operations in Fine's statespace semantics defined above, rather than choice or combination in some selection space.

We first note some structural principles governing the interaction of our operations. It will often be convenient to appeal to the fact that $\bigsqcup a$ is the *least* upper bound of a, so that, if t bounds a, then $\bigsqcup a \sqsubseteq t$. We will abbreviate our appeal by saying that $\bigsqcup a \sqsubseteq t$ *by leastness*. Thus, $\bigsqcup a \sqsubseteq \bigsqcup b$ whenever $a \subseteq b$ by leastness.

Lemma 4.2 *(Associativity of* $\bigsqcup$*)* $\bigsqcup\{\bigsqcup a^1, \bigsqcup a^2, \ldots\} = \bigsqcup(a^1 + a^2 + \cdots)$.

Proof It is enough to show that every bound of $\{\bigsqcup a^1, \bigsqcup a^2, \ldots\}$ is a bound of $(a^1 + a^2 + \cdots)$, and vice versa. Suppose that t bounds $\{\bigsqcup a^1, \bigsqcup a^2, \ldots\}$. For each i, $\bigsqcup a^i$ bounds a^i. So, t bounds $(a^1 + a^2 + \cdots)$. Suppose, then, that t bounds $a^1 + a^2 + \cdots$. By leastness, $(\bigsqcup a^i \sqsubseteq t)$. So, t bounds $\{\bigsqcup a^1, \bigsqcup a^2, \ldots\}$.

Say that a is *complete* iff, for every non-empty subset b of a, $\bigsqcup b \in a$. Complete sets of states are closed under fusion. Say that a is *convex* if whenever there are r, s, and t such that $r, t \in a$ and $r \sqsubseteq s \sqsubseteq t$, s is also in a. Convex sets of states contain no "gaps": any state between two states in the set is also in the set. So, if we have three states r, s, and t, the set $\{r \sqcup s, t\}$ may turn out to be incomplete because it does not contain its own fusion $r \sqcup s \sqcup t$. If we were to throw the fusion in, then we get the set $\{r \sqcup s, t, r \sqcup s \sqcup t\}$. Now, this new set may turn out to be non-convex if it does not contain $s \sqcup t$, which lies between t and $r \sqcup s \sqcup t$. The next two lemmas justify calling $[a]$ the "complete, convex" closure of a, as we did in Definition 4.1.

Lemma 4.3 $[a]$ *is complete and convex. [10, L1, p. 648]*

Proof The result is trivial if $a = \emptyset$. For completeness, suppose $s^1, s^2, \dots \in [a]$, so that $(t^i \sqsubseteq s^i \sqsubseteq \bigsqcup a)$, for some $(t^i) \subseteq a$. $\bigsqcup a$ bounds $\{s^1, s^2, \ldots\}$, so, by leastness, $\bigsqcup\{s^1, s^2, \ldots\} \sqsubseteq \bigsqcup a$. Also, $t^1 \sqsubseteq \bigsqcup\{s^1, s^2, \ldots\}$, so $\bigsqcup\{s^1, s^2, \ldots\} \in [a]$. For convexity, suppose $s^1, s^2 \in [a]$ and $s^1 \sqsubseteq t \sqsubseteq s^2$. Then $(\exists t' \in a) t' \sqsubseteq s^1 \sqsubseteq t \sqsubseteq s^2 \sqsubseteq \bigsqcup a$. So, $s \in [a]$.

Lemma 4.4 *If a is complete and convex, then* $[a] = a$.

Proof Suppose a is complete and convex. If $a = \emptyset$, then there are no $t \in a$, so $[a] = \emptyset$. Suppose $s \in a$. Then $s \sqsubseteq s \sqsubseteq \bigsqcup a$, so $s \in [a]$. Suppose $s \in [a]$. Then $t \sqsubseteq s \sqsubseteq \bigsqcup a$ for some $t \in a$. Since a is complete and non-empty, $\bigsqcup a \in a$. So, since a is convex, $s \in a$.

The following utility lemma, due to Fine [9, pp. 207-8], is helpful for proving identity claims among closures of subsets of S.

Lemma 4.5 $[a] = [b]$ *iff (i)* $\bigsqcup a = \bigsqcup b$*; (ii)* $(\forall s \in a)(\exists t \in b) t \sqsubseteq s$*; and (iii)* $(\forall t \in b)(\exists s \in a) s \sqsubseteq t$.

Proof $\Rightarrow$: Suppose $[a] = [b]$. If $a = \emptyset$, then $[a] = \emptyset = [b]$, so $b = \emptyset$, and (i)-(iii) are trivially true. Suppose both a and b are non-empty. Then $\bigsqcup a \in [a] = [b]$, and so $\bigsqcup a \sqsubseteq \bigsqcup b$. By symmetry, $\bigsqcup b \sqsubseteq \bigsqcup a$, so, by anti-symmetry of $\sqsubseteq$, $\bigsqcup a = \bigsqcup b$, establishing (i). Suppose $s \in [a] = [b]$. Then $(\exists t \in b) t \sqsubseteq s \sqsubseteq \bigsqcup b$, establishing (ii) and (by symmetry) (iii).

$\Leftarrow$: Suppose (i)-(iii) are each true. Suppose $s \in [a]$. Then $(\exists t \in a) t \sqsubseteq s \sqsubseteq \bigsqcup [a] \overset{((\mathrm{I}))}{=} \bigsqcup [b]$. By (2), $\exists t' \in [b] t' \sqsubseteq t \sqsubseteq s$, and we already have $s \sqsubseteq \bigsqcup b$. So, $s \in [b]$. The result follows by symmetry.

The next two lemmas establish that our two operations are associative and idempotent when applied to non-empty subsets of states.

Lemma 4.6

1. *If* $a^1, a^2, \ldots$ *are each non-empty, then* $\bigsqcup (a^1.a^2.\cdots) = \bigsqcup (a^1 + a^2 + \cdots)$.
2. $\bigsqcup a = \bigsqcup [a]$.
3. $\bigsqcup ([a^1] + [a^2] + \cdots) = \bigsqcup (a^1 + a^2 + \cdots)$.
4. *If* $a^1, a^2, \ldots$ *are each non-empty, then* $\bigsqcup (a^1.a^2.\cdots) = \bigsqcup \{\bigsqcup a^1, \bigsqcup a^2, \ldots\}$.

Proof

1. Suppose $a^1, a^2, \ldots$ are each non-empty. It is enough to show that every bound of $(a^1.a^2.\cdots)$ also bounds $(a^1 + a^2 + \cdots)$ and vice versa. Suppose t bounds $(a^1.a^2.\cdots)$, and that $s \in a^i$, for some i. Then, because each of (a^j) is non-empty, $s \sqsubseteq s^1 \sqcup s^2 \sqcup \cdots \sqcup s^i (= s) \sqcup \ldots$, for some (s^j) such that $(s^j \in a^j)$. So, since t bounds $(a^1.a^2.\cdots)$, $s \sqsubseteq s^1 \sqcup s^2 \sqcup \cdots \sqsubseteq t$. So t bounds $(a^1 + a^2 + \cdots)$. Suppose, then, that t bounds $(a^1 + a^2 + \cdots)$. Consider any (s^i) where $(s^i \in a^i)$. Since $\{s^1, s^2, \ldots\} \subseteq (a^1 + a^2 + \cdots)$, $s^1 \sqcup s^2 \sqcup \cdots \sqsubseteq t$ by leastness. So, t bounds $(a^1.a^2.\cdots)$.
2. Suppose t' bounds a. By leastness, $\bigsqcup a \sqsubseteq t'$. Suppose $s \in [a]$, so that $t \sqsubseteq s \sqsubseteq \bigsqcup a$, for some $t \in a$. Then $s \sqsubseteq \bigsqcup a \sqsubseteq t'$, so t' bounds $[a]$. Conversely, since $a \subseteq [a]$, any bound t' of $[a]$ also bounds a.
3.
$$\bigsqcup ([a^1]+[a^2]+\cdots) \overset{(\text{Lemma } 4.2)}{=} \bigsqcup \{\bigsqcup [a^1], \bigsqcup [a^2], \cdots\} \overset{(2)}{=} \bigsqcup \{\bigsqcup a^1, \bigsqcup a^2, \cdots\} \overset{(\text{Lemma } 4.2)}{=} \bigsqcup (a^1+a^2+\cdots).$$
4. Suppose $a^1, a^2, \ldots$ are each non-empty.
$$\bigsqcup (a^1.a^2.\cdots) \overset{(1)}{=} \bigsqcup (a^1 + a^2 + \cdots) \overset{(\text{Lemma } 4.2)}{=} \bigsqcup \{\bigsqcup a^1, \bigsqcup a^2, \ldots\}$$

Lemma 4.7 ***(Associativity and Idempotence)***

1. $[[a^1].[a^2].\cdots] = [a^1.a^2.\cdots]$.
2. $[[a^1] + [a^2] + \cdots] = [a^1 + a^2 + \cdots]$.
3. $[a.a.\cdots] = [a]$.
4. $[a + a + \cdots] = [a]$
5. $[a^1.a^2.\cdots.[b^1.b^2.\cdots].[c^1.c^2.\cdots].\cdots] = [a^1.a^2.\cdots.b^1.b^2.\cdots.c^1.c^2.\cdots]$.
6. $[a^1 + a^2 + \cdots + [b^1 + b^2 + \cdots] + [c^1 + c^2 + \cdots] + \cdots] = [a^1 + a^2 + \cdots + b^1 + b^2 + \cdots + c^1 + c^2 + \cdots]$.
7. $[a.[b.c]] = [[a.b].c]$.
8. $[a + [b + c]] = [[a + b] + c]$.

Proof (4) is trivial, since $(a + a + \cdots) = a$. Since $[b^1.b^2.\cdots] = [(b^1.b^2.\cdots)]$ has the form $[b]$, (5) follows from (1) and Lemma 4.2 (Associativity of $\bigsqcup$). Similarly,

(6) follows from (2) and the associativity of the set-union operation. (7) follows from (5) and the commutativity of $\bigsqcup$, and (8) from (6) by the commutativity of of $\bigsqcup$ and set-union. We prove each of (1), (2), and (3) by Lemma 4.5.

1. Suppose that $a^i = \emptyset$, for some i. Then $[a^i] = \emptyset \Rightarrow [[a^1].[a^2].\cdots] = \emptyset = [a^1.a^2.\cdots]$. Suppose, then, that $a^1, a^2, \ldots$ are each non-empty.

   (i): $\bigsqcup[[a^1].[a^2].\cdots] \overset{\text{(Lemma 4.6(2))}}{=} \bigsqcup([a^1].[a^2].\cdots) \overset{\text{(Lemma 4.6(1))}}{=} \bigsqcup([a^1] + [a^2] + \cdots) \overset{\text{(Lemma 4.6(3))}}{=} \bigsqcup(a^1 + a^2 + \cdots) \overset{\text{(Lemma 4.6(1))}}{=} \bigsqcup(a^1.a^2.\cdots) \overset{\text{(Lemma 4.6(2))}}{=} \bigsqcup[a^1.a^2.\cdots]$

   (ii): Suppose $s \in ([a^1].[a^2].\cdots)$. Then s has the form $s^1 \sqcup s^2 \sqcup \ldots$ where $(s^i \in [a^i])$. So, for each i, $(\exists t^i \in a^i) t^i \sqsubseteq s^i \sqsubseteq s$. So, s bounds $\{t^1, t^2, \ldots\}$ for some $t^1, t^2, \ldots$ such that $(t^i \in a^i)$. By leastness, $t^1 \sqcup t^2 \sqcup \cdots \sqsubseteq s$. So, there is a $t' = t^1 \sqcup t^2 \sqcup \cdots \in (a^1.a^2.\cdots)$ such that $t' \sqsubseteq s$.

   (iii): Suppose $s \in (a^1.a^2.\cdots)$. Then s has the form $s^1 \sqcup s^2 \sqcup \ldots$ where $(s^i \in a^i)$. For each i, since $a^i \subseteq [a^i]$, $s^i \in [a^i]$. So, $s = s^1 \sqcup s^2 \sqcup \cdots \in ([a^1].[a^2].\cdots)$ and $s \sqsubseteq s$.

2.

   (i): $\bigsqcup[[a^1] + [a^2] + \cdots] \overset{\text{(Lemma 4.6(2))}}{=} \bigsqcup([a^1] + [a^2] + \cdots) \overset{\text{(Lemma 4.6(3))}}{=} \bigsqcup(a^1 + a^2 + \cdots)$

   $$\overset{\text{(Lemma 4.6(2))}}{=} \bigsqcup[a^1 + a^2 + \cdots]$$

   (ii): Suppose $s \in [a^i]$, for some i. By construction, there is a $t \in a^i$ such that $t \sqsubseteq s$.

   (iii): Suppose $s \in a^i$, for some i. Since $a^i \subseteq [a^i]$, $s \in [a^i]$. $s \sqsubseteq s$.

3.

   (i): Suppose t bounds a. Then, for any set $\{s^1, s^2, \ldots\} \subseteq a$, t bounds $\{s^1, s^2, \ldots\}$. By leastness, $s^1 \sqcup s^2 \sqcup \cdots \sqsubseteq t$, so t bounds $(a.a.\cdots)$. Suppose t bounds $(a.a.\cdots)$, and let $s \in a$. Then $s \sqsubseteq s \sqcup s \sqcup \cdots \subseteq t$, so t bounds a.

   (ii): Suppose $s \in a$. Then $s = s \sqcup s \sqcup \cdots \in (a.a.\ldots)$, and $s \sqsubseteq s$.

   (iii): Suppose $s \in (a.a.\ldots)$. Then there are $s^1, s^2, \cdots \subseteq a$ such that $s = s^1 \sqcup s^2 \sqcup \ldots$. So, $s^1 \in a$ and $s^1 \sqsubseteq s^1 \sqcup s^2 \sqcup \cdots = s$.

Now we can prove the requisite distributivity laws.

Lemma 4.8 ***(Distributivity)*** *If a, b, and c are each non-empty, then*

1. $[a + [b.c]] = [[a + b].[a + c]]$*; and*
2. $[a.[b + c]] = [[a.b] + [a.c]]$.

Proof We prove both results by Lemma 4.5.

1. Suppose a, b, and c are each non-empty.

   (i): It's easy to see that by Lemma 4.6 and Lemma 4.2 (Associativity of $\bigsqcup$), $\bigsqcup[a+[b.c]] = \bigsqcup(a+(b+c))$. Similarly, $\bigsqcup[[a+b].[a+c]] = \bigsqcup((a+b)+(a+c))$. But $(a + (b + c)) = (a + b + c) = ((a + b) + (a + c))$.

(ii): Suppose $s' \in (a + [b.c])$. Suppose $s' \in a$.

$$s' \in a \Rightarrow s' \in (a+b), (a+c) \Rightarrow s' \in [a+b], [a+c] \Rightarrow s' = s' \sqcup s' \in ([a+b].[a+c]) \\ \Rightarrow (\exists t = s' \in ([a+b].[a+c]))t \sqsubseteq s'.$$

Suppose, instead, that $s' \notin a$, and so $s' \in [b.c]$. Then there are $s^b \in b$ and $s^c \in c$ such that $s^b \sqcup s^c \sqsubseteq s'$. $s^b \in b \Rightarrow s^b \in (a+b) \Rightarrow s^b \in [a+b]$. Similarly, $s^c \in [a+c]$. So, there is a $t(= s^b \sqcup s^c) \in ([a+b].[a+c])$ such that $t \sqsubseteq s'$.

(iii): Suppose $s \in ([a+b].[a+c])$. Then there are $s'_b \in [a+b]$ and $s'_c \in [a+c]$ such that $s = s'_b \sqcup s'_c$. So, $(\exists s^b \in (a+b))s^b \sqsubseteq s'_b$ and $(\exists s^c \in (a+c))s^c \sqsubseteq s'_c$. Suppose $s^b \in a$. Then $s^b \in (a+[b.c]) \Rightarrow (\exists t(= s^b) \in (a+[b.c]))t \sqsubseteq s'_b \sqsubseteq s'_b \sqcup s'_c = s$. Similarly, if $s^c \in a$, then we're done. Suppose, then, that $s^b \in b$ and $s^c \in c$. Then $s^b \sqcup s^c \in (b.c) \subseteq [b.c] \subseteq (a + [b.c])$. Since $s'_b \sqcup s'_c$ bounds $\{s'_b, s'_c\}$, it also bounds $\{s^b, s^c\}$, and so, by leastness, $s^b \sqcup s^c \sqsubseteq s'_b \sqcup s'_c = s$.

2. Suppose a, b, and c are each non-empty.

(i): Similar to (1)(i).

(ii): Suppose $s \in (a.[b+c])$. Then there are $s^a \in a$ and $s'_b \in [b+c]$ such that $s = s^a \sqcup s'_b$. Since $s'_b \in [b+c]$, for some $s^b \in (b+c)$, $s^b \sqsubseteq s'_b$. By symmetry, we may (wlog) assume $s^b \in b$. Then $s^a \sqcup s^b \in [a.b] \subseteq ([a.b] + [a.c])$. Also, s bounds $\{s^a, s^b\}$, so, by leastness, $s^a \sqcup s^b \sqsubseteq s$.

(iii): Suppose $s \in ([a.b] + [a.c])$. By symmetry, we may (wlog) assume $s \in [a.b]$. Then, for some $s^a \in a$ and $s^b \in b$, $s^a \sqcup s^b \sqsubseteq s$. Also, $s^b \in b \subseteq (b+c) \subseteq [b+c]$. So, there is a $t = s^a \sqcup s^b$, such that $t \in (a.[b+c])$ and $t \sqsubseteq s$.

With these preliminaries concerning features of statespaces out of the way, we can now specify Fine's notion of a model for a language $\mathscr{L}$. A *Fine-model* (F-model) $\mathfrak{M}$ for a language $\mathscr{L}$ is a triple $\langle S, \sqsubseteq, |\cdot|\rangle$, where $\langle S, \sqsubseteq\rangle$ is a statespace, and $|\cdot|$ takes every atomic sentence of $\mathscr{L}$ to a pair (a, b) of non-empty subsets of S. If $|\phi| = (a, b)$, then I denote a by $\phi_\oplus$ (with no bars over the top or on the sides) and b by $\phi_\ominus$. Intuitively, $\phi_\oplus$ is the set of *verifiers for* ϕ, that is, the set of states in which ϕ is true. Similarly, $\phi_\ominus$ is, intuitively, the set of *falsifiers* for ϕ. Recursively extend $|\cdot|$ to molecular sentences:

1. $|\neg\phi| = (\phi_\ominus, \phi_\oplus)$;
2. $|\phi \wedge \psi| = ((\phi_\oplus.\psi_\oplus), (\phi_\ominus + \psi_\ominus))$; and
3. $|\phi \wedge \psi| = ((\phi_\oplus + \psi_\oplus), (\phi_\ominus.\psi_\ominus))$.

On this definition, the falsifiers for ϕ are exactly the verifiers for $\neg\phi$. Notice that there is no requirement that the set of verifiers or falsifiers for a sentence be closed. Still, Fine's semantics ultimately interprets equivalence by appeal to the closures of semantic values. Sentences ϕ and ψ are equivalent when their respective sets of verifiers have the same complete, convex closure: $\phi \approx \psi$ is true in an F-model iff $[\phi_\oplus] = [\psi_\oplus]$ [9, pp. 208, 210]. When we construct an A-model (which is, recall, a certain kind of selection-space model) from a given F-model, we will assign to ϕ a pair comprising the closures of ϕ's verifiers and falsifiers, respectively. We will define choice and combination using the operations + and . defined in Definition 4.1 above. We will then see that the resulting function is an interpretation in the sense defined in Section 1 above. So, for instance, the assigning the closure of the set of $(\phi \wedge \psi)$'s verifiers as that sentence's truth condition turns out to be equivalent to assigning the combination, in the defined sense, of the contents of ϕ and ψ.

Fine [9, T23, p. 216] proves that F-models are weakly complete for AC:

Theorem 4.9 ***(F-Completeness)*** *If $\phi \approx \psi$ is true in every F-model, then $\vdash \phi \approx \psi$.*[14]

We will show how, given an arbitrary F-model $\mathfrak{M}$ for a language $\mathscr{L}$, to construct an A-model $\langle F^{\mathfrak{M}}, \Sigma^{\mathfrak{M}}, \Pi^{\mathfrak{M}}, \bar{\cdot}^{\mathfrak{M}}\rangle$ in which the very same equivalences $\phi \approx \psi$ are true. Let's suppose for the remainder of the section that we have fixed an arbitrary F-model $\mathfrak{M} = \langle S, \sqsubseteq, |\cdot|\rangle$. Recall that $\phi_\oplus$ is the truth-condition for atomic ϕ: a non-empty subset of the set of states S in the statespace of $\mathfrak{M}$. We start by defining, for every sentence ϕ of $\mathscr{L}$, a function $\bar{\cdot}$ mapping sentences to pairs of subsets of S. The mapping $\bar{\cdot}$ will turn out to be our interpretation in the A-model we are defining, and, as we said above, it will become clear that it meets requirements specified in Section 1 for being an *interpretation* in selection space semantics. To make this easy to see, we define $\bar{\cdot}$ to mirror the specification of the extension of a selection-space semantics interpretation:

Definition 4.10

1. $\bar{\phi} = ([\phi_\oplus], [\phi_\ominus])$, for atomic ϕ;
2. $\overline{\neg\phi} = (\bar{\phi}_\ominus, [\bar{\phi}_\oplus])$;
3. $\overline{(\phi \wedge \psi)} = ([\bar{\phi}_\oplus.\bar{\psi}_\oplus], [\bar{\phi}_\ominus + \bar{\psi}_\ominus])$; and
4. $\overline{(\phi \vee \psi)} = ([\bar{\phi}_\oplus + \bar{\psi}_\oplus], [\bar{\phi}_\ominus.\bar{\psi}_\ominus])$.

Bear in mind that $[v + w + \cdots]$ indicates an application of the operation of Fine's statespace semantics defined in Definition 4.1: the closure of the union of $v, w, \ldots$. It is not meant here to indicate a choice in some selection space, though we will soon see that it can do double duty as such a choice in a space, defined in Definition 4.12, which satisfies (C+U). Similar remarks apply to $[v.w.\cdots]$.

A simple induction on complexity of formulae, using Lemma 4.3, easily confirms that, for any formula ϕ, $\bar{\phi}_\oplus$ and $\bar{\phi}_\ominus$ are each complete and convex. So, Lemma 4.4 applies: $\bar{\phi}_\oplus = [\bar{\phi}_\oplus]$ and $\bar{\phi}_\ominus = [\bar{\phi}_\ominus]$. Also, for all ϕ, Lemma 4.7(1) applies:

$$[\bar{\phi}_\oplus.\bar{\psi}_\oplus] = [[\bar{\phi}_\oplus].[\bar{\psi}_\oplus]].$$

$$[\bar{\phi}_\ominus.\bar{\psi}_\ominus] = [[\bar{\phi}_\ominus].[\bar{\psi}_\ominus]],$$
$$[\bar{\phi}_\oplus + \bar{\psi}_\oplus] = [[\bar{\phi}_\oplus] + [\bar{\psi}_\oplus]], \text{ and}$$
$$[\bar{\phi}_\ominus + \bar{\psi}_\ominus] = [[\bar{\phi}_\ominus] + [\bar{\psi}_\ominus]].$$

So, a simple induction on complexity of formulas shows that, if we think of $\bar{\phi}_\oplus$ as the truth condition for ϕ, then $\bar{\cdot}$ identifies the truth condition for ϕ with the closure of the set of its verifiers:

[14] This is a *weak* completeness result. By contrast, AC is *strongly* complete for the class of F-models iff, if every F-model in which every member of a set S of equivalence claims is true is also a model in which $\phi \approx \psi$ is true, then $\phi \approx \psi$ is derivable from S in AC. The question of whether AC is strongly complete for the class of F-models (and so, by Lemma 4.15 below, the class of A-models) is an interesting open problem.

Lemma 4.11 $[\phi_\oplus] = \bar{\phi}_\oplus$ *and* $[\phi_\ominus] = \bar{\phi}_\ominus$.

We are now ready to define our selection space and corresponding A-model $\langle F^{\mathfrak{M}}, \Sigma^{\mathfrak{M}}, \Pi^{\mathfrak{M}}, \bar{\cdot}^{\mathfrak{M}} \rangle$:

Definition 4.12 The *A-structure corresponding to* $\mathfrak{M}$ is the quadruple $\langle F^{\mathfrak{M}}, \Sigma^{\mathfrak{M}}, \Pi^{\mathfrak{M}}, \bar{\cdot}^{\mathfrak{M}} \rangle$ where

1. $F^{\mathfrak{M}}$ is the set of all complete, convex closures of subsets of S;
2. $\Sigma^{\mathfrak{M}}(\langle (a^1, b^1), (a^2, b^2), ...\rangle) = \begin{cases} [a^1 + a^2 + \cdots]', & \text{if } a^1, a^2, \ldots \text{ are each non-empty; and} \\ \emptyset, & \text{otherwise;} \end{cases}$
3. $\Pi^{\mathfrak{M}}(\langle (a^1, b^1), (a^2, b^2), ...\rangle) = \begin{cases} [a^1.a^2.\cdots]', & \text{if } a^1, a^2, \ldots \text{ are each non-empty; and} \\ \emptyset, & \text{otherwise;} \end{cases}$
4. $\Sigma^{\mathfrak{M}}(\emptyset) = \emptyset$;
5. $\Pi^{\mathfrak{M}}(\emptyset) = \{\bigsqcup \emptyset\}$;
6. $\Pi^{\mathfrak{M}}(\langle (a, b)\rangle) = \Sigma^{\mathfrak{M}}(\langle (a, b)\rangle) = [a]'$; and
7. $\bar{\phi}^{\mathfrak{M}} = ([\phi_\oplus]', [\phi_\ominus]')$, for atomic ϕ.

We will show that the A-structure corresponding to $\mathfrak{M}$ is a model. $\langle F^{\mathfrak{M}}, \Sigma^{\mathfrak{M}}, \Pi^{\mathfrak{M}}\rangle$ is clearly a selection system, since $\Sigma^{\mathfrak{M}}$ and $\Pi^{\mathfrak{M}}$ are defined on all sequences of members of $F^{\mathfrak{M}}$, and converge on singletons.

We need to show that the A-structure corresponding to $\mathfrak{M}$ meets the four constraints for a selection space to be an A-frame. We first show that it meets (C+U).

Lemma 4.13 *If* $(a^i) = (b^j)$*, then* $\Pi^{\mathfrak{M}}(\langle (a^1, c^1), (a^2, c^2), ...\rangle) = \Pi^{\mathfrak{M}}(\langle (b^1, c^1), (d^2, d^2), ...\rangle)$ *and* $\Sigma^{\mathfrak{M}}(\langle (a^1, c^1), (a^2, c^2), ...\rangle) = \Sigma^{\mathfrak{M}}(\langle (b^1, d^1), (b^2, d^2), ...\rangle)$

Proof Suppose $(a^i) = (b^j)$. Suppose $(a^i) = \emptyset$. Then $\Sigma^{\mathfrak{M}}(\langle (a^1, c^1), (a^2, c^2), ...\rangle) = \emptyset = \Sigma^{\mathfrak{M}}(\langle (b^1, d^1), (b^2, d^2), ...\rangle)$ and $\Pi^{\mathfrak{M}}(\langle (a^1, c^1), (a^2, c^2), ...\rangle) = \{\bigsqcup \emptyset\} = \Pi^{\mathfrak{M}}(\langle (b^1, d^1), (b^2, d^2), ...\rangle)$. Suppose, instead, that (a^i) is non-empty. Suppose $a^i = \emptyset$, for some i. Then $\Sigma^{\mathfrak{M}}(\langle (a^1, c^1), (a^2, c^2), ...\rangle) = \emptyset = \Sigma^{\mathfrak{M}}(\langle (b^1, d^1), (b^2, d^2), ...\rangle)$ and $\Pi^{\mathfrak{M}}(\langle (a^1, c^1), (a^2, c^2), ...\rangle) = \emptyset = \Pi^{\mathfrak{M}}(\langle (b^1, d^1), (b^2, d^2), ...\rangle)$. If $\langle (a^1, c^1), (a^2, c^2), ...\rangle$ is a singleton $\langle (a^1, c^1)\rangle$, then $(a^i) = (b^j) = \{a\}$, for some a, and Lemma 4.7(3) and (4) imply the result. By symmetry, we are also done if $\langle (b^1, d^1), b^2, d^2), ...\rangle$ is a singleton. So, assume that neither $\langle (a^1, c^1), (a^2, c^2), ...\rangle$ nor $\langle (b^1, d^1), b^2, d^2), ...\rangle$ is a singleton, and that (a^i) are each non-empty. Then $\Pi^{\mathfrak{M}}(\langle (a^1, c^1), (a^2, c^2), ...\rangle) = [a^1.a^2.\ldots]'$. Now, since $(a^i) = (b^j)$, a^i has the form $\{a^1_1, a^1_2, \ldots, a^2_1, a^2_2, \ldots\}$, where $a^j_{k_j} = b^j$ for all j, k_j. Suppose $c \in (a^1.a^2.\cdots)$, so that c has the form $s^1 \sqcup s^2 \sqcup \ldots$, where $(s^i \in a^i)$, for all i. Then (s^i) has the form $\{s^1_1, s^1_2, \ldots, s^2_1, s^2_2, \ldots\}$, where $s^j_{k_j} \in a^j_{k_j} = b^j$ for all j, k_j. Since, for all j, $b^j (\in F^{\mathfrak{M}})$ is complete, $s^j_1 \sqcup s^j_2 \sqcup \cdots \in b^j$. So,

$$c = \bigsqcup\{s^1, s^2, \ldots\} = \bigsqcup\{s^1_1, s^1_2, \ldots, s^2_1, s^2_2, \ldots\}$$

$$\overset{\text{(Lemma 4.2)}}{=} (s^1_1 \sqcup s^1_2 \sqcup \ldots) \sqcup (s^2_1 \sqcup s^2_2 \sqcup \ldots) \sqcup \ldots) \in (b^1.b^2.\cdots)$$

By symmetry, if $c \in (b^1.b^2.\cdots)$, then $c \in (a^1.a^2.\cdots)$. So, $(a^1.a^2.\cdots) = (b^1.b^2.\cdots) \Rightarrow [a^1.a^2.\cdots]' = [b^1.b^2.\cdots]'$. A similar argument shows $[a^1 + a^2 + \cdots]' = [b^1 + b^2 + \cdots]'$.

$\overline{\phi}^{\mathfrak{M}}$ is clearly an interpretation, and can be extended to molecular sentences in the way specified in Section 1. Moreover, by Lemma 4.13 the A-structure corresponding to $\mathfrak{M}$ satisfies (C+U). So we can define the choice $[a^1 + a^2 + \cdots]$ and combination $[a^1.a^2.\cdots]$ operations on *conditions* (a^i) (rather than *contents*, i.e., pairs of such conditions) as previously specified in Section 3 just below the statement of (C+U). A routine induction on the complexity of formulas then yields

Lemma 4.14 $\overline{\phi}^{\mathfrak{M}}_{\oplus} = \bar{\phi}_{\oplus}$ *and* $\overline{\phi}^{\mathfrak{M}}_{\ominus} = \bar{\phi}_{\ominus}$

Thus, by Lemma 4.11, the interpretation of any sentence ϕ in our A-structure can be obtained either directly, by just taking the closures of the sets of verifiers and falsifiers for ϕ in the original F-model, or inductively, by building them in the way specified in Section 1, using the choice and combination operations defined in Definition 4.12.

Notice that, if (a^i) are each non-empty, then $[a^1.a^2.\cdots] = [a^1.a^2.\cdots]'$ and $[a^1 + a^2 + \cdots] = [a^1 + a^2 + \cdots]'$. So, if (a^i) are each non-empty, we can, without ambiguity, drop the superscript $'$, and simply regard the choice $[a^1 + a^2 + \cdots]$ and the combination $[a^1.a^2.\cdots]$ as the complete, convex closures, respectively, of $(a^1 + a^2 + \cdots)$ and $(a^1.a^2.\cdots)$.

We now have everything we need to show that the A-structure we have defined from our given F-model meets the remaining constraints (INVOLUTION), (ASSOCIATIVITY), and (DISTRIBUTION) for being an A-model, and that the F-model and the A-structure countenance exactly the same equivalences among sentences.

Lemma 4.15 *The A-structure* $\langle F^{\mathfrak{M}}, \Sigma^{\mathfrak{M}}, \Pi^{\mathfrak{M}}, \bar{\cdot}^{\mathfrak{M}} \rangle$ *corresponding to* $\mathfrak{M}$ *is an A-model, and* $\phi \approx \psi$ *is true in* $\mathfrak{M}$ *iff it is true in* $\langle F^{\mathfrak{M}}, \Sigma^{\mathfrak{M}}, \Pi^{\mathfrak{M}}, \bar{\cdot}^{\mathfrak{M}} \rangle$.

Proof (C+U): Lemma 4.11

(Involution): If $a = \emptyset$, $[a] = \emptyset$. Suppose, then, that a is non-empty. Then, since for all $a \in F^{\mathfrak{M}}$, a is complete and convex, $[a] = [a]' \overset{\text{(Lemma 4.4)}}{=} a$.

(Associativity): Suppose at least one of a, b, and c is empty. Then $[a.[b.c]] = \emptyset = [[a.b].c]$, and $[a + [b + c]] = \emptyset = [[a + b] + c]$. Suppose, then, that a, b, and c are each non-empty. Lemma 4.7(7) and (8) imply the result.

(Distribution): As in the previous case, the result is trivial when a, b, or c is empty. If a, b, and c are each non-empty, then Lemma 4.8 implies the result.

Finally, $\phi \approx \psi$ is true in $\mathfrak{M} \Leftrightarrow [\phi_{\oplus}] = [\psi_{\oplus}] \overset{\text{(Lemma 4.11)}}{\Leftrightarrow} \bar{\phi}_{\oplus} = \bar{\psi}_{\oplus} \overset{\text{(Lemma 4.13)}}{\Leftrightarrow} \overline{\phi}^{\mathfrak{M}}_{\oplus} = \overline{\psi}^{\mathfrak{M}}_{\oplus} \Leftrightarrow \phi \approx \psi$ is true in the A-structure corresponding to $\mathfrak{M}$.

Completeness follows straightforwardly.

Theorem 4.16 ***(Completeness)*** *If* $\phi \approx \psi$ *is true in every A-model, then* $\vdash \phi \approx \psi$.

Proof Theorem 4.9 and Lemma 4.15.

Given the fairly straightforward construction in Definition 4.12 of a selection-space from an F-model, it is plausible to regard the statespace semantics for AC as presenting in simplified form a special case of the selection space semantics: the class of F-models offers a simplified specification of the class of selection-space models given by Definition 4.12. In this sense, we may regard the statespace semantics for AC as a special case of selection-space semantics. This result is unsurprising: of course we can constrain choice and combination so they behave the way that the corresponding relations in Fine's semantics do, yielding the "flatter" conception of content required by AC.

We will now establish a more surprising result: the conception of ground we defined above (Definition 2.1) in the context of $\leq$-models automatically corresponds exactly, in any A-frame, to the completely different definition of ground deployed in the "flatter" treatments [3, 11]. This suggests that the "flatter" conception of ground specifies exactly the same idea as the "raised" conception, with the differences in extension in the two treatments completely explained by differences in the individuation of content.

5 "Flatter" Ground

As I have indicated, the Angellic conception of content pairs naturally with a certain view of ground which defines ground in terms of disjunction, conjunction, and equivalence. For convenience, we will confine ourselves to finitary grounding claims, with only finite numbers of sentences appearing on the LHS of any grounding claim. Then, on this view, $\phi^1, \phi^2, \ldots, \phi^n \leq \phi$ iff $(\phi^1 \wedge \phi^2 \wedge \ldots \phi^n) \vee \phi \approx \phi$, that is, iff the conjunction is a *disjunctive part* of ϕ.[15] The notions of weak partial ground ($\preceq$), strict partial ground ($\prec$), and strict full ground ($<$) are defined in terms of $\leq$ in the standard way:

$\psi \preceq \phi$ iff there is a χ such that $\psi, \chi \leq \phi$;
$\psi \prec \phi$ iff $\psi \preceq \phi$ and $\phi \npreceq \psi$; and
$\Delta < \phi$ iff $\Delta \leq \phi$ and $(\forall \delta \in \Delta)\delta \prec \phi$ (for finite Δ).

This is the view of ground (on an Angellic conception of content) that is characterized semantically in [11] and semantically and axiomatically in [3]. Astonishingly, so long as equivalence is a logical notion, then ground, on this conception, also turns out to be a logical notion [11, p. 686]. We now verify that the interpretation of $\leq$ in A-models exactly corresponds to this view of ground.

We start by reviewing how to capture the definition of weak ground in terms of disjunctive parthood. Intuitively, A is a disjunctive part of B when there is some C such that the truth-condition for $(A \vee C)$ is the truth-condition for B. In an A-model, we would characterize the idea by appeal to choice. If a and c are conditions, then a is a disjunctive part of c just in case $[a + b] = c$, for some b. Then, for any content (a, d), (a, d) is a disjunctive part of (c, e) iff a is a disjunctive part of c. Similarly, we can define a correlative notion of conjunctive part: a condition a is a conjunctive part of c iff $[a.b] = c$, for some condition b, and (a, d) is a conjunctive part of (c, e) iff a is a conjunctive part of c. It conveniently turns out that, given the four constraints on A-models, the quantification over conditions b in these definitions are dispensable.

[15] In the limit case in which there is only one occurrence of any sentence ϕ' on the LHS, we exploit the fact that the grounding claim $\phi' \leq \phi$ is identical to the claim $\phi', \phi' \leq \phi$, and, applying the previous truth condition, $\phi' \leq \phi$ iff $(\phi' \wedge \phi') \vee \phi \approx \phi$ iff $\phi' \vee \phi \approx \phi$ iff ϕ' itself is a disjunctive part of ϕ. In the further limit case in which there are no sentences on the LHS of $\leq$, we appeal to a "null fact" $\Box$, which is an identity element for conjunction: for all ϕ, $(\Box \wedge \phi) \approx \phi$. Then $\emptyset \leq \phi$ iff $\Box \vee \phi \approx \phi$.

Lemma 5.1 *[3, pp. 265-6]*

1. $[a+b]=c \Rightarrow [a+c]=c$,
2. $[a.b]=c \Rightarrow [a.c]=c$, *and*
3. $[[a.b]+c]=c \Rightarrow [[a.c]+c]=c$

Proof

1. Suppose

   ($\bigstar$) $[a+b]=c$.

   $c \overset{\text{(INVOL)}}{=} [c] \overset{\text{(C+U)}}{=} [c+c] \overset{(\bigstar)}{=} [[a+b]+[a+b]] \overset{\text{(ASSOC)}}{=} [a+b+a+b] \overset{\text{(C+U)}}{=} [a+a+b] \overset{\text{(ASSOC)}}{=} [a+[a+b]] \overset{(\bigstar)}{=} [a+c]$.

2. Similar to (1).
3. Suppose

   ($\maltese$) $[[a.b]+c]=c$.

   $c \overset{(\maltese)}{=} [[a.b]+c] \overset{\text{(DIST.)}}{=} [[a+c].[b+c]] \overset{(2)}{=} [[a+c].c] \overset{\text{(DIST.)}}{=} [[a.c]+[c.c]] \overset{\text{(C+U)}}{=} [[a.c]+[c]] \overset{\text{(INVOL)}}{=} [[a.c]+c]$.

An immediate corollary of Lemma 5.1 is that $(\exists b)[a+b]=c$ iff $[a+c]=c$, $(\exists b)[a.b]=c$ iff $[a.c]=c$, and $(\exists b)[[a.b]+c]=c$ iff $[[a.c]+c]=c$. So, for A-models, disjunctive parthood, conjunctive parthood, and being a conjunctive part of some disjunctive part can be all be defined without quantification.

We can now show that, in every A-frame, $\leq_1$ corresponds to the view of ground we have described. In what follows, Lemma 3.1 is generally assumed, so that we can move back and forth between use of + and . for operations on sequences of contents and their correlative use for operations on sequences of conditions, and we will write $a \ll b$ (where a is a *condition*) whenever any *content* (a, c) whose truth-condition is a is an immediate selection from b.

Lemma 5.2 *Let* $\mathfrak{F} = \langle F, \Sigma, \Pi\rangle$ *be an A-frame. Then* $v^1, v^2, \ldots v^n \leq_1 v \Rightarrow [[v^1.v^2.\cdots]+v_\oplus]=v_\oplus$.

Proof We prove the result by induction on $\leq_1$.

(Basis) Suppose $v^1, v^2, \cdots \ll v_\oplus$. There are two cases:

$\boldsymbol{v_\oplus = [v^1.v^2.\cdots]}$: Then $v_\oplus \overset{\text{(INVOL)}}{=} [v_\oplus] \overset{\text{(C+U)}}{=} [v_\oplus+v_\oplus] \overset{\text{(supp.)}}{=} [[v^1.v^2.\cdots]+v]$. This argument handles the case in which $\langle v^1, v^2, \ldots\rangle = \langle v\rangle$ and the case in which $\langle v^1, v^2, \ldots\rangle = \emptyset$.

$\boldsymbol{v_\oplus = [w+w^1+\cdots]}$ and $\boldsymbol{v^1, v^2, \cdots = w}$: Then $v_\oplus \overset{\text{(INVOL)}}{=} [v_\oplus] \overset{\text{(C+U)}}{=} [v_\oplus+v_\oplus] \overset{\text{(supp.)}}{=} [[w+w^1+\cdots]+v_\oplus] \overset{\text{(C+U)}}{=} [[w+w+w^1+\cdots]+v_\oplus] \overset{\text{(ASSOC.)}}{=} [[w+[w+w^1+\cdots]]+v_\oplus] \overset{\text{(supp.)}}{=} [[w+v_\oplus]+v_\oplus] \overset{\text{(ASSOC.)}}{=} [w+[v_\oplus+v_\oplus]] \overset{\text{(C+U)}}{=} [w+[v_\oplus]] \overset{\text{(INVOL)}}{=} [w+v_\oplus]$

Suppose, now, that $v^1, v^2, \cdots \ll [v] \overset{\text{(IDEM)}}{=} v_\oplus$. Since $[v] \overset{\text{(C+U)}}{=} [v_\oplus] \overset{\text{(IDEM)}}{=} v_\oplus$, the argument above establishes the result.]

(Cut): Since we are restricting ourselves to the finitary grounding claims, it is enough to prove the result when the application of (CUT) has a single minor premise. Suppose, then, that $w^1, w^2, \cdots \leq_1 w$ and $w, v^1 v^2, \cdots \leq_1 v$. By IH, $[[w^1.w^2.\cdots] + w_\oplus] = w_\oplus$ and $[[w.v^1.v^2.\cdots] + v_\oplus] = v_\oplus$. Then $v_\oplus \overset{\text{(supp.)}}{=} [[w.v^1.v^2.\cdots] + v_\oplus] \overset{\text{(ASSOC)}}{=} [[w.[v^1.v^2.\cdots]] + v_\oplus] \overset{\text{(SUPP.)}}{=} [[[[w^1.w^2.\cdots] + w_\oplus].[v^1_\oplus.v^2_\oplus.\cdots]] + v_\oplus] \overset{\text{(DIST.)}}{=} [[[[[w^1.w^2.\cdots].[v^1_\oplus.v^2_\oplus.\cdots]] + [w_\oplus.[v^1_\oplus.v^2_\oplus.\cdots]]]] + v_\oplus] \overset{\text{(ASSOC)}}{=} [[[w^1.w^2.\cdots.v^1.v^2.\cdots] + [w.v^1.v^2.\cdots]] + v_\oplus] \overset{\text{(ASSOC)}}{=} [[w^1.w^2.\cdots.v^1.v^2.\cdots] + [w.v^1.v^2.\cdots] + v_\oplus] \overset{\text{(ASSOC)}}{=} [[w^1.w^2.\cdots.v^1.v^2.\cdots] + [[w.v^1.v^2.\cdots] + v_\oplus]] \overset{\text{(SUPP.)}}{=} [[w^1.w^2.\cdots.v^1.v^2.\cdots] + v_\oplus]$

(Level): Suppose $((w^i_j) \ll v^i_\oplus)$ and $(v^i) \leq_1 ([v], b)$. By IH, $[[v^1.v^2.\cdots] + [v]] = [v] \overset{\text{(INVOL)}}{\Rightarrow} [[v^1.v^2.\cdots] + v_\oplus] = v_\oplus$. By the argument in the case of (BASIS), $[[w^i_1.w^i_2.\cdots] + v^i_\oplus] = v^i_\oplus$. So, by the argument in the case of (CUT), $[[w^1_1.w^1_2.\cdots.w^2_1.w^2_2.\cdots] + v_\oplus] = v_\oplus$.

We can now show that, in any A-model, the relevant disjunctive part relation is just our old friend $\leq_1$:

Theorem 5.3 $[[v^1.v^2.\cdots] + v_\oplus] = v_\oplus \Leftrightarrow v^1, v^2, \cdots \leq_1 v$.

Proof Lemma 5.2 establishes the right-to-left direction. Suppose that $[[v^1.v^2.\cdots] + v_\oplus] = v_\oplus$. Then $[v^1.v^2.\cdots] \ll v_\oplus$. $[v^1.v^2.\cdots] \ll v_\oplus \Longrightarrow ([v^1.v^2.\cdots], b) \leq_1 v$, and $v^1.v^2.\cdots \ll [v^1.v^2.\cdots] \Longrightarrow v^1.v^2.\cdots \leq_1 ([v^1.v^2.\cdots], b)$, for all b. So, by (CUT), $v^1, v^2, \cdots \leq_1 v$.

It follows immediately from Theorem 5.3 and Lemma 5.1 that we can define the other grounding relations in terms of choice and combination:

Corollary 5.4

1. $(a, b) \preceq_1 (c, d)$ *iff* $[[a.c] + c] = c$;
2. $(a, b) \prec_1 (c, d)$ *iff* $[[a.c] + c] = c$, *but* $[[c.a] + a] \neq a$; *and*
3. $(a^1, b^2), (a^2, b^2), \cdots < (c, d)$ *iff* $[[a^1.a^2.\ldots] + c] = c$ *and, for all* a^i, $[[c.a^i] + a^i] \neq a^i$.

Clearly, we can express corresponding claims using the object language expressions $\vee$, $\wedge$, and $\approx$: $\phi^1, \phi^2, \cdots \leq \psi$ iff $((\phi^1 \wedge \phi^2 \wedge \ldots) \vee \psi) \approx \psi$, $\phi \preceq \psi$ iff $(\phi \wedge \psi) \vee \psi \approx \psi$, *etc.*

Finally, a second corollary of Theorem 5.3 is that there is no difference between v's having a weak selection $w^1, w^2, \ldots$, and its having an immediate selection $[w^1.w^2.\cdots]$:

Corollary 5.5 $w^1, w^2, \cdots \leq_1 v$ *iff* $[w^1.w^2.\cdots] \ll v_\oplus$.

This corollary illustrates the extreme "flatness" of the structure of selection according to the Angellic theory of content: every selection from v is at most a single level down.

So, a single framework, employing a single conception of ground, can capture a "raised" conception of content encapsulated in GG when it is constrained by ($\leq$-MAXIMALITY), and can also capture the much "flatter" conception of content encapsulated in AC when it is constrained by (C+U), (INVOLUTION), (ASSOCIATIVITY), and (DISTRIBUTION).

It is worth dwelling on a few notable features of the results. First, of course, we have a general framework appealing to a single underlying conception of ground and yielding different logics of ground. So, the existence of these natural, but incompatible logics of ground provides no reason to doubt the cogency or univocality of a generic notion of ground, rather than indicating theoretical disputes over the individuation of content.

The cost is that the underlying conception of ground is, it seems, somewhat less natural and familiar than the alternative conceptions appropriate to GG and AC, respectively. deRosset and Fine's definition of strict ground in terms of selection (Definition 1.1), is highly natural, specifying the notion inductively by giving a very natural basis case and closing under various fairly familiar chaining operations. By contrast, the alternative definition (Definition 2.1) directly defines the relatively unfamiliar notion of weak ground. Despite the unfamiliarity of the idea, the notion is central to the logic of ground. Fine [8, pp. 52-3] contends that weak ground is more fundamental than strict ground, despite its relative unfamiliarity. New support for this contention is provided by the fact that articulating a conception of ground that covers both "raised" and "flatter" logics requires taking weak ground to be more fundamental.

It must be admitted, however, that the notion of weak ground specified in Definition 2.1 is less intuitive than both the specification of strict ground in [6] and the specification of weak ground in the "flatter" treatments [3, 11]. This feature is familiar from other efforts in mathematics to generalize an idea to cover a wider range of cases. It is standard to find that generalization requires appealing to the less familiar and intuitive of two ways of specifying the notion to be generalized. The present study demonstrates that the logic of ground fits this general pattern.

There are a large number of competing views concerning the conditions under which sentences are equivalent. A familiar intensionalist view, for instance, holds that cointensional sentences express the same fact [13]. A slightly more liberal view, *booleanism*, identifies boolean equivalents, but allows inequivalences among other cointensional claims [2]. Booleanism allows some hyperintensionality, but it allows less than the Angellic view captured by A-models. A *Dorric* view, on which only the equivalence of A and $\neg\neg A$ is required [7], allows still more hyperintensionality. We have already seen that GG requires distinctions among contents disallowed by each of these views. Finally, there is a russellian view on which no logically complex sentence is equivalent to any other logically complex sentence, unless there is a content- and structure-preserving mapping from the atomic sentences of one to the constituents of the other.[16] Obviously, many other views could be formulated.[17] We have shown that a single semantic framework, with a single conception of ground, can be adapted to yield the two denizens of this zoo for which a formal treatment of grounding claims has been suggested. It is not clear that it can be adapted to yield the others, nor is it clear what logic of ground these others might yield. But our results so far indicate reason for optimism on this score.[18]

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[16] Proponents of this russellian view face the problem posed by the Russell-Myhill paradox; see [7] for a statement and discussion.

[17] Correia [5, Section 4], for instance, provides a semantics for a view which accepts all of the Angellic equivalences but DISTRIB($\vee/\wedge$).

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