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A Sound and Complete Tableaux Calculus for Reichenbach's Quantum Mechanics Logic

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Abstract

In 1944 Hans Reichenbach developed a three-valued propositional logic (RQML) in order to account for certain causal anomalies in quantum mechanics. In this logic, the truth-value *indeterminate* is assigned to those statements describing physical phenomena that cannot be understood in causal terms. However, Reichenbach did not develop a deductive calculus for this logic. The aim of this paper is to develop such a calculus by means of First Degree Entailment logic (FDE) and to prove it sound and complete with respect to RQML semantics. In Section 1 we explain the main physical and philosophical motivations of RQML. Next, in Sections 2 and 3, respectively, we present RQML and FDE syntax and semantics and explain the relation between both logics. Section 4 introduces $\mathcal{Q}$ calculus, an FDE-based tableaux calculus for RQML. In Section 5 we prove that $\mathcal{Q}$ calculus is sound and complete with respect to RQML three-valued semantics. Finally, in Section 6 we consider some of the main advantages of $\mathcal{Q}$ calculus and we apply it to Reichenbach's analysis of causal anomalies.

Keywords Causal anomaly · First degree entailment · Indeterminate · Reichenbach's quantum mechanics logic · Three-valued logic · Semantic tableaux

1 Introduction

In 1944 Reichenbach developed a three-valued logic in order to account for causal anomalies that arise in the standard interpretation of quantum mechanics. A "causal law

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of nature", according to Reichenbach, is defined under two conditions (Reichenbach [19], 117):

a) "it is required that the cause determine the effect univocally [...];
b) it is demanded that the effect spread continuously through space, following the principle of action by contact."

The type of causality that satisfies *a*) and *b*) is called "normal". Non-compliance with *b*) constitutes a *causal anomaly*.

Reichenbach considers that the main problem of quantum mechanics can be summed up in the following question: *is there an exhaustive interpretation* —that is, one that gives value assignments to all physical quantities in all physical states (Reichenbach [21])— *of quantum mechanics that satisfies the requirements of "normal causality"?* (Reichenbach [22]). Since an exhaustive interpretation provides a value for every physical quantity in all physical states, it includes a complete description of *interphenomena*, that is, of unobservable entities that do not obey the usual classical laws –such as the relative motion of an electron (see Turquette [26]).

However, since an exhaustive interpretation assigns certain values to *interphenomena*, it leads to causal anomalies. For example, in the case of the double slit experiment (the explanation of which can be found in Albert [1], Norsen [12] or Maudlin [9]), it is possible to modify the distribution pattern of the particles that reach the screen through the first slit by changing the state of the second slit (opening or closing it), located at a distance from these particles. This is a clear example of action at a distance that implies the breach of normal causality in any exhaustive interpretation of quantum mechanics. Therefore, given the definition of "exhaustive interpretation", and since *interphenomena* intervene in quantum mechanics –which present actions at a distance–, it follows that there is no exhaustive interpretation of quantum mechanics without causal anomalies (Reichenbach [20, 22]).

One way out of the bind is to admit only statements about phenomena and to regard all statements about *interphenomena* as "nonsense". This is the interpretation that Reichenbach associates with Bohr and Heisenberg (Reichenbach [22]). Naturally, the Bohr-Heisenberg interpretation lies in moving from an exhaustive interpretation to a restrictive one in which only statements about phenomena are allowed. This implies that the causal anomalies are dissolved, precisely because the pretence of giving definite values to all physical quantities is abandoned (see Reichenbach [19]). Nevertheless, the idea that all statements about *interphenomena* are "nonsense", as Reichenbach points out, means giving up knowledge about such entities because no verification conditions are assigned to their corresponding statements (Reichenbach [19]).

On the contrary, Reichenbach's proposal consists of changing the laws of classical logic by adding a third truth-value, the indeterminate (i) (Reichenbach [19]). Any statement that expresses something about *interphenomena* must have value i. Thus, a statement will make sense if, and only if (*iff* onwards), it is true, false or indeterminate. Since i is a value analogous to the others, we substitute "nonsense" for "indeterminate" or currently "neither verified nor falsified", in such a way that statements about *interphenomena* make sense again (Reichenbach [19]; Hardegree [5]). It is not necessary to give up the statement "the particle travels through the first slit": it is possible that it has a meaning, even though its truth value is not "true", and thus the corresponding causal anomaly is suppressed (see Reichenbach [19] and Section 6 of this paper for further details).

Although the analysis of the physical and philosophical aspects in Reichenbach's proposal is not our main objective,[1] it is necessary to present the issues developed above as motivating Reichenbach's logic, since the latter attempts to explain causal anomalies without allowing meaningless statements. Reichenbach provides a semantics for this logic, including new connectives and using the method of truth-tables. However, he does not develop a strictly deductive calculus, nor, by extension, the soundness and completeness theorems for it, something which has been implicitly or explicitly criticised, for instance, by Hempel [6], Nagel [10], Feyerabend [4] or Nilson [11]. For this reason, the aim of this paper is to implement the method of semantic-tableaux for First Degree Entailment logic (FDE)[2] to Reichenbach's logic in order to obtain the corresponding tableaux-based deductive calculus ($\mathcal{Q}$) and to prove it sound and complete. In Section 2, we present the formalism of quantum logic as expounded by Reichenbach [19]. We establish the relationship between Reichenbach's quantum mechanics logic and FDE in Section 3. In Section 4, we elaborate the tableau rules for RQML that we must add to the ones for FDE. This allows us to obtain $\mathcal{Q}$ calculus and to achieve our main goal (Section 5): to develop a deductive calculus which is sound and complete with respect to RQML semantics. Finally, in Section 6 we consider some of the main advantages of $\mathcal{Q}$ calculus over natural deduction calculi[3] for Reichenbach's analysis of causal anomalies.

2 Reichenbach's Three-valued Quantum Logic

The following rule defines the concept of a *well-formed-formula* (WFF) in RQML[4]:

$$\text{WFF} \ ::= \ p \ | \ \neg A \ | \sim A \ | \ A \vee B$$

We include two negations as primitive monary connectives: $\neg$, the *classical negation*, and $\sim$, the *cyclical negation* (Reichenbach [19]). The set $\{\neg, \sim, \vee\}$ is functionally complete, as proved in Corolary 5.2 (Appendix A). We define $A \wedge B$ as $\neg(\neg A \vee \neg B)$.

Secondly, Reichenbach points out that a statement with truth value i is obviously not true (it is neither false nor true). It follows that an inference from a true statement to an indeterminate statement, which is, therefore, not true, involves moving from a statement where the property of "being true" holds to one where it does not. Consequently, inferences from true statements to indeterminate statements do not guarantee

1 See Hempel [6], Nagel [10] or Feyerabend [4] for some the most important criticisms.

2 For further references on FDE logic, see Section 3.

3 In Section 6 we consider two natural deduction systems for a logic equivalent to RQML.

4 Although Reichenbach takes every connective introduced in this section as primitive, we only take $\neg$, $\sim$ and $\vee$ as primitive, since they are sufficient to define the remaining connectives (see Appendix A, where functional completeness of the set $\{\neg, \sim, \wedge\}$ is proved).

Table 1 Truth-table for $\neg$, $\sim$ and $\overline{}$

A	$\neg A$	$\sim A$	$\overline{A}$
1	0	i	i
i	i	0	1
0	1	1	1

truth preservation, and so cannot be valid. Only 1 is a designated value[5] in RQML (Reichenbach [19]).

We now present RQML semantics by means of truth-tables. First of all, we define a new monary connective, the *complete negation* ($\overline{A}$), as follows (Reichenbach [19]): $\overline{A} =_{def} \sim A \vee \sim\sim A$. The truth-table (Table 1) corresponds to the three negations.

On the other hand, for the order $0 < i < 1$, $v(A \vee B) = max[v(A), v(B)]$ and $v(A \wedge B) = min[v(A), v(B)]$. Also, among the possible implications in RQML, Reichenbach uses the following three conditionals, all of which are defined in the following page: the *standard conditional* ($\supset$), the *alternative implication*[6] ($\rightarrow$) and the *quasi-implication* ($\rightsquigarrow$) (Reichenbach [19]). Finally, we introduce the *classical equivalence* ($\equiv$) and the *alternative equivalence* ($\cong$), also as defined symbols (Reichenbach [19]). The semantics of these connectives is given in Table 2.

Although Reichenbach does not provide a definition for $\rightarrow$, $\supset$ and $\rightsquigarrow$ (only for $\cong$ and $\equiv$) (Reichenbach [19]), we define them in the following way[7]:

$$\begin{aligned}
A \rightarrow B &=_{def} \sim \neg(\overline{A} \vee B) \\
A \rightsquigarrow B &=_{def} (A \wedge B) \vee \sim \overline{A} \\
A \cong B &=_{def} (A \rightarrow B) \wedge (B \rightarrow A) \wedge (\neg A \rightarrow \neg B) \wedge (\neg B \rightarrow \neg A) \\
A \supset B &=_{def} (\neg A \vee B) \vee (A \cong B) \\
A \equiv B &=_{def} (A \supset B) \wedge (B \supset A)
\end{aligned}$$

In the following section we introduce FDE syntax and semantics. Its relation to RQML, the logic explained in this section, will be clarified in Section 3.1.

3 First Degree Entailment

In classical propositional calculus, an interpretation is a function from formulas to the truth values 1 and 0. But this involves that every formula is either true or false –never neither, and never both. By contrast, FDE is a logic where interpretations are formulated as relations rather than functions. Thus, a formula may relate to 1; it may relate to 0; it may relate to both; or it may relate to neither. Therefore, the fact that a propositional parameter is untrue does not mean that it is false; and the fact that a

[5] Remember that a *designated* truth value is the one preserved in valid inferences.

[6] As a special feature, we remark in Appendix B that deduction theorem (Theorem 6) only holds for this conditional.

[7] Note that $\supset$ corresponds to Łukasiewicz three-valued conditional (Łukasiewicz, [7]; Łukasiewicz [8]). RQML could then be regarded as an extension of **Ł3** (however, see footnote 10 on some differences between RQML and both **Ł3** and **K3**).

Table 2 Truth-table for $\wedge$, $\vee$, $\rightarrow$, $\rightsquigarrow$, $\supset$, $\cong$ and $\equiv$

A	B	$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \rightsquigarrow B$	$A \supset B$	$A \cong B$	$A \equiv B$
1	1	1	1	1	1	1	1	1
1	i	i	1	0	i	i	0	i
1	0	0	1	0	0	0	0	0
i	1	i	1	1	i	1	0	i
i	i	i	i	1	i	1	1	1
i	0	0	i	1	i	i	0	i
0	1	0	1	1	i	1	0	0
0	i	0	i	1	i	1	0	i
0	0	0	0	1	i	1	1	1

propositional parameter is false does not mean that it is untrue (hereafter, we mainly follow Priest [18]; for further developments of FDE, Belnap-Dunn semantics and its extensions, see Routley & Routley [24], Rivieccio [23], Albuquerque, Přenosil & Rivieccio [2], Omori & Wansing [14] and Přenosil [17]).

Hence, an FDE interpretation is a relation ρ between propositional parameters and the values 1 and 0, that is, $\rho \subseteq \mathcal{P} \times \{1, 0\}$, where $\mathcal{P}$ is the set of propositional parameters. For example, $p\rho 1$ means that p relates to 1. Given an interpretation ρ, this can be extended to a relation between all formulas and truth values by the following recursive clauses (Priest [18]) –we take $\neg$ and $\vee$ as primitive connectives in FDE, and define $A \wedge B$ as $\neg(\neg A \vee \neg B)$[8]:

$A \vee B\rho 1$ *iff* $A\rho 1$ or $B\rho 1$
$A \vee B\rho 0$ *iff* $A\rho 0$ and $B\rho 0$
$\neg A\rho 1$ *iff* $A\rho 0$
$\neg A\rho 0$ *iff* $A\rho 1$

These clauses are the same as the classical truth conditions, stripped of the assumption that truth and falsity are exclusive and exhaustive.

Let us now consider the tableaux for FDE. Each entry of a tableau has the form $A, +$ or $A, -$, where $A, +$ means that A is related to 1 (A is true) and $A, -$ means that it is not. So, $\neg A, +$ means that A is related to 0 (A is false) and $\neg A, -$ that it is not. Hence, if both $A, -$ and $\neg A, -$ occur on a branch, it means that A is indeterminate (Priest [18]). To test the inference $A_1, ..., A_n \vdash_{\text{FDE}} B$, we start with an initial list of the form

$$A_1, +$$
$$\vdots$$
$$A_n, +$$
$$B, -$$

[8] While the language of FDE, as expounded by Priest, contains the connectives $\wedge$, $\vee$ and $\neg$ as primitive (Priest [18]), we define $\wedge$ in terms of $\neg$ and $\vee$.

A branch of a tableau closes if it contains nodes of the form $A, +$ and $A, -$. The basic FDE rules given by Priest [18] are as follows –although $\wedge$ is taken as a defined connective here, we introduce the corresponding tableau rules for FDE (as Priest does) for practical reasons:

Box 1 FDE BASIC RULES

$$\begin{array}{cccc} \neg\neg A, + & \neg\neg A, - & A \wedge B, + & A \vee B, - \\ \downarrow & \downarrow & \downarrow & \downarrow \\ A, + & A, - & A, + & A, - \\ & & B, + & B, - \end{array}$$

$$\begin{array}{ccc} & A \wedge B, - & \\ \swarrow & & \searrow \\ A, - & & B, - \end{array} \qquad \begin{array}{ccc} & A \vee B, + & \\ \swarrow & & \searrow \\ A, + & & B, + \end{array}$$

$$\begin{array}{cccc} \neg(A \wedge B), + & \neg(A \wedge B), - & \neg(A \vee B), + & \neg(A \vee B), - \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \neg A \vee \neg B, + & \neg A \vee \neg B, - & \neg A \wedge \neg B, + & \neg A \wedge \neg B, - \end{array}$$

3.1 First Degree Entailment and Reichenbach's Three-valued Logic

For any formula A and any interpretation ρ in FDE, there are four possibilities: A is true and not false; A is false and not true; A is both true and false; A is neither true nor false. It is then possible to think of FDE as a four-valued logic. However, consider an FDE interpretation satisfying the following constraint:

EXCLUSION: *For no* A, $A\rho 1$ *and* $A\rho 0$

No formula is both true and false in this FDE-interpretation, while a propositional parameter can still be neither true nor false. Our thesis is that the logic defined in terms of truth preservation over all interpretations satisfying this constraint could be RQML. However, RQML uses more primitive connectives than FDE, so we cannot say that RQML semantics strictly corresponds to an FDE interpretation satisfying the previous constraint. In fact, $\sim A$ is not definable in terms of $\neg$ and $\vee$, given that they preserve value i. Hence, since RQML connectives must also be interpretable in terms of FDE relational semantics, we introduce cyclical negation as a primitive symbol in FDE and give its corresponding FDE interpretation:

$\sim A\rho 1$ *iff* $\neg A\rho 1$ *iff* $A\rho 0$
$\sim A\rho 0$ *iff* $A\not\rho 1$ and $\neg A\not\rho 1$ *iff* $A\not\rho 1$ and $A\not\rho 0$

Since both logics now share the same syntax it is possible to use FDE formalism to give a calculus for RQML (remember that the definitions of the RQML conditionals and equivalences, that is, $\rightarrow$, $\rightsquigarrow$, $\supset$, $\equiv$ and $\cong$, are in Section 2). Onwards, we will write FDE* for FDE satisfying the exclusion constraint and including $\neg$, $\sim$ and $\vee$ as primitive constants.

As we have seen in Section 4, RQML is a three-valued logic in which, for any given formula A, there are three possibilities: $v(A) = 1$ (A is true); $v(A) = 0$ (A is false); $v(A) = i$ (A is indeterminate). These are the same possibilities as in FDE*: $A\rho 1$ (A is true); $A\rho 0$ (A is false); $A\not\rho 1$ and $A\not\rho 0$ (A is indeterminate). In fact, in RQML, as in FDE*, there can be *gaps*, but not *gluts*.[9]

Now, we can use the FDE-tableaux formalism ($A, +; A, -; \neg A, +; \neg A, -$) to make a proper calculus for RQML. We must pay attention that the exclusion constraint adds a closing rule: a branch of a tableau also closes if it contains nodes of the form $A, +$ and $\neg A, +$. Given the new connectives, in Section 4 we will introduce new closing and tableau rules for RQML in addition to the rules already introduced for FDE in this section (Box 1).

We first prove Theorem 1, which allows to connect FDE* relational semantics with RQML truth-functional semantics, but previously we give the following definition:

Definition 1 We write $\vDash_{\text{FDE*}}$ for validity and logical consequence according to FDE* semantics and $\vDash_{\text{RQML}}$ for validity and logical consequence according to RQML semantics, and define $\vDash_{\text{FDE*}}$ and $\vDash_{\text{RQML}}$ as follows:

a) $\Sigma \vDash_{\text{FDE*}} A$ *iff*, if $B\rho 1$ for all $B \in \Sigma$, then $A\rho 1$
b) $\Sigma \vDash_{\text{RQML}} A$ *iff*, if $v(B) = 1$ for all $B \in \Sigma$, then $v(A) = 1$

Theorem 1 [10] $\Sigma \vDash_{FDE*} A$ *iff* $\Sigma \vDash_{RQML} A$

Proof From left to right, suppose $\Sigma \vDash_{\text{FDE*}} A$. Hence, for an FDE* interpretation ρ, if $B\rho 1$ for every $B \in \Sigma$, then $A\rho 1$. But since

$A\rho 1$ in FDE* *iff* $v(A) = 1$ in RQML
$A\not\rho 1$ and $A\not\rho 0$ in FDE* *iff* $v(A) = i$ in RQML
$A\rho 0$ in FDE* *iff* $v(A) = 0$ in RQML

whenever $v(B) = 1$ for every $B \in \Sigma$, then $v(A) = 1$ in RQML, which means that $\Sigma \vDash_{\text{RQML}} A$. Similarly, from right to left. □

The above theorem depends on the introduction of $\sim$ (together with $\neg$ and $\vee$) as a primitive connective in FDE* and the definition of $\overline{}$, $\supset$, $\rightsquigarrow$, $\rightarrow$, $\equiv$ and $\cong$ in terms of the primitive connectives –it is clear that if FDE* and RQML did not share the same language, the relationship stipulated in Theorem 1 would not hold. The aforementioned theorem will play an important role in proving $\mathcal{Q}$ sound and complete with respect to RQML semantics in Section 5.

[9] In other words, it can be the case that A is neither true nor false, but not both (Priest [18]).

[10] A similar proof of this theorem can be found in Section 3 of (Estrada-Gonzales & Cano-Jorge, [3]), in which Dunn's semantics is applied to Reichenbach's connectives to obtain a relational semantics which excludes the subset {0, 1} of the set of truth values {0, 1} (this is equivalent to our Exclusion constraint). Although both our proposal and that of the above paper use the same relational semantics, in order to define FDE* we take only ¬, ∼ and ∨ as primitive connectives and define the remaining new connectives. By contrast, Estrada-Gonzales & Cano-Jorge take Reichenbach's new negations and conditionals as primitive connectives, and regard Reichenbach's logic as an expansion of the fragment {¬, ∨, ∧} of Kleene's **K3** logic by means of five new connectives. Note, however, that our set {¬, ∼, ∨} of primitive connectives is functionally complete (see Appendix A), while that of the primitive connectives of **K3** is not. This suffices to state that **K3** is not equivalent to RQML. The same applies to **Ł3**, containing ¬, ∧, ∨ and ⊃ as primitive connectives, which do not form a functionally complete set. Moreover, ∼ is not definable with the connectives of **Ł3**.

Table 3 Truth-table for $\neg\sim A$

A	$\sim A$	$\neg\sim A$
1	i	i
i	0	1
0	1	0

4 $\mathcal{Q}$ Calculus: an FDE*–based Calculus for RQML

In order to develop the table rules for $\mathcal{Q}$ calculus, in this section we proceed in the following way. Firstly, according to RQML semantics, we elaborate the truth table corresponding to every connective. Next, we consider the cases in which the corresponding formula is true and the cases in which it is not: in both cases, this will allow us to check the truth-values to which the propositional parameters that are included in the formula relates. Finally, with this result, we develop the FDE* rules for the cases in which the formula relates to 1 and those in which it does not. For instance, to get the rule for $\neg\sim A, +$, we first construct the table (Table 3). We check that when $\neg\sim A$ is true, A is i. Hence, after applying the rule for $\neg\sim A, +$, the nodes $A, -$ and $\neg A, -$ must be on the extended branch. Similarly for $\neg\sim A, -$. Note that in the latter case, when $\neg\sim A$ is not true, there are two possibilities: A is true or false. Therefore, we must add two branches, one of them with $A, +$, and the other with $\neg A, +$, as shown in Box 4 below.

We add the rules in Box 2 to 6 to the ones in Box 1 to develop $\mathcal{Q}$ calculus. First, the closing rules for FDE* are the following:

Box 2 CLOSING RULES

A branch closes whenever one of these formulas or pairs of formulas appears on it:

(1) $A, +$ and $A, -$

(2) $A, +$ and $\neg A, +$

(3) $\sim \overline{A}, +$

(4) $\neg\overline{A}, +$

(1) is contradictory both in FDE with and without exclusion. Also, it immediately follows from the exclusion constraint for FDE* that **(2)** entails a contradiction. On the other hand, the reason why a branch closes if $\sim \overline{A}, +$ or $\neg\overline{A}, +$ occurs on it (rules **(3)** and **(4)**, respectively) is the following: the truth-table for $\sim \overline{A}, +$ and $\neg\overline{A}, +$ (Table 4) show that they can never be true, so none of them relates to 1 in an FDE* interpretation.

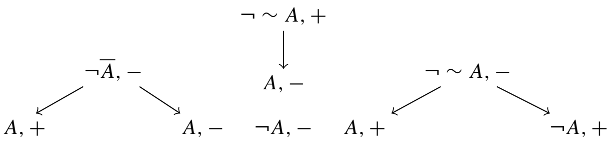
Table 4 Truth-table for $\sim \overline{A}$ and $\neg\overline{A}$

A	$\neg A$	$\sim A$	$\overline{A}$	$\sim \overline{A}$	$\neg\overline{A}$
1	0	i	i	0	i
i	i	0	1	i	0
0	1	1	1	i	0

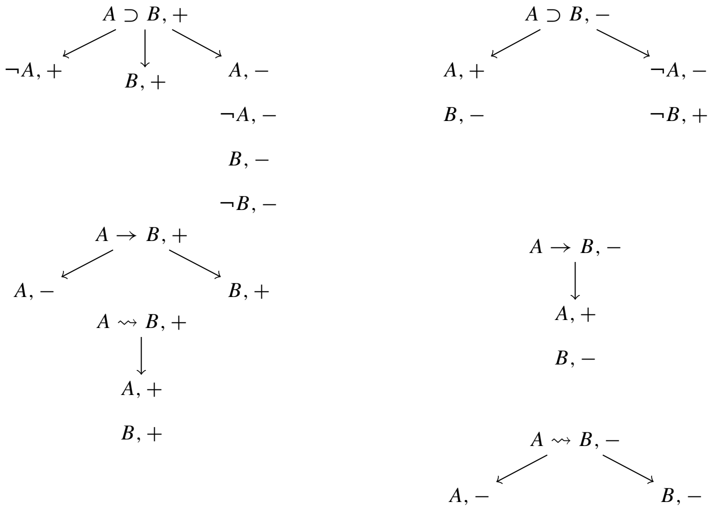
Since the rules for $\wedge$ in Box 1, the ones for $\overline{}$ in Box 3 and all the rules in Box 5 and Box 6 correspond to defined connectives (see Section 2), they all could be omitted. However, we include them in $\mathcal{Q}$ for practical reasons.

Box 3 NON-CLASSICAL NEGATIONS ELIMINATION

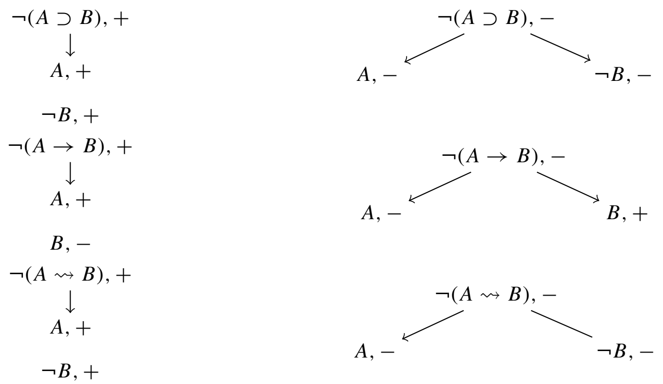
Box 4 COMBINED NEGATIONS ELIMINATION



Box 5 CONDITIONALS ELIMINATION



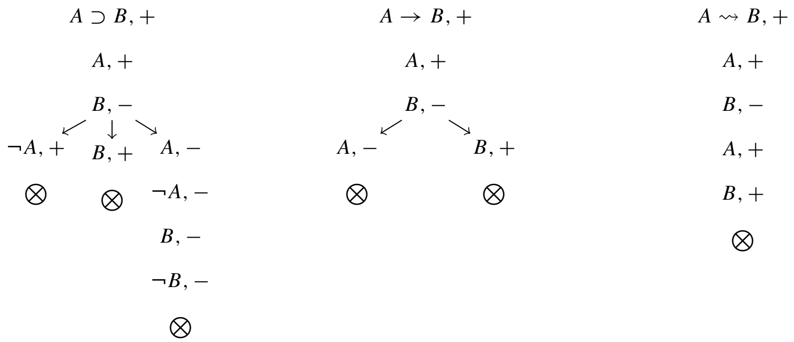
Box 6 NEGATED CONDITIONALS ELIMINATION



Onwards, we write $\vdash_{\mathcal{Q}}$ for deduction by means of $\mathcal{Q}$ calculus.

4.1 Examples of Valid Inferences in RQML and Deductions in $\mathcal{Q}$

Firstly, *Modus ponens* is a valid inference in RQML (whatever the conditional used) as we can check in Table 2. We also prove that (1) $A, A \to B \vdash_{\mathcal{Q}} B$ (2) $A, A \rightsquigarrow B \vdash_{\mathcal{Q}} B$ and (3) $A, A \supset B \vdash_{\mathcal{Q}} B$:



As expected, *tertium non datur* does not generally hold in RQML, except for the form $A \vee \overline{A}$[11] (see Table 5). We now prove that $\vdash_{\mathcal{Q}} A \vee \overline{A}$:

[11] The validity of this last formula allows us to define a *pseudo tertium non datur* (Reichenbach [19]). The formula $A \vee \overline{A}$ does not have the classical properties of excluded middle, since, by analogy, complete negation also does not have the genuine properties of classical negation. It does not allow us, for instance, to infer the truth value of A if we know the truth value of $\overline{A}$.

Table 5 Tertium non datur

A	$\neg A$	$\sim A$	$\overline{A}$	$A \vee \neg A$	$A \vee \sim A$	$A \vee \overline{A}$
1	0	i	i	1	1	1
i	i	0	1	i	i	1
0	1	1	1	1	1	1

$$
\begin{array}{c}
A \vee \overline{A}, - \\
A, - \\
\overline{A}, - \\
A, + \\
\otimes
\end{array}
$$

We have already checked that the *tertium non datur* is not valid in RQML because there is a *tertium*, an intermediate value between 0 and 1. However, the formula $A \vee \sim A \vee \sim\sim A$ is a valid formula in RQML (Table 6) and a theorem of our calculus. So, we have something like a *quartum non datur* (see Reichenbach [19] for further details), which is also a theorem of $\mathcal{Q}$:

$$
\begin{array}{ccc}
 & (A \vee \sim A) \vee \sim\sim A, - & \\
 & A \vee \sim A, - & \\
 & \sim\sim A, - & \\
 & A, - & \\
 & \sim A, - & \\
 & \neg A, - & \\
 & \neg \sim A, - & \\
\swarrow & & \searrow \\
A, + & & \neg A, + \\
\otimes & & \otimes
\end{array}
$$

The *law of non-contradiction* (LNC) holds in RQML in three forms (Reichenbach [19]), as shown in Table 7.

Table 6 Quartum non datur

A	$\sim A$	$\sim\sim A$	$A \vee \sim A \vee \sim\sim A$
1	i	0	1
i	0	1	1
0	1	i	1

Table 7 Law of non-contradiction

A	$\neg A$	$\sim A$	$\overline{A}$	$A \wedge \neg A$	$A \wedge \sim A$	$A \wedge \overline{A}$	$\overline{(A \wedge \neg A)}$	$\overline{(A \wedge \sim A)}$	$\overline{(A \wedge \overline{A})}$
1	0	i	i	0	i	i	1	1	1
i	i	0	1	i	0	i	1	1	1
0	1	1	1	0	0	0	1	1	1

The three last formulas in Table 7 are theorems of $\mathcal{Q}$ calculus:

$$\begin{array}{ccc} \overline{A \wedge \sim A}, - & \overline{A \wedge \neg A}, - & \overline{A \wedge \overline{A}}, - \\ A \wedge \sim A, + & A \wedge \neg A, + & A \wedge \overline{A}, + \\ A, + & A, + & A, + \\ \sim A, + & \neg A, + & \overline{A}, + \\ \neg A, + & \otimes & A, - \\ \otimes & & \otimes \end{array}$$

However, $\neg(A \wedge \neg A)$, the classical LNC, is neither valid in RQML nor a theorem in $\mathcal{Q}$.

The *principle of explosion* holds in RQML (Table 8). Therefore, $A \wedge \neg A \vDash_{\text{RQML}} B$, $A \wedge \sim A \vDash_{\text{RQML}} B$ and $A \wedge \overline{A} \vDash_{\text{RQML}} B$ are valid inferences (note that the premises never take the value 1). The inferences above hold in $\mathcal{Q}$, so $A \wedge \neg A \vdash_{\mathcal{Q}} B$, $A \wedge \sim A \vdash_{\mathcal{Q}} B$ and $A \wedge \overline{A} \vdash_{\mathcal{Q}} B$:

$$\begin{array}{ccc} A \wedge \neg A, + & A \wedge \sim A, + & A \wedge \overline{A}, + \\ B, - & B, - & B, - \\ A, + & A, + & A, + \\ \neg A, + & \sim A, + & \overline{A}, + \\ \otimes & \neg A, + & A, - \\ & \otimes & \otimes \end{array}$$

The distribution of $\wedge$ over $\vee$ holds in RQML. In fact,

$$\vDash_{\text{RQML}} [A \wedge (B \vee C)] \equiv [(A \wedge B) \vee (A \wedge C)]$$

and

$$\vDash_{\text{RQML}} [A \wedge (B \vee C)] \cong [(A \wedge B) \vee (A \wedge C)]$$

Table 8 Principle of explosion

A	$\neg A$	$\sim A$	$\overline{A}$	$A \wedge \neg A$	$A \wedge \sim A$	$A \wedge \overline{A}$
1	0	i	i	0	i	i
1	0	i	i	0	i	i
1	0	i	i	0	i	i
i	i	0	1	i	0	i
i	i	0	1	i	0	i
i	i	0	1	i	0	i
0	1	1	1	0	0	0
0	1	1	1	0	0	0
0	1	1	1	0	0	0

as one can check by observing the truth-tables for $\equiv$ and $\cong$: since $v(A \equiv B) = v(A \cong B) = 1$ *iff* $v(A) = v(B)$, the formulas $A \wedge (B \vee C)$ and $(A \wedge B) \vee (A \wedge C)$ are equivalent. Then, $A \wedge (B \vee C) \vdash_{\mathcal{Q}} (A \wedge B) \vee (A \wedge C)$.

Finally, *reductio ad absurdum* holds in RQML only in the two following forms (Table 9): $\vDash_{\text{RQML}} (A \to \overline{A}) \to \overline{A}$ and $\vDash_{\text{RQML}} (A \supset \overline{A}) \supset \overline{A}$. We now prove that $\vdash_{\mathcal{Q}} (A \to \overline{A}) \to \overline{A}$ and $\vdash_{\mathcal{Q}} (A \supset \overline{A}) \supset \overline{A}$:

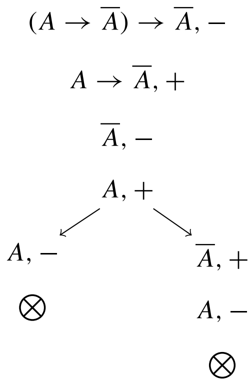
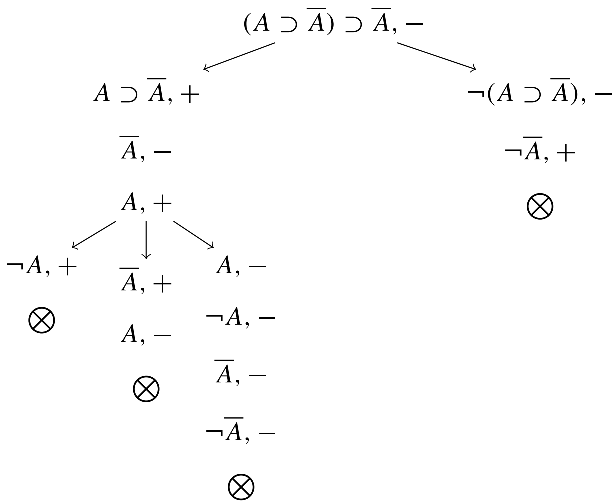
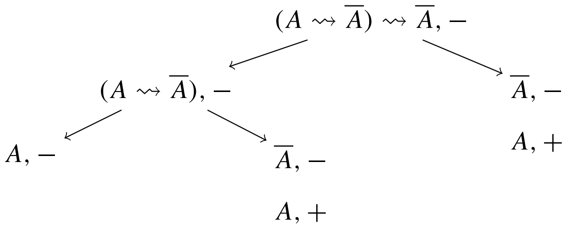


Table 9 Reductio ad absurdum for $\to$ and $\supset$

A	$\overline{A}$	$A \supset \overline{A}$	$A \to \overline{A}$	$(A \to \overline{A}) \to \overline{A}$	$(A \supset \overline{A}) \supset \overline{A}$
1	i	i	0	1	1
i	1	1	1	1	1
0	1	1	1	1	1



However, $\nvDash_{\mathcal{Q}} (A \rightsquigarrow \overline{A}) \rightsquigarrow \overline{A}$ and $\nvdash_{\mathcal{Q}} (A \rightsquigarrow \overline{A}) \rightsquigarrow \overline{A}$ (Table 10):



5 Soundness and Completeness of $\mathcal{Q}$ Calculus

In this section we mainly use the proof method given by Priest for FDE tableaux (Priest [18]) to show that $\mathcal{Q}$ is sound and complete with respect to both FDE* semantics and Reichenbach's three-valued semantics.

Definition 2 If ρ is a relational interpretation and b is a branch of a tableau, we say ρ is faithful to b *iff* for every node $A, +$ on b, $A\rho 1$, and for every node $A, -$ on b, $A \not\rho 1$.

Table 10 Reductio ad absurdum for $\rightsquigarrow$

A	$\overline{A}$	$A \rightsquigarrow \overline{A}$	$(A \rightsquigarrow \overline{A}) \rightsquigarrow \overline{A}$
1	i	i	i
i	1	i	i
0	1	i	i

Lemma 1 *If ρ is faithful to a branch b of a tableau and a tableau rule is applied to b, then ρ is faithful to at least one of the branches generated.*

Proof The proof is by a case-by-case examination of the tableau rules. For example, suppose $\sim A, +$ is on the branch and we apply the corresponding rule: since ρ is faithful to the branch b, $\sim A\rho 1$, then $\neg A\rho 1$, so ρ is faithful to the extended branch. Now suppose $\sim A, -$ is on the branch. Suppose we apply the corresponding rule: since ρ is faithful to the branch, $\sim A\not\rho 1$, then $\neg A\not\rho 1$, so ρ is faithful to the extended branch. Now, assume $\overline{A}, +$ is on b and we apply the corresponding rule. Since ρ is faithful to the branch, $\overline{A}\rho 1$. Then, $A\not\rho 1$, so ρ is faithful to the extended branch. Next, assume $\neg\sim A, +$ is on the branch. Since ρ is faithful to the branch, $\neg\sim A\rho 1$. Now suppose we apply the rule for $\neg\sim A, +$: then, $A, -$ and $\neg A, -$ will be on the extended branch, so, since ρ is faithful to $\neg\sim A, +$, both $A\not\rho 1$ and $A\not\rho 0$. Therefore, ρ is faithful to the extended branch. Finally, assume $A \rightarrow B, +$ is on the branch. Since ρ is faithful to the branch, $A \rightarrow B\rho 1$. Assume we apply the rule for $A \rightarrow B, +$: then, either $A\not\rho 1$ or $B\rho 1$. Hence, ρ is faithful to either the left branch or the right branch. Similarly for the other tableaux rules. □

Lemma 2 *Every tableau in $\mathcal{Q}$ can be completed*[12] *after a finite number of steps.*

Proof First, note that every tableau rule, when applied to a formula, generates a finite extension of the corresponding branch. Second, $\mathcal{Q}$ rules eventually allow us to decompose any complex formula into atomic formulas or classical negations ($\neg$) of atomic formulas, with its corresponding $+$ or $-$. Therefore, every tableau is completed after a finite number of steps. □

Theorem 2 (Soundness) *For finite Σ, if $\Sigma \vdash_{\mathcal{Q}} A$ then $\Sigma \vDash_{FDE^*} A$.*

Proof Suppose that $\Sigma \nvDash_{\text{FDE}^*} A$. Then there is a interpretation, ρ, such that $B\rho 1$ for every $B \in \Sigma$ but $A\not\rho 1$. Now consider a completed tableau for the inference. It follows from Lemma 1 that ρ is faithful to, at least, one branch, b, of the completed tableau. Now if b were closed, it would contain one of the formulas or pair of formulas that we can find in the closing rules (Box 2). But this is impossible since ρ is faithful to the branch, according to Lemma 1 (otherwise, ρ would not be faithful to any generated branch, contrary to the assumption). Hence, the tableau is open, i.e., $\Sigma \nvdash_{\mathcal{Q}} A$. Therefore, if $\Sigma \vdash_{\mathcal{Q}} A$, then $\Sigma \vDash_{\text{FDE}^*} A$. □

Definition 3 Let b be an open branch of a tableau. The interpretation induced by b is the interpretation, ρ, such that for every propositional parameter, A:

$A\rho 1$ *iff* $A, +$ occurs on b
$A\rho 0$ *iff* $\neg A, +$ occurs on b

12 A branch of a tableau is *completed* (regardless of whether it is open or closed) *iff* no further rules can be applied to it. A table is *completed iff* all its branches are completed.

Lemma 3 *Let b be an open completed branch of a tableau. Let A be any formula and ρ the interpretation induced by b. Then:*

if $A, +$ occurs on b, then $A\rho 1$
if $A, -$ occurs on b, then $A\not\rho 1$
if $\neg A, +$ occurs on b, then $A\rho 0$
if $\neg A, -$ occurs on b, then $A\not\rho 0$

Proof The proof is by induction on the complexity of A. Suppose A is a propositional parameter: if $A, +$ occurs on b, then $A\rho 1$ by definition, and if $A, -$ occurs on b, then $A, +$ does not occur on b, since it is open. Hence, by definition, it is not the case that $A\rho 1$, so $A\not\rho 1$. The cases for 0 are similar. Suppose $\sim A, +$ occurs on the branch b. Then $\neg A, +$ occurs on b. By induction hypothesis, $A\rho 0$. Hence, $\sim A\rho 1$, as required. Now assume $\sim A, -$ occurs on the branch. Then $\neg A, -$ is on b. By induction hypothesis, $\neg A\not\rho 1$, so $\sim A\not\rho 1$, as required. For $\overline{A}$: if $\overline{A}, +$ occurs on b, then $A, -$ occurs on b. By induction hypothesis, $A\not\rho 1$. Therefore, $\overline{A}\rho 1$, as required. Now suppose $\neg \sim A, +$ occurs on b. Then, $A, -$ and $\neg A, -$ occur on b. By induction hypothesis, $A\not\rho 1$ and $A\not\rho 0$. Hence, $\neg \sim A\rho 1$. Now assume $A \rightarrow B, +$ occurs on b. Then, $A, -$ or $B, +$ occur in b. By induction hypothesis, $A\not\rho 1$ or $B\rho 1$. Therefore, $A \rightarrow B\rho 1$, as required. Similarly for the others rules. □

Theorem 3 (Completeness) *For finite Σ, if $\Sigma \vDash_{FDE*} A$ then $\Sigma \vdash_{\mathcal{Q}} A$.*

Proof Suppose that $\Sigma \nvdash_{\mathcal{Q}} A$. Consider a completed open tableau for the inference, and choose an open branch. The interpretation that the branch induces makes all the members of Σ true, and A false, by Lemma 3. Hence, $\Sigma \nvDash_{\text{FDE*}} A$. Therefore, if $\Sigma \vDash_{\text{FDE*}} A$, then $\Sigma \vdash_{\mathcal{Q}} A$. □

Corollary 3.1 (Consistency) *$\mathcal{Q}$ calculus is consistent, that is, $\nvdash_{\mathcal{Q}} A \wedge \neg A$, $\nvdash_{\mathcal{Q}} A \wedge \sim A$ and $\nvdash_{\mathcal{Q}} A \wedge \overline{A}$ for any formula A.*

Proof Suppose the opposite, i. e., that there exist some formula A such that either $\vdash_{\mathcal{Q}} A \wedge \neg A$, or $\vdash_{\mathcal{Q}} A \wedge \sim A$ or $\vdash_{\mathcal{Q}} A \wedge \overline{A}$. Hence, by Theorem 2, either $\vDash_{\text{FDE*}} A \wedge \neg A$, or $\vDash_{\text{FDE*}} A \wedge \sim A$ or $\vDash_{\text{FDE*}} A \wedge \overline{A}$. But none of these can be the case, since if $A\rho 1$, then $\neg A\not\rho 1$, $\sim A\not\rho 1$ and $\overline{A}\not\rho 1$ in FDE*, so none of the three conjunctions above can be valid (and therefore cannot be a theorem, by Theorem 2). Hence, $\mathcal{Q}$ is consistent.[13] □

Theorem 4 (Main Theorem) *$\mathcal{Q}$ calculus is sound and complete with respect to Reichenbach's three-valued semantics, that is, for finite Σ, $\Sigma \vDash_{RQML} A$ iff $\Sigma \vdash_{\mathcal{Q}} A$.*

Proof It follows directly from Theorems 1, 2, and 3. □

[13] See Post [16], where the consistency of the propositional axiomatic system in *Principia Mathematica* is proved by means of a similar method.

6 Logico-philosophical Advantages of $\mathcal{Q}$ Calculus

As explained in the introduction, one of the main goals of RQML is to provide an exhaustive interpretation[14] of quantum mechanics which allows to avoid causal anomalies that arise, for instance, in the double slit experiment. Having shown that $\mathcal{Q}$ calculus is sound and complete with respect to RQML semantics, we will now show how this calculus can be applied to Reichenbach's solution to causal anomalies. Furthermore, we will compare $\mathcal{Q}$ with other calculi developed for many-valued logics and present some main formal advantages of $\mathcal{Q}$.

First, we offer a brief outline of Reichenbach's proposal to avoid the causal anomaly that arises in the classical interpretation of the double slit experiment –a detailed explanation of which can be found in (Estrada-González & Cano-Jorge [3]). For a system of two slits, let B_1 and B_2 respectively state that a given particle passes through slits 1 and 2, and let C represent the causal anomaly that arises in the classical interpretation of the double slit experiment. Now, let $B_1 \cong \neg B_2$ express that the particle passes through exactly one of the slits, and let $(B_1 \vee B_2) \rightarrow C$ express that, if the particle is known to have passed through one of the slits, the causal anomaly obtains.[15] As can be easily checked, C follows from the premises above if only classical inputs are considered (in this case, if we assume that $B_1 \cong \neg B_2$, then $B_1 \vee B_2$ is necessarily true even if no observation is made on the slits). Nevertheless, in RQML it is possible that $v(B_1 \cong \neg B_2) = v((B_1 \vee B_2) \rightarrow C) = 1$ with $v(B_1) = v(B_2) = i$ and either $v(C) = i$ or $v(C) = 0$, so $B_1 \cong \neg B_2$, $(B_1 \vee B_2) \rightarrow C \nvDash_{\text{RQML}} C$.[16]

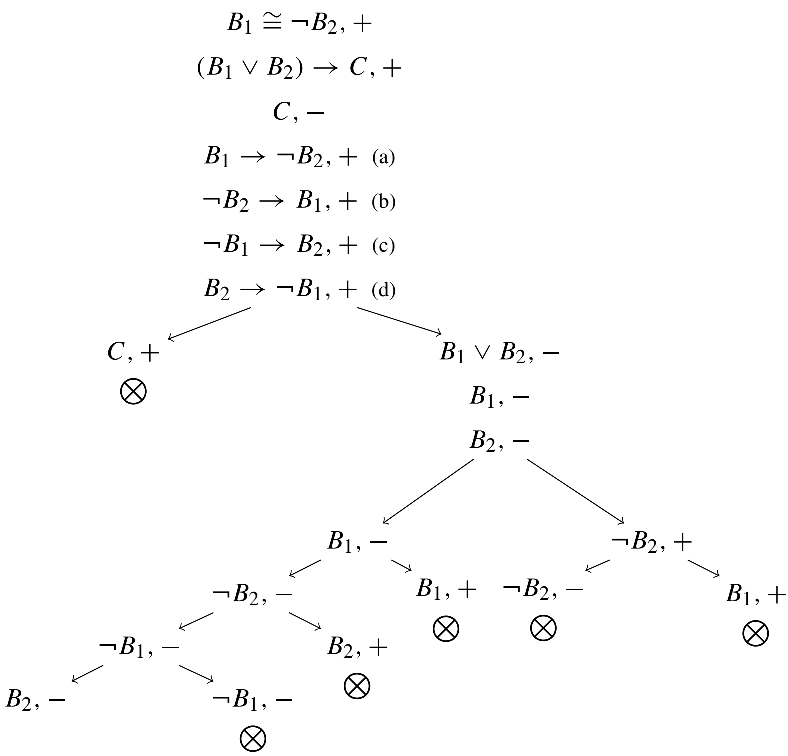
It can be checked that the above solution makes use of the meaning of i as "neither verified nor falsifies", and which is applied to both B_1 and B_2 in order to state that it is not known whether or not the particle has passed through any of the slits. Through $\mathcal{Q}$ calculus, however, the proof that the causal anomaly does not follow in RQML can be deductively obtained by means of the following proof[17] (from which the above countermodels can be easily inferred):

[14] In the sense already indicated in the introduction.

[15] A classical interpretation of the double-slit experiment implies that, if something is observed on the detection screen, then either B_1 is true or B_2 is true.

[16] Roughly, this means that, even if something is observed on the detection screen, it does not follow that either B_1 or B_2 is true, since both B_1 and B_2 could be indeterminate, as indicated in the counter-model above. In this case, since neither B_1 nor B_2 can be asserted, neither can it be asserted that the state of slit 2 affects the passage of the particle through slit 1, simply because we cannot affirm that the particle has passed through slit 1.

[17] Steps (a), (b), (c) and (d) follow from the definition of $\cong$. In steps (c) and (d) we would obtain $\neg B_1 \rightarrow \neg\neg B_2, +$ and $\neg\neg B_2 \rightarrow \neg B_1, +$, although we have directly eliminated the double negations.



As expected, from the only open branch we can read off the two following countermodels (by Theorem 1):

$$v(B_1) = v(B_2) = i \qquad v(B_1) = v(B_2) = i$$
$$v(C) = i \qquad v(C) = 0$$

We now turn to the formal advantages of this analysis of Reichenbach's solution. With regard to the development of deductive calculi for many-valued logics, although no tableaux-based calculus has yet been developed for RQML, in (Omori & Wansing [15]) two natural deduction systems are provided for a logic (called $\mathbf{K3}^4$ by the authors[18]) equivalent to RQML. The systems in question are provided in definitions 13 and 17 of the above-mentioned paper, and they are proved to be sound and complete with respect to the three-valued semantics in Theorem 27. Nevertheless, it must be noted that, contrary to tableaux-based calculus, natural deduction systems neither allow to sintactically prove that

$$B_1 \cong \neg B_2, (B_1 \vee B_2) \rightarrow C \nvdash C$$

nor to infer a countermodel, which would provide the situation in which the causal anomaly does not obtain. By contrast, $\mathcal{Q}$ calculus allows both to prove that the inference

[18] $\mathbf{K3}^4$ is a variation of Kleene's three-valued logic $\mathbf{K3}$ which adds the negation $\sim_4$ (equivalent to Reichenbach's $\sim$) as a primitive connective.

from $B_1 \cong \neg B_2$ and $(B_1 \vee B_2) \rightarrow C$ to C is not valid[19] and to read off from the open branch of the tableau a counter-model corresponding to Reichenbach's solution to the causal anomaly that arises in the double slit experiment (as shown on the proof above). Hence, $\mathcal{Q}$ calculus allows to obtain a proper counter-model (showing the expected physical situation, in which no observation is made on the slits and the causal anomaly does not obtain) which cannot be inferred from a natural deduction calculus.

7 Conclusion

In this paper we have developed an FDE–based tableaux calculus ($\mathcal{Q}$) for RQML and proved it sound and complete with respect to Reichenbach's three-valued semantics. Theorem 1 (Section 3) relates FDE* and RQML making it possible to develop $\mathcal{Q}$ calculus. Precisely, this theorem allows to move from FDE* relational semantics to RQML truth-functional semantics, and vice versa –otherwise, it would not have been possible to prove that $\mathcal{Q}$ (an FDE–based calculus) is sound and complete *with respect to* RQML semantics.

Theorems 2 and 3 (Section 5), together with Theorem 1 (Section 3), allow to prove Theorem 4 (Section 5), the main objective of our paper. This theorem emphasises the relationship between RQML, FDE* and $\mathcal{Q}$ calculus, and converts any consequence relation $\Sigma \vDash_{\text{RQML}} A$ in RQML into a deduction $\Sigma \vdash_{\mathcal{Q}} A$ in $\mathcal{Q}$, and vice versa. This makes it unnecessary to refer to FDE* relational semantics. $\mathcal{Q}$ calculus and Reichenbach's semantics are, therefore, sufficient to provide a sound and complete quantum logic system.

Appendix A

Definition 4 A set S of truth-functional connectives is functionally complete in an n-valued logic L (for finite n) *iff* any truth-function f in L different from the ones in S can be defined in terms of S (Theorem 5 provides a functional completeness criterion).

Theorem 5 [20] *If* $A = \{0, i, 1\}$, *with order* $0 < i < 1$, *the set containing the monadic functions*

$$f_{000}, f_{iii}, f_{111}, f_{100}, f_{010}, f_{001}$$

and the binary functions $\wedge$ *and* $\vee$ *is functionally complete.*

Proof Any n-ary connective g defined over $A = \{1, i, 0\}$ can be expressed by means of the following normal form:

$$g(x_1, ..., x_n) = \bigvee_{(a_1,...,a_n) \in A^n} [g(a_1, ..., a_n) \wedge \bigwedge_{j \in A} L_{a_j}(x_j)]$$

[19] It is possible to know precisely when a tableaux is completed (see footnote 12).

[20] We are grateful to Prof. José Pedro Úbeda Rives for his personal communication of Theorem 5 –see also Úbeda [27] for more information on this topic. Corollary 5.2 below easily follows from Theorem 5.

(for $a_k \in A$), where $L_1 = f_{100}$, $L_i = f_{010}$ and $L_0 = f_{001}$ (f_{jkm} –for $j, k, m \in \{0, i, 1\}$– represents the unary connective h such that $h(1) = j$, $h(i) = k$ and $h(0) = m$), and $g(a_1, ..., a_n)$ for any $a_1, ..., a_n$ is 0, i or 1, that can be obtained from f_{111}, f_{iii} and f_{000} applied to any variable $x_1, ..., x_n$. □

Remark 5.1 Another general criterion of functional completeness for many-valued logics is provided by Słupecki in (Słupecki [25]). Also, see Theorem 3.4 in (Omori & Sano [13]) for further developments on Słupecki's criterion for functional completeness. Moreover, in (Omori & Wansing [15]), functional completeness of $\mathbf{K3}^4$ (equivalent to RQML) is given in Theorem 39, and the functional completeness of $\{\sim, \vee\}$ is given in (Post [16]) –see footnote 21.

Corollary 5.2 *The set* $\{\neg, \sim, \vee\}$ *of the primitive connectives in RQML is functionally complete.*

Proof The function $\vee$ is a primitive connective in RQML (see Section 2), while $\wedge$ is definable in terms of $\neg$ and $\vee$; also, the six monary functions in Theorem 5 can be defined in terms of $\neg$, $\sim$ and $\vee$:

$$
\begin{aligned}
f_{111} &= A \vee (\sim A \vee \sim\sim A) \\
f_{iii} &= \sim f_{111} \\
f_{000} &= \neg f_{111} \\
f_{100} &= \neg \sim\sim [\sim (\sim A \vee \sim\sim A) \vee \sim\sim (\sim A \vee \sim\sim A)] \\
f_{010} &= \neg \sim \neg(A \vee \sim A) \\
f_{001} &= \neg(A \vee \neg \sim A)
\end{aligned}
$$

Since $\vee$ is a primitive connective in RQML and, on the other hand, the truth-functions $\wedge$, f_{000}, f_{iii}, f_{111}, f_{100}, f_{010} and f_{001} are definable in terms of our primitive connectives, the set $\{\neg, \sim, \vee\}$ is functionally complete in RQML.[21] Naturally, it follows that the ten connectives taken as primitives by Reichenbach also form a functionally complete set. □

Appendix B

Theorem 6 (Deduction Theorem) *If* $\Sigma \vdash_Q A$*, where* $\Sigma = \{B_1, ... B_n\}$*, then*

$$\vdash_Q \bigwedge_{i \leq n} B_i \rightarrow A$$

Proof First, suppose $\Sigma \vdash_Q A$. This corresponds to a closed tableau of type

[21] Yet, since $\neg$ can be defined as $\sim [\sim (\sim\sim A \vee A) \vee \sim\sim (\sim A \vee A)]$, the set $\{\neg, \sim, \vee\}$ is not independent. However, we include $\neg$ as a primitive connective, given as it is commonly taken as an element of the formalism used in the calculus for FDE and FDE*. A proof that $\{\sim, \vee\}$ is functionally complete was given by Post [16]. Given that RQML and Post's three-valued logic with $\sim$ and $\vee$ as primitive connectives (hereafter **P3**) are both functionally complete three-valued logics and take 1 as the only designated value, then $\Sigma \vDash_{\text{RQML}} A$ *iff* $\Sigma \vDash_{\mathbf{P3}} A$ (where Σ can be empty). This entails that RQML and **P3** are equivalent.

$$B_1$$
$$\vdots$$
$$B_i$$
$$A, -$$
$$\vdots$$
$$\otimes$$

Now consider a tableau starting with $\bigwedge_{i \leq n} B_i \rightarrow A, -$. Given that $\Sigma \vdash_{\mathcal{Q}} A$, the tableau

$$\bigwedge_{i \leq n} B_i \rightarrow A, -$$
$$\bigwedge_{i \leq n} B_i, +$$
$$A, -$$
$$\vdots$$
$$\otimes$$

closes too, since it will contain $B_i, +$ for every $B_i \in \Sigma$ and $A, -$ (that is, the same elements as the tableau for $\Sigma \vdash_{\mathcal{Q}} A$). Hence, if $\Sigma \vdash_{\mathcal{Q}} A$, where $\Sigma = \{B_1, \ldots B_n\}$, then

$$\vdash_{\mathcal{Q}} \bigwedge_{i \leq n} B_i \rightarrow A$$

as required. □

Theorem 7 *There exists deductions $\Sigma \vdash_{\mathcal{Q}} A$, where $\Sigma = \{B_1, \ldots B_n\}$, such that*

$$\nvdash_{\mathcal{Q}} \bigwedge_{i \leq n} B_i \rightsquigarrow A$$

or

$$\nvdash_{\mathcal{Q}} \bigwedge_{i \leq n} B_i \supset A$$

Proof By means of $\mathcal{Q}$ calculus one can check that

$A \vee A \vdash_{\mathcal{Q}} A$, but $\nvdash_{\mathcal{Q}} (A \vee A) \rightsquigarrow A$
$A \supset \sim A \vdash_{\mathcal{Q}} \sim A$, but $\nvdash_{\mathcal{Q}} (A \supset \sim A) \supset \sim A$

Hence, Deduction Theorem only applies to $\rightarrow$, and neither to $\rightsquigarrow$ nor to $\supset$. □

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