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Reasoning about Dependence, Preference and Coalitional Power

Qian Chen[1,2] · Chenwei Shi[1] · Yiyan Wang[3]

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Abstract
This paper presents a logic of preference and functional dependence (LPFD) and its hybrid extension (HLPFD), both of whose sound and strongly complete axiomatization are provided. The decidability of LPFD is also proved. The application of LPFD and HLPFD to modelling cooperative games in strategic form is explored. The resulted framework provides a unified view on Nash equilibrium, Pareto optimality and the core. The philosophical relevance of these game-theoretical notions to discussions of collective agency is made explicit. Some key connections with other logics are also revealed, for example, the coalition logic, the logic of functional dependence and the logic of ceteris paribus preference.

Keywords Coalitional power · Ceteris paribus preference · Functional dependence · Pareto optimality · Collective agency.

1 Introduction

On each of the three concepts, dependence, preference and coalitional power, there have been logical works. To name but a few, for dependence, the dependence logic [18] has been studied in various ways (cf.[9]) and a simple logic of functional dependence is recently proposed in [3]; for coalitional power, the coalition logic [14] and the alternating-time temporal logic (ATL) [1, 10] are representative; for preference, good surveys can be found in [11] and [12, Chapter 1.1]. Despite not being explicitly emphasized, the concept of dependence permeates the analyses of the other two con-

✉ Chenwei Shi
scw@mail.tsinghua.edu.cn

1 Tsinghua - UvA Joint Research Centre for Logic, Department of Philosophy, Tsinghua University, 100084 Beijing, China

2 Institute for Logic, Language and Computation, University of Amsterdam, 1012 WX Amsterdam, The Netherlands

3 School of Philosophy, Shanxi University, 030006 Taiyuan, China

cepts, for example, in [6, 14]. However, as far as we know, there is hardly any logic explicitly modeling all of these three concepts, especially making dependence the hub to which the other two concepts join. In this paper, we provide such a logic, which characterizes the interaction between the three concepts. Moreover, we show that by making the role of dependence explicit, our logical analysis leads to a unified view of several key concepts in game theory, namely Nash equilibrium, Pareto optimality and the core. We also explore a philosophical implication about collective agency of our logical analysis. We take the stability of a group to be an essential aspect of collective agency. Instead of focusing on intentionality as in the philosophical literature [16], we elaborate on our understanding in a game theoretical context.

Our main work in this paper centers on introducing preference into the logic of functional dependence [3] by adding preference relations in the original semantic model and a new modal operator in the original language for the intersection of different kinds of relations, including equivalence relations, preorders and strict preorders. By taking a game theoretic interpretation of the semantic setting, the new operator enables us to express not only Nash equilibrium but also Pareto optimality.

While Nash equilibrium is taken to be a benchmark for modern logics of games and many logics have been demonstrated to be able to express it (see [6, section 7.1] and the reference in it), Pareto optimality as an equally important notion in game theory [1] seems to receive less attention in logical literature than Nash equilibrium. As shown in this paper, to express Pareto optimality, the new modal operator is critical. In fact, given the operator, we can express a relativized version of Nash equilibrium and Pareto optimality, that is, "given the current strategies of some players, the current strategy profile of the other players would be a Nash equilibrium/Pareto optimality." Moreover, by taking dependence relation into consideration, our logic shows that Nash equilibrium can be defined by Pareto optimality.

As Pareto optimality is rarely studied by logicians, compared to the *non-cooperative* game theory, the *cooperative* game theory [15] seems not very salient to logicians either. [2] We will demonstrate that our logic of preference and functional dependence (LPFD) can also be adapted to model a qualitative version of cooperative games in strategic form [15, Section 11]. We will also show that a hybrid extension of LPFD can express the core, an essential solution concept in the cooperative games analogous to Nash equilibrium in the non-cooperative games. The core characterizes a coalition's stability as a state where none of its subcoalitions has any incentive to deviate even if they can. The three concepts, dependence, preference and coalitional power, crystallize in the core. Through the lens framed by the three concepts, a unified view of the core, Nash equilibrium and Pareto optimality is revealed by our logics.

In addition to the logics and their application to a unified analysis of key game theoretical concepts, our contributions include several technical results about the logics themselves. We provide a sound and strongly complete axiomatization respectively for LPFD and its hybrid extension HLPFD. Moreover, we also prove that the satisfiability

[1] For example, in the prisoners' dilemma, the Nash equilibrium is not Pareto optimal.

[2] The review on modal logic for games and information [19, Chapter 20] is exclusively about *non-cooperative* game theory; the book [5] touches on few issues on *cooperative* game theory either. The only exception we know is the work in [21], where two different logics are proposed to reason about cooperative games.

problem of LPFD is decidable. While the proof for the completeness result of HLPFD is standard, the proof for the completeness of LPFD is much harder and requires new techniques. Our proof modifies the classical unraveling method [7, Chapter 4.5] and combines it with a special way of selecting the tree branches.

The Structure of the Paper is summarized as follows. The background on the logic of functional dependence (LFD) are presented in Section 2. In Section 3, we introduce the logic of preference and functional dependence and show how it can naturally express Nash equilibrium and Pareto optimality. Section 4 contains sound and strongly complete axiomatization of LPFD and its hybrid extension and the decidability of LPFD's satisfiability problem. For those who are not interested in the proof details, Section 4.3 and Section 4.4 can be safely skipped. In Section 5, we turn to our modelling of cooperative games in strategic form in LPFD and analyze the core. In Section 5.3, we show how the core can be relevant to philosophical discussions of collective agency. Before conclusion, we compare our work with the logical works in [6, 14, 21].

Notations The following notations will be used throughout this paper. Let A, B be two sets. We will use B^A to denote the set of mappings from A to B. Specially, for each $n \in \omega$, let A^n denote the set of all sequences on A with length n, i.e., $A^n = \{(x_0, \cdots, x_{n-1}) : x_0, \cdots, x_{n-1} \in A\}$. Let $[A]^{<\omega}$ denote the set of all finite subsets of A. For each sequence $\vec{x} = (x_i : i \in I)$, we write $\mathsf{ran}(\vec{x})$ for the set $\{x_i : i \in I\}$. For every language $\mathcal{L}$ and class $\mathscr{C}$ of mathematical structures, let $\mathsf{Log}_{\mathcal{L}}(\mathscr{C})$ denote the set of all valid formulas in $\mathcal{L}$ w.r.t $\mathscr{C}$.

2 LFD Interpreted in Games

In this section, we introduce LFD and take a game-theoretical view on it.

LFD starts with a relational vocabulary $(\mathsf{V}, \mathsf{Pred}, \mathsf{ar})$, where V is a countable set of variables, Pred is a countable set of predicate symbols and $\mathsf{ar} : \mathsf{Pred} \to \mathbb{N}$ is an arity map, associating to each predicate $P \in \mathsf{Pred}$ a natural number $\mathsf{ar}(P)$. In what follows, unless otherwise specified, in a vocabulary $(\mathsf{V}, \mathsf{Pred}, \mathsf{ar})$, $|\mathsf{V}| = \aleph_0$ and $|\{P \in \mathsf{Pred} : \mathsf{ar}(P) = n\}| = \aleph_0$ for each $n \in \omega$.

To view LFD from a game-theoretical perspective, we take the variables to represent players in games and the dependence models of LFD become models for different players' actions or strategies in static games in strategic form.

Definition 1 **(Dependence models)** *A model is a pair $M = (O, I)$, where O is a non-empty set of actions and I is a mapping that assigns to each predicate $P \in \mathsf{Pred}$ a subset of $O^{\mathsf{ar}(P)}$. A dependence model $\mathbf{M}$ is a pair $\mathbf{M} = (M, A)$, where $M = (O, I)$ is a model and $A \subseteq O^{\mathsf{V}}$ is a set of strategy profiles.*

For each $X \in [\mathsf{V}]^{<\omega}$, we define a binary relation $=_X$ on A such that for all $a, a' \in A$, $a =_X a'$ if and only if $a{\upharpoonright}X = a'{\upharpoonright}X$, i.e., the action of x in a is the same as her action in a' for each $x \in X$.

Note that we use O to denote all available actions rather than outcomes as usually done in game theoretical literature. When $A \neq O^{\mathsf{V}}$, some strategy profiles are missing, which gives rise to restrictions on how players can act together. Suppose a strategy profile s is not in A. Then the players cannot act according to s simultaneously.[3]

Next, we turn to the syntax and semantics of LFD. To capture functional dependence, LFD uses two operators $\mathbb{D}$ and D in its language.

Definition 2 *The language $\mathcal{L}$ of* LFD *is given by*

$$\mathcal{L} \ni \varphi ::= P\vec{x} \mid D_X y \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbb{D}_X \varphi$$

where $P \in \mathsf{Pred}$, $\vec{x} \in \mathsf{V}^{\mathsf{ar}(P)}$, $X \in [\mathsf{V}]^{<\omega}$ and $y \in \mathsf{V}$.

$\mathbb{D}_X \varphi$ is meant to express that whenever the players in X take their current actions, φ is the case; $D_X y$ says that whenever the players in X take their current actions, y also takes its current action.

Definition 3 *Truth of a formula $\varphi \in \mathcal{L}$ in a dependence model $\mathbf{M} = (M, A)$ at a strategy profile $a \in A$ is defined as follows:*

$\mathbf{M}, a \models P\vec{x}$	*iff*	$a(\vec{x}) \in I(P)$
$\mathbf{M}, a \models D_X y$	*iff*	$a =_y a'$ *for all* $a' \in A$ *with* $a =_X a'$
$\mathbf{M}, a \models \neg\varphi$	*iff*	$\mathbf{M}, a \not\models \varphi$
$\mathbf{M}, a \models \varphi \wedge \psi$	*iff*	$\mathbf{M}, a \models \varphi$ *and* $\mathbf{M}, a \models \psi$
$\mathbf{M}, a \models \mathbb{D}_X \varphi$	*iff*	$\mathbf{M}, a' \models \varphi$ *for all* $a' \in A$ *with* $a =_X a'$

Note that $=_X$ is an equivalence relation on A and $a =_\emptyset a'$ holds for all $a, a' \in A$. So $\mathbb{D}_\emptyset$ is a universal operator and we define $\mathsf{A}\varphi := \mathbb{D}_\emptyset \varphi$ and $\mathsf{E}\varphi := \neg\mathsf{A}\neg\varphi$.

3 Logic of Preference and Functional Dependence

In this section, we extend LFD to LPFD.

3.1 Syntax and Semantic for LPFD

Definition 4 **(Syntax)** *The language $\mathcal{L}^{\preceq}$ of LPFD is given by:*

$$\mathcal{L}^{\preceq} \ni \varphi ::= P\vec{x} \mid D_X y \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\![X, Y, Z]\!]\varphi$$

where $P \in \mathsf{Pred}$, $\vec{x} \in \mathsf{V}^{\mathsf{ar}(P)}$, $y \in \mathsf{V}$ and $X, Y, Z \in [\mathsf{V}]^{<\omega}$.

$\mathcal{L}^{\preceq}$ only differs from the language of LFD in the new operator $[\![X, Y, Z]\!]\varphi$, which is an operator for ceteris paribus group preference. In the language $\mathcal{L}^{\preceq}$, the formula $\mathbb{D}_X \varphi$ is defined as $[\![X, \emptyset, \emptyset]\!]\varphi$, capturing "centeris paribus". Y and Z in $[\![X, Y, Z]\!]\varphi$ are used to capture group preferences. We define $\langle\!\langle X, Y, Z\rangle\!\rangle\varphi := \neg[\![X, Y, Z]\!]\neg\varphi$ and $D_X Y := \bigwedge_{y \in Y} D_X y$ for each $X, Y, Z \in [\mathsf{V}]^{<\omega}$ and $\varphi \in \mathcal{L}^{\preceq}$.

Next we turn to the semantics of LPFD.

[3] Such restrictions serve as basis for representing dependence but do not necessarily imply dependence between the players. For example, for $O = \{a_1, a_2, b_1, b_2\}$ and $\mathsf{V} = \{x, y\}$, if we take $A = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$, the restriction of each player's range of actions is not due to what the other player does.

Definition 5 **(PD-models).** *A* preference dependence model (PD-model) *is a pair* $\mathbb{M} = (\mathbf{M}, \preceq)$ *in which* $\mathbf{M} = (M, A)$ *is a dependence model and* $\preceq: \mathsf{V} \to \mathcal{P}(A \times A)$ *is a mapping assigning to each* $x \in \mathsf{V}$ *a pre-order* $\preceq_x$ *on* A. *Let* Mod *denote the class of all PD-models.*

For each $x \in \mathsf{V}$, we define the binary relation $\prec_x = \{(a, b) \in \preceq_x : (b, a) \notin \preceq_x\}$. For all $a, b \in A$, we write $a \preceq_X b$ ($a \prec_X b$) if $a \preceq_x b$ ($a \prec_x b$) for each $x \in X$. Moreover, we write $s \simeq_X t$ if $s \preceq_X t$ and $t \preceq_X s$.

Definition 6 *Truth of PD-formulas of the form* $P\vec{x}$, $D_X y$, $\neg\varphi$ *or* $\varphi \wedge \psi$ *is defined as in Definition 3. For formulas of the form* $[\![X, Y, Z]\!]\varphi$, *we say* $[\![X, Y, Z]\!]\varphi$ *is true at* a *in* $\mathbb{M}$, *notation:* $\mathbb{M}, a \models [\![X, Y, Z]\!]\varphi$, *if*

$$\mathbb{M}, a' \models \varphi \text{ for all } a' \in A \text{ satisfying } a =_X a',\ a \preceq_Y a' \text{ and } a \prec_Z a'.$$

A formula $\varphi \in \mathcal{L}^{\preceq}$ *is valid if* $\mathbb{M}, a \models \varphi$ *for all PD-model* $\mathbb{M} = (M, A, \preceq)$ *and* $a \in A$. *Let* LPFD *denote the set of all valid formulas, i.e.,* $\mathsf{LPFD} = \mathsf{Log}_{\mathcal{L}^{\preceq}}(\mathsf{Mod})$.

Note that $[\![\emptyset, \{x\}, \emptyset]\!]\varphi$ and $[\![\emptyset, \emptyset, \{x\}]\!]\varphi$ are standard modal operators defined on $\preceq_x$ and $\prec_x$ respectively. Thus $[\![X, Y, Z]\!]\varphi$ is in fact a standard modal operator defined on the intersection of the relations $=_X$, $\preceq_Y$ and $\prec_Z$. It concerns the preferences of the players in Y and Z conditional on the actions of the players in X.

There is a close connection between LPFD and the work in [6] on ceteris paribus preference. We will discuss this connection in Section 6.2. Next, we show how some key game theoretical notions can be expressed in LPFD.

3.2 Pareto Optimality and Nash Equilibrium in LPFD

Having laid out the basics of LPFD, we turn to questions concerning expressing and reasoning about Pareto optimality and Nash equilibrium in LPFD. One important assumption we will adopt is that the group of players V is finite. In LPFD, there is no such restriction on V. However, it is worth noting that in the language of LPFD, all subscripts in the two operators need to be finite. So to express something like $[\![-X, \emptyset, X]\!]\varphi$ in LPFD where $-X := \mathsf{V} - X$, which is frequently referred to in game theory, we have to ensure that X and $-X$ are both finite.

We start with recalling what Nash equilibrium and weak/strong Pareto optimality mean.

Definition 7 *Let* $\mathbb{M}$ *be a PD-model and* $X \subseteq \mathsf{V}$.

- s *is a* ***Nash equilibrium*** *for* X *if for all* $x \in X$ *there is no* $t =_{-\{x\}} s$ *such that* $s \prec_x t$;
- s *is* ***strongly Pareto optimal*** *for* X *if there is no* $t =_{-X} s$ *such that (a) for all* $x \in X$, $s \preceq_x t$ *and (b) there is one* $x \in X$ *such that* $s \prec_x t$;
- s *is* ***weakly Pareto optimal*** *for* X *if there is no* $t =_{-X} s$ *such that for all* $x \in X$, $s \prec_x t$.

Note that such a way of defining the notions of Nash equilibrium, weak and strong Pareto optimality in a PD-model applies to all subgroups of V rather than only the

Table 1

go out			stay home		
	go out	stay home		go out	stay home
go out	(4,4,4)	(1,0,1)	go out	(1,1,0)	(1,2,2)
stay home	(0,1,1)	(2,2,1)	stay home	(2,1,2)	(2,2,2)

whole group of players V. According to which subgroup the definition applies, it requires the actions of players outside the subgroup to be fixed, after which the normal definition then applies. For example, in the case of Nash equilibrium for X, the above definition actually says that after fixing the actions of players in $-X$, the current action profile of X satisfies the conditions of Nash equilibrium.

Example 1 Table 1 shows three students' preferences on staying home or going out. The row is for student a; the column is for student b; the left and right division is for student c.

We can check that going out together and all staying home are both Nash equilibrium; going out together is also Pareto optimal. Moreover, (stay home, stay home, go out) is a Nash equilibrium for student a and b given student c goes out, although it is not a Nash equilibrium for the whole group. (stay home, stay home, stay home) is Pareto optimal for student a and b given student c stays home, but not for student a, b and c together.

It is relatively easy to get how Nash equilibrium and weak Pareto optimality can be expressed in LPFD, as the following fact shows.

Fact 1 *Let $\mathbb{M} = (M, A, \preceq)$ be a PD-model and $s \in A$. Then*

- *s is a Nash equilibrium for $X \subseteq \mathsf{V}$ given that the players in $-X$ have acted according to s, if and only if, $\mathbb{M}, s \models \bigwedge_{x \in X} [\![-\{x\}, \emptyset, \{x\}]\!]\bot$;*
- *s is weakly Pareto optimal for $X \subseteq \mathsf{V}$ given that the players in $-X$ have acted according to s, if and only if, $\mathbb{M}, s \models [\![-X, \emptyset, X]\!]\bot$.*

In the case of weak Pareto optimality, because the truth condition of the operator $[\![-X, \emptyset, X]\!]$ depends on what formulas are satisfied on all elements in the set $\{t \in A \mid s =_{-X} t, s \prec_X t\}$, if it is an empty set and thus $\bot$ can be vacuously satisfied on all elements in it, then s is weakly Pareto optimal for X.

To express strong Pareto optimality in LPFD, we need to express the following model theoretical fact, namely, the set $\{t \in A \mid s =_{-X} t, s \preceq_X t \text{ and } t \not\preceq_X s\}$ $= \bigcup_{x \in X} \{t \in A \mid s =_{-X} t, s \preceq_{X - \{x\}} t, s \prec_x t\}$ is empty.

Since $s \models [\![-X, X - \{x\}, \{x\}]\!]\bot$ iff $\{t \in A \mid s =_{-X} t, s \preceq_{X-\{x\}} t, s \prec_x t\} = \emptyset$, we can define strong Pareto optimality as follows.

Fact 2 *In a PD-model $\mathbb{M}$, s is strongly Pareto optimal for $X \subseteq \mathsf{V}$ given that the players in $-X$ have acted according to s iff $\mathbb{M}, s \models \bigwedge_{x \in X} [\![-X, X - \{x\}, \{x\}]\!]\bot$.*

To facilitate our discussion, we define weak and strong Pareto optimality and Nash equilibrium in LPFD as

$$\mathsf{wPa}\, X := [\![-X, \emptyset, X]\!]\bot \tag{1}$$

$$\mathsf{sPa}\, X := \bigwedge_{x\in X} [\![-X, X-\{x\}, \{x\}]\!]\bot \tag{2}$$

$$\mathsf{Na}\, X := \bigwedge_{x\in X} [\![-\{x\}, \emptyset, \{x\}]\!]\bot \tag{3}$$

An easy but important observation is that Nash equilibrium can be defined via Pareto optimality.

Theorem 1 $\mathsf{Na}\, X = \bigwedge_{x\in X} \mathsf{sPa}\{x\} = \bigwedge_{x\in X} \mathsf{wPa}\{x\}$.

4 Calculus of LPFD and its Hybrid Extension

In this section, a relational semantics of LPFD shall be introduced. We show the relation between this semantics and the standard semantics given in Section 3.1. The new semantics provides us with a modal view, which facilitates our calculus $\mathsf{C}_{\mathrm{LPFD}}$ and the proof of its soundness and strongly completeness. We show that LPFD is decidable while it lacks the finite model property. Moreover, we extend it with nominals and give also a sound and complete calculus $\mathsf{C}_{\mathrm{HLPFD}}$. In Section 5.3, this hybrid extension will be useful in expressing a key game theoretic concept.

4.1 Relational Semantics

In this part, we introduce the relational semantics for LPFD and show the relation between this semantics and the standard one.

Definition 8 *A* relational PD-frame (RPD-frame) *is a pair $\mathfrak{F} = (W, \sim, \leq)$, where W is a non-empty set, $\sim: \mathsf{V} \to \mathcal{P}(W \times W)$ and $\leq: \mathsf{V} \to \mathcal{P}(W \times W)$ are maps such that $\sim_x$ is an equivalence relation and $\leq_x$ is a pre-order for all $x \in \mathsf{V}$. For all $x \in \mathsf{V}$ and $X, Y, Z \in [\mathsf{V}]^{<\omega}$, let $<_x = \{(w, u) \in \leq_x : (u, w) \notin \leq_x\}$ and*

$$R(X, Y, Z) = \bigcap_{x\in X} \sim_x \cap \bigcap_{y\in Y} \leq_y \cap \bigcap_{z\in Z} <_z .$$

A relational PD-model (RPD-model) *is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F} = (W, \sim, \leq)$ is an RPD-frame and V is a valuation associating to each formula of the form $P\vec{x}$ a subset $V(P\vec{x})$ of W. The valuation V is required to satisfy the following condition for all $w, u \in W$, $P \in \mathsf{Pred}$ and $\vec{x} \in \mathsf{V}^{\mathsf{ar}(P)}$:*

if $w \sim_{\mathsf{ran}(\vec{x})} u$, then $w \in V(P\vec{x})$ if and only if $u \in V(P\vec{x})$. (Val)

Let RMod *denote the class of all RPD-models. Truth of a formula $\varphi \in \mathcal{L}^{\preceq}$ in $\mathfrak{M} = (W, \sim, \leq, V)$ at $w \in W$ is defined as follows:*

$$
\begin{array}{lll}
\mathfrak{M}, w \models P\vec{x} & \textit{iff} & w \in V(P\vec{x}) \\
\mathfrak{M}, w \models D_X y & \textit{iff} & w \sim_y v \textit{ for all } v \sim_X w \\
\mathfrak{M}, w \models \neg\varphi & \textit{iff} & \mathfrak{M}, w \not\models \varphi \\
\mathfrak{M}, w \models \varphi \wedge \psi & \textit{iff} & \mathfrak{M}, w \models \varphi \textit{ and } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models [\![X, Y, Z]\!]\varphi & \textit{iff} & \mathfrak{M}, v \models \varphi \textit{ for all } v \in R(X, Y, Z)(w)
\end{array}
$$

Validity is defined as usual. Let $\mathsf{RLPFD} = \mathsf{Log}_{\mathcal{L}^{\preceq}}(\mathsf{RMod})$.

We now show that under the assumption V is infinite, the relational semantics is equivalent to the standard one in the sense that $\mathsf{RLPFD} = \mathsf{LPFD}$.

Definition 9 *Let* $\mathbb{M} = (O, I, A, \preceq)$ *be a PD-model. Then we define the RPD-model* $rel(\mathbb{M}) = (A, \sim, \leq, V)$ *induced by* $\mathbb{M}$ *as follows:*

- $V(P\vec{x}) = \{a \in A : a(\vec{x}) \in I(P)\}$ *for all* $P \in \mathsf{Pred}$ *and* $\vec{x} \in \mathsf{V}^{\mathsf{ar}(P)}$.
- $\sim_x = (=_x)$, $\leq_x = \preceq_x$ *for each* $x \in \mathsf{V}$.

Proposition 1 *Let* $\mathbb{M} \in \mathsf{Mod}$ *and* $rel(\mathbb{M})$ *the RPD-model induced by* $\mathbb{M}$*. Then for each* $a \in A$ *and formula* $\varphi \in \mathcal{L}^{\preceq}$,

$$\mathbb{M}, a \models \varphi \textit{ if and only if } rel(\mathbb{M}), a \models \varphi.$$

Proof By induction on the complexity of φ. □

While it is straightforward to induce an RPD-model from a PD-model, some care needs to be taken to induce a suitable PD-model from a RPD-model.

Definition 10 *Let* $\mathfrak{M} = (W, \sim, \leq, V)$ *be an RPD-model. Then*

- $\mathfrak{M}$ *is a* differential model, *if* $\bigcap_{x \in \mathsf{V}} \sim_x = Id_W = \{(w, w) : w \in W\}$.
- $\mathfrak{M}$ *is a* pre-differential model, *if for all* $w, u \in W$, $w \sim_{\mathsf{V}} u$ *implies* $w \leq_{\mathsf{V}} u$.

Let RMod_d *and* RMod_{pd} *denote the class of all differential RPD-models and pre-differential RPD-models, respectively.*

Definition 11 *Let* $\mathfrak{M} = (W, \sim, \leq, V)$ *be a differential model. Then we define the PD-model* $dp(\mathfrak{M}) = (O, I, A, \preceq)$ *induced by* $\mathfrak{M}$ *as follows:*

- $O = \{(x, |w|_x) : x \in \mathsf{V}, w \in W$ *and* $|w|_x = \{v \in W : w \sim_x v\}\}$.
- $A = \{w^* : w \in W\}$, *where* $w^*(x) = (x, |w|_x)$ *for each* $x \in \mathsf{V}$.
- $\preceq_x = \{(w^*, v^*) : w \leq_x v\}$ *for each* $x \in \mathsf{V}$.
- I *is the interpretation mapping each n-ary predicate* P *to the set*

$$I(P) = \{w^*(\vec{x}) : w \in W, \vec{x} \in \mathsf{V}^n \textit{ and } w \in V(P\vec{x})\}.$$

$I(P)$ is well-defined for each predicate P since $x = y$ and $w \sim_x v$ whenever $w^*(x) = v^*(y)$. To see that $\preceq_x$ is a pre-order for each $x \in \mathsf{V}$, it suffices to show that $(W, \leq)$ is isomorphic to $(A, \preceq)$. Since $\mathfrak{M}$ is a differential model, for all $w, v \in W$, $w \neq v$ implies $w \nsim_y v$ for some $y \in \mathsf{V}$. Thus $w \neq v$ implies $w^* \neq v^*$ and so the function $(\cdot)^* : W \to A$ is an isomorphism. Hence $\preceq_x$ is a pre-order for all $x \in \mathsf{V}$. It is clearly that $\prec_x = \{(w^*, v^*) : w <_x v\}$ for all $x \in \mathsf{V}$.

Proposition 2 *Let $\mathfrak{M}$ be a differential RPD-model and $dp(\mathfrak{M})$ the PD-model induced by $\mathfrak{M}$. Then for each w in $\mathfrak{M}$ and formula $\varphi \in \mathcal{L}^{\preceq}$,*

$$\mathfrak{M}, w \models \varphi \text{ if and only if } dp(\mathfrak{M}), w^* \models \varphi.$$

Proof By induction on the complexity of φ. □

Theorem 2 RLPFD = LPFD.

Proof Suppose $\varphi \notin$ LPFD. Then $\neg\varphi$ is satisfied by some PD-model. By Proposition 1, φ is satisfied by some RPD-model, which entails $\varphi \notin$ RLPFD and so RLPFD $\subseteq$ LPFD. Suppose $\varphi \notin$ RLPFD. Then there is an RPD-model $\mathfrak{M} = (W, \sim, \leq, V)$ and $w \in W$ such that $\mathfrak{M}, w \not\models \varphi$. Since V is infinite, there is $x \in$ V which does not occur in φ. Then let $\mathfrak{M}' = (W, \sim', \leq, V)$ be an RPD-model where

$$\sim'_y = \begin{cases} \sim_y & \text{, if } y \neq x; \\ \{(w, w) : w \in W\} & \text{, otherwise.} \end{cases}$$

Now we can readily check that $\mathfrak{M}'$ is a differential model with $\mathfrak{M}', w \models \varphi$. By Proposition 2, we have $dp(\mathfrak{M}'), w^* \not\models \varphi$. Hence RLPFD = LPFD. □

Relation Between the Two Semantics with Finite Variables

The assumption $|\mathsf{V}| \geq \aleph_0$ is crucial in the proof of Theorem 2. The readers can check that the transformations rel and dp provide a correspondence between classes of models Mod and RMod_d in the sense that for all $\mathfrak{M} \in \mathsf{RMod}_d$ and $\mathbb{M} \in$ Mod,

$$dp(rel(\mathbb{M})) \cong \mathbb{M} \text{ and } rel(dp(\mathfrak{M})) \cong \mathfrak{M}.$$

A simple example is given in the left part of Fig. 1, where dotted lines stand for $\sim$ relations and arrows for preferences. Under the assumption that V is infinite, we can see from the proof of Theorem 2 that every satisfiable formula is satisfied by some differential RPD-model. However, Theorem 2 does not hold when V is finite.

Suppose now that V is finite. Let $\mathfrak{M}'_0$ be the RPD-model shown in the right part of Figure 1. Then the readers can verify that $\mathfrak{M}'_0, a_3 \models [\![\varnothing, \varnothing, \mathsf{V}]\!]\bot \wedge \langle\!\langle \mathsf{V}, \varnothing, \varnothing\rangle\!\rangle\langle\!\langle \varnothing, \varnothing, \mathsf{V}\rangle\!\rangle\top$. On the other hand, for any PD-model $\mathbb{M} = (M, A, \preceq)$ and

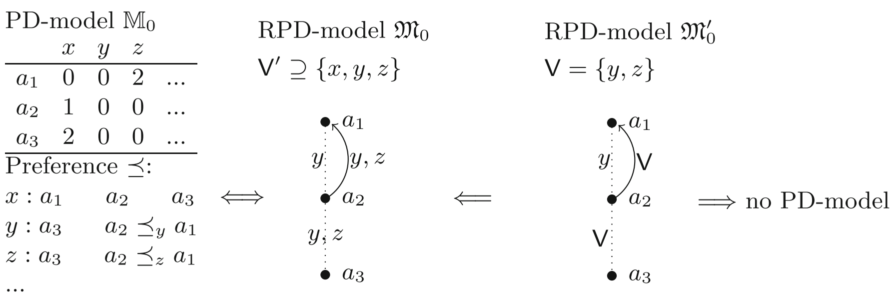

Fig. 1 Translation between PD-models and RPD-models

$a, b \in A$, $a =_{\mathsf{V}} b$ implies $a = b$. Thus $\varphi \to [\![\mathsf{V}, \varnothing, \varnothing]\!]\varphi$ is valid and so $[\![\varnothing, \varnothing, \mathsf{V}]\!]\bot \wedge \langle\!\langle \mathsf{V}, \varnothing, \varnothing\rangle\!\rangle\langle\!\langle \varnothing, \varnothing, \mathsf{V}\rangle\!\rangle\top$ cannot be satisfied by any PD-model for the vocabulary $(\mathsf{V}, \mathsf{Pred}, \mathsf{ar})$. Thus $\mathsf{RLPFD} \subsetneq \mathsf{LPFD}$.

To obtain a clearer view of the relation between the two semantics, the class RMod_{pd} of RPD-models plays an important role. The readers can readily check that the following facts hold:

Fact 3 *Let $\mathfrak{F} = (W, \sim, \leq)$ be an RPD-frame and $\mathfrak{M} = (\mathfrak{F}, V)$ an RPD-model. Then*

$$\mathfrak{M} \in \mathsf{RMod}_{pd} \text{ if and only if } \mathfrak{F} \models \langle\!\langle \mathsf{V}, \varnothing, \varnothing\rangle\!\rangle\varphi \to \langle\!\langle \mathsf{V}, \mathsf{V}, \varnothing\rangle\!\rangle\varphi.$$

Fact 4 *Let $\mathfrak{M} = (W, \sim, \leq, V)$ be a pre-differential RPD-model and we define the RPD-model $\mathfrak{M}/{\sim_{\mathsf{V}}} = (W', \sim', \leq', V,)$ as follows:*

- $W' = \{[w] : w \in W\}$ *where* $[w] = \{u : w \sim_{\mathsf{V}} u\}$;
- $\sim'_x = \{\langle [w], [u]\rangle : w \sim_x u\}$, $\leq'_x = \{\langle [w], [u]\rangle : w \leq_x u\}$;
- $V'(P\vec{x}) = \{[w] : w \in V(P\vec{x})\}$ *for all* $P \in \mathsf{Pred}$ *and* $\vec{x} \in \mathsf{V}^{\mathsf{ar}(P)}$.

Then $\mathfrak{M}/{\sim_{\mathsf{V}}} \in \mathsf{RMod}_d$. Moreover, for all $\varphi \in \mathcal{L}^{\prec}$ and $w \in W$,

$$\mathfrak{M}, w \models \varphi \text{ if and only if } \mathfrak{M}/{\sim_{\mathsf{V}}}, [w] \models \varphi.$$

By Fact 3 and Fact 4, we obtain immediately that

$$\mathsf{Log}_{\mathcal{L}^{\preceq}}(\mathsf{RMod}_d) = \mathsf{Log}_{\mathcal{L}^{\preceq}}(\mathsf{RMod}_{pd}) = \mathsf{RLPFD} \oplus \langle\!\langle \mathsf{V}, \varnothing, \varnothing\rangle\!\rangle\varphi \to \langle\!\langle \mathsf{V}, \mathsf{V}, \varnothing\rangle\!\rangle\varphi.$$

Note that $\mathsf{Log}_{\mathcal{L}^{\preceq}}(\mathsf{RMod}_d) = \mathsf{Log}_{\mathcal{L}^{\preceq}}(\mathsf{Mod}) = \mathsf{LPFD}$, we have

Theorem 3 *If* V *is finite, then* $\mathsf{RLPFD} \oplus \langle\!\langle \mathsf{V}, \varnothing, \varnothing\rangle\!\rangle\varphi \to \langle\!\langle \mathsf{V}, \mathsf{V}, \varnothing\rangle\!\rangle\varphi = \mathsf{LPFD}$.

4.2 Hilbert-style Calculus $\mathsf{C}_{\mathsf{LPFD}}$

In this part, we present a calculus $\mathsf{C}_{\mathrm{LPFD}}$ of LPFD and show that $\mathsf{C}_{\mathrm{LPFD}}$ is sound, by which some key axioms are semantically explained. In what follows, we write C for $\mathsf{C}_{\mathrm{LPFD}}$ if there is no danger of confusion.

(Tau) Axioms and rules for classical propositional logic;
(Nec) from φ infer $[\![X, Y, Z]\!]\varphi$;
(K) $[\![X, Y, Z]\!](\varphi \to \psi) \to ([\![X, Y, Z]\!]\varphi \to [\![X, Y, Z]\!]\psi)$;
(Ord) Axioms for preference relations:
(a) $[\![X, Y, \varnothing]\!]\varphi \to \varphi$;
(b) $\langle\!\langle X, Y, Z\rangle\!\rangle\langle\!\langle X', Y', Z'\rangle\!\rangle\varphi \to \langle\!\langle X \cap X', Y \cap Y', (Z \cap Y') \cup (Z \cap Z') \cup (Y \cap Z')\rangle\!\rangle\varphi$;
(c) $[\![X, Y, Z]\!]\varphi \to [\![X', Y', Z']\!]\varphi$, provided $X \subseteq X'$, $Y \subseteq Y'$ and $Z \subseteq Z'$.
(d) $\langle\!\langle X, Y, Z\rangle\!\rangle\varphi \to \langle\!\langle X, Y \cup Z, Z\rangle\!\rangle\varphi$;
(e) $(\varphi \wedge \langle\!\langle X, Y, Z\rangle\!\rangle\psi) \to \langle\!\langle X, Y, Z\rangle\!\rangle(\psi \wedge \langle\!\langle X, Y, \varnothing\rangle\!\rangle\varphi) \vee \bigvee_{y \in Y}\langle\!\langle X, Y, Z \cup \{y\}\rangle\!\rangle\psi$.
(Dep) Axioms and rules for dependence:
(a) $D_X X$;
(b) $\varphi \to \mathbb{D}_X\varphi$, provided $\varphi \in \{P\vec{x} : \mathsf{ran}(\vec{x}) \subseteq X\} \cup \{D_Y z : Y \subseteq X\}$;
(c) $D_X S \wedge D_S T \to D_X T$;

(d) $D_X S \wedge [\![S, Y, Z]\!]\varphi \rightarrow [\![X, Y, Z]\!]\varphi$.

The definition of derivations in C are as usual. If there is a derivation from Γ to φ in C, then we write $\Gamma \vdash_{\mathsf{C}} \varphi$.

Theorem 4 **(Soundness)** *For each $\varphi \in \mathcal{L}^{\preceq}$, $\vdash_{\mathsf{C}} \varphi$ implies $\varphi \in$ LPFD.*

Proof We take (Ord,b) and (Ord,e) as two examples, showing their validity and giving some intuitions. Other axioms and rules can be easily checked to be valid. Let $\mathfrak{M} = (W, \sim, \leq, V)$ be an RPD-model and $w \in W$ a point.

For (Ord,b), it characterizes some kind of generalized transitivity. Suppose $\mathfrak{M}, w \models \langle\!\langle X, Y, Z\rangle\!\rangle\langle\!\langle X', Y', Z'\rangle\!\rangle\varphi$. Then there are points $u, v \in W$ such that $u \in R(X, Y, Z)(w)$, $v \in R(X', Y', Z')(u)$ and $\mathfrak{M}, v \models \varphi$. Let $T = (Z \cap Y') \cup (Z \cap Z') \cup (Y \cap Z')$. It is obvious that $w \sim_{X\cap X'} v$ and $w \leq_{Y\cap Y'} v$ hold. It suffices to show that $w <_T v$. Suppose $x \in Z \cap Y'$. Then $w \leq_x u$, $u \nleq_x w$ and $u \leq_x v$. By the transitivity of $\leq_x$, we see $w \leq_x v$ and $v \nleq_x w$, i.e., $w <_x v$. Similarly, we see $w <_x v$ whenever $x \in Y \cap Z'$ or $x \in Z \cap Z'$. Hence $\mathfrak{M}, w \models (Ord, b)$.

For (Ord,e), it characterizes to some degree the definition of $<$. Suppose $\mathfrak{M}, w \models \varphi \wedge \langle\!\langle X, Y, Z\rangle\!\rangle\psi$. Then there is a point $u \in R(X, Y, Z)(w)$ such that $\mathfrak{M}, u \models \psi$. If $u \leq_Y w$, then clearly $\mathfrak{M}, u \models \psi \wedge \langle\!\langle X, Y, \varnothing\rangle\!\rangle\varphi$, which entails $\mathfrak{M}, w \models \langle\!\langle X, Y, Z\rangle\!\rangle(\psi \wedge \langle\!\langle X, Y, \varnothing\rangle\!\rangle\varphi)$. Suppose $u \nleq_Y w$. Then there is $y \in Y$ such that $u \nleq_y w$ and so $w <_y u$. Recall that $u \in R(X, Y, Z)(w)$, we obtain $u \in R(X, Y, Z \cup \{y\})$ and so $\mathfrak{M}, w \models \langle\!\langle X, Y, Z \cup \{y\}\rangle\!\rangle\psi$. Hence $\mathfrak{M}, w \models (Ord, e)$. □

4.3 Strong Completeness of $\mathsf{C}_{\mathsf{LPFD}}$

For the proof of completeness, a special kind of unraveling method is used. The main reason we take such a method is that the 'canonical model' need not be an RPD-model, and modification is needed. To construct an RPD-model satisfying some given consistent set of formulas, we first pick out those so-called saturated formulas, which are sufficient to determine the preference relations in the model. Then we take 'paths' as the domain of the desired model instead of using just maximal consistent sets, which helps us deal with the intersections of relations. The relations in this model are closures of some 'one-step' relations, which help solve the problems that arise from dependence formulas. With such a model, we prove the Truth Lemma and so the Completeness Theorem.

To define a model for some satisfiable set of formulas Γ, we first define the canonical quasi-frame and investigate some properties of it:

Definition 12 **(Canonical Quasi PD-Frame)** *Let Δ be a set of $\mathcal{L}^{\preceq}$-formulas. We say that Δ is consistent if $\Delta \nvdash_{\mathsf{C}} \bot$. We say that Δ is a maximal consistent set (MCS) if Δ is consistent and every proper extension of Δ is not consistent. The canonical quasi PD-frame $\mathfrak{F}^q = (W^q, R^q)$ is defined as follows:*

- *W^q is the set of all MCSs;*
- *for all $X, Y, Z \in [\mathsf{V}]^{<\omega}$, we define the binary relation $R^q(X, Y, Z)$ on W^q by:*

$$wR^q(X, Y, Z)u \text{ if and only if } \{\varphi \in \mathcal{L}^{\preceq} : [\![X, Y, Z]\!]\varphi \in w\} \subseteq u.$$

Proposition 3 *For all* $\Delta_1, \Delta_2, \Delta_3 \in W^q$ *and* $X, Y, Z \in [\mathsf{V}]^{<\omega}$*:*

(1) $R^q(X, Y, \varnothing)$ *is reflexive;*
(2) If $\Delta_1 R^q(X, Y, Z)\Delta_2$*, then* $\Delta_1 R^q(X', Y', Z')\Delta_2$ *for all* $X' \subseteq X$*,* $Y' \subseteq Y \cup Z$ *and* $Z' \subseteq Z$*;*
(3) For all $Z' \in [\mathsf{V}]^{<\omega}$*, if* $Z, Z' \subseteq Y$*,* $\Delta_1 R^q(X, Y, Z)\Delta_2$ *and* $\Delta_2 R^q(X, Y, Z')\Delta_3$*, then* $\Delta_1 R^q(X, Y, Z \cup Z')\Delta_3$*;*
(4) If $D_X S \in \Delta_1$ *and* $\Delta_1 R^q(X, Y, Z)\Delta_2$*, then* $\Delta_1 R^q(S, Y, Z)\Delta_2$ *and* $D_X S \in \Delta_2$*.*

Proof (1) follows form axiom (Ord,a), (2) follows from Axiom (Ord,c,d), (3) follows from axiom (Ord,b) and (4) follows from axiom (Dep,b,d) immediately. □

Definition 13 *Let* Σ *be a MCS and* $\langle\!\langle X, Y, Z\rangle\!\rangle\varphi \in \Sigma$*. We say that* $\langle\!\langle X, Y, Z\rangle\!\rangle\varphi$ *is a saturated formula in* Σ *if* $\bigvee_{y \in Y}\langle\!\langle X, Y, Z \cup \{y\}\rangle\!\rangle\varphi \notin \Sigma$ *and* $Y \cap Z = \varnothing$*. Let* $S(\Sigma)$ *denote the set of all saturated formulas in* Σ*.*

Lemma 1 *Let* $\Sigma \in W^q$ *be a MCS,* $\langle\!\langle X, Y, Z\rangle\!\rangle\varphi \in \Sigma$ *and* $S = Y \cup Z$*. Then there is* $T \in \mathcal{P}(\mathsf{V})$ *such that* $\langle\!\langle X, T, (Y \cup Z) \setminus T\rangle\!\rangle\varphi \in S(\Sigma)$*.*

Proof The proof proceeds by induction on the size n of $Y \setminus Z$. When $n = 0$, one obtains $Z = Y \cup Z$. By axiom (Ord,c), $\langle\!\langle X, \varnothing, Z\rangle\!\rangle\varphi \in \Sigma$. Note that $\bigvee \varnothing = \bot \notin \Sigma$, $\varnothing$ is the desired set. Suppose $n > 0$ and $Y \setminus Z = \{y_0, \cdots, y_{n-1}\}$. If $\langle\!\langle X, Y, Z \cup \{y_i\}\rangle\!\rangle\varphi \notin \Sigma$ for any $i < n$, then $T = Y \setminus Z$ satisfies the requirement. Suppose $\langle\!\langle X, Y, Z \cup \{y_i\}\rangle\!\rangle\varphi \in \Sigma$ for some $i < n$. Then we see $|Y \setminus (Z \cup \{y_i\})| < n$ and by induction hypothesis, there is $T \in \mathcal{P}(\mathsf{V})$ such that $\langle\!\langle X, T, (Y \cup Z) \setminus T\rangle\!\rangle\varphi \in S(\Sigma)$. Since $Y \cup Z = Y \cup (Z \cup \{y_i\})$, T satisfies the requirement. □

Lemma 2 *Let* $\Sigma \in W^q$ *and* $\langle\!\langle X, Y, Z\rangle\!\rangle\varphi \in S(\Sigma)$*. Then there is a MCS* $\Delta \in R^q(X, Y, Z)(\Sigma)$ *such that* $\varphi \in \Delta$ *and* $\Delta R^q(X, Y, \varnothing)\Sigma$*.*

Proof We write $\Box$ for $[\![X, Y, Z]\!]$ and $\blacklozenge$ for $\langle\!\langle X, Y, \varnothing\rangle\!\rangle$ in this proof. It is sufficient to show that $\Delta_0 = \{\psi : \Box\psi \in \Sigma\} \cup \{\blacklozenge\gamma : \gamma \in \Sigma\} \cup \{\varphi\}$ is consistent. Otherwise, there are formulas $\Box\psi_1, \cdots, \Box\psi_n, \gamma_1, \cdots, \gamma_m, \in \Sigma$ such that

$$\vdash_{\mathsf{C}} \psi_1 \wedge \cdots \wedge \psi_n \wedge \blacklozenge\gamma_1 \wedge \cdots \wedge \blacklozenge\gamma_m \wedge \varphi \to \bot.$$

Let $\gamma = \gamma_1 \wedge \cdots \wedge \gamma_m$ and $\psi = \psi_1 \wedge \cdots \wedge \psi_n$. Clearly, $\gamma \in \Sigma$. By axiom (Nec) and (K), we have $\vdash_{\mathsf{C}} \blacklozenge\gamma \to (\blacklozenge\gamma_1 \wedge \cdots \wedge \blacklozenge\gamma_m)$. Thus $\vdash_{\mathsf{C}} \psi \wedge \blacklozenge\gamma \wedge \varphi \to \bot$, which entails $\vdash_{\mathsf{C}} \Box\psi \to \neg\Diamond(\varphi \wedge \blacklozenge\gamma)$. Note that $\Box\psi \in \Sigma$, we have $\neg\Diamond(\varphi \wedge \blacklozenge\gamma) \in \Sigma$. Since $\gamma \wedge \Diamond\varphi \in \Sigma$ and $(\gamma \wedge \Diamond\varphi) \to \Diamond(\varphi \wedge \blacklozenge\gamma) \vee \bigvee_{y \in Y}\langle\!\langle X, Y, Z \cup \{y\}\rangle\!\rangle\varphi$ is an instant of axiom (Ord,e), we obtain $\bigvee_{y \in Y}\langle\!\langle X, Y, Z \cup \{y\}\rangle\!\rangle \in \Sigma$, which contradicts that $\langle\!\langle X, Y, Z\rangle\!\rangle\varphi \in S(\Sigma)$. □

With the help of Lemma 1 and Lemma 2, we are now able to define the paths in W^q, which constitute the domain of our desired model.

Definition 14 *A path in* W^q *is a sequence* $\pi = \langle \Sigma_0, \psi_0, \cdots, \Sigma_{n-1}, \psi_{n-1}, \Sigma_n\rangle$ *in which the following conditions hold for all* $i < n$*:*

– $\psi_i = \langle\!\langle X_i, Y_i, Z_i\rangle\!\rangle\varphi_i \in S(\Sigma_i)$ *is a saturated formula in* $\Sigma_i \in W^q$*;*

– $\varphi_i \in \Sigma_{i+1} \in W^q$, $\Sigma_{i+1} R^q(X_i, Y_i, \varnothing)\Sigma_i$ *and* $\Sigma_i R^q(X_i, Y_i, Z_i)\Sigma_{i+1}$.

We denote Σ_0 *by start*(π), Σ_n *by last*(π) *and the set of all paths by Path.*

In what follows, let Γ be some fixed consistent set. Without loss of generality, suppose Γ is a MCS. We now construct a model for Γ.

Definition 15 *(*Γ*-***Canonical PD-model***) The* Γ*-canonical PD-model* $\mathfrak{M}^c_\Gamma = (\mathfrak{F}^c_\Gamma, V^c)$*, in which* $\mathfrak{F}^c_\Gamma = (W^c_\Gamma, \leq^c, \sim^c)$*, is defined as follows:*

– $W^c_\Gamma = \{\pi \in Path : start(\pi) = \Gamma\}$*, and we write* W^c *for* W^c_Γ *in what follows;*
– *for all* $y \in \mathsf{V}$ *and* $\pi, \pi' \in W^c$, $\pi \leq_y \pi'$ *iff one of the following holds:*

  - $\pi' = \langle \pi, \langle\!\langle X, Y, Z\rangle\!\rangle\varphi, \Sigma\rangle$ *and* $y \in Y \cup Z$;
  - $\pi = \langle \pi', \langle\!\langle X, Y, Z\rangle\!\rangle\varphi, \Sigma\rangle$ *and* $y \in Y$;
  - $\pi = \pi'$.

  Let $\leq^c_y$ *be the transitive closure of* $\leq_y$.
– *for all* $s \in \mathsf{V}$ *and* $\pi, \pi' \in W^c$, $\pi \rightharpoonup_s \pi'$ *if and only if* $\pi' = \langle \pi, \langle\!\langle X, Y, Z\rangle\!\rangle\varphi, \Sigma\rangle$ *and* $D_X s \in last(\pi)$*. Let* $\rightleftharpoons_s$ *be the reflexive-symmetric closure of* $\rightharpoonup_s$.
  Let $\sim^c_s$ *be the transitive closure of* $\rightleftharpoons_s$.
– *for all* $P\vec{x} \in \mathcal{L}$, $V^c(P\vec{x}) = \{\pi \in W^c : P\vec{x} \in last(\pi)\}$.

For all $X, Y, Z \in [\mathsf{V}]^{<\omega}$*, the binary relations* $R^c(X, Y, Z)$, $\sim^c_X$, $\leq^c_Y$ *and* $<^c_Z$ *are defined in the natural way. By Axiom (Dep,a),* $D_X X$ *always holds. Thus for each* $\pi \in W^c_\Gamma$ *and* $\pi' = \langle \pi, \langle\!\langle X, Y, Z\rangle\!\rangle\varphi, \Delta\rangle$*, we have* $\pi R^c(X, Y, Z)\pi'$.

To characterize the structure of W^c, we define $T \subseteq W^c \times W^c$ as follows:

$$\pi T \pi' \text{ if and only if } \pi \text{ is of the form} \langle \pi, \langle\!\langle X, Y, Z\rangle\!\rangle\varphi, \Sigma\rangle.$$

It is clear that (W^c, T) is a tree. Then for all $\pi, \pi' \in W^c$, there is a shortest T-sequence $\langle \pi_0, \cdots, \pi_n\rangle$ such that $\pi = \pi_0$, $\pi' = \pi_n$ and for all $i < n$, $\pi_i T \pi_{i+1}$ or $\pi_{i+1} T \pi_i$. We denote the shortest sequence by $T^\pi_{\pi'}$.

Fact 5 *Let* $\pi, \pi' \in W^c$, $T^\pi_{\pi'} = \langle \pi_0, \cdots, \pi_n\rangle$ *and* $y, s \in \mathsf{V}$*. Then*

(1) $\pi \sim^c_s \pi'$ *iff* $\pi_i \rightleftharpoons_s \pi_{i+1}$ *for all* $i < n$.
(2) $\pi \leq^c_y \pi'$ *iff* $\pi_i \leq_y \pi_{i+1}$ *for all* $i < n$.

Proof Since $\rightleftharpoons_s, \leq_y \subseteq (T \cup T^{-1})$, $\sim^c_s$ is the transitive closure of $\rightleftharpoons_s$ and $\leq^c_y$ the transitive closure of $\leq_y$, the proof can be done by induction on n easily. □

In what follows, we show that the relations $R^c(X, Y, Z)$ are consistent with the relations $R^q(X, Y, Z)$.

Lemma 3 *Let* $\pi, \pi' \in W^c$, $X, Y, Z \subseteq \mathsf{V}$, $\pi \rightleftharpoons_X \pi'$, $\pi \leq_Y \pi'$ *and* $\pi <_Z \pi'$*. Then*

(1) $last(\pi) R^q(X, Y, Z) last(\pi')$.
(2) if $D_X S \in last(\pi)$*, then* $\pi \rightleftharpoons_S \pi'$.

Proof Suppose $\pi \rightleftharpoons_X \pi'$, $\pi \leq_Y \pi'$ and $\pi <_Z \pi'$. Then we have three cases:

- $\pi = \pi'$. Then $Z = \varnothing$. By Proposition 3(1), $R^q(X, Y, \varnothing)$ is reflexive and $\text{last}(\pi) R^q(X, Y, Z) \text{last}(\pi')$.
- $\pi = \langle \pi', \langle\!\langle X', Y', Z' \rangle\!\rangle \psi, \Delta \rangle$. Then $Z = \varnothing$, $Y \subseteq Y'$ and $D_{X'}X \in \text{last}(\pi')$. Clearly, $\text{last}(\pi') R^q(X', Y', Z') \text{last}(\pi)$. by Proposition 3(4), $D_{X'}X \in \text{last}(\pi)$. Recall that one has $\text{last}(\pi) R^q(X', Y', \varnothing) \text{last}(\pi')$, by Proposition 3(2,4), we see $\text{last}(\pi) R^q(X, Y, Z) \text{last}(\pi')$.
- $\pi' = \langle \pi, \langle\!\langle X', Y', Z' \rangle\!\rangle \psi, \Delta \rangle$. Then $Z \subseteq Z'$, $Y \subseteq Y' \cup Z'$ and $D_{X'}X \in \text{last}(\pi)$. Note that $\text{last}(\pi) R^q(X', Y', Z') \text{last}(\pi')$, by Proposition 3(2,4), we see $\text{last}(\pi') R^q(X, Y, Z) \text{last}(\pi')$.

Hence $\text{last}(\pi) R^q(X, Y, Z) \text{last}(\pi')$ and (1) holds.

For (2), suppose $D_X S \in \text{last}(\pi)$. Then we have also three cases:

- $\pi = \pi'$. Note that $\rightleftharpoons_S$ is reflexive, $\pi \rightleftharpoons_S \pi'$.
- $\pi \rightharpoonup_X \pi'$. Then π' is of the form $\langle \pi, \langle\!\langle X', Y', Z' \rangle\!\rangle \psi, \Delta \rangle$ and $D_{X'}X \in \text{last}(\pi)$. By axiom (Dep,c), $D_{X'}S \in \text{last}(\pi)$. Thus $\pi \rightharpoonup_S \pi'$.
- $\pi' \rightharpoonup_X \pi$. Then π is of the form $\langle \pi', \langle\!\langle X', Y', Z' \rangle\!\rangle \psi, \Delta \rangle$ and $D_{X'}X \in \text{last}(\pi')$. By (1), $D_X S \in \text{last}(\pi')$. By axiom (Dep,c), $D_{X'}S \in \text{last}(\pi)$. Thus $\pi' \rightharpoonup_S \pi$.

Hence $\pi \rightleftharpoons_S \pi'$ and (2) holds. □

Lemma 4 *Let* $\pi, \pi' \in W^c$, $X, Y, Z \in [\mathsf{V}]^{<\omega}$ *and* $\pi R^c(X, Y, Z) \pi'$. *Then*

(1) $\mathit{last}(\pi) R^q(X, Y, Z) \mathit{last}(\pi')$.
(2) $D_X S \in \mathit{last}(\pi)$ *implies* $\pi R^c(S, Y, Z) \pi'$.

Proof Suppose $\pi R^c(X, Y, Z) \pi'$. Then $\pi \sim^c_X \pi'$, $\pi \leq^c_{Y \cup Z} \pi'$ and $\pi <^c_Z \pi'$. Let $T^{\pi}_{\pi'} = \langle \pi_0, \cdots, \pi_n \rangle$. By Fact 5, for all $i < n$, $\pi_i \rightleftharpoons_X \pi_{i+1}$ and $\pi_i \leq_{Y \cup Z} \pi_{i+1}$. Moreover, for each $z \in Z$, there is $i_z \in n$ such that $\pi_{i_z} <_z \pi_{i_z+1}$. Then by Lemma 3(1), $\text{last}(\pi_i) R^q(X, Y \cup Z, \varnothing) \text{last}(\pi_{i+1})$ for all $i \in n$ and for all $z \in Z$, $\text{last}(\pi_{i_z}) R^q(X, Y \cup Z, \{z\}) \text{last}(\pi_{i_z+1})$. Then by Proposition 3(2,3), we see $\text{last}(\pi) R^q(X, Y, Z) \text{last}(\pi')$ and (1) holds. Suppose $D_X S \in \text{last}(\pi)$. Note that $\pi \sim^c_X \pi_i$ for all $i \leq n$, by (1), $D_X S \in \text{last}(\pi_i)$ for all $i \leq n$. Then by Lemma 3(2), $\pi_i \rightleftharpoons_S \pi_{i+1}$ for all $i \in n$, which entails $\pi \sim^c_S \pi'$. □

The final step is to show that $\mathfrak{M}^c$ is a PD-model in which Γ is satisfiable.

Lemma 5 $\mathfrak{M}^c$ *is a PD-model.*

Proof It suffices to show that V^c satisfies (Val). Let $\pi, \pi' \in W^c$ be points such that $\pi \sim^c_X \pi'$. By Lemma 4, $\text{last}(\pi) R^q(X, \varnothing, \varnothing) \text{last}(\pi')$. Assume $P\vec{x} \in \text{last}(\pi)$, then by axiom (Dep,b), $\mathbb{D}_X P\vec{x} \in \text{last}(\pi)$, which entails $P\vec{x} \in \text{last}(\pi')$. Similarly, we can verify that $P\vec{x} \in \text{last}(\pi')$ implies $P\vec{x} \in \text{last}(\pi)$. Thus V^c satisfies (Val) and so $\mathfrak{M}^c_\Gamma$ is a PD-model. □

Lemma 6 **(Truth Lemma)** *For each formula* $\varphi \in \mathcal{L}^{\preceq}$ *and path* $\pi \in W^c$, $\mathfrak{M}^c, \pi \models \varphi$ *if and only if* $\varphi \in \mathit{last}(\pi)$.

Proof The proof proceeds by induction on the complexity of φ. The case when φ is of the form $P\vec{x}$ is trivial. The Boolean cases are also trivial. Let φ be of the form

$D_X s$. Suppose $D_X s \in \text{last}(\pi)$. Let $\pi' \in W^c$ such that $\pi \sim^c_X \pi'$. By Lemma 4, $\text{last}(\pi) R^q(X, \varnothing, \varnothing)\text{last}(\pi')$. Then by Proposition 3(2,4), $\pi \sim^c_s \pi'$. Thus $\mathfrak{M}^c, \pi \models D_X s$. Suppose $D_X s \notin \text{last}(\pi)$. Let $\pi' = \langle \pi, \langle\!\langle X, \varnothing, \varnothing \rangle\!\rangle \top, \text{last}(\pi)\rangle$. Then $\pi \not\rightharpoonup_s \pi'$ and so $\pi \not\leftrightharpoons_s \pi'$. Clearly, $T^{\pi}_{\pi'} = \langle \pi, \pi' \rangle$. By Fact 5, $\pi \not\sim^c_s \pi'$. Note that $\pi \sim^c_X \pi'$, we see $\mathfrak{M}^c, \pi \not\models D_X s$. Let $\varphi = \langle\!\langle X, Y, Z \rangle\!\rangle \psi$. Suppose $\mathfrak{M}^c, \pi \models \varphi$. Then there is $\pi' \in R^c(X, Y, Z)\pi$ such that $\mathfrak{M}^c, \pi' \models \psi$. By induction hypothesis, $\psi \in \text{last}(\pi')$. By Lemma 4, $\text{last}(\pi) R^q(X, Y, Z)\text{last}(\pi')$. Then $\varphi \in \text{last}(\pi)$. Suppose $\varphi \in \text{last}(\pi)$. Without loss of generality, assume that $\varphi \in S(\text{last}(\pi))$. Then by Lemma 2, there is a Δ such that $\pi' = \langle \pi, \varphi, \Delta \rangle$ is a path with $\psi \in \text{last}(\pi')$. By induction hypothesis, $\mathfrak{M}^c, \pi' \models \psi$. Note that $\pi R^c(X, Y, Z)\pi'$, we have $\mathfrak{M}^c, \pi \models \varphi$. □

Theorem 5 *For each $\Gamma \subseteq \mathcal{L}^{\preceq}$, if Γ is consistent, then Γ is satisfiable.*

4.4 Properties of LPFD

In this part, we prove that LPFD lacks the finite model property. The decidability of LPFD shall also be shown.

Theorem 6 *LPFD lacks the finite model property, that is, there exists a formula $\varphi \in \mathcal{L}^{\preceq}$ which is only satisfiable in infinite RPD-models.*

Proof Let $\varphi = \neg([\![\varnothing, \varnothing, \{z\}]\!]\bot \vee \langle\!\langle \varnothing, \varnothing, \{z\} \rangle\!\rangle [\![\varnothing, \varnothing, \{z\}]\!]\bot)$. Note that for each PD-frame $\mathfrak{F} = (W, \sim, \leq)$ and $z \in \mathsf{V}$, $<_z$ is irreflexive and transitive. Thus for each finite PD-frame $\mathfrak{G}$, we have $\mathfrak{G} \models [\![\varnothing, \varnothing, \{z\}]\!]\bot \vee \langle\!\langle \varnothing, \varnothing, \{z\} \rangle\!\rangle [\![\varnothing, \varnothing, \{z\}]\!]\bot$. Clearly, φ is satisfiable in $(\omega, \sim, \leq)$, where $\leq_z$ is the usual $\leq$ relation on ω. □

In what follows, let α be some fixed formula, V_α the set of variables occur in α and Pred_α the set of predicates occur in α. Without loss of generality, we assume that the modal depth of α is not 0. Then we define $\text{Vo} = (V_\alpha, \mathsf{Pred}_\alpha, ar \restriction V_\alpha)$ as the vocabulary restricted to α. Let $\mathcal{L}_\alpha$ be the fragment of $\mathcal{L}^{\preceq}$ based on Vo, in which every formula is of modal degree no more than α. It can be easily verified that up to modal equivalence, $\mathcal{L}_\alpha$ contains only finitely many formulas.

Definition 16 *A set Γ of $\mathcal{L}_\alpha$-formulas is said to be a $\mathcal{L}_\alpha$-maximal consistent set if $\Gamma \nvdash_{\mathsf{C}} \bot$ and $\Gamma' \vdash_{\mathsf{C}} \bot$ for all Γ' such that $\Gamma \subsetneq \Gamma' \subseteq \mathcal{L}_\alpha$. Let MCS_α denote the set of all $\mathcal{L}_\alpha$-maximal consistent sets. For all $X, Y, Z \subseteq V_\alpha$ and $\Delta, \Sigma \in MCS_\alpha$, we write $\Delta R^p_\alpha(X, Y, Z)\Sigma$ if*

$$\{\langle\!\langle X \cap X', Y \cap Y', (Z \cap Y') \cup (Z' \cap Y) \cup (Z \cap Z') \rangle\!\rangle \varphi \in \mathcal{L}_\alpha : \langle\!\langle X', Y', Z' \rangle\!\rangle \varphi \in \Sigma\} \subseteq \Delta.$$

One may find that the definition of $R^p_\alpha(X, Y, Z)$ is modified from the Lemmon filtration. Given that $\langle\!\langle X, Y, Z \rangle\!\rangle \varphi \in \mathcal{L}_\alpha$ and $\Delta R^p_\alpha(X, Y, Z)\Sigma$, we see $\varphi \in \Sigma$ implies $\langle\!\langle X, Y, \varnothing \rangle\!\rangle \varphi \in \Sigma$ and so $\langle\!\langle X, Y, Z \rangle\!\rangle \varphi \in \Delta$. Then we have the following proposition:

Proposition 4 *For all $\Delta_1, \Delta_2, \Delta_3 \in MCS_\alpha$ and $X, Y, Z \subseteq V_\alpha$,*

(1) $R^p_\alpha(X, Y, \varnothing)$ is reflexive;

(2) If $\Delta_1 R^p_\alpha(X, Y, Z)\Delta_2$*, then* $\Delta_1 R^p_\alpha(X', Y', Z')\Delta_2$ *for all* $X' \subseteq X$, $Y' \subseteq Y \cup Z$ *and* $Z' \subseteq Z$*;*
(3) If $D_X S \in \Delta_1$ *and* $\Delta_1 R^p_\alpha(X, Y, Z)\Delta_2$*, then* $\Delta_1 R^p_\alpha(S, Y, Z)\Delta_2$ *and* $D_X S \in \Delta_2$
(4) For all $Z' \in [\mathsf{V}_\alpha]^{<\omega}$*, if* $Z, Z' \subseteq Y$, $\Delta_1 R^p_\alpha(X, Y, Z)\Delta_2$ *and* $\Delta_2 R^p_\alpha(X, Y, Z')\Delta_3$*, then* $\Delta_1 R^p_\alpha(X, Y, Z \cup Z')\Delta_3$*.*

Proof (1) and (2) are trivial. For (3), $\Delta_1 R^p_\alpha(S, Y, Z)\Delta_2$ follows from axiom (Dep,d). Recall that the modal depth of α is not 0, we see $\mathbb{D}_X D_X S \in \Delta_1$ and so $D_X S \in \Delta_2$. For (4), suppose $\langle\!\langle X_0, Y_0, Z_0\rangle\!\rangle\varphi \in \Delta_3$. Then $\langle\!\langle X \cap X_0, Y \cap Y_0, (Z' \cap Y_0) \cup (Z_0 \cap Y) \cup (Z' \cap Z_0)\rangle\!\rangle\varphi \in \Delta_2$. Recall that $Z, Z' \subseteq Y$, it follows that

$$\langle\!\langle X \cap X_0, Y \cap Y_0, ((Z \cup Z') \cap Y_0) \cup (Z_0 \cap Y) \cup ((Z \cup Z') \cap Z_0)\rangle\!\rangle\varphi \in \Delta_1.$$

Thus $\Delta_1 R^p_\alpha(X, Y, Z \cup Z')\Delta_3$ and (4) holds. □

Definition 17 **($\mathcal{L}_\alpha$-Pre-model)** *An* $\mathcal{L}_\alpha$*-pre-model is a set* F *of* $\mathcal{L}_\alpha$*-MCSs such that for all* $X, Y, Z \subseteq V_\alpha$ *and* $\Delta \in F$*, the following statement holds:*

(†) If $\langle\!\langle X, Y, Z\rangle\!\rangle\varphi$ *is a saturated formula in* Δ*, then there is* $\Sigma \in F$ *such that* $\Delta R^p_\alpha(X, Y, Z)\Sigma$, $\varphi \in \Sigma$ *and* $\Sigma R^p_\alpha(X, Y, \varnothing)\Delta$*.*

We say φ *is satisfied in* F *if there is some* $\Delta \in F$ *such that* $\varphi \in \Delta$*.*

Lemma 7 *For each satisfiable* $\varphi \in \mathcal{L}^{\preceq}$, φ *is satisfied in some pre-model.*

Proof Let $\mathfrak{M} = (W, \sim, \leq, V)$ be an RPD-model and $w \in W$ such that $\mathfrak{M}, w \models \varphi$. Then we define $F_{\mathfrak{M}} = \{\Delta_w : w \in \mathfrak{M} \text{ and } \Delta_w = \{\varphi \in \mathcal{L}_\alpha : \mathfrak{M}, w \models \varphi\}\}$. It suffices to show that $F_{\mathfrak{M}}$ satisfies (†). Suppose $\langle\!\langle X, Y, Z\rangle\!\rangle\varphi$ is a saturated formula in Δ_w. Then $\mathfrak{M}, w \models \langle\!\langle X, Y, Z\rangle\!\rangle\varphi$ and there is $u \in R(X, Y, Z)(w)$ such that $\mathfrak{M}, u \models \varphi$. Note that $\langle\!\langle X, Y, Z\rangle\!\rangle\varphi$ is a saturated formula, we have $w \in R(X, Y, \varnothing)(u)$. Then it is not hard to verify that $\Delta_w R^p_\alpha(X, Y, Z)\Delta_u$ and $\Delta_u R^p_\alpha(X, Y, \varnothing)\Delta_w$. Recall that $\varphi \in \Delta_u$, we see that (†) holds for $F_{\mathfrak{M}}$. □

Definition 18 **(Induced Model)** *Let* F *be a pre-model. An* F*-path is a tuple* $\langle \Sigma_0, \psi_0, \cdots, \Sigma_{n-1}, \psi_{n-1}, \Sigma_n\rangle$ *where the following conditions hold for all* $i < n$*:*

- $\psi_i = \langle\!\langle X_i, Y_i, Z_i\rangle\!\rangle\varphi_i$ *is a saturated formula in* $\Sigma_i \in F$*;*
- $\varphi_i \in \Sigma_{i+1} \in F$, $\Sigma_{i+1} R^p_\alpha(X_i, Y_i, \varnothing)\Sigma_i$ *and* $\Sigma_i R^p_\alpha(X_i, Y_i, Z_i)\Sigma_{i+1}$*.*

The RPD-model $\mathfrak{M}^F = (W^F_\Gamma, \leq^F, \sim^F, V^F)$ *induced by* $\Gamma \in F$ *is defined by:*

- W^F_Γ *is the set of all paths in* F *begins with* φ*.*
- *for all* $y \in V_\alpha$ *and* $\pi, \pi' \in W^F_\Gamma$, $\pi \leq_y \pi'$ *iff one of the following holds:*
  - $\pi' = \langle \pi, \langle\!\langle X, Y, Z\rangle\!\rangle\varphi, \Sigma\rangle$ *and* $y \in Y \cup Z$*;*
  - $\pi = \langle \pi', \langle\!\langle X, Y, Z\rangle\!\rangle\varphi, \Sigma\rangle$ *and* $y \in Y$*.*

  Let $\leq^F_y$ *be the reflexive-transitive closure of* $\leq_y$*.*
- *for all* $s \in V_\alpha$ *and* $\pi, \pi' \in W^F_\Gamma$, $\pi \rightharpoonup_s \pi'$ *if and only if* $\pi' = \langle \pi, \langle\!\langle X, Y, Z\rangle\!\rangle\varphi, \Sigma\rangle$ *and* $D_X s \in last(\pi)$*. Let* $\sim^F_s$ *the reflexive-symmetric-transitive closure of* $\rightleftharpoons_s$*.*
- *for all* $P\vec{x} \in \mathcal{L}_\alpha$, $V^F(P\vec{x}) = \{\pi \in W^F : P\vec{x} \in last(\pi)\}$*.*

One may notice now that the construction of the desired model is almost the same as the one we used in the proof of Completeness Theorem. And similar to the proof of Completeness Theorem, with the help of Fact 4 and the definition of pre-models, we can verify that the following lemma holds:

Lemma 8 **(Truth Lemma)** *For each formula $\varphi \in \mathcal{L}_\alpha$ and path $\pi \in W^F$,*

$$\mathfrak{M}^F, \pi \models \varphi \text{ if and only if } \varphi \in last(\pi).$$

As a consequence, for each $\varphi \in \mathcal{L}^{\preceq}$, φ is satisfiable if and only if φ is satisfied in some $\mathcal{L}_\varphi$-pre-model. Recall that up to modal equivalence, $\mathcal{L}_\varphi$ contains finitely many formulas, MCS_φ is finite for each $\varphi \in \mathcal{L}^{\preceq}$, we obtain the following theorem:

Theorem 7 *The satisfiability problem of LPFD is decidable.*

4.5 The Hybrid Extension of LPFD

In this subsection, we extend LPFD with nominals. We will use this hybrid extension to express an important solution concept for cooperative games – the core – in Section 5.

By a vocabulary with nominals we mean a tuple $(\mathsf{V}, \mathsf{Pred}, \mathsf{Nom}, \mathsf{ar})$ where $(\mathsf{V}, \mathsf{Pred}, \mathsf{ar})$ is a vocabulary and $\mathsf{Nom} = \{i_k : k \in \omega\}$ a set of nominals.

The language $\mathcal{L}^{\preceq}_{\mathsf{Nom}}$ with nominals is given by:

$$\mathcal{L}^{\preceq}_{\mathsf{Nom}} \ni \varphi ::= P\vec{x} \mid D_X y \mid i \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\![X, Y, Z]\!]\varphi,$$

where $P \in \mathsf{Pred}$, $\vec{x} \in \mathsf{V}^{\mathsf{ar}(P)}$, $i \in \mathsf{Nom}$, $y \in \mathsf{V}$ and $X, Y, Z \in [\mathsf{V}]^{<\omega}$. The language $\mathcal{L}^{\preceq}_{\mathsf{Nom}}$ only differs from the language of LPFD in those nominals. For all $i \in \mathsf{Nom}$ and $\varphi \in \mathcal{L}^{\preceq}_{\mathsf{Nom}}$, we write $@_i\varphi$ for the formula $\langle\!\langle \varnothing, \varnothing, \varnothing \rangle\!\rangle(i \wedge \varphi)$.

A hybrid PD-model with nominals (HPD-model) is a tuple $\mathbb{M} = (O, I, A, \preceq)$ such that $\mathbb{M}' = (O, I{\upharpoonright}\mathsf{Pred}, A, \preceq)$ is a PD-model and $I{\upharpoonright}\mathsf{Nom}$ is a partial function from Nom to A. For each nominal $i \in \mathsf{Nom}$, $\mathbb{M}, a \models i$ if and only if $a = I(i)$. Similarly, a hybrid RPD-model (HRPD-model) is a tuple $\mathfrak{M} = (W, \sim, \leq, V)$ such that $\mathfrak{M} = (W, \sim, \leq, V{\upharpoonright}\mathsf{Pred})$ is an RPD-model and $V{\upharpoonright}\mathsf{Nom}$ is a partial function from Nom to W. For each nominal i, $\mathfrak{M}, w \models i$ if and only if $w = V(i)$.

For all class $\mathscr{C}$ of models mentioned in Section 4.1, we write $\mathscr{C}^h$ for the corresponding class of models with nominals. For example, Mod^h denotes the class of all HPD-models. Note that $|\mathsf{V}|$ is infinite, the equivalence between Mod^h and RMod^h can be established as in Section 4.1.

As usual, we call LPFD with nominals 'hybrid LPFD', abbreviated to HLPFD. Let Nom be a fixed set of nominals. We present here the calculus $\mathsf{C}_{\mathsf{Nom}}$ for HLPFD and show its soundness and completeness with respect to RMod^h. The axioms and rules of $\mathsf{C}_{\text{HLPFD}}$ are as follows:

(Tau) Axioms and rules for classical propositional logic;
(Nec) from φ infer $[\![X, Y, Z]\!]\varphi$;

(K) $[\![X, Y, Z]\!](\varphi \to \psi) \to ([\![X, Y, Z]\!]\varphi \to [\![X, Y, Z]\!]\psi)$;
(Dep) $\varphi \to \mathbb{D}_X\varphi$, provided $\varphi \in \{P\vec{x} : \mathsf{ran}(\vec{x}) \subseteq X\}$;
(Nom) $@_i\varphi \to [\![\varnothing, \varnothing, \varnothing]\!](i \to \varphi)$, provided $i \in \mathsf{Nom}$;
(Name) from $i \to \varphi$ infer φ, provided that $i \notin \varphi$, i.e., i does not occur in φ;
(Paste) from $@_i\langle\!\langle X, Y, Z\rangle\!\rangle j \to @_j\varphi$ infer $@_i[\![X, Y, Z]\!]\varphi$, provided $i \neq j$ and $j \notin \varphi$;

(DD) Axioms and rules for $[\![\ ]\!] - D$ interaction:
(1) $D_X s \wedge [\![\{s\}, \varnothing, \varnothing]\!]\varphi \to [\![X, \varnothing, \varnothing]\!]\varphi$;
(2) $i \wedge \neg D_X s \to \langle\!\langle X, \varnothing, \varnothing\rangle\!\rangle[\![s, \varnothing, \varnothing]\!]\neg i$.

(Ord) Axioms for the preference orders:
(1) $[\![X, Y, \varnothing]\!]\varphi \to \varphi$;
(2) $\varphi \to [\![\{v\}, \varnothing, \varnothing]\!]\langle\!\langle\{v\}, \varnothing, \varnothing\rangle\!\rangle\varphi$;
(3) $\langle\!\langle X, Y, Z\rangle\!\rangle\langle\!\langle X', Y', Z'\rangle\!\rangle\varphi \to \langle\!\langle X \cap X', Y \cap Y', (Z \cap Y') \cup (Z \cap Z') \cup (Y \cap Z')\rangle\!\rangle\varphi$;
(4) $@_i\langle\!\langle\varnothing, \varnothing, \{v\}\rangle\!\rangle j \leftrightarrow @_i\langle\!\langle\varnothing, \{v\}, \varnothing\rangle\!\rangle j \wedge @_j\neg\langle\!\langle\varnothing, \{v\}, \varnothing\rangle\!\rangle i$, provided $i, j \in \mathsf{Nom}$;
(5) $\langle\!\langle X, Y, Z\rangle\!\rangle i \wedge \langle\!\langle X', Y', Z'\rangle\!\rangle i \leftrightarrow \langle\!\langle X \cup X', Y \cup Y', Z \cup Z'\rangle\!\rangle i$, provided $i \in \mathsf{Nom}$.

Comparing $\mathsf{C}_{\mathsf{Nom}}$ with C, in addition to the standard axioms and rules for nominals, axioms (Ord,4,5) and (DD,2) are new, which characterize RPD-models in a more refined way. Note also that some old axioms in C are presented in $\mathsf{C}_{\mathsf{Nom}}$ in a different way. For example, axiom (DD,1) in $\mathsf{C}_{\mathsf{Nom}}$ are bottom-up versions of axioms (Dep,d) in C.

With the above mentioned changes in $\mathsf{C}_{\mathsf{Nom}}$ due to the addition of nominals, the completeness of $\mathsf{C}_{\mathsf{Nom}}$ can be proved by directly using the canonical model, which is a standard method and relatively routine. So we relegate the details of the following theorem's proof in the appendix.

Theorem 8 $\mathsf{C}_{\mathsf{Nom}}$ *is sound and strongly complete w.r.t* RMod^h *and* Mod^h.

As for the decidability of HLPFD, we cannot prove it by directly following the strategy used in the proof of LPFD's decidability. We will not attack this problem in this paper but rather leave it for future work.

Remark 1 When V is finite, the readers can easily verify that the formula $\varphi_d = i \to [\![\mathsf{V}, \varnothing, \varnothing]\!]i$ characterizes the class of differential HRPD-frames. As in Section 4.1, we see that $\mathsf{C}_{\mathsf{Nom}}$ is sound and strongly complete w.r.t RMod^h and $\mathsf{C}_{\mathsf{Nom}} \oplus \varphi_d$ is the desired calculus for Mod^h.

5 Cooperative Games and the Core in HLPFD

Should there be any difference between a collective action and an agglomeration of actions? This is a key issue in the philosophical analysis of collective agency [16]. In this section, we provide a game theoretical perspective on this issue by modeling cooperative games in strategic form [15, Section 11] and characterizing one of its solution concepts, the core, in HLPFD.

5.1 LPFD for Coalitional Power in Cooperative Games

Different from non-cooperative games, in cooperative games in strategic form [15, Section 11], players can not only act individually but also choose to join a coalition and act as a part of the coalition. In such games, the players in a coalition can do something together in agreement rather than separately. So collective actions and power are different from an agglomeration of individual actions and its effectiveness. In this part, we propose a framework based on LPFD to represent cooperative games and make the difference explicit. For simplicity, we restrict ourselves to the cases where the set of the players is finite.

To explicitly model coalitions as a different part of each player's choices from strategies, we distinguish between the terms "strategy"(or equivalently "action") and "choice".

Definition 19 (Players' Choices and Choices Merging)

- ***Players' Choices:*** *The set of the players' choices is defined as follows:*

$$O := \{f : I \to \Sigma \mid I \subseteq \mathsf{V}\}$$

  where Σ is the set of all possible strategies of all players.
- ***Choices Merging:*** *For $f, f' \in O$ with $\mathsf{dom}(f) \cap \mathsf{dom}(f') = \emptyset$, $f \oplus f' := f \cup f'$.*

For example, given three players $\mathsf{V} = \{1, 2, 3\}$ and the players' possible strategies in $\Sigma = \{\alpha, \beta\}$, $f = \{(1, \alpha), (3, \beta)\} \in O$ denotes a possible choice of the players 1 and 3 as a coalition; $f' = \{(2, \alpha)\} \in O$ denotes a possible choice of the player 2. Then $f \oplus f' = \{(1, \alpha), (2, \alpha), (3, \beta)\}$.

In a PD-model, there is no requirement on $A \subseteq O^{\mathsf{V}}$. This is not the case any longer when the players' choices concern forming coalitions. We impose three conditions on a *realizable* choice profile. First of all, a player cannot choose to form a coalition she is not in. Second, a player cannot choose to form a coalition without the others in the coalition making the same choice. Third, once a coalition forms, it acts as a whole, which means that its members act according to a unique strategy sequence. This strategy sequence can be seen as a collective plan which is made effective by common consent.

To make the definition of realizable choice profiles precise, we make use of the following notations.

Notation 20 – $\Pi(\mathsf{V})$ *is the set of all partitions of* V. [4]
- *Given $a \in O^{\mathsf{V}}$,*
  - *a_i denotes the ith element of a, which is a function;*
  - $a_{\mathsf{rng}} := \{a_i \in O \mid i \in \mathsf{V}\}$*;*
  - $a_{\mathsf{dom}} := \{\mathsf{dom}(a_i) \subseteq \mathsf{V} \mid i \in \mathsf{V}\}$*;*

Definition 21 (Realizable Choice and Strategy Profiles)

[4] A partition of V is a set of non-empty subsets of V whose union is V and which do not intersect each other.

– *A choice profile* $a \in O^N$ *is realizable if and only if it satisfies the following three conditions:*

1. $i \in \mathsf{dom}(a_i)$*;*
2. $a_{\mathsf{dom}} \in \Pi(\mathsf{V})$
3. $\mathsf{dom}(a_i) = \mathsf{dom}(a_j)$ *implies that* $a_i = a_j$ *for all* $i, j \in \mathsf{V}$.

– *Let* Ξ *denote the set of all realizable choice profiles.*
Let $a_{\mathsf{merge}} := \bigoplus_{f \in a_{\mathsf{rng}}} f$ *for* $a \in \Xi$*. Given* $A \subseteq \Xi$*, the set of all realizable strategy profiles of a partition* $\pi \in \Pi(\mathsf{V})$ *in* A *is*

$$\sigma_A(\pi) := \{a_{\mathsf{merge}} \mid a \in A \text{ and } a_{\mathsf{dom}} = \pi\} \ .$$

When there is no danger of ambiguity, we will leave out the subscript A*.*

Having defined O and Ξ, we define a class of PD-models we will work with.

Definition 22 (Coalition-preference-dependence (CPD) models) *A coalition-preference-dependence model is a PD-model* $\mathbb{M} = ((O, I), A)$ *in which* O *is defined in Definition 19 and* A *and* $\preceq_i$ *satisfy the following conditions:*

1. $A \subseteq \Xi$*;*
2. $\{a_{\mathsf{dom}} \mid a \in A\} = \Pi(\mathsf{V})$*;*
3. *if* $\pi \in \Pi(\mathsf{V})$ *is finer than* $\pi' \in \Pi(\mathsf{V})$,[5] *then* $\sigma_A(\pi) \subseteq \sigma_A(\pi')$*;*
4. *if* $a_{\mathsf{merge}} = a'_{\mathsf{merge}}$*, then* $a \simeq_i a'$ *for all* $i \in \mathsf{V}$*;*
5. $\preceq_i$ *is total for all* $i \in \mathsf{V}$.

The first condition says that A should contain realizable choice profiles. The second condition says that the players can form coalitions according to all possible partitions of N. The third condition requires bigger coalitions to have no less strategies than smaller coalitions. The fourth condition requires that the players' preference relations depend directly on strategy profiles. The players' choices of coalitions can only influence the players' preferences by affecting their strategies. The last condition requires the players' preference relations to be total, which is a standard assumption in game theory.

The following example illustrates our notations and the CPD-models.

Example 2 Let $\mathsf{V} = \{1, 2, 3\}$ and $\Sigma = \{\alpha, \beta, \gamma\}$. A is given in Table 2.

According to our notation,

– $a_{\mathsf{merge}} = a'_{\mathsf{merge}} = a^{3\prime}_{\mathsf{merge}} = a^{5\prime}_{\mathsf{merge}} = a^{7\prime}_{\mathsf{merge}} = \{(1, \alpha), (2, \beta), (3, \alpha)\}$;
– $\sigma(\{\{1\}, \{2\}, \{3\}\}) = \{\{(1, \alpha), (2, \beta), (3, \alpha)\}\}$ and $\sigma(\{\{1, 2\}, \{3\}\}) = \{\{(1, \alpha), (2, \beta), (3, \alpha)\}, \{(1, \alpha), (2, \gamma), (3, \beta)\}\}$.

As the readers can verify, all the requirements of a CPD-model concerning A are satisfied here. For example, $\sigma(\{\{1\}, \{2\}, \{3\}\}) \subseteq \sigma(\{\{1, 2\}, \{3\}\}) \subseteq \sigma(\{N\})$. To make sure $\preceq_i$ satisfy the requirements, $a \simeq_i a' \simeq_i a^{3\prime} \simeq_i a^{5\prime} \simeq_i a^{7\prime}$ needs to be the case.

[5] That is, for all $X \in \pi$ there is $X' \in \pi'$ such that $X \subseteq X'$.

Table 2 A in Example 2

	1	2	3
a	$\{(1,\alpha)\}$	$\{(2,\beta)\}$	$\{(3,\alpha)\}$
a'	$\{(1,\alpha),(2,\beta)\}$	$\{(1,\alpha),(2,\beta)\}$	$\{(3,\alpha)\}$
a''	$\{(1,\alpha),(2,\gamma)\}$	$\{(1,\alpha),(2,\gamma)\}$	$\{(3,\beta)\}$
$a^{3\prime}$	$\{(1,\alpha)\}$	$\{(2,\beta),(3,\alpha)\}$	$\{(2,\beta),(3,\alpha)\}$
$a^{4\prime}$	$\{(1,\beta)\}$	$\{(2,\beta),(3,\gamma)\}$	$\{(2,\beta),(3,\gamma)\}$
$a^{5\prime}$	$\{(1,\alpha),(3,\alpha)\}$	$\{(2,\beta)\}$	$\{(1,\alpha),(3,\alpha)\}$
$a^{6\prime}$	$\{(1,\gamma),(3,\alpha)\}$	$\{(2,\alpha)\}$	$\{(1,\gamma),(3,\alpha)\}$
$a^{7\prime}$	$\{(1,\alpha),(2,\beta),(3,\alpha)\}$	$\{(1,\alpha),(2,\beta),(3,\alpha)\}$	$\{(1,\alpha),(2,\beta),(3,\alpha)\}$
$a^{8\prime}$	$\{(1,\alpha),(2,\gamma),(3,\beta)\}$	$\{(1,\alpha),(2,\gamma),(3,\beta)\}$	$\{(1,\alpha),(2,\gamma),(3,\beta)\}$
$a^{9\prime}$	$\{(1,\beta),(2,\beta),(3,\gamma)\}$	$\{(1,\beta),(2,\beta),(3,\gamma)\}$	$\{(1,\beta),(2,\beta),(3,\gamma)\}$
$a^{10\prime}$	$\{(1,\gamma),(2,\alpha),(3,\alpha)\}$	$\{(1,\gamma),(2,\alpha),(3,\alpha)\}$	$\{(1,\gamma),(2,\alpha),(3,\alpha)\}$
$a^{11\prime}$	$\{(1,\gamma),(2,\gamma),(3,\gamma)\}$	$\{(1,\gamma),(2,\gamma),(3,\gamma)\}$	$\{(1,\gamma),(2,\gamma),(3,\gamma)\}$

As can be easily spotted in the above example, coalitions are explicitly incorporated into the players' choices in the CPD-models. Once a coalition forms, the players in it act as a whole. Moreover, a coalition could possibly do more than its constituent parts.

The coalition partition formed in a game directly affects each player's strategy. Hence it has a substantial influence on the final outcome of the game. Can the language of LPFD express what partition is formed in a realizable choice profile? The following proposition gives a partially positive answer.

Proposition 5 *Let $\mathbb{M} = ((M, A), \preceq)$ be a CPD-model with $\mathbb{M}, a' \models \neg D_X(-X)$ for all $a' \in A$ satisfying $a'_{\mathsf{dom}} = \{X, -X\}$. Then for all $a \in A$ and non-empty subset $X \subseteq \mathsf{V}$, the following two are equivalent:*

1. *$X \in a_{\mathsf{dom}}$;*
2. *$\mathbb{M}, a \models \bigwedge_{i \in X} D_i X \wedge \bigwedge_{j \notin X} \neg D_X j$.*

Proof **From 1 to 2.**

Assume $X \in a_{\mathsf{dom}}$. Suppose $a' \in A$ and $a =_i a'$ for some $i \in X$. Then $X \in a'_{\mathsf{dom}}$. Since $A \subseteq \Xi$, $a_i = a_j$ and $a'_i = a'_j$ for all $i, j \in X$. Note that $a =_i a'$ for some $i \in X$, we see $a_j = a_i = a'_i = a'_j$ for all $j \in X$, i.e. $a =_X a'$. Thus $\mathbb{M}, a \models D_i X$. By the arbitrariness of $i \in X$, we see $\mathbb{M}, a \models \bigwedge_{i \in X} D_i X$.

When $X = \mathsf{V}$, we see that $\bigwedge_{j \notin X} \neg D_X j$ is $\top$ and $\mathbb{M}, a \models \bigwedge_{j \notin X} \neg D_X j$. Suppose $X \neq \mathsf{V}$. Take an arbitrary $j \notin X$. Then we have the following cases:

- $a_{\mathsf{dom}} \neq \{X, -X\}$. Let $\pi = \{X, -X\}$. Note that $\sigma_A(a_{\mathsf{dom}}) \subseteq \sigma_A(\pi)$, there must be $b \in A$ such that $b_{\mathsf{dom}} = \pi$ and $a_{\mathsf{merge}} = b_{\mathsf{merge}}$. Then it must be the case that $\mathsf{dom}(b_j) = -X \neq \mathsf{dom}(a_j)$ and so $a \neq_j b$.
- $a_{\mathsf{dom}} = \{X, -X\}$. Since $\mathbb{M}, a \models \neg D_X(-X)$, there must be $b \in A$ such that $a =_X b$ and $a \neq_{-X} b$. If $a_{\mathsf{dom}} \neq b_{\mathsf{dom}}$, then $\mathsf{dom}(a_j) = -X \neq \mathsf{dom}(b_j)$ and so $a \neq_j b$. Suppose $a_{\mathsf{dom}} = b_{\mathsf{dom}}$. Then a_k are all the same for $k \in -X$ and b_h are all the same for $h \in -X$. Since $a \neq_{-X} b$, we see $a_j \neq b_j$.

Hence $\mathbb{M}, a \models \neg D_X j$. By the arbitrariness of j, we see $\mathbb{M}, a \models \bigwedge_{j \notin X} \neg D_X j$.

From 2 to 1.

Assume that $X \notin a_{\mathsf{dom}}$ and $\mathbb{M}, a \models \bigwedge_{i \in X} D_i X \wedge \bigwedge_{j \notin X} \neg D_X j$. Let $x \in X$.

- $X \subsetneq \mathsf{dom}(a_x)$. Then there is $j \in \mathsf{dom}(a_x) \setminus X$ such that $a_j = a_i$ for all $i \in \mathsf{dom}(a_x)$. So for all $a' =_X a$, $a'_j = a_i = a'_i$ for all $i \in X$. Then we have $\mathbb{M}, a \models D_X j$ where $j \notin X$. Contradiction!
- Otherwise, there is $j \in X \setminus \mathsf{dom}(a_x)$. Since $\mathbb{M}, a \models D_x X$, we see $\mathbb{M}, a \models D_{\mathsf{dom}(a_x)} j$. By the direction we have proved above, $\mathbb{M}, a \models \neg D_{\mathsf{dom}(a_x)} j$, which is a contradiction.

□

The assumption of the above proposition that $\mathbb{M}, a \models \neg D_X(-X)$ for all $a \in A$ satisfying $a'_{\mathsf{dom}} = \{X, -X\}$ requires that no coalition can completely decides what its complementary coalition chooses to do. If X can completely control what $-X$ chooses, then the division of X and $-X$ is senseless, because $\mathbb{M}, a \models D_X \mathsf{V}$ follows from $\mathbb{M}, a \models D_X(-X)$. As the readers can verify, the CPD-model in Example 2 does not satisfy the assumption at a'', $a^{4\prime}$, $a^{6\prime}$.

To avoid vacuous coalitions division, we will work with the CPD-models with the above assumption.

Definition 23 **(Real CPD-models)** *A real CPD-model (RCPD-model)* $\mathbb{M}$ *is a CPD-model that satisfies the assumption that* $\mathbb{M}, a \models \neg D_X(-X)$ *for all* $a \in A$ *satisfying* $a_{\mathsf{dom}} = \{X, -X\}$.

In an RCPD-model, $\bigwedge_{i \in X} D_i X \wedge \bigwedge_{j \notin X} \neg D_X j$ expresses that X is in the coalition partition. We will use the abbreviation

$$p_X := \bigwedge_{i \in X} D_i X \wedge \bigwedge_{j \notin X} \neg D_X j$$

for convenience in the next section.

5.2 The Core in HLPFD

Having set up the LPFD framework for representing cooperative games in strategic and coalitional form, in this part, we show that the core, an important solutions concept in the cooperative game theory, can be expressed in HLPFD. Moreover, by considering functional dependence explicitly, we generalize the core and show how it is related to Nash equilibrium and Pareto optimality.

Just as Nash equilibrium in non-cooperative games captures stability of a strategy profile, the concept of the core, as a basic solution concept in cooperative games, also captures stability of a strategy profile in cooperative games. The difference is that the core takes the stability of a coalition into consideration. There are other notions for characterizing stability in cooperative games, for example, stable set, bargaining set and so on. We focus on the core.

The concept of the core is formulated in CPD-models as follows. [6]

Definition 24 **(Core in CPD-Model)** *Given a CPD-model $\mathbb{M}$, a choice profile $a \in A$ is in the core of $\mathbb{M}$ if and only if*

1. $a_{\mathsf{dom}} = \{\mathsf{V}\}$*; and*
2. *there is no $X \subseteq$ and $a' \in A$ such that*
   (a) $X \in a'_{\mathsf{dom}}$*; and*
   (b) *for all $a'' =_X a'$ and all $i \in X$, $a \prec_i a''$.*

Let $Co_{\mathbb{M}}$ denote the core of $\mathbb{M}$.

If the set of the players V arrives at a choice profile a, which is in the core, then no $X \subset \mathsf{V}$ has any incentive to deviate from the coalition V, because forming the coalition X cannot guarantee all players in X end up with a better outcome. Coalitional power plays a key role in the basic idea of the core, because whether X has any incentive to deviate depends on whether X as a coalition can force a choice profile that all of its members prefer to the current choice profile.

Note that according to the definition of the core, if $X = \mathsf{V}$, there is no other choice profile with the coalition partition $\{\mathsf{V}\}$ which is strictly preferred by every player in V. Namely, a is weakly Pareto optimal among the choice profiles with the coalition partition $\{\mathsf{V}\}$. In fact, the following proposition holds.

Proposition 6 *Given a CPD-model $\mathbb{M}$, if a choice profile $a \in A$ is in the core of $\mathbb{M}$ then a is weakly Pareto optimal.*

Proof Since $\mathbb{M}$ satisfies the condition that $\sigma_A(\pi) \subseteq \sigma_A(\{\mathsf{V}\})$ for all $\pi \in \Pi(\mathsf{V})$, by the fourth condition of Definition 22, the weak Pareto optimality of a within the choice profiles having $\{\mathsf{V}\}$ as their coalition partition can be generalized trivially to all choice profiles. □

The following example illustrates the concept of core and how it differs from Nash equilibrium and Pareto optimality.

Example 3 Let $\mathsf{V} = \{1, 2\}$ and $\Sigma = \{\alpha, \beta\}$. A and the preference relations are given in Table 3. The preference relations are given in the form of a pair of ordinal utilities where the first element is for player 1 and the second for player 2.

Readers familiar with game theory can recognize that without the last four rows the table represents the prisoners' dilemma. $a^{3'}$ is a Nash equilibrium but a is not as in the original prisoners' dilemma. Now our coalitional version allows player 1 and player 2 to form a coalition by whatever means, for example, a binding agreement or switching to the mode of team reasoning simultaneously. So there are four extra profiles in which both players explicitly choose to join the coalition. Among these four extra profiles, although all of them are trivially Nash equilibria, $a^{4'}$ is the only element in the core.

Note that in the example $\{1, 2\}$ as a coalition does not expand what each of the players can choose, namely $\sigma(\{1, 2\}) = \sigma(\{\{1\}, \{2\}\})$. But the coalition still makes some difference. Each member of the coalition anchors their actions to the coalition, which may bring extra stability.

[6] The definition of the core can vary in different settings. Our definition is based on [8, Definition 2.2], which is a relatively general version.

Table 3 A in Example 3

	1	2	Ordinal Utility
a	$\{(1,\alpha)\}$	$\{(2,\alpha)\}$	(9,9)
a'	$\{(1,\alpha)\}$	$\{(2,\beta)\}$	(0,10)
a''	$\{(1,\beta)\}$	$\{(2,\alpha)\}$	(10,0)
$a^{3\prime}$	$\{(1,\beta)\}$	$\{(2,\beta)\}$	(1,1)
$a^{4\prime}$	$\{(1,\alpha),(2,\alpha)\}$	$\{(1,\alpha),(2,\alpha)\}$	(9,9)
$a^{5\prime}$	$\{(1,\alpha),(2,\beta)\}$	$\{(1,\alpha),(2,\beta)\}$	(0,10)
$a^{6\prime}$	$\{(1,\beta),(2,\alpha)\}$	$\{(1,\beta),(2,\alpha)\}$	(10,0)
$a^{7\prime}$	$\{(1,\beta),(2,\beta)\}$	$\{(1,\beta),(2,\beta)\}$	(1,1)

Next, we show that the core can be expressed in HLPFD with respect to the class of RCPD-models (Definition 23) with nominals.

Proposition 7 *Given a RCPD-model* $\mathbb{M}$ *with nominals* Nom, *the current choice profile* a *with name* n, *i.e.,* $a = I(\mathsf{n}) \in A$, *is in the core of* $\mathbb{M}$, *if and only if*

$$\mathbb{M}, a \models \mathsf{n} \wedge p_{\mathsf{V}} \wedge \bigwedge_{\emptyset \neq X \subseteq \mathsf{V}} \mathsf{A}(p_X \to \langle\!\langle X, \emptyset, \emptyset \rangle\!\rangle \bigvee_{x \in X} \langle\!\langle \emptyset, \{x\}, \emptyset \rangle\!\rangle \mathsf{n}).$$

In the above HLPFD formula, p_{V} specifies the first condition of the core. The second condition is specified by the big conjunction, which says that for any subgroup X of V, no matter what X as a coalition chooses to do, it cannot guarantee that everyone in X ends up with a better outcome. In other words, given the choice of X, there is always a possibility where someone in X would not become better than he does currently as a member of the coalition V in the choice profile n.

To generalize the concept of the core, we can have its relativized version as in the case of Nash equilibrium and Pareto optimality.

$$\mathsf{Core}_X \mathsf{n} := \mathsf{n} \wedge p_X \wedge \bigwedge_{\emptyset \neq C \subseteq X} \mathbb{D}_{-X}(p_C \to \langle\!\langle -X \cup C, \emptyset, \emptyset \rangle\!\rangle \bigvee_{c \in C} \langle\!\langle -X, \{c\}, \emptyset \rangle\!\rangle \mathsf{n})$$

which expresses that given the choices of the players in $-X$ fixed, all members of X choose to join X as a group, act according to a collective strategy profile and no subgroup of X has any incentive to cooperate and deviate. Note that when taking $X = \mathsf{V}$, we get the original definition of the core as expressed in Proposition 7. The relativized version of the core enables us to express some interesting relationships between coalitions. For example,

$$\mathsf{Core}_X \mathsf{n} \wedge \mathsf{Core}_{-X} \mathsf{n}$$

which says that in the current choice profile n, both X and $-X$ form coalitions and are in their relativized cores.

More generally, we can define the following concept:

$$\mathsf{Core}_\pi \mathsf{n} := \bigwedge_{X \in \pi} \mathsf{Core}_X \mathsf{n}$$

where π is a partition of N. It characterizes the stability of a collection of coalitions at a choice profile n. The core is a special case of it where $\pi = \{\mathsf{V}\}$. Moreover, Nash equilibrium $\mathsf{Na}\,\mathsf{V}$ is also a special case of it where $\pi = \{\{1\}, \{2\}, \ldots, \{n\}\}$.

Theorem 9 *Given a RCPD-model* $\mathbb{M}$ *with nominals* $\mathsf{n} \in \mathsf{Nom}$*, and* $a \in A$ *with* $a_{\mathsf{dom}} = \pi = \{\{1\}, \{2\}, \ldots, \{n\}\}$*,*

$$\mathbb{M}, a \models \mathsf{Core}_\pi \mathsf{n} \leftrightarrow (\mathsf{n} \wedge \mathsf{Na}\,\mathsf{V}) \ .$$

As a corollary to this proposition, we see that unlike the core, $\mathsf{Core}_\pi \mathsf{n}$ does not necessarily imply the weak Pareto optimality of n for V. But the following generalization of Proposition 6 holds.

Theorem 10 *Given a RCPD-model* $\mathbb{M}$ *with nominals* $\mathsf{n} \in \mathsf{Nom}$ *and* $a \in A$*,*

$$\mathbb{M}, a \models \mathsf{Core}_\pi \mathsf{n} \rightarrow \bigwedge_{X \in \pi} \mathsf{wPa}\, X \ .$$

Therefore, in the sense of the above two theorems, our generalization of the core can be seen as a notion that unifies the core, Nash equilibrium and Pareto optimality. Moreover, it is worthwhile to emphasize that $\mathsf{Core}_\pi \mathsf{n}$ is more than a simple combination of each coalition's stability, because each coalition's stability is actually depends on other coalitions' stability. The overall stability reflected in $\mathsf{Core}_\pi \mathsf{n}$ lies in the interdependence of each coalition's stability.

5.3 Collective Action and Stability

We shall end this section by elaborating on how our way of modeling cooperative games makes clear the difference between a collective action and an agglomeration of actions.

RCPD-models enable us to distinguish the following three levels of group actions. First, coalitional effectiveness, that is, what an agglomeration of a group of players' actions can force. We can use $\mathbb{D}_X \varphi$ in our logic to express that a group of agent X can force φ to be the case if they act according to their current choices. Second, collective effectiveness, expressed by $p_X \wedge \mathbb{D}_X \varphi$. Different from coalitional effectiveness, collective effectiveness requires that each player in X explicitly chooses to join coalition X and act as its member. Third, the core effectiveness, $\mathsf{Core}_X \mathsf{n} \wedge \mathbb{D}_X \varphi$. Compared with collective effectiveness, it requires not only that each player of X acts as a member of coalition X but also that the collective action should be sustainable or stable given what the players in $-X$ choose to do.

We contend that collective agency would emerge at no lower level than the core effectiveness.

6 Related Works and Conclusion

Before conclusion, we compare our work with three closely related works, the colitional logic [14], the modal coalitional game logic (MCGL) in [21][7] and the logic of ceteris paribus preference (LCP) in [6]. Moreover, we explicate the relation between our representation of cooperative games in strategic forms (namely the CPD-models) and its formulation in [15, Definition 2.1.1].

6.1 Comparison with the Coalition Logic

The coalitional effectiveness that the coalition logic aims to reason about is formally characterized by an effectivity function E_G. Based on this effectivity function, the main operator of the coalition logic $[C]\varphi$ is defined, expressing that the set of agents C can force φ to be the case at their current state.

The effective function, when adapted in a dependence model $\mathbf{M} = (M, A)$, can be defined as $E_{\mathbf{M}} : \mathcal{P}^{<\aleph_0}(\mathsf{V}) \to \mathcal{P}(\mathcal{P}(A))$ satisfying

$$S \in E_{\mathbf{M}}(X) \text{ iff } \exists a \in A, \forall a' \in A \text{ if } a' =_X a \text{ then } a' \in S \ .$$

Here, $S \in E_{\mathbf{M}}(X)$ means that the coalition X can force the game to be in S. We can express $S \in E_{\mathbf{M}}(X)$ in LFD as $\mathsf{E}\mathbb{D}_X\varphi$ assuming that $S = [\![\varphi]\!]$, because

$$\mathbf{M} \models \mathsf{E}\mathbb{D}_X\varphi \text{ iff } [\![\varphi]\!] \in E_{\mathbf{M}}(X) \ .$$

The operator $[C]\varphi$ in the coalition logic essentially has the same semantic meaning despite being interpreted in the neighborhood semantics.

We will not go into a detailed comparison between LFD and the coalition logic, but only point out a substantial difference between $\mathsf{E}\mathbb{D}_X\varphi$ and $[C]\varphi$ with regard to the characteristic axiom of the coalition logic, superadditivity:

$$([C_1]\varphi_1 \wedge [C_2]\varphi_2) \to [C_1 \cup C_2](\varphi_1 \wedge \varphi_2) \text{ where } C_1 \cap C_2 = \emptyset \ .$$

Superadditivity fails for $\mathsf{E}\mathbb{D}_X\varphi$, because in a dependence model it is possible that there is $a, a' \in A$ such that, for $C_1 \cap C_2 = \emptyset$, there is no $a'' \in A$ satisfying both $a =_{C_1} a''$ and $a' =_{C_2} a''$.

As the readers who are familiar with the coalition logic can verify, except for superadditivity, its other axioms are all valid for $\mathsf{E}\mathbb{D}_X$ in LFD.

6.2 Comparison with LCP and MCGL

Both LCP and MCGL use modal operators for characterizing preorders. Given a preorder $\preceq$ in its semantic model, LCP only includes one modal operator for $\preceq$ and one for $\prec$. MCGL concerns a multi-agent setting where for each agent there is a preorder.

[7] There are two logics in [21]. MCGL is the second one. The first one is more customized and limited than the second one. For example, it only considers finite games where both players and states need to be finite.

Besides modal operators for individual agents, MCGL includes group operators, one for the intersections of a set of preorders and one for the intersection of a set of strict preorders. It also includes modal operators for the inverse of the preorders and a difference operator. Nevertheless, it does not have any operator for the intersection of strict and non-strict preorders. Our logic has such operators and we show that they are critical for expressing strong Pareto optimality.

Next, with each of these two other logics, the comparison will focus on different aspects.

Comparison with [21] on different formulations of the core It is shown in [21] that MCGL can express not only the core in coalitional games but also the stable set and the bargaining set. However, the setting they adopt for representing coalitional games is not general enough to model the coalitional games formalized by the CPD-models. The limitation is due to their way of defining the cooperative effective function or the characteristic function as they call it. In a CPD-model $\mathbb{M}$, their characteristic function can be understood as $f : 2^N \setminus \{\emptyset\} \to \mathcal{P}(A)$, a function assigning a set of choice profiles to each coalition. Their formulation of the core only requires that the current choice profiles are strictly preferred to all the choice profiles in $f(X)$ for all $X \subseteq \mathsf{V}$. But in our formulation of the core in Definition 24, what matters is the following set for each $X \subseteq \mathsf{V}$

$$E(X) := \{a(X) \subseteq A \mid a \in A \text{ and } X \in a_{\mathsf{dom}}\}$$

where $a(X) := \{a' \in A \mid a =_X a'\}$. $E : 2^N \to \mathcal{P}(\mathcal{P}(A))$ is a function assigning to each coalition a set of sets of choices profiles. Our formulation of the core requires a comparison between the current choice profile and each of the set in $E(X)$. Note that the compartmentalization of what a coalition X can force is essential for our formulation of the core, because what a coalition X can enforce depends on what X would do. This subtlety is not captured by the characteristic function in [21].

Comparison with [6] on different ways of characterizing dependence We have seen that in LPFD variables are taken to partition the space of possible assignments according to their possible values. The dependence relation is the relation between different partitions. In LCP, what partitions the space of possible states are all possible sets of formulas of its base language. If we think of a formula as a binary variable with its values 0 or 1, then the operators $[\Gamma]$, $[\Gamma]^{\preceq_x}$ and $[\Gamma]^{\prec_x}$ in LCP correspond to our operators $[\![\Gamma, \emptyset, \emptyset]\!]$, $[\![\Gamma, x, \emptyset]\!]$ and $[\![\Gamma, \emptyset, x]\!]$ respectively. It has also been shown in [6] how these operators can be used to express the notion of Nash equilibrium.

The above comparison raises an interesting question: if we only allow binary variables, what is the difference between using variables (as in LFD) and formulas (as in LCP) to capture the functional dependence between variables? Furthermore, do we really lose anything in LFD if we only allow binary variables? A systematic study of these two questions would require future work.

6.3 Relation Between CPD Models and Cooperative Games in Strategic Form

CPD models are used to represent cooperative games in strategic form as formulated in the following definition.

Definition 25 *([15, Definition 11.1.1]) A cooperative game in strategic form is* $(N, (\Sigma(S))_{\emptyset \neq S \subseteq N}, (u^i)_{i \in N})$ *where*

1. *N is a finite nonempty set of players;*
2. *for each coalition $\emptyset \neq S \subseteq N$, $\Sigma(S)$ is a nonempty set of strategies of S;*
3. *if $\emptyset \neq S, T \subset N$, $S \cap T = \emptyset$, then $\Sigma(S \cup T) \supset \Sigma(S) \times \Sigma(T)$;*
4. *for every $i \in N$, $u^i : \Sigma(N) \to \mathbb{R}$ is i's payoff function.*

There is an obvious difference between the representation in CPD models and the above definition. While the definition uses a function to assign to each coalition its available strategies, CPD models start with each player's possible choices of coalitions and actions. Nevertheless this difference is not essential.

On the one hand, We can get a CPD model from a given cooperative game in strategic form. Taking any element c in $\Sigma(S)$ as the constant function from S to $\{c\}$, the set A in the corresponding CPD model is defined as

$$A := \{a \mid \exists \pi \in \Pi(V) \forall S \in \pi \exists c \in \Sigma(S) \forall i \in S : a_i = c\}$$

where a is a sequence of constant functions of length $|N|$. O can be derived from A and I as the interpretation of predicates can be given arbitrarily. The preference relation can be derived from the payoff function u^i. On the other hand, given a CPD model, it is easy to get a cooperative game in strategic form, although the payoff function u^i cannot be defined uniquely given i's preference relation. We just need to define $\Sigma(S)$ as $\{a_i \mid i \in S \in a_{\mathsf{dom}}$ and $a \in A\}$. For instance, in Example 2, $\Sigma(\{1, 2\}) = \{\{(1, \alpha), (2, \beta)\}, \{(1, \alpha), (2, \gamma)\}\}$ and $\Sigma(\{3\}) = \{\{(3, \alpha)\}, \{(3, \beta)\}\}$.

It is worth noting that in Example 2 $\{(1, \alpha), (2, \beta), (3, \beta)\}$ is not in $\Sigma(\{1, 2, 3\})$ but $\{(1, \alpha), (2, \beta)\}$ is in $\Sigma(\{1, 2\})$ and $\{(3, \beta)\}$ is in $\Sigma(\{3\})$. This means that condition 3 in Definition 25 does not hold. The reason is that $\{(1, \alpha), (2, \beta), (3, \beta)\}$ is not included in the set of available choice profiles in this example of CPD models. This reveals a characteristic feature of CPD models which it inherits from the dependence models for LFD. That is, A in the model does not need to be the set of all realizable choice profiles. This is different from the standard assumption in game theory, namely the completeness of the space of strategies profiles.

If $\{(1, \alpha), (2, \beta), (3, \beta)\}$ were in A of Example 2, condition 3 in Definition 22 would force it to be in $\Sigma(\{1, 2, 3\})$. In this sense, our condition 3 plays the role of condition 3 in Definition 25 when the standard assumption does not hold. So, for the analysis in Section 5, it makes no difference whether we require condition 3 of Definition 25 to be the case. In particular, Proposition 6 tells us that the relation between the core and the weak Pareto optimality does not rely on condition 3 in Definition 25. Instead, the proof shows that condition 3 in Definition 22 plays a crucial role. Example 2 shows that to ensure condition 3 in Definition 22 it is not necessary to assume condition 3 of Definition 25.

An analogous result to Proposition 6 can be found in [15, Lemma 12.1.2]. However, the analogy is not very precise. After all, the definition of the core in [15] relies on each player's payoff function, which is different from our qualitative way of representing each player's preference. We believe that a further comparison between the qualitative definition of the core as adopted in this paper and the quantitative definition of the core in [15, Section 12.1] would be meaningful. But this would require us to introduce more definitions, which go beyond the scope of this paper.

6.4 Conclusion and More Future Work

We have proposed two logics by extending LFD and studying their axiomatizations and other properties. We have also demonstrated how our logics can help reason about the notions of dependence, preference and coalitional power in a game theoretical setting and provide a unified view on three key concepts in game theory, i.e., Nash equilibrium, Pareto optimality and the core. On the basis of the two logics, we bring insights to the general discussion on collective agency.

More work on collective agency from a cooperative-game-theoretical perspective needs to be done as we have instigated. The core effectiveness we have shown in subsection 5.3 highlights the stability of both the coalition and its collective action. Suppose we further abstractly understand the core effectiveness as a specific pattern for relations between the members within a coalition. In that case, our account interprets collective agency as a relatively stable state of relations, which is in line with the call for a relationalist account (cf. [2, 13, 17, 20]).

The connection between LFD and the coalition logic we have revealed indicates that it may be fruitful to explore the relationship between LPFD and ATL [10]. Some work has been done on exploring the temporal dimension of dependence [4]. Further work in these directions could make a logical analysis of extensive games more full-fledged.

Appendix

Strong Completeness of $\mathsf{C}_{\mathsf{Nom}}$

The strategic of our proof is to show that every $\mathsf{C}_{\mathsf{Nom}}$-consistent set Γ of formulas is satisfiable. Let Γ be a fixed consistent set. We first show the follow lemma:

Lemma 9 *Let Γ be a $\mathsf{C}_{\mathsf{Nom}}$-consistent set and $\mathsf{Nom}' = \mathsf{Nom} \cup \{j_n : n \in \omega\}$. Then Γ can be extended to a maximal $\mathsf{C}_{\mathsf{Nom}'}$-consistent set Γ^+ of formulas satisfying the following conditions:*

(Named) $\Gamma^+ \cap \mathsf{Nom}' \neq \varnothing$;
(Pasted) *For all $@_i\langle\langle X, Y, Z\rangle\rangle\varphi \in \Gamma^+$, there is a nominal $j \in \mathsf{Nom}'$ such that $@_i\langle\langle X, Y, Z\rangle\rangle j \wedge @_j\varphi \in \Gamma^+$.*

The proof of Lemma 9 is standard. Since Nom is countable, we can assume that Γ itself is a named and pasted $\mathsf{C}_{\mathsf{Nom}'}$-MCS without loss of generality.

For each $i \in \mathsf{Nom}$ such that $@_i\top \in \Gamma$, we define $\Delta_i = \{\varphi : @_i\varphi \in \Gamma\}$. The readers can check that Δ_i is a MCS for each $i \in \mathsf{Nom}$.

Definition 26 *The canonical model $\mathfrak{M}_\Gamma = (W_\Gamma, \sim_\Gamma, \leq_\Gamma, V_\Gamma)$ is defined by:*

- *$W_\Gamma = \{\Delta_i : @_i\top \in \Gamma\}$;*
- *for each $v \in \mathsf{V}$, $\Delta_i \sim_v \Delta_j$ if and only if $@_i\langle\langle \{v\}, \varnothing, \varnothing\rangle\rangle j \in \Gamma$;*

– *for each* $v \in \mathsf{V}$, $\Delta_i \leq_v \Delta_j$ *if and only if* $@_i\langle\langle \varnothing, \{v\}, \varnothing\rangle\rangle j \in \Gamma$*;*
– $V(P\vec{x}) = \{\Delta_i : @_i P\vec{x} \in \Gamma\}$ *and* $V(i) = \Delta_i$*.*

Lemma 10 $\mathfrak{M}_\Gamma = (W, \sim, \leq, V)$ *is an HRPD-model.*

Proof Since Γ is named, $W_\Gamma \neq \varnothing$. Let $v \in \mathsf{V}$. By axiom (Ord,1,2,3), $\sim_v$ is a pre-order and $\leq_v$ is an equivalence relation. Note that $V(i) \in W$ for each $i \in \mathsf{Nom} \cap \mathrm{dom}(V)$. To show that $\mathfrak{M}_\Gamma$ is an HRPD-model, it suffices to show that V satisfies (Val). Let $P \in \mathsf{Pred}$ and $\vec{x} \in \mathsf{V}^{\mathsf{ar}(P)}$. Suppose $\Delta_i \sim_{\mathsf{ran}(\vec{x})} \Delta_j$ and $\Delta_i \in V(P\vec{x})$. Then $P\vec{x} \in \Delta_i$. By (Dep), $\mathbb{D}_X P\vec{x} \in \Delta_i$, which entails $P\vec{x} \in \Delta_j$. □

Lemma 11 *Let* $\mathfrak{M}_\Gamma = (W, \sim, \leq, V)$*,* $i \in \mathsf{Nom}$ *and* $\Delta_i \in W$*. Then*

(1) If $\langle\langle X, Y, Z\rangle\rangle j \in \Delta_i$*, then* $\Delta_i R(X, Y, Z)\Delta_j$*;*
(2) If $@_i\langle\langle X, Y, Z\rangle\rangle\varphi \in \Gamma$*, then there is* $j \in \mathsf{Nom}$ *with* $\varphi \in \Delta_j$ *and* $\Delta_i R(X, Y, Z)\Delta_j$*.*
(3) $D_X s \in \Delta_i$ *if and only if* $\mathfrak{M}_\Gamma, \Delta_i \models D_X s$*.*
(4) For all $\varphi \in \mathcal{L}_{\mathsf{Nom}}$*,* $\varphi \in \Delta_i$ *if and only if* $\mathfrak{M}_\Gamma, \Delta_i \models \varphi$*.*

Proof For (1), suppose $\langle\langle X, Y, Z\rangle\rangle j \in \Delta_i$. By axiom (Ord,5), we see $\langle\langle \{x\}, \varnothing, \varnothing\rangle\rangle j$, $\langle\langle \varnothing, \{y\}, \varnothing\rangle\rangle j$, $\langle\langle \varnothing, \varnothing, \{z\}\rangle\rangle j \in \Delta_i$ for all $x \in X$, $y \in Y$ and $z \in Z$, which entails by axiom (Ord,4) that $\Delta_i \sim_X \Delta_j$, $\Delta_i \leq_Y \Delta_j$ and $\Delta_i <_Z \Delta_j$. Thus $\Delta_i R(X, Y, Z)\Delta_j$.

For (2), suppose $@_i\langle\langle X, Y, Z\rangle\rangle\varphi \in \Gamma$. Since Γ is pasted, there is $j \in \mathsf{Nom}$ such that $@_i\langle\langle X, Y, Z\rangle\rangle j \wedge @_j\varphi \in \Gamma$. Thus $\varphi \in \Delta_j$ and $\Delta_i R(X, Y, Z)\Delta_j$.

For (3), suppose $D_X s \in \Delta_i$ and $\Delta_i \sim_X \Delta_j$. We show that $\langle\langle \{s\}, \varnothing, \varnothing\rangle\rangle j \in \Delta_i$. Assume $\langle\langle \{s\}, \varnothing, \varnothing\rangle\rangle j \notin \Delta_i$. Then by axiom (DD,1), we see $\mathbb{D}_X \neg j \in \Delta_i$, which contradicts to $\Delta_i \sim_X \Delta_j$. Thus $\mathfrak{M}_\Gamma, \Delta_i \models D_X s$. Suppose $D_X s \notin \Delta_i$. Then $i \wedge \neg D_X s \in \Delta_i$. By axiom (DD,2), we see $@_i\langle\langle X, \varnothing, \varnothing\rangle\rangle\mathbb{D}_s\neg i \in \Gamma$. Since Γ is pasted, there is $j \in \mathsf{Nom}$ such that $@_i\langle\langle X, \varnothing, \varnothing\rangle\rangle j \wedge @_j\mathbb{D}_s\neg i \in \Gamma$. Thus $\Delta_i \sim_X \Delta_j$ and $\Delta_i \nsim_s \Delta_j$. Note that $\sim_s$ is symmetric, $\Delta_j \nsim_s \Delta_i$. Thus $\mathfrak{M}_\Gamma, \Delta_i \not\models D_X s$.

For (4), the proof proceeds by induction on the complexity of φ. The case when $\varphi = D_X s$ follows from (3). The case $\varphi = P\vec{x}$ or $\varphi \in \mathsf{Nom}$ is trivial. The Boolean cases are also trivial. Let $\varphi = [\![X, Y, Z]\!]\psi$. Assume $[\![X, Y, Z]\!]\psi \notin \Delta_i$. Then $\langle\langle X, Y, Z\rangle\rangle\neg\psi \in \Delta_i$ and so $@_i\langle\langle X, Y, Z\rangle\rangle\neg\psi \in \Gamma$. By (2), $\neg\psi \in \Delta_j$ for some $\Delta_j \in R(X, Y, Z)(\Delta_i)$. Then $\psi \notin \Delta_j$ and by induction hypothesis, $\mathfrak{M}_\Gamma, \Delta_j \not\models \psi$, which entails $\mathfrak{M}_\Gamma, \Delta_i \not\models [\![X, Y, Z]\!]\psi$. Assume that $\mathfrak{M}_\Gamma, \Delta_i \not\models [\![X, Y, Z]\!]\psi$. Then there is $\Delta_j \in R(X, Y, Z)(\Delta_i)$ such that $\mathfrak{M}_\Gamma, \Delta_j \not\models \psi$. By induction hypothesis, $\psi \notin \Delta_j$ and so $\neg\psi \wedge j \in \Delta_j$. Note that $\langle\langle X, Y, Z\rangle\rangle j \in \Delta_i$, we see $\langle\langle X, Y, Z\rangle\rangle\neg\psi \in \Delta_i$, which entails $[\![X, Y, Z]\!]\psi \notin \Delta_i$. □

Theorem. $\mathsf{C}_{\mathsf{Nom}}$ *is sound and strongly complete.*

Proof Soundness is not hard to verify. By Lemma 11, $\mathfrak{M}_\Gamma, \Gamma \models \Gamma$. By the arbitrariness of Γ, we obtain the strong completeness. □

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Declarations

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