Journal of Philosophical Logic (2024) 53:331–349
https://doi.org/10.1007/s10992-023-09726-3

An Algebraic View of the Mares-Goldblatt Semantics

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Received: 13 January 2023 / Accepted: 3 October 2023 / Published online: 26 January 2024

Abstract
An algebraic characterisation is given of the Mares-Goldblatt semantics for quantified extensions of relevant and modal logics. Some features of this more general semantic framework are investigated, and the relations to some recent work in algebraic semantics for quantified extensions of non-classical logics are considered.

Keywords Algebraic logic · Quantifiers · Relevant logic

1 Introduction

There is a curious phenomenon in the history of relevant logics (at least in the *Anderson-Belnap* tradition [1]). Relevant logics were introduced with deep and, I think, compelling philosophical motivations concerning the relations of entailment and logical consequence, and the invalidity of intuitively implausible argument forms such as explosion. However, it has, time and again, turned out that in order to study these, rather nice, logics we must build new mathematical tools, as it has often turned out that the existing tools were inadequate. Furthermore, when the new tools were developed, they have usually been found to be *substantially more general* than the pre-existing tools, and so of use for a range of logics outside of the scope of the relevant logic enterprise.[1]

One example of this phenomenon is that standard Gentzen-style sequent systems were found to be inadequate for systems like **R**, once we try to include all the vocabulary.[2] This led Belnap to develop the *display calculus*, presented in [2, §62], which, it turns out, is applicable to a very wide range of systems, as discussed, for instance, in

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[1] This phenomenon suggests to me a good reason why relevant logics should be of perennial interest. They are both philosophically motivated and mathematically rich: what more could one possibly want?

[2] Leading to the development of "lattice-**R**" or "**LR**" in [27].

[43]. Another example, especially salient here, is that Kripke semantics for modal and intuitionistic logic (for details of which see [4], or any other modal logic textbook) turned out to be inadequately expressive to accommodate relevant logics. This motivated the development of alternate frame-based semantics, most famously the ternary relation semantics due to Sylvan (né Routley) and Meyer [35]. This, in turn, lead to the development by Dunn of *gaggle theory* [3, 10], which is an extremely powerful semantic framework accommodating a very wide range of systems.[3]

On the point of frame-based semantics, a very general toolkit developed to accommodate first order relevant logics is the *Mares-Goldblatt* semantics for quantifiers. Quantified extensions of relevant logics have presented (in)famous difficulties when it comes to semantic presentations. The *Kripkean* approach, either in constant or variable domain, commonly investigated in the context of modal logics, and initially conjectured to work in the context of relevant logics [37], was eventually shown, by Fine [16], to fail. In particular, quantified extensions of the stronger relevant logics, such as **R**, **E**, and **T**, were shown to be incomplete with respect to the extensions of their propositional frames by machinery to accommodate the Tarskian truth conditions for the quantifiers. By *Kripkean* here, I mean that interpretation which takes a universal quantified formula to be interpreted by the *generalised intersection* of the interpretations of its instances, and an existentially quantified one in terms of the *generalised union* thereof. This approach, it turns out, does not work on a pretty deep level. In order to avoid the incompleteness phenomenon, an alternate truth condition for quantifiers was introduced by Fine [15], and another by Mares and Goldblatt [26]. Once again, the curious phenomenon has shown itself, as this approach to quantifiers is, it seems, really rather general, as further investigations [14, 18, 19, 41] have shown.

The Mares-Goldblatt semantics is, as mentioned, a frame-based semantics, but the formal properties of the frames which do the work of supporting their alternate truth condition seems to be more general. In particular, there is, to be found with just a bit of scratching, a rather natural algebraic structure underlying Mares-Goldblatt frames and models. The aim of this paper is to pull out this algebraic structure, investigate some of its properties, and compare it to a recent algebraic approach (detailed in [7, Ch. 7]) to interpreting quantified extensions of a range of logics. The upshot is a rather general algebraic semantics which both underlies the original Mares-Goldblatt presentation, and, as I'll discuss in the final section, promises interesting connections with a range of other approaches to studying quantified extensions of non-classical logics.

Before diving in, let me get some necessary preliminaries out of the way.

2 Preliminaries

2.1 Languages

Throughout we'll deal with a range of different languages, which I'll uniformly refer to using $\mathcal{L}$. Generally these will be abstract, consisting of a set of formulas, $\mathcal{L}$, with

[3] As was shown by Restall [32], the generality of the display calculus and that of gaggle theory are intimately related. As I'll mention in the final section, this connection motivates some work extending that which I'll do in this paper.

propositional connectives of varying (finite) arities, a *denumerable* set of individual variables $Var = \{x_n\}_{n\in\omega}$, and a pair of quantifiers $\forall$ and $\exists$.[4] I'll deal with one particular language, when dealing with the relevant logic **R** and its quantified extensions. This will include the propositional constant $\mathtt{t}$, the unary connective $\neg$, and the binary connectives $\wedge$, $\vee$, and $\rightarrow$. In order to omit writing some parentheses, I'll assume that $\rightarrow$ is the most weakly binding connective. Furthermore, we take $\leftrightarrow$ to be defined as $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$.

A language *signature* will be a pair $\langle Con, Pred\rangle$ where Con is a set of name constants, and $Pred$ a set of predicate letters of varying arities (the set of n-ary predicate letters will be written $Pred^n$, so that $Pred = \bigcup_{n\in\omega} Pred^n$). The set of terms is $Term = Con \cup Var$. From a signature, a language $\mathcal{L}$ is defined the usual way. I'll use capital letters from the beginning of the Latin alphabet as metavariables over formulas (perhaps with free variables occurring therein). A formula is *closed* when no free variables occur in it.[5]

When writing axioms and rules which have side conditions specifying that a certain variable cannot occur free in some subformula, I'll write that subformula with the variable in question as a superscript: for example, when specifying that in the axiom $\forall x(A \vee B) \rightarrow A \vee \forall x B$, the variable x cannot occur free in A, I'll write $\forall x(A^x \vee B) \rightarrow A^x \vee \forall x B$.

2.2 Logics

I'll deal, here and there, with logics in two flavours. The first flavour is the more familiar, where logics are FMLA systems, in Humberstone's [22] terminology (or "assertional systems" in Dunn and Hardegree's [11]). Here logics are sets of formulas, and I'll often write something like $\vdash_{\mathbf{L}} A$ in place of $A \in \mathbf{L}$ to indicate that A is valid with respect to **L**. One class of FMLA logics that will be of particular importance are called, by Cintula [6], *weakly implicative* logics. A weakly implicative logic is one including at least a binary connective $\rightarrow$ such that, where $C(A/B)$ is a formula C in which some occurrences of A are replaced by B:

(Refl)	$A \rightarrow A \in \mathbf{L}$
(rMP)	$\{A \rightarrow B, A\} \subseteq \mathbf{L} \Rightarrow B \in \mathbf{L}$
(rTran)	$\{A \rightarrow B, B \rightarrow C\} \subseteq \mathbf{L} \Rightarrow A \rightarrow C \in \mathbf{L}$
(rCong)	$\{A \rightarrow B, B \rightarrow A\} \subseteq \mathbf{L} \Rightarrow C \rightarrow C(A/B) \in \mathbf{L}$

These constraints indicate that $\rightarrow$ expresses a partial order, from which we can obtain a congruence with respect to the connectives of **L** (by taking, e.g., the Leibniz congruence as discussed in [17]), and satisfies a rule form of *modus ponens*. These are plausibly minimal conditions for $\rightarrow$ to count as a kind of *logical implication*, as

[4] I avoid identity because it introduces many additional complexities, though Mares-Goldblatt style treatments of identity have been studied by Ferenz [13] and Standefer [39].

[5] This language is presented without function symbols, in order both to cut down on technical complexity and to avoid tricky philosophical issues concerning the interpretation of *identity* in a relevance-friendly way. The latter issue has recently come up for renewed discussion, notably in [13, 39], but I leave aside the issue of choosing how to proceed in this general setting for future work.

argued in [36], and Cintula and Noguera [7] show that this class of logic also exhibit reasonably nice behaviour when extended to a quantificational setting. In particular, given a weakly implicative logic **L**, let us take its *basic quantified extension* **QL** to be characterised by the following axioms and rules:

(∀E) $\forall x A \to A[x/\tau]$ for any τ substitutable for x in A
(∃I) $A[\tau/x] \to \exists x A$ for any τ substitutable for x in A

$$(\text{r}\forall\text{I})\ \frac{A^x \to B}{A^x \to \forall x B} \qquad (\text{r}\exists\text{E})\ \frac{A \to B^x}{\exists x A \to B^x}$$

So much for FMLA logics. A FMLA-FMLA logic is a set of *pairs* of formulas, or sequents. I'll often write $\vdash_{\mathbf{L}} A \prec B$ in place of $\langle A, B\rangle \in \mathbf{L}$. As in the case of weakly implicative FMLA logics, I'll assume some basic properties for FMLA-FMLA systems, so that $\prec$ is adequate to be read as an *order* the symmetrisation of which gives rise to a congruence on the algebra of formulas of $\mathcal{L}$, namely I'll require that $\vdash_{\mathbf{L}} A \prec A$, and that the following rules preserve validity:

$$(\text{rTran}_\prec)\ \frac{A \prec B \qquad B \prec C}{A \prec C} \qquad (\text{rCong}_\prec)\ \frac{A \prec B \qquad B \prec A}{C \prec C(A/B)}$$

Note that the use of "rules" here is in accordance with Smiley's [38] "rules of proof" – so that the rules are required to preserve *validity*.[6]

Given a FMLA-FMLA logic **L**, we'll define **QL** in a way similar to that for a FMLA logic, taking the following axioms and rules:

($\forall\text{E}_\prec$) $\forall x A \prec A[x/\tau]$ for any τ substitutable for x in A
($\exists\text{I}_\prec$) $A[\tau/x] \prec \exists x A$ for any τ substitutable for x in A

$$(\text{r}\forall\text{I}_\prec)\ \frac{A^x \prec B}{A^x \prec \forall x B} \qquad (\text{r}\exists\text{E}_\prec)\ \frac{A \prec B^x}{\exists x A \prec B^x}$$

When context makes clear which versions of these we mean, I'll elide the occurrence of the subscripted $\prec$.

I introduce these two notions of logic in order to facilitate (1) introducing the Mares-Goldblatt semantics for **QR**, which is treated as a FMLA system while (2) providing for the most natural algebraic generalisation of the Mares-Goldblatt, which concerns FMLA-FMLA systems.

3 Mares-Goldblatt Semantics for Quantified Extensions of R

Before going on to consider the algebraic structure of Mares-Goldblatt frames, let me introduce the target logic **R**, and the frame semantics itself, along with some of its salient properties.

6 So they are more similar to the rule of necessitation in normal modal logics than the rule of modus ponens in, say, classical logic.

3.1 Axioms for Quantified Extensions of R

The propositional relevant logic **R** can be axiomatised as follows (where in ($\wedge$E) and ($\vee$I), $i \in \{1, 2\}$):

$$
\begin{array}{rl}
\text{(Id)} & A \to A \\
\text{(B)} & (A \to B) \to ((C \to A) \to (C \to B)) \\
\text{(W)} & (A \to (A \to B)) \to (A \to B) \\
\text{(C)} & (A \to (B \to C)) \to (B \to (A \to C)) \\
(\wedge\text{E}) & A_1 \wedge A_2 \to A_i \\
(\wedge\text{I}) & (A \to B) \wedge (A \to C) \to (A \to B \wedge C) \\
(\vee\text{I}) & A_i \to A_1 \vee A_2 \\
(\vee\text{E}) & (A \to C) \wedge (B \to C) \to (A \vee B \to C) \\
\text{(Dist)} & A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C) \\
\text{(Cont)} & (A \to \neg B) \to (B \to \neg A) \\
\text{(DNE)} & \neg\neg A \to A \\
(\texttt{t}) & A \leftrightarrow (\texttt{t} \to A)
\end{array}
$$

$$
\text{(rMP)}\ \frac{A \to B \qquad A}{B} \qquad\qquad \text{(rAdj)}\ \frac{A \qquad B}{A \wedge B}
$$

Note that these rules are rules of *proof*, in the sense of [22].[7]

QR is the basic quantified extension of **R**, as defined in the previous section[8], and the system **RQ** is the extension of **QR** with the additional axiom:

$$
\text{(EC)} \quad \forall x(A^x \vee B) \to A^x \vee \forall x B
$$

The axioms governing $\exists$ in **QR** are redundant, as we can take $\exists x A := \neg\forall x \neg A$. I keep them separate above for the sake of generalisation. Some of the salient consequences of **QR** are presented in [26, §6].

[7] This fact makes things a bit complicated when comparing directly with treatments of consequences for non-classical logics, such as in [7]. What is required for such a treatment to make sense is, like in the case of normal modal logics, to consider the *global* consequence relation associated with the logic.

[8] Note that because we can define additional connectives $\circ$ and $\leftarrow$ in **R**, forming the residuated triple $\{\leftarrow, \circ, \to\}$, closure under (r$\forall$I) and (r$\exists$E) ensures that we can prove the associated axiom forms:

$$
\begin{array}{rl}
(\forall\text{I}) & \forall x(A^x \to B) \to (A^x \to \forall x B) \\
(\exists\text{E}) & \forall x(A \to B^x) \to (\exists x A \to B^x)
\end{array}
$$

but, as shown in [41], this may not be the case in systems where these connectives are not definable. In addition, because of the addition of t to the language, we also get, as a special case of (r$\forall$I), the derivability of the universal generalisation (UG) rule, as follows:

$$
\cfrac{\cfrac{\cfrac{A}{\texttt{t} \to A}\ (\texttt{t}), \text{(rMP)}}{\texttt{t} \to \forall x A}\ (\text{r}\forall\text{I})}{\forall x A}\ (\texttt{t}), \text{(rMP)}
$$

So in the more general setting of weaker relevant logics, we will need to add these connectives to the propositional logic **L** in order to get the ‘usual’ basic quantified extension **QL** thereof out of the basic quantified extensions discussed here.

3.2 Mares-Goldblatt Frames

In giving the frame semantics, it makes sense to start with the definition of ternary relation frames and complex algebras thereon, as both of these play an important role in the definition of Mares-Goldblatt frame.

Definition 3.1 A Ternary Relation (TR) frame is a tuple $F = \langle W, N, R, *\rangle$ such that $\varnothing \neq N \subseteq W$, $R \subseteq W \times W \times W$, and $* : W \longrightarrow W$, and furthermore, fixing the definitions:

$$\leq \, = \{\langle a, b\rangle \in W \times W \mid \exists c \in N(Rcab)\}$$
$$\mathcal{P}(W)^{\uparrow} = \{X \subseteq W \mid \forall b \in W(\exists a \in X(a \leq b) \Rightarrow b \in X)\}$$

the following constraints are satisfied:

(1) $\langle W, \leq\rangle$ is a partially ordered set
(2) $N \in \mathcal{P}(W)^{\uparrow}$
(3) If $a' \leq a$, $b' \leq b$, $c \leq c'$, and $Rabc$, then $Ra'b'c'$
(4) $a \leq b \Rightarrow b^* \leq a^*$ and $a^{**} = a$

Definition 3.2 Given a TR frame F, we define the following operations on $\mathcal{P}(W)^{\uparrow}$, alongside $\cap$ and $\cup$ defined as usual:

$$\neg X = \{a \in W \mid a^* \notin X\}$$
$$X \rightarrow Y = \{a \in W \mid \forall c \in W(\exists b \in X(Rabc) \Rightarrow c \in Y)\}$$

A *complex algebra* of F is a tuple $\langle Prop, N, \neg, \cap, \cup, \rightarrow\rangle$ generated by a $Prop \subseteq \mathcal{P}(W)^{\uparrow}$. The *full* complex algebra of F is that where $Prop = \mathcal{P}(W)^{\uparrow}$.

Note that in any complex algebra of a TR frame F, we have $X \cup Y = \neg(\neg X \cap \neg Y)$.

Definition 3.3 A Mares-Goldblatt (MG) frame $F = \langle W, N, R, *, Prop, D, PropFun\rangle$ satisfying the following constraints:

(1) $F' = \langle W, N, R, *\rangle$ is a *ternary relation* (TR) frame.
(2) $Prop$ is the carrier set of a *complex* algebra of F'.
(3) $D \neq \varnothing$
(4) $PropFun$ is a collection of functions of type $D^{\omega} \longrightarrow Prop$, such that:
   (a) There is a $\varphi_N \in PropFun$ such that for any $f \in D^{\omega}$, $\varphi_N f = N$.
   (b) For $\{\varphi, \psi\} \subseteq PropFun$ and $\otimes \in \{\cap, \cup, \rightarrow\}$, there is a $\varphi \otimes \psi \in PropFun$ such that for any $f \in D^{\omega}$, $(\varphi \otimes \psi) f = \varphi f \otimes \psi f$.
   (c) For $\varphi \in PropFun$ and $n \in \omega$, there is an $\forall_n \varphi \in PropFun$ such that for any $f \in D^{\omega}$:

$$(\forall_n \varphi) f = \bigcup \Big\{X \in Prop \mid X \subseteq \bigcap_{f' \sim_n f} \varphi f'\Big\}$$

(d) For $\varphi \in PropFun$ and $n \in \omega$, there is an $\exists_n \varphi \in PropFun$ such that for any $f \in D^\omega$:

$$(\exists_n \varphi) f = \bigcap \Big\{ X \in Prop \mid \bigcup_{f' \sim_n f} \varphi f' \subseteq X \Big\}$$

Definition 3.4 A model on an MG frame F is a tuple $\langle F, M \rangle$ where M is a multitype function, of types $Con \longrightarrow D$ and $Pred^n \longrightarrow (D^n \longrightarrow \mathbb{A}^{\mathfrak{N}})$. Furthermore, given $f \in D^\omega$, we form $M_f : Term \longrightarrow D$ by fixing:

$$M_f(\tau) = \begin{cases} M(\tau) & \text{if } \tau \in Con \\ fn & \text{if } \tau = x_n \in Var \end{cases}$$

We obtain a valuation $[\![\cdot]\!]^M : \mathcal{L} \times D^\omega \longrightarrow Prop$ using the following recursive clauses (sometimes writing "$[\![A]\!]^M_f$" for "$([\![A]\!]^M) f$"):

$$[\![P(\tau_1, \ldots, \tau_n)]\!]^M_f = M(P)(M_f(\tau_1), \ldots, M_f(\tau_n)) \qquad [\![\neg A]\!]^M_f = \neg([\![A]\!]^M_f)$$
$$[\![A \wedge B]\!]^M_f = [\![A]\!]^M_f \cap [\![B]\!]^M_f \qquad [\![A \to B]\!]^M_f = [\![A]\!]^M_f \to [\![B]\!]^M_f$$
$$[\![\forall x_n A]\!]^M_f = (\forall_n [\![A]\!]^M) f \qquad [\![\exists x_n A]\!]^M_f = (\exists_n [\![A]\!]^M) f$$

A formula A is satisfied on $\langle F, M, f \rangle$, written $\langle F, M, f \rangle \vDash A$, just in case $N \subseteq [\![A]\!]^M_f$. A is satisfied by $\langle F, M \rangle$, written $\langle M, F \rangle \vDash A$, just in case $\langle F, M, f \rangle \vDash A$ holds for every $f \in D^\omega$. A is valid on F, $F \vDash A$, just in case $\langle F, M \rangle \vDash A$ holds for every M on F. Given a class $\mathbf{F}$ of MG frames, $\vDash_{\mathbf{F}} A$ holds iff $F \vDash A$ holds for every $F \in \mathbf{F}$.

A logic $\mathbf{L}$ is *the logic* of a class $\mathbf{F}$ of MG frames just in case $\vdash_{\mathbf{L}} A \iff \vDash_{\mathbf{F}} A$.

Note that this definition provides for complex algebras which are *matrices*, where the set of designated values is the *cone* of N: i.e., $T = \{X \in Prop \mid N \subseteq X\}$.[9]

Definition 3.5 A TR frame is an $\mathbf{R}$ frame just in case it satisfies the following constraints:

(cCont)	$Rabc \Rightarrow Rac^*b^*$
(cB)	$Rabcd \Rightarrow Ra(bc)d$
(cWI)	$Raaa$
(cCI)	$Rabc \Rightarrow Rbac$

where we fix the notation:[10]

$$Rabcd \iff \exists x(Rabx \;\&\; Rxcd)$$
$$Ra(bc)d \iff \exists x(Rbcx \;\&\; Raxd)$$

---

9 Matrix semantics are discussed in a number of places, for instance [17, Ch. 4].

10 Some of these complexities in defining $\mathbf{R}$ frames can be traded in for other complexities, up to the reader's preference, by adopting the *collection frame* approach recently introduced in [34].

Theorem 3.6 (Mares and Goldblatt, 2006) **QR** is the logic of MG frames whose underlying TR frame is an **R** frame.

The proof employs a more or less standard canonical model construction for the completeness direction, some of the details of which we'll recapitulate later. Related results have been given for a range of quantified extensions of relevant logics [12, 41], of modal relevant logics [14], and quantified modal logics [18, 19].

3.3 MG Frames for RQ

A simple maneuver allows us to obtain frames for **RQ**, building in a constraint on $PropFun$ which delivers exactly the validity of (EC).

Definition 3.7 Given a TR frame F and $\{X, Y\} \subseteq \mathcal{P}(W)^{\uparrow}$, we fix:

$$X \setminus Y = \{a \in W \mid \exists b \leq a(b \in X \ \&\ b \notin Y)\}$$

Note that $\mathcal{P}(W)^{\uparrow}$ is closed under $\setminus$.

Definition 3.8 An MG frame F is a **RQ** frame just in case (1) its underlying TR frame is an **R** frame and (2) for any $\{X, Y\} \subseteq Prop$, $\varphi \in PropFun$, and $f \in D^{\omega}$:

$$X \setminus Y \subseteq \bigcap_{f' \sim_n f} \varphi f' \Rightarrow X \setminus Y \subseteq (\forall_n \varphi) f$$

Theorem 3.9 (Mares and Goldblatt, 2006) **RQ** is the logic of **RQ** frames.

We'll discuss a bit further why this strategy works in Section 4.2.

4 MG Structures

With this specified, let us cut to the chase. The following definition seeks to specify the essential properties of Mares-Goldblatt frames, when we consider these in terms of the behaviour of their complex algebras. The *duality* between frames and algebras, in the case between TR frames for **R** and De Morgan monoids (to be introduced in the next section), is a special case of the broader representation theory studied in the context of gaggles [3, 10], a point to which we return later.[11]

I'll start with the abstract definition, and some of its properties, and then come back to consider how this abstracts away from the frame situation, and how, in general, completeness proofs go.

Definition 4.1 An MG structure is a five-tuple $\mathfrak{A} = \langle \mathcal{A}^{\mathfrak{N}}, \mathcal{A}^{\mathfrak{C}}, D, PF, h \rangle$ where:

(1) $\mathcal{A}^{\mathfrak{N}} = \langle \mathbb{A}^{\mathfrak{N}}, \leq^{\mathfrak{N}}, \{\otimes_i^{\mathfrak{N}}\}_{i \in I} \rangle$ is a partially ordered algebra (or a "po-algebra").

[11] Related results in the context of modal logic can be found in [5, Ch. 8].

(2) $\mathcal{A}^{\mathfrak{C}} = \langle \mathbb{A}^{\mathfrak{C}}, \leq^{\mathfrak{C}}, \{\otimes_i^{\mathfrak{C}}\}_{i \in I} \rangle$ is a complete lattice.[12]
(3) $D \neq \varnothing$
(4) $PF \subseteq \{\varphi \mid \varphi : D^{\omega} \longrightarrow \mathbb{A}^{\mathfrak{N}}\}$ is such that:

- (a) If $\otimes$ is a 0-ary operation, then there is a $\varphi_{\otimes} \in PF$ such that for every $f \in D^{\omega}$, $\varphi_{\otimes} f = \otimes$.
- (b) If $\otimes$ is an m-ary ($m \geq 1$) operation and $\{\varphi_i\}_{i \leq m} \subseteq PF$, then there is a $\otimes(\varphi_1, \ldots, \varphi_m) \in PF$ such that for every $f \in D^{\omega}$,

$$\otimes(\varphi_1, \ldots, \varphi_m) f = \otimes^{\mathfrak{N}}(\varphi_1 f, \ldots, \varphi_m f)$$

- (c) For each $\varphi \in PF$ and $n \in \omega$, there are $\forall_n \varphi, \exists_n \varphi \in PF$.

(5) $h : \mathbb{A}^{\mathfrak{N}} \longrightarrow \mathbb{A}^{\mathfrak{C}}$ is a map such that:

- (a) $a \leq^{\mathfrak{N}} b \iff ha \leq^{\mathfrak{C}} hb$
- (b) For each m-ary operation $\otimes$:

$$h(\otimes^{\mathfrak{N}}(a_1, \ldots, a_m)) = \otimes^{\mathfrak{C}}(ha_1, \ldots, ha_m)$$

- (c) $h((\forall_n \varphi) f) = \bigvee^{\mathfrak{C}} \left\{ a \in ran(h) \mid a \leq^{\mathfrak{C}} \bigwedge^{\mathfrak{C}}_{f' \sim_n f} h(\varphi f') \right\}$
- (d) $h((\exists_n \varphi) f) = \bigwedge^{\mathfrak{C}} \left\{ a \in ran(h) \mid \bigvee^{\mathfrak{C}}_{f' \sim_n f} h(\varphi f') \leq^{\mathfrak{C}} a \right\}$

An MG structure $\mathfrak{A}$ belongs to a class of po-algebras $\mathbf{A}$ iff $\{\mathcal{A}^{\mathfrak{N}}, \mathcal{A}^{\mathfrak{C}}\} \subseteq \mathbf{A}$.

We could shorten this statement by noting that condition (5)(a) requires that h be an *order-embedding* of $\mathcal{A}^{\mathfrak{N}}$ into $\mathcal{A}^{\mathfrak{C}}$, and that (5)(b) requires this to be a *homomorphism*. That $\mathcal{A}^{\mathfrak{C}}$ is a complete lattice, and h an order-embedding of $\mathcal{A}^{\mathfrak{N}}$ into $\mathcal{A}^{\mathfrak{C}}$, can be expressed by saying that $\mathcal{A}^{\mathfrak{C}}$ is a *completion* of $\mathcal{A}^{\mathfrak{N}}$, mediated by h.[13] So, in essence, these two conditions indicate that $\mathcal{A}^{\mathfrak{C}}$ be a completion of $\mathcal{A}^{\mathfrak{N}}$, mediated by a homomorphic order-embedding. Then the only distinctive properties are those concerning the quantifiers, whose behaviour is specified in terms of the *interaction* between $\mathcal{A}^{\mathfrak{N}}$ and $\mathcal{A}^{\mathfrak{C}}$. To characterise this interaction, let's note some key facts about MG structures.

Proposition 4.2 When $\mathfrak{A} = \langle \mathcal{A}^{\mathfrak{N}}, \mathcal{A}^{\mathfrak{C}}, D, PF, h \rangle$ is an MG structure, then for each $\varphi \in PF$ and $n \in \omega$, then:

(1) $\{(\forall_n \varphi) f, (\exists_n \varphi) f\} \subseteq \mathbb{A}^{\mathfrak{N}}$
(2) $(\forall_n \varphi) f$ is the greatest lower bound of $\{\varphi f' \mid f' \sim_n f\}$.
(3) $(\exists_n \varphi) f$ is the least upper bound of $\{\varphi f' \mid f' \sim_n f\}$.

Proof Point (1) is guaranteed by the constraints on MG structures, according to which each $\varphi \in PF$ is a function of type $D^{\omega} \longrightarrow \mathbb{A}^{\mathfrak{N}}$. For point (2), first note that:

[12] That is, for any set $X \subseteq \mathbb{A}^{\mathfrak{C}}$, we have a meet $\bigwedge^{\mathfrak{C}} X$ and join $\bigvee^{\mathfrak{C}} X$, defined w.r.t. $\leq^{\mathfrak{C}}$, inhabiting $\mathbb{A}^{\mathfrak{C}}$. Salient definitions and results are drawn from [8], but can be found in any treatment of lattice theory.

[13] Further discussion of completions can be found in [21].

$$h((\forall_n\varphi)f) \leq^{\mathfrak{C}} \bigwedge_{f' \sim_n f}^{\mathfrak{C}} h(\varphi f')$$

and so for each $f' \sim_n f$, we have that

$$h((\forall_n\varphi)f) \leq^{\mathfrak{C}} h(\varphi f')$$

and so by condition (5)(a), for each $f' \sim_n f$:

$$(\forall_n\varphi)f \leq^{\mathfrak{N}} \varphi f'.$$

Thus $(\forall_n\varphi)f$ is a lower bound of $\{\varphi f' \mid f' \sim_n f\}$. To show that it is the greatest such, suppose that $a \leq^{\mathfrak{N}} \varphi f'$ holds for each $f' \sim_n f$. It follows, by (5)(a), that:

$$ha \leq^{\mathfrak{C}} \bigwedge_{f' \sim_n f}^{\mathfrak{C}} h(\varphi f')$$

and thus:

$$ha \leq^{\mathfrak{C}} h((\forall_n\varphi)f)$$

and thus:

$$a \leq^{\mathfrak{N}} (\forall_n\varphi)f.$$

The proof of point (3) is similar to that for point (2). □

I'll express points (2) and (3) of the above by writing $(\forall_n\varphi)f = \inf\{\varphi f' \mid f' \sim_n f\}$ and $(\exists_n\varphi)f = \sup\{\varphi f' \mid f' \sim_n f\}$, leaving implicit that inf and sup concern the order $\leq^{\mathfrak{N}}$.

We can adapt the notion of *safe structure* from [7, Ch. 7], roughly an interpreted algebraic structure which has enough points to interpret quantified formulas as meets and joins of their instances, and note that the latter two points of the above result indicate that the nugget of an MG structure is always safe.

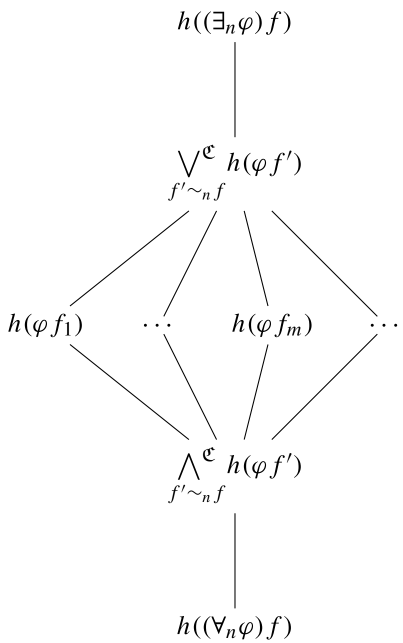
Definition 4.3 A triple $\langle \mathcal{A}, D, PF\rangle$, where $\mathcal{A}$ is a po-algebra, $D \neq \varnothing$, and $PF \subseteq \{\varphi \mid \varphi : D^\omega \longrightarrow \mathbb{A}\}$ satisfies condition (5) of Definition 4.1, is a *safe structure* just in case for any $\varphi \in PF$ and $f \in D^\omega$:

$$\{\inf\{\varphi f' \mid f' \sim_n f\}, \sup\{\varphi f' \mid f' \sim_n f\}\} \subseteq \mathbb{A}$$

Proposition 4.4 If $\mathcal{A}$ is the nugget of an MG structure $\langle \mathcal{A}, \mathcal{B}, D, PF, h\rangle$, then $\langle \mathcal{A}, D, PF\rangle$ is a safe structure.

This suggests that the reason MG structures *work* is precisely because the additional machinery enforces the nugget to be safe. In the case of algebras we may always just enforce safety, but the advantage of building in this extra machinery seems to be that it transfers nicely to structures of complex algebras of frames.

However, this isn't the whole picture: while we can be sure that $(\forall_n\varphi)f$ and $(\exists_n\varphi)f$ will be the inf and sup of $\{\varphi f' \mid f' \sim_n f\}$, we have no such guarantee that $h((\forall_n\varphi)f)$ and $h((\exists_n\varphi)f)$ will be the meet and join, respectively, of $\{h(\varphi f') \mid f' \sim_n f\}$ (which are guaranteed to exist just because $\mathcal{A}^{\mathfrak{C}}$ is a complete lattice). That is, we might have a situation like the following:



in which only the extremal elements and the various $h(\varphi f)$'s need occupy $\mathcal{A}^{\mathfrak{N}}$.[14]

A concrete example of this sort of situation is given in [26, §5]: the point there is that the *real* meet and join of instances may not exist in $\mathcal{A}^{\mathfrak{N}}$, in which case we have $(\forall_n \varphi) f$ and $(\exists_n \varphi) f$ as *approximations* thereof. So, while $\mathcal{A}^{\mathfrak{N}}$ will be a safe structure, as will $\mathcal{A}^{\mathfrak{C}}$ be, the order-embedding relating them need not (and often will not) *preserve existing meets and joins*. That is, this completion often will not be *regular*, in the terminology of [21]. If lattice operators are in the type of the algebras in question, we will have that h preserves finite meets and joins, but in general, it may well be that the following hold, even where $\bigwedge^{\mathfrak{N}} X$ and $\bigvee^{\mathfrak{N}} X$ exist:

$$\bigwedge{}^{\mathfrak{C}} hX \neq h(\bigwedge{}^{\mathfrak{N}} X) \qquad\qquad \bigvee{}^{\mathfrak{C}} hX \neq h(\bigvee{}^{\mathfrak{N}} X)$$

So what we have in an MG structure is a homomorphism from a safe structure into a complete lattice, preserving *particular* meets and joins, namely those interpreting quantified formulas. The reason this works is because the preservation of those particular meets and joins is just what we need in order to obtain completeness results.

Before getting to that, let's define models over MG structures. First, given a class of algebras $\mathbf{A}$, I'll say that an MG structure $\mathfrak{A} \in \mathbf{A}$ iff both the nugget and seam of $\mathfrak{A}$ belong to $\mathbf{A}$.[15]

Since we're taking po-algebras as basic, it makes sense to consider logics as FMLA-FMLA systems, where we can take the sequent separator symbol to be interpreted in

[14] The diagram is not meant to indicate that the number of n variants of f need be enumerable.

[15] So, for instance, $\mathfrak{A}$ being a matrix requires that both nugget and seam are matrices, and whatever properties the designated elements are required to satisfy w.r.t. the operations of the algberas will be satisfied in both cases.

terms of the partial order. In the case of FMLA logics, the natural requirement is that the MG structure in question be a *matrix*, with a set T of designated elements. I'll present both options in the definition.

Definition 4.5 A *model* is a tuple $\langle \mathfrak{A}, M \rangle$ where M and M_f are as in Definition 3.4. Using this, we define $[\![\cdot]\!]^M : \mathcal{L} \times D^\omega \longrightarrow \mathbb{A}^{\mathfrak{N}}$ as follows:

$$
\begin{aligned}
[\![P(\tau_1, \ldots, \tau_n)]\!]^M_f &= M(P)(M_f(\tau_1), \ldots, M_f(\tau_n)) \\
[\![\otimes(A_1, \ldots, A_n)]\!]^M_f &= \otimes^{\mathfrak{N}}([\![A_1]\!]^M_f, \ldots, [\![A_n]\!]^M_f) \\
[\![\forall x_n A]\!]^M_f &= (\forall_n [\![A]\!]^M) f \\
[\![\exists x_n A]\!]^M_f &= (\exists_n [\![A]\!]^M) f
\end{aligned}
$$

If $\mathfrak{A}$ is a matrix, then $\langle \mathfrak{A}, M, f \rangle \vDash A$ holds just in case $[\![A]\!]^M_f \in T^{\mathfrak{N}}$. Furthermore, $\langle \mathfrak{A}, M \rangle \vDash A$ just in case $\langle \mathfrak{A}, M, f \rangle \vDash A$ holds for each $f \in D^\omega$. Also, $\mathfrak{A} \vDash A$ holds just in case $\langle \mathfrak{A}, M \rangle \vDash A$ holds for each M on $\mathfrak{A}$.

If $\mathfrak{A}$ is not a matrix, then $\langle \mathfrak{A}, M, f \rangle A \prec B$ holds iff $[\![A]\!]^M_f \leq^{\mathfrak{N}} [\![B]\!]^M_f$. The related notions, satisfaction not relative to a variable assignment e.g., are defined as in the matrix case.

Let's consider a worked example which indicates how the features of MG structures recapitulate those of frames.

4.1 Example: MG Structures Induced by MG Frames

Definition 4.6 A *De Morgan monoid* is an algebra $\langle \mathbb{A}, e, \neg, \wedge, \vee, \circ \rangle$ where:

(1) $\langle \mathbb{A}, \wedge, \vee \rangle$ is a distributive lattice
(2) $\neg$ is an order-inverting involution with respect to the lattice order $\leq$
(3) $\circ$ is commutative, associative, and monotone w.r.t. the lattice order
(4) $a \leq a \circ a$
(5) $e \circ a = a$
(6) $a \circ b \leq c \iff a \circ \neg c \leq \neg b$

Propositional **R** is the logic of De Morgan monoids, first proved by Dunn [9] relying on the fact that $\rightarrow$ can be understood as a *residual* of $\circ$ (when fixed by the definition $a \rightarrow b := \neg(a \circ \neg b)$ – we could also have gone the other way, taking $\rightarrow$ as primitive and defining $a \circ b := \neg(a \rightarrow \neg b)$). Furthermore, it is easy to verify:

Proposition 4.7 Any complex algebra of an **R** frame is a De Morgan monoid.

Using this fact, we can obtain a completeness proof for **QR** with respect to MG structures which have De Morgan monoids as both nugget and seam – what I'll call "MG De Morgan monoids", or "MGDM's". To work our way up, let us verify, also by way of a sanity check, that we can construct an MG structure from an MG frame in the natural way.

Theorem 4.8 Each MG frame $F = \langle W, N, R, *, Prop, D, PropFun\rangle$ gives rise to an MG structure $\mathfrak{A}^F = \langle \mathcal{A}^P, \mathcal{A}^F, D, PF, \iota\rangle$ where:

(1) $\mathcal{A}^P = \langle Prop, N, \neg, \cap, \cup, \rightarrow\rangle$
(2) $\mathcal{A}^F = \langle \mathcal{P}(W)^\uparrow, N, \neg, \cap, \cup, \rightarrow\rangle$
(3) D is as in F and $PF = PropFun$
(4) ι is the inclusion map: i.e., $\iota : X \mapsto X$ for $X \in Prop$.

Furthermore, any model M satisfying the conditions of Definition 3.4 will be a model on $\mathfrak{A}^F$ satisfying just the same formulas there as on F.

Proof We must verify that $\mathfrak{A}^F$ is, indeed, and MG structure, and that any model M on F will by such that $\langle \mathfrak{A}^F, M\rangle \vDash A \iff \langle F, M\rangle \vDash A$.

For the first bit, note that it is immediate that $\mathcal{A}^P$ and $\mathcal{A}^F$ are po-algebras of the same type, that $\mathcal{A}^F$ is a complete lattice, and that $\mathcal{A}^P$ is a subalgebra of $\mathcal{A}^F$. Thus, ι is, indeed, an order-embedding homomorphism. We just need to note, then, that:

$$\iota((\forall_n\varphi)f) = (\forall_n\varphi)f = \bigcup\{X \in Prop \mid X \subseteq \bigcap_{f'\sim f} \varphi f'\} = \bigcup\{X \in Prop \mid \iota(X) \subseteq \bigcap_{f'\sim f} \iota(\varphi f')\}$$

and that a similar fact holds for $\exists_n\varphi$. This guarantees that ι satisfies the required properties, and that D, $PF = PropFun$ do is immediate from the fact that F is an MG frame.

For the latter part, given an M satisfying Definition 3.4, it suffices to note that our complex algebras are matrices, with $T^{\mathcal{A}^F} = \{X \in Prop \mid N \subseteq X\}$, and so it follows that $N \subseteq [\![A]\!]^M_f \iff [\![A]\!]^M_f \in T^{\mathcal{A}^F}$ holds for any $f \in D^\omega$. Thus $\langle F, M\rangle \vDash A \iff \langle \mathfrak{A}^F, M\rangle \vDash A$, as desired, follows. □

With this in mind, we may prove:

Theorem 4.9 **QR** is the logic of MG De Morgan monoids.

Proof The canonical MG frame for **QR** induces an MG structure satisfying all and only theorems of **QR**, by Theorem 4.8. To obtain the completeness direction it remains to verify that the canonical MG structure built from an MG frame is a De Morgan monoid, but this follows from the fact that any complex algebra of an **R** frame is a De Morgan monoid. Thus, any invalidity of **QR** will be falsified in at least this MG De Morgan monoid.

This leaves the soundness direction, but for this it suffices to verify that the quantifier axioms of **QR** are satisfied in any MG De Morgan monoid, and that the rules preserve satisfaction in such structures, as any De Morgan monoid will satisfy the propositional axioms (and that the rules will preserve satisfaction therein). In fact, this follows from Proposition 4.2, as the additional axioms (∀E) and (∃I) and rules (r∀I) and (r∃E) will be satisfied, and preserve satisfaction, whenever the quantifiers are interpreted as meets and joins of the sets of instances, which is provided by the fact that these are interpreted as meets and joins of $\{\varphi f' \mid f' \sim_n f\}$. □

Let's consider in a bit more detail how to prove completeness using the canonical model in a more concrete manner. In effect, what we are doing is starting from the

Lindenbaum-Tarski algebra for **QR**, and using the canonical model construction to find a complete lattice (the complex algebra of the canonical frame) into which the former algebra embeds in a way satisfying the desired properties. Let me define "Lindenbaum-Tarski algebra" for FMLA-FMLA logics, as a prelude to more general discussion later.

Definition 4.10 Let $\mathbf{L}$ be a FMLA-FMLA logic. The Lindenbaum-Tarski algebra $\mathcal{A}^{\mathbf{L}}$ of $\mathbf{L}$ is the tuple $\langle \mathcal{L}^{\mathbf{L}}, \leq^{\mathbf{L}}, \{\otimes_i^{\mathbf{L}}\}_{i\in I}\rangle$ where, given the congruence

$$\equiv^{\mathbf{L}} = \{\langle A, B\rangle \in \mathcal{L}^2 \mid \{A \prec B, B \prec A\} \subseteq \mathbf{L}\}$$

and fixing $[A]^{\mathbf{L}} = \{B \in \mathcal{L} \mid A \equiv^{\mathbf{L}} B\}$, we define:

(1) $\mathcal{L}^{\mathbf{L}} = \{[A]^{\mathbf{L}} \mid A \in \mathcal{L}\}$
(2) $\leq^{\mathbf{L}} = \{\langle [A]^{\mathbf{L}}, [B]^{\mathbf{L}}\rangle \in (\mathcal{L}^{\mathbf{L}})^2 \mid A \prec B \in \mathbf{L}\}$
(3) For each n-ary connective, $\otimes^{\mathbf{L}}([A_1]^{\mathbf{L}}, \ldots, [A_n]^{\mathbf{L}}) = [\otimes_i(A_1, \ldots, A_n)]^{\mathbf{L}}$

It is easy to see, following Rasiowa [30] and Cintula and Noguera [7], that whenever we consider $\mathcal{A}^{\mathbf{QL}}$, required to include FMLA-FMLA versions of the basic quantifier rules, that:

Proposition 4.11 Given any formula A, the set $\{[A[\tau/x_n]]^{\mathbf{QL}} \mid \tau \in Term\}$ has an infimum w.r.t. $\leq^{\mathbf{L}}$, namely $[\forall x_n A]^{\mathbf{QL}}$, and a supremum $[\exists x_n A]^{\mathbf{QL}}$.

Proof The axiom (∀E) guarantees that $[\forall x_n A]^{\mathbf{QL}}$ is a lesser bound of the target set, and (r∀I) guarantees that it is a greatest lower bound. A similar remark applies for $[\exists x_n A]^{\mathbf{QL}}$. □

Thus, $\mathcal{A}^{\mathbf{QR}}$ is a safe structure, and is, furthermore, a De Morgan monoid. In order to use this fact to obtain completeness w.r.t. MG structures, we just need to find some D and PF for which we have (1) a complete lattice $\mathcal{B}$ and (2) an embedding h of $\mathcal{A}^{\mathbf{QR}}$ into $\mathcal{B}$ which preserves the values of $\forall x_n A$ and $\exists x_n A$. However, we have just such a thing if we work with the *full complex algebra* of the canonical frame $F^{\mathbf{QR}}$, which has been defined in [26].

So what is needed, when considering quantified logics in general, is that we can find some complete lattice into which the Lindenbaum-Tarski algebra of our target quantified logic can homomorphically embed (and some domain and class of propositional functions that suit our needs). In the case that we work with a logic which is complete w.r.t. a class of (general) frames, and which is *canonical* in the usual sense (see [5], e.g.), we can just take the full complex algebra of the canonical frame and the identity map as specifying the desired completion.[16]

When one has a logic which is complete w.r.t. a class of frames for which the logic is not canonical, this simple argument form may not work. In that case, we'd need to show that each of the frames (or perhaps their filtrations, or some similarly simple

[16] For example, this kind of construction works in the modal setting of [18, 19], when we consider *canonical* systems – a general completeness result for quantified extensions of canonical logics is proved in [18]. The only point to note is that if we are working with some classical modal logic, we construct complex algebras somewhat differently, as there the frames are only *discretely* ordered (i.e., ordered by =). This allows us to avoid certain complexities, but the overall construction strategy remains the same.

objects) can be homomorphically embedded into complete lattices of the appropriate sort, or at least that one per invalid formula/sequent can be so embedded. A general setting where we can be sure that the appropriate embeddings are possible is in logics which are complete w.r.t. classes of *gaggles*, as in such a setting we know that we can obtain the desired representations and completeness proofs via canonical model constructions - detailed information can be found in [3]. So we we can be sure that the basic quantifier extension of a wide range of logics which are complete w.r.t. algebraic semantics of this kind will have quantified extensions which are complete w.r.t. MG structures over those algebras. Let's return once again to the difference between **QR** and **RQ** to see an example of when we can obtain a completeness result for something beyond the basic quantified extension.

4.2 MG Structures for RQ

As in the frame case, we can add a simple principle to obtain a class of structures for which **RQ** is the logic. We employed a definition of $X \setminus Y$ in the frame case, but note that we did not require $Prop$ to be closed under this operation, but rather just exploited the fact that $\mathcal{P}(W)^{\uparrow}$ is so closed.[17] We can translate this requirement into the setting of MG structures by noting that in any complete De Morgan monoid, we can define $a \setminus b := \bigwedge\{c \mid a \leq b \vee c\}$, obtaining the result that:

$$a \leq b \vee c \iff a \setminus b \leq c$$

so long as

$$(\star) \qquad \bigwedge_{i \in I}(b \vee c_i) \leq b \vee \bigwedge_{i \in I} c_i$$

obtains, for then $\bigwedge\{d \mid a \leq b \vee d\} \leq c$ implies $b \vee \bigwedge\{d \mid a \leq b \vee d\} \leq b \vee c$, and so, by ($\star$), $a \leq b \vee c$ holds (the converse direction is immediate).

($\star$) certainly holds in the *full* complex algebra of any frame, as any such thing is *completely distributive*, which is why we can help ourselves to $\setminus$ in the frame case. Then, by enforcing the familiar principle, stated for any $\{a, b\} \in ran(h)$:

$$(\star\star) \qquad a \setminus b \leq^{\mathfrak{C}} \bigwedge^{\mathfrak{C}}_{f' \sim_n f} \Rightarrow a \setminus b \leq^{\mathfrak{C}} h((\forall_n \varphi) f)$$

Then the fact that this results in (EC) being valid can be seen straightforwardly by noting (eliding the superscripted $\mathfrak{C}$) that if $\varphi f = \varphi f'$, for any $f' \sim_n f$:

$$h((\forall_n(\varphi \vee \psi))f) \leq \bigwedge_{f' \sim_n f} h((\varphi \vee \psi)f') = h(\varphi f) \vee \bigwedge_{f' \sim_n f} h(\psi f')$$

17 The operation $\setminus$ is *intuitionistic biimplication*, as in [31] (note this operation plays an interesting, and apparently related, role in [33]). Note that intuitionistic implication is also definable in $\mathcal{P}(W)^{\uparrow}$, but need not be in an arbitrary choice of $Prop$.

from which it follows, by using \, that:

$$h((\forall_n(\varphi \vee \psi))f) \setminus h(\varphi f) \leq \bigwedge_{f' \sim_n f} h(\psi f')$$

and thus

$$h((\forall_n(\varphi \vee \psi))f) \setminus h(\varphi f) \leq h((\forall_n \psi) f)$$

and from this we can infer $h(\forall_n(\varphi \vee \psi) f) \leq h(\varphi \vee \forall_n \psi) f$, so that we have, as desired, that $(\forall_n(\varphi \vee \psi)) f \leq^{\mathfrak{N}} (\varphi \vee \forall_n \psi) f$.

So we can obtain the further result, essentially an adaptation of that from [26]:

Theorem 4.12 **RQ** is the logic of MG De Morgan monoids satisfying ($\star\star$).

A more boring way to proceed is to define the salient class of MG De Morgan monoids as those satisfying $(\forall_n(\varphi \vee \psi)) f \leq^{\mathfrak{N}} (\varphi \vee \forall_n \psi) f$, but using ($\star\star$) is somewhat more enlightening. In general the procedure of just writing out the 'semantic analogue' of the desired axioms/rules one wishes to add has been taken in [41], when dealing with weak relevant logics, and in [12, 14] the *only* semantic clauses specified for the relevant modal Barcan formulas are such *homophonic* conditions.

5 Future Directions

Perhaps the most obvious way forward is to attempt to use this framework to investigate the relationship between Mares-Goldblatt models and alternative semantics for quantified relevant logics, such as Fine's semantics [15, 23] and Logan and Leach-Krouse's hyperdoctrine semantics for **RQ** [24, 25]. The latter is especially natural as the hyperdoctrine semantics has a structure which is *prime facie* similar to that of MG structures, involving maps between De Morgan Monoids in the interpretation of quantifiers. Ideally, the resulting relations would support the kind of 'duality-style' results reported in [29, 40].

Another natural line to follow would involve generalising the work in [32] showing the intimate relationship between the gaggle-theoretic representation of logics and their display calculi formulations. In particular, it seems natural to attempt to relate the present line of investigation to work by Tzimoulis [42] in providing display calculus formulations of quantifiers (which, not quite incidentally, involves the category theoretic machinery of *adjoints*, closely related to the abstract residuation machinery of gaggle theory).

Another avenue is to use this construction in investigating extensions of quantified relevant logics, and whether or not they are conservatively extended by Boolean negation. Such extensions hold in many propositional relevant logics, but the question is more difficult in the quantified setting.

Finally, while the gaggle-theoretic focus has provided us a rather nice, general completeness proof, more general results would seem to be available. In particular, in [41] a Mares-Goldblatt treatment was employed to provide complete *general neighbourhood* semantics for quantified extensions of some logics which do not fit the gaggle-theoretic criterion (in particular, in including operations with no tonicity or distribution type). It seems that in these cases we must be employing a kind of completion in some way other than the kind delivered to us by gaggle-theoretic techniques, and it's a natural question to ask what the properties of such completions must be. I'll leave this question for future investigation here, along with those above, and rest hoping that the results here are indicative of the power and flexibility of the MG machinery, and how they provide a nice bridge from algebraic to frame theoretic semantics for a wide range of quantified logics.

Of course, my hope is that the curious phenomenon in relevant logic mentioned at the beginning will continue here, and these avenues for future work provide some reason to believe that this is the case.

Acknowledgements Thanks to Marta Bilková, Nicholas Ferenz, Shay Allen Logan, Franci Mangraviti, Hrafn Oddsson, Francesco Paoli, Grace Paterson, Greg Restall, Shawn Standefer, Yde Venema, Jamie Wannenburg, and audiences in Utrecht, Louvain-la-Neuve, St. Andrews, and at the Australasian Association of Logic and Logica conferences for helpful discussion, as well as to an anonymous referee for the journal. I gratefully acknowledge the Humboldt foundation for fellowship funding.

Author Contributions Not applicable.

Funding Open access funding provided by University of Vienna. Tedder gratefully acknowledges fellowship funding from the Alexander von Humboldt foundation.

Data Availability Statement Not applicable.

Declarations

Ethical Approval Not applicable.

Competing Interest Statement Not applicable.

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