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On the Metainferential Solution to the Semantic Paradoxes

Rea Golan[1,2]

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Abstract

Substructural solutions to the semantic paradoxes have been broadly discussed in recent years. In particular, according to the non-transitive solution, we have to give up the metarule of Cut, whose role is to guarantee that the consequence relation is transitive. This concession—giving up a *meta*rule—allows us to maintain the entire consequence relation of classical logic. The non-transitive solution has been generalized in recent works into a hierarchy of logics where classicality is maintained at more and more metainferential levels. All the logics in this hierarchy can accommodate a truth predicate, including the logic at the top of the hierarchy—known as CM_ω—which presumably maintains classicality at all levels. CM_ω has so far been accounted for exclusively in model-theoretic terms. Therefore, there remains an open question: how do we account for this logic in proof-theoretic terms? Can there be found a proof system that admits each and every classical principle—at all inferential levels—but nevertheless blocks the derivation of the liar? In the present paper, I solve this problem by providing such a proof system and establishing soundness and completeness results. Yet, I also argue that the outcome is philosophically unsatisfactory. In fact, I'm afraid that in light of my results this metainferential solution to the paradoxes can hardly be called a "solution," let alone a good one.

Keywords Metainferences · Proof theory · ST hierarchy · Semantics paradoxes

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✉ Rea Golan
reagolan@gmail.com

1 Institut FR Philosophie, Freie Universitat Habelschwerdter Allee 30 14195 Berlin, Germany

2 Present address: Philosophy Department, Ben Gurion University of the Negev, David Ben Gurion Blvd 1, Be'er Sheva 8410501, Israel

1 Introduction

Substructural solutions to the semantic paradoxes have been broadly discussed in recent years (see, e.g., [11, 15, 19, 21, 23, 25, 26]). Roughly put, the rationale of such solutions is to preserve all the operational rules while facing the paradoxes, at the expense of some structural rules. In particular, according to the non-transitive solution (as presented, e.g., in [6, 20, 21]), we have to give up the structural metarule known as Cut:

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta} Cut$$

whose role is to guarantee that the consequence relation is transitive. By giving up this rule, it becomes possible to introduce into our language a *transparent* truth predicate, i.e., a predicate T governed by the rules:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, T(\langle A \rangle) \Rightarrow \Delta} TL \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow T(\langle A \rangle), \Delta} TR$$

where $\langle A \rangle$ is the name, or Gödel code, of A.[1]

Now, giving up Cut doesn't exactly amount to giving up transitivity. In effect, the non-transitive solution and the logic it espouses— known as ST— remains transitive unless the language is enriched with the truth predicate, along with its rules. Thus, ST has the same consequence relation as that of classical logic (CL). Moreover, even when the truth predicate is added, transitivity still holds "almost everywhere," so to speak; it fails to hold only in the context of "liar-like" sentences.[2]

In light of this result, the non-transitive solution seems as cost-effective as possible. But what does it mean? As I implied, Cut is a *meta*rule, namely, a rule that governs inferences between sequents, or *metainference*. At the end of the day, that is why giving it up doesn't affect much the lower level of inferences. This notion of "metainference" will be central to my discussion. Roughly speaking, metainferences are inferences between inferences (like instances of Cut), but they are also inferences between previously defined metainferences. For example, if $\frac{A \Rightarrow B}{C \Rightarrow D}$, $\frac{C \Rightarrow D}{E \Rightarrow F}$, and $\frac{A \Rightarrow B}{E \Rightarrow F}$ are previously defined metainferences, then so is

$$\frac{\frac{A \Rightarrow B}{C \Rightarrow D} \quad \frac{C \Rightarrow D}{E \Rightarrow F}}{\frac{A \Rightarrow B}{E \Rightarrow F}}$$

To be precise, here is a formal, recursive definition:

Definition 1 Let $SEQ^0(\mathcal{L})$ be the set of all inferences in our language ($\mathcal{L}$), namely, sequents of the form $\Gamma \Rightarrow \Delta$ where $\Gamma, \Delta \subseteq \mathcal{L}$. For all $n \in \mathbb{N}$, we define $SEQ^n(\mathcal{L})$ to

[1]For the purposes of the present paper, there is no need to make up our minds either on whether A is conceived of as a sentence, proposition, formula, etc., or on whether $\langle\ \rangle$ is understood as a disquotational device. I thus allow myself to talk about "sentences" throughout this paper without undertaking any relevant philosophical commitment. Likewise, I shall refer to $\langle A \rangle$ simply as the name of A.

[2]See [20, 21] for discussions of the precise extent to which ST remains transitive even after the truth predicate is added.

be the set of metainferences of level n, namely, sequents of the form $\Gamma \Rightarrow_n A$ where $\Gamma \subseteq SEQ^{n-1}(\mathcal{L})$, $A \in SEQ^{n-1}(\mathcal{L})$. The members of Γ are called the *premise metainferences* and A is the *conclusion metainference*.

For the sake of clarity, I shall sometimes denote a metainference of level n, $\Gamma \Rightarrow_n A$ (where $\Gamma = \{\gamma_1, ..., \gamma_m\} \subseteq SEQ^{n-1}(\mathcal{L})$, $A \in SEQ^{n-1}(\mathcal{L})$) by ${}^{(n)}\ \frac{\gamma_1,...,\gamma_m}{A}$, where the level of that metainference may be optionally mentioned inside brackets to the left of the horizontal proof line.[3]

Let us go back to ST. We saw that to countenance the paradoxes, this logic only gives up Cut, which is a metarule governing metainferences between sequents. In this way, ST manages to preserve all of classical logic at a lower inferential level, that of inferences. A suspicion then arises that ST may only be a starting point in this direction. For, if it is possible both to form a solution to the paradoxes and to preserve all classical inferences by giving up a metainferential principle, why wouldn't it be possible both to form such a solution and to preserve all classical inferences *and* metainferences, simply by giving up a metametainferential principle? Indeed, why wouldn't it be possible both to form such a solution and to preserve all classical inferences and metainferences up to an arbitrary degree, simply by giving up a higher-order metainferential principle?

That suspicion has recently been proven. In [1] (see also [2] for a philosophical discussion), Barrio, Pailos, and Szmuc (BPS) generalized the ST phenomenon by constructing a hierarchy of infinitely many logics. The way they see it, the higher we climb up their hierarchy, the more "classical" the logics we come across, in the sense that each such logic preserves classicality at more metainferential levels than its predecessors. Moreover, the logic at the top of the hierarchy, CM_ω—defined as the union of all these logics—seems to fully agree with classical logic, at each and every metainferential level. Most strikingly, Pailos has proven in [19] that each logic in the hierarchy—including CM_ω—can accommodate a transparent truth predicate. CM_ω thus seems to offer an ideal solution to the paradoxes: all of classical logic, at all inferential levels, combined with transparent truth.

Confronted with such a fantastic result, one cannot help but suspect that it is a magic trick. After all, Tarski taught us that CL and transparent truth are mutually exclusive. To be specific, we know that once we enrich our language with the truth predicate, there will be a sentence λ that is intersubstitutable with $\neg T(\langle\lambda\rangle)$. Presumably, one could then reason as follows to derive the empty sequent (from which point, given the structural rule of Weakening, anything follows):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{T(\langle\lambda\rangle) \Rightarrow T(\langle\lambda\rangle)}{\neg T(\langle\lambda\rangle) \Rightarrow \neg T(\langle\lambda\rangle)}}{\lambda \Rightarrow \lambda}}{T(\langle\lambda\rangle) \Rightarrow \lambda}}{\Rightarrow \neg T(\langle\lambda\rangle), \lambda}}{\Rightarrow \lambda} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{T(\langle\lambda\rangle) \Rightarrow T(\langle\lambda\rangle)}{\neg T(\langle\lambda\rangle) \Rightarrow \neg T(\langle\lambda\rangle)}}{\lambda \Rightarrow \lambda}}{\lambda \Rightarrow T(\langle\lambda\rangle)}}{\neg T(\langle\lambda\rangle), \lambda \Rightarrow}}{\Rightarrow \lambda}}{\Rightarrow} \qquad (1)$$

[3]Below, I shall use this notation mainly in proof trees involving metainferences of different levels.

Now, if CM_ω contains each and every rule that figures in derivation (1)—which it does—how can it block this derivation? How come it accommodates transparent truth along with all the principles required to show that transparent truth leads to disaster? Those questions have so far remained open, as CM_ω has been accounted for only in model-theoretic terms. After all, the way CM_ω manages to do away with derivation (1) is in the domain of proof theory.

In this paper, I intend to solve this problem, both technically and philosophically. Drawing on a previous work of mine where I provided proof systems for the logics in the BPS hierarchy (except for CM_ω), I shall introduce below a proof system for CM_ω and establish soundness and completeness results. On the face of it, proponents of the metainferential approach should be happy with such results, but I shall argue that they shouldn't be. Rather, I shall contend, the proof-theoretic account of CM_ω also sheds philosophical light on the nature of the solution this logic provides to the paradoxes, and the outcome is unsatisfactory, to say the least. Actually, it will turn out that by going metainferential all the way through, we give up perhaps the most basic notion of logic, which is that of inference. I take it that this can hardly be called a "solution," let alone a good one.

But first things first, let me sketch the structure of the paper. In Section 2, I describe the logic ST and its solution to the paradoxes. In Section 3, I turn to the BPS hierarchy, show how it is constructed model-theoretically, and why each logic in it accommodates a truth predicate. Next, I do the same for CM_ω. In Section 4, I turn to proof-theory. First, based on a previous work of mine, I present proof systems for all the logics in the hierarchy. Second, I prove a new result about these systems—a generalization of Gentzen's *Hauptsatz*—with the help of which I explain how each logic deals with the paradoxes. In Section 5, I provide a proof system for CM_ω, and establish soundness and completeness results. In addition, I explain how CM_ω blocks derivations like (1). In Section 6, I conduct a philosophical discussion of the metainferential solution to the paradoxes. My conclusion is that the metainferential solution to the semantic paradoxes comes at a grave, unbearable cost.

2 The logic ST and its solution to the paradoxes

Let $\mathcal{L}$ be a formal language with the connectives $\neg, \vee, \wedge$. The connectives $\supset, \leftrightarrow$ will be defined derivatively, in the usual way: $A \supset B \equiv \neg A \vee B$, $A \leftrightarrow B \equiv (A \supset B) \wedge (B \supset A)$.[4] Here is the sequent calculus for ST I shall be working with:[5]

[4] I assume, throughout this paper, that our language is first-order (as it may be enriched with a truth predicate). For the sake of simplicity, though, I shall focus in my proof-theoretic account on the propositional fragment. Nonetheless, the results below clearly extend to the entire first-order case.

[5] This sequent calculus is taken from [16]. As opposed to the sequent calculus introduced by Ripley in [21], ST is formulated here with double-line rules. These double-line rules make many metarules derivable that would be merely admissible if we used single-line rules. This is important: as proven in [9], only the double-line rules formulation is sound and complete with respect to metainferences of level 1. See my discussion below.

Axioms:

$$A \Rightarrow A$$

Structural Rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \textit{Weakening}$$

Operational Rules:

$$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}\ L\neg \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}\ R\neg$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ R\vee$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\ R\wedge$$

As stated before and as the informed reader may now see, this sequent calculus results from formulating the classical operational rules of CL (as they are presented, e.g., in [21]) with double lines, while removing the rule of Cut. On the face of it, this is a slight difference, since the operational rules are known to be invertible in the presence of Cut, and since Gentzen's *Hauptsatz* tells us that Cut is *admissible* in CL. Thus, dispensing with Cut presumably comes cheap: the resultant system ST has the same consequence relation as that of CL.

From a model-theoretic perspective, ST is based on *strong Kleene valuations*.[6] A valuation function $v : \mathcal{L} \to \{0, \frac{1}{2}, 1\}$ is strong Kleene if it is given by the following truth tables:

	$\neg$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

These strong Kleene valuations will be regarded here as models of ST. We will say that a sequent $\Gamma \Rightarrow \Delta$ is satisfied by some valuation v ($v \models_{ST} \Gamma \Rightarrow \Delta$) if it is not the case that both $v(\gamma) = 1$ for all $\gamma \in \Gamma$, and $v(\delta) = 0$ for all $\delta \in \Delta$. We will say that $\Gamma \Rightarrow \Delta$ is satisfied by ST ($\models_{ST} \Gamma \Rightarrow \Delta$) if $v \models_{ST} \Gamma \Rightarrow \Delta$ for all v. Notice that ST sets different standards for premise-satisfaction and conclusion-satisfaction: a sequent $\Gamma \Rightarrow \Delta$ is ST-valid if, given that the premises are all "strictly" satisfied—namely, assigned the value 1—at least one conclusion is "tolerantly" satisfied—namely, assigned either 1 or $\frac{1}{2}$. This aspect of ST will be of importance later on.

[6] ST is presented from this perspective in [6].

It is well known that a valuation v provides an ST-counterexample to a given sequent $\Gamma \Rightarrow \Delta$ ($v \nvDash_{ST} \Gamma \Rightarrow \Delta$) iff v provides a CL-counterexample to that sequent ($v \nvDash_{CL} \Gamma \Rightarrow \Delta$).[7] Yet, even though the additional truth value doesn't make a difference in consequence relation, it does make a difference at a higher level, that of metainferences of the first level. In particular, the additional truth value $\frac{1}{2}$ lets ST have counterexample models for Cut. Consider, for instance, a model where $v(\gamma) = 1$ for all $\gamma \in \Gamma$, and $v(\delta) = 0$ for all $\delta \in \Delta$, and $v(A) = \frac{1}{2}$. Then $v \models_{ST} \Gamma, A \Rightarrow \Delta$ (because not all the premises are assigned the value 1), and $v \models_{ST} \Gamma \Rightarrow A, \Delta$ (because not all the conclusions are assigned the value 0), but clearly $v \nvDash_{ST} \Gamma \Rightarrow \Delta$.[8]

Interestingly, the counterexample in the last paragraph presupposes a notion of satisfaction applying not only to formulas and sequents, but also to metainferences: a metainference is said to be "dissatisfied" by some valuation v because v satisfies the metainference's premise-sequents, but not its conclusion-sequent. This notion is sometimes called the "local" notion of metainferential validity: we say that some valuation v locally satisfies (or just satisfies) a metainference of level n $\Gamma \Rightarrow_n A$ ($v \models \Gamma \Rightarrow_n A$) if either v doesn't satisfy some $\gamma \in \Gamma$, or v satisfies A. I shall adopt this local notion of satisfaction, following the literature on this matter.[9]

It is also well known (see, e.g., [6, 20, 21]) that ST can be coherently extended with a transparent truth predicate: derivations like (1) are blocked at the last stage, due to the absence of Cut. Moreover, the extension is conservative, and the consequence relation of the extended system remains classical: if Γ, Δ are T-free then $\vdash_{CL} \Gamma \Rightarrow \Delta$ iff $\vdash_{ST} \Gamma \Rightarrow \Delta$, and, moreover, if $\vdash_{CL} \Gamma \Rightarrow \Delta$ then $\vdash_{ST} \Gamma^* \Rightarrow \Delta^*$ where Γ^*, Δ^* are the result of applying *any* uniform substitution $*$ (on the entire language) to all the formulas in Γ, Δ, respectively. It's just that Cut fails to hold in cases of sequents involving liar-like sentences. From a model-theoretic perspective, it turns out that one can apply Kripke's fixed-point construction from [17] to ST models, thereby enriching them with transparent truth. A liar sentence λ gets in such enriched models the value $\frac{1}{2}$, which indicates (as we saw) formulas on which it is impossible to cut. Let us now see how to generalize this interesting phenomenon.

3 The Hierarchy, from a Model-Theoretic Perspective

The hierarchy begins with ST.[10] To introduce the second logic in the hierarchy, we first need to introduce the logic TS. A sequent $\Gamma \Rightarrow \Delta$ is TS-satisfied by a valuation

[7] See [20, p. 358] for a proof.

[8] See [6] for a further discussion of this counterexample.

[9] There is an alternative, "global" way to define metainferential validity, according to this which a metainference is valid if, given that its premises all hold (namely, satisfied in all models) its conclusion also holds. See [1, 9, 13] for discussions of why the local notion is preferable to the global one.

[10] Not exactly. In [1], the hierarchy begins with the paraconsistent logic LP, that is less classical than ST: whereas ST agrees with CL at the level of inferences, LP agrees with CL only on theorems. Since I'm concerned here with *metainferential* solutions to the paradoxes—solution that are, as will turn out later, substructural in nature—I prefer to begin the hierarchy with ST, as in [19]. After all, ST offers a substructural solution to the the paradoxes whereas LP manifests the typical paraconsistent approach.

v if either $v(\gamma) = 0$ for some $\gamma \in \Gamma$, or $v(\delta) = 1$ for some $\delta \in \Delta$. As a result, a valuation v satisfies TS-satisfies an inference $\Gamma \Rightarrow \Delta$ ($v \models_{TS} \Gamma \Rightarrow \Delta$) iff v CL-satisfies that inference ($v \models_{CL} \Gamma \Rightarrow \Delta$). For, in both cases, satisfaction amounts to one of two cases: either there is $\gamma \in \Gamma$ such that $v(\gamma) = 0$, or there is some $\delta \in \Delta$ such that $v(\delta) = 1$. [19, p. 256]

With TS and ST in mind, we can introduce the logic TS/ST, whose satisfaction standards come apart at the first metainferential level: premise-sequents are TS/ST-satisfied if they are TS-satisfied, and conclusion-sequents are TS/ST-satisfied if they are ST-satisfied. That is to say, a metainference $\Gamma \Rightarrow_1 A$ (where $\Gamma \subseteq SEQ^0(\mathcal{L})$, $A \in SEQ^0(\mathscr{L})$) is TS/ST-satisfied by a valuation v ($v \models_{TS/ST} \Gamma \Rightarrow_1 A$) iff either there is some $\gamma \in \Gamma$ such that $v \not\models_{TS} \gamma$, or $v \models_{ST} A$. We say that $\Gamma \Rightarrow_1 A$ is TS/ST-valid ($\models_{TS/ST} \Gamma \Rightarrow_1 A$) if, for all v, $v \models_{TS/ST} \Gamma \Rightarrow_1 A$.

It is worth pointing out that a sequent $\Gamma \Rightarrow \Delta$ may be identified with a metainference of level 1 with no premises, namely, $\frac{\emptyset}{\Gamma \Rightarrow \Delta}$. Therefore, the TS/ST satisfaction criterion for sequents turns out to be that of ST. Consequently, like ST, TS/ST agrees with CL on all inferences and theorems. More importantly, observe that a metainference is *invalid* according to TS/ST if all of its premises-sequents are TS-valid, but its conclusion-sequent is ST-invalid. Yet, as we just saw, a valuation v TS-satisfies an inference $\Gamma \Rightarrow \Delta$ iff it classically satisfies that inference, and that a valuation v ST-dissatisfies an inference $\Gamma \Rightarrow \Delta$ iff it classically dissatisfies that inference. It thus follows that TS/ST agrees with CL not only at the level of inferences, but also at the level of metainferences. [19, p. 256-257] That is to say, TS/ST appears to be classical up to the first metainferential level.

The models of TS/ST are exactly those of ST and TS: strong Kleene valuations. It's just that these logics set different standards for "validity" at various metainferential levels, but over the same models. Therefore, similarly to ST models, one can apply Kripke's fixed-point construction to TS/ST models, thereby enriching them with transparent truth [19]. From a proof-theoretic perspective, though, this fact calls for explanation: after all, all the rules involved in derivation (1) are TS/ST valid, as this logic agrees with CL not only about inferences, but also about metainferences, including Cut. So how does TS/ST block the paradox?

The answer to that question will have to wait until the next section. For now, I wish to point out that this is just the beginning of the story. For, we can define a new logic, ST/TS/TS/ST, whose satisfaction standards diverge for metametainferences, and likewise we can define other logics in a similar way. In general, given two logics L_1, L_2, let us define the logic L_1/L_2 as the logic whose premise-satisfaction standard for metainferences (of some level) is given by L_1 and whose conclusion-satisfaction standard for metainferences (of the same level) by L_2. BPS use this definition to construct their hierarchy in the following way:

Definition 2 The collection $\mathbb{ST} = \{L_i | i \in \mathbb{N}\}$ of logical systems is recursively defined so that $L_1 = ST$, and for $j \geq 2$, $L_j = \overline{L_{j-1}}/L_{j-1}$ (where $\overline{L_j} = L_n/L_m$ if $L_j = L_m/L_n$).

Definition 3 For $j \geq 2$ and $L_j \in \mathbb{ST}$ and a metainference $\Gamma \Rightarrow_{j-1} A$, where $\Gamma \subseteq SEQ^{(j-2)}(\mathcal{L})$, $A \in SEQ^{(j-2)}(\mathcal{L})$, a valuation v satisfies $\Gamma \Rightarrow_{j-1} A$ in L_j ($v \models_{L_j} \Gamma \Rightarrow_{j-1} A$) if $v \not\models_{\overline{L_{j-1}}} \gamma$ for some $\gamma \in \Gamma$, or $v \models_{L_{j-1}} A$. [1, p. 15]

These definitions tell us how metainferences of various levels are evaluated by each logic in the hierarchy. For each n, the logic L_n sets different satisfaction standards for premises and conclusions of metainferences of level $n-1$: $v \models_{L_n} \Gamma \Rightarrow_{n-1} A$ if $v \not\models_{\overline{L_{n-1}}} \gamma$ for some $\gamma \in \Gamma$, or $v \models_{L_{n-1}} A$. For $j \geq n$, however, metainferences are evaluated by L_n uniformly, i.e., $v \models_{L_n} \Gamma \Rightarrow_j A$ if either $v \not\models_{L_n} \gamma$ for some $\gamma \in \Gamma$, or $v \models_{L_n} A$. Likewise, for $j < n-1$, a metainference $\Gamma \Rightarrow_j A$ is evaluated uniformly by L_n, namely, by the standards of L_{n-1}. Thus, the logics in the hierarchy basically differ on the level at which the standard for premise satisfaction and the standard for conclusion satisfaction diverge: the higher we climb up the hierarchy, the higher this level is.

As the reader might have expected, the logics in the hierarchy appear to be progressively classical. That is, each such logic appears to be in agreement with CL at more metainferential levels than its predecessors. This is expressed in a precise manner in the following theorem, proven by BPS:

Theorem 4 *For all $n \geq 1$, for all $j > n$, for all $\Gamma \subseteq SEQ^{n-1}(\mathcal{L})$, $A \in SEQ^{n-1}(\mathcal{L})$:*
$\models_{L_{n+1}} \Gamma \Rightarrow_n A$ *iff* $\models_{L_j} \Gamma \Rightarrow_n A$ *and* $\models_{L_{n+1}} \Gamma \Rightarrow_n A$ *iff* $\models_{CL} \Gamma \Rightarrow_n A$. [1, p.17]

The way BPS put it, "[E]ach system of the hierarchy mimics ST in coinciding with CL up to a certain inferential point." [1, pp. 17-18]

Now, I believe that this way of putting things is inaccurate. Later on, I shall explain why this is so. For now, let me point out that classicality, in its full scope, is not reached at any finite point in the hierarchy. To achieve full classicality, we need to introduce the logic CM_ω, which results from taking the union of all the logics in the hierarchy: a metainference $\Gamma \Rightarrow_n A$ of level n (for any n) is said to be CM_ω-valid if it is L_m-valid for some m. As Pailos proves [19, p. 264] CM_ω reaches full agreement with CL, at all levels: a metainference $\Gamma \Rightarrow_n A$ (for any n) is CM_ω-valid iff it is CL-valid.

However, unlike CL, CM_ω has the same models of ST, TS/ST, and the rest of the logics in the hierarchy: strong Kleene valuations. It's just that CM_ω-validity is defined differently over the same models. Thus, there is no problem in applying Kripke's fixed-point construction to these models, and so even CM_ω can accommodate a transparent truth predicate. In sum, CM_ω is both comprehensively classical—as it agrees with CL at each and every inferential level—and capable of accommodating transparent truth. Ostensibly, CM_ω has achieved the impossible. After all, Tarski thought us that CL and transparent truth are impossible to combine. But here we stand, with a logic that is presumably indistinguishable from CL—at all inferential levels—and yet has no problem accommodating transparent truth.

Yet miracles are rare in philosophy, and even more so in logic. And where some see a miracle, a skeptical mind may look for explanation. It is quite clear that the place to look for such an explanation is proof theory. At the end of the day, it all comes down to the question of how CM_ω blocks liar derivations like (1). Therefore, to solve the problem, we have to have a sound and complete proof system for CM_ω. In effect, what we first need is a proof theoretic account of all the logics in the BPS hierarchy, along with an explanation as to how each such logic blocks the paradox. Only then will it be possible to provide CM_ω with a sound and complete proof system, explain how it solves the paradox, and why, in my opinion, it does so at a grave, unbearable cost.

4 The Hierarchy, from a Proof-Theoretic Perspective

In a previous work [14], I provided proof systems for all the logics in the hierarchy, at all inferential levels.[11] In this section, I introduce some of the results proved there, conduct a philosophical discussion about them, and prove a new result—a generalization of Gentzen's *Hauptsatz* for these systems—that will help us understand how each such logic blocks liar derivations like (1) and thereby accommodates a truth predicate.

To start off, I shall introduce such a metainferential proof system for CL. We have already discussed a "regular" sequent calculus for CL, namely, that of ST with the addition of Cut. Let us call it CL^0. To get a classical proof system for all inferential levels, we need to enrich CL^0 with higher-order axioms and rules. These axioms and rules are supposed to allow us to produce, for each n, proof trees with "sequents" in the form of metainferences of level n and "rules" in the form of metainferences of level $n+1$. Moreover, each instance of a "rule" of level n will become an instance of an "axiom" that can serve as a "leaf" in a proof tree of metainferences of level $n+1$.

Here is a list of the higher-order rules and axioms that we need:[12]

[11] It is worthwhile mentioning other proof systems, to which my philosophical discussion below (Section 6) does not seem to apply. In particular, one can find in the literature labelled and nested sequent calculi such as [7, 10]. However, labelled and nested sequent calculi sneak in (to the syntax) *model-theoretic* notions such as the truth values. Consequently, I believe that when it comes to appreciating the metainferential solution to the paradoxes from a *proof-theoretic* perspective, a more "regular" proof system (such as mine) is preferable to those calculi. Likewise, any philosophical approach to the hierarchy that draws on those labelled and nested calculi (e.g., [3]) is just as problematic. It is also worth mentioning the non-labelled proof system given in [8]. The latter system is similar to my own system presented below, but it is a bit more complex.

[12] Note that in cases where $n = 1$ the metainferences designated by $\Rightarrow_{n-1}$ are simply ordinary sequents. In that case, the consequents of those sequents (denoted by English capital letters) have to be understood as *sets of formulas* rather than metainferences (there are no metainferences of level -1).

1. Transitivity rules - $Tran_n = tran_n^p \cup tran_n^c$, governing the transitivity of premise metainferences and conclusion metainferences, respectively:[13]

$$(n+1)\frac{(n)\dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_{i-1} \Rightarrow_{n-1} A_{i-1}, \Gamma_i \Rightarrow_{n-1} A_i, ..., \Gamma_k \Rightarrow_{n-1} A_k}{\Sigma \Rightarrow_{n-1} B} \quad (n)\dfrac{\Pi \Rightarrow_{n-1} C}{\Gamma_i \Rightarrow_{n-1} A_i}}{(n)\dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_{i-1} \Rightarrow_{n-1} A_{i-1}, \Pi \Rightarrow_{n-1} C, ..., \Gamma_k \Rightarrow_{n-1} A_k}{\Sigma \Rightarrow_{n-1} B}} Tran_n^p \quad (1 \leq i \leq k)$$

$$(n+1)\frac{(n)\dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_k \Rightarrow_{n-1} A_k}{\Sigma_1 \Rightarrow_{n-1} B_1}, ..., (n)\dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_k \Rightarrow_{n-1} A_k}{\Sigma_l \Rightarrow_{n-1} B_l} \quad (n)\dfrac{\Sigma_1 \Rightarrow_{n-1} B_1, ..., \Sigma_l \Rightarrow_{n-1} B_l}{\Pi \Rightarrow_{n-1} C}}{(n)\dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_k \Rightarrow_{n-1} A_k}{\Pi \Rightarrow_{n-1} C}} Tran_n^c$$

2. Monotonicity rules - $Weak_n$, for weakening a given metainference of level n with additional premise metainferences (of level $n-1$):[14]

$$(n+1)\frac{(n)\dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_k \Rightarrow_{n-1} A_k}{\Sigma \Rightarrow_{n-1} B}}{(n)\dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_k \Rightarrow_{n-1} A_k, \Gamma_{k+1} \Rightarrow_{n-1} A_{k+1}}{\Sigma \Rightarrow_{n-1} B}} Weak_n$$

3. Reflexivity axioms: $(n)\dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_i \Rightarrow_{n-1} A_i, ..., \Gamma_k \Rightarrow_{n-1} A_k}{\Gamma_i \Rightarrow_{n-1} A_i} Ref_n$ $(1 \leq i \leq k)$.

4. A set Aux_n of auxiliary axioms of the form $(n)\dfrac{\emptyset^{n-1}}{\Gamma \Rightarrow_{n-1} A}$ for all $\Gamma \subseteq SEQ^{n-2}(\mathcal{L})$, $A \in SEQ^{n-2}(\mathcal{L})$, such that $\Gamma \Rightarrow_{n-1} A$ is derivable via a proof tree of metainferences of level $n-1$, and where $\emptyset^{n-1}$ is the empty metainference of level $n-1$, namely, the metainference of level $n-1$ whose premises and conclusions are all empty: $\emptyset^0 = \emptyset$, $\emptyset^1 = \frac{\emptyset}{\emptyset}$, $\emptyset^2 = \frac{\frac{\emptyset}{\emptyset}}{\frac{\emptyset}{\emptyset}}$, etc.[15]

Definition 5 We recursively define the sequent rules and axioms for classical metainferences of level $n+1$ to be

$$CL^{n+1} = CL^n \cup Tran_{n+1} \cup Weak_{n+1} \cup Ref_{n+1} \cup Aux_{n+1},$$

and $CL^\infty = \bigcup_{n=1}^{\infty} CL^n$. Observe that

$$CL^\infty = CL^0 \cup \bigcup_{n=1}^{\infty} (Tran_n \cup Weak_n \cup Ref_n \cup Aux_n).$$

[13] Note that Cut can be regarded as a $Tran_0$ rule, along these lines. In what follows, I shall sometimes refer to Cut in this way.

[14] In [14] I also had monotonicity rules for weakening with premises and conclusions at the bottom sequent of a given metainference. But as proven there, those rules are redundant, in the sense that they are derivable from the others rules. Moreover, to introduce these rules we need more notation. For simplicity, I omit these rules from the account given here.

[15] Observe that $\emptyset^0$ denotes lack of premises rather than a set-theoretic object. $\emptyset^n$ (for $n \geq 1$) should be understood analogously. It is worth pointing out that these auxiliary axioms are required for uniformity, as they state that lower-level derivations are regarded as derivable at higher levels. See the discussion in [14].

In other words, each CL^n consists of CL^0 along with all the structural rules up to level n. We say that $\vdash_{CL^\infty} \Gamma \Rightarrow_n A$ iff there is some m such that $\Gamma \Rightarrow_n A$ is derivable in CL^m ($\vdash_{CL^m} \Gamma \Rightarrow_n A$).

Let us see how to work with these rules. Suppose, for example, that we would like to derive in CL^∞ not a sequent, but a metainference, say, the one known as meta-Cut:

$$(2)\ \frac{(1)\ \frac{\Sigma\Rightarrow\Pi}{\Gamma,A\Rightarrow\Delta} \quad (1)\ \frac{\Sigma\Rightarrow\Pi}{\Gamma\Rightarrow A,\Delta}}{(1)\ \frac{\Sigma\Rightarrow\Pi}{\Gamma\Rightarrow\Delta}} \tag{2}$$

We do so by applying $Tran_1^c$ to the premise metainferences of (2) together with Cut, which is taken as an axiom at the second metainferential level:

$$(2)\ \frac{(1)\ \frac{\Sigma\Rightarrow\Pi}{\Gamma,A\Rightarrow\Delta} \quad (1)\ \frac{\Sigma\Rightarrow\Pi}{\Gamma\Rightarrow A,\Delta} \quad (1)\ \frac{\Gamma,A\Rightarrow\Delta \quad \Gamma\Rightarrow A,\Delta}{\Gamma\Rightarrow\Delta}Cut}{(1)\ \frac{\Sigma\Rightarrow\Pi}{\Gamma\Rightarrow\Delta}} \tag{3}$$

Likewise, we may construct other proof trees of various kinds, at all levels.

I also establish in [14] that CL^∞ is sound and complete, at all levels, with respect to the standard models of classical logics. That is:

Theorem 6 *For all* $n \in \mathbb{N} \cup \{0\}$ *and for every metainference* $\Gamma \Rightarrow_n A$, $\vdash_{CL^\infty} \Gamma \Rightarrow_n A$ *iff* $\models_{CL} \Gamma \Rightarrow_n A$.

It is just as easy to provide sequent rules and axioms for ST, at all inferential levels. In effect, the sequent rules of ST for all levels higher than 0 are exactly those of CL^∞. That is:

Definition 7 Let ST^0 be the regular sequent calculus of ST given in Section 2. We recursively define

$$ST^{n+1} = ST^n \cup Tran_{n+1} \cup Weak_{n+1} \cup Ref_{n+1} \cup Aux_{n+1}$$

and $ST^\infty = \bigcup_{n=1}^{\infty} ST^n$. Observe that

$$ST^\infty = ST^0 \cup \bigcup_{n=1}^{\infty} (Tran_n \cup Weak_n \cup Ref_n \cup Aux_n).$$

We say that $\vdash_{ST^\infty} \Gamma \Rightarrow_n A$ iff there is some m such that $\Gamma \Rightarrow_n A$ is derivable in ST^m ($\vdash_{ST^m} \Gamma \Rightarrow_n A$).

Theorem 8 $\vdash_{ST^\infty}$ *is sound and complete with respect to* $\models_{ST}$*: for all* $n \in \mathbb{N} \cup \{0\}$ *and for every metainference* $\Gamma \Rightarrow_n A$, $\vdash_{ST^\infty} \Gamma \Rightarrow_n A$ *iff* $\models_{ST} \Gamma \Rightarrow_n A$.

As for the rest of the logics in the hierarchy, we first need to consider the following substitution rules $Sub_n = \{Sub_n^p, Sub_n^c\}$:

$$
{\scriptstyle (n+1)}\ \frac{{\scriptstyle (n)}\ \dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_{i-1} \Rightarrow_{n-1} A_{i-1}, \Gamma_i \Rightarrow_{n-1} A_i, ..., \Gamma_k \Rightarrow_{n-1} A_k}{\Sigma \Rightarrow_{n-1} B} \qquad {\scriptstyle (n)}\ \dfrac{\Pi \Rightarrow_{n-1} C}{\overline{\Gamma_i \Rightarrow_{n-1} A_i}}}{{\scriptstyle (n)}\ \dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_{i-1} \Rightarrow_{n-1} A_{i-1}, \Pi \Rightarrow_{n-1} C, ..., \Gamma_k \Rightarrow_{n-1} A_k}{\Sigma \Rightarrow_{n-1} B}}\ {\scriptstyle Sub_n^p} \quad (1 \leq i \leq k)
$$

$$
{\scriptstyle (n+1)}\ \frac{{\scriptstyle (n)}\ \dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_k \Rightarrow_{n-1} A_k}{\Sigma \Rightarrow_{n-1} B} \qquad {\scriptstyle (n)}\ \dfrac{\Sigma \Rightarrow_{n-1} B}{\overline{\Pi \Rightarrow_{n-1} C}}}{{\scriptstyle (n)}\ \dfrac{\Gamma_1 \Rightarrow_{n-1} A_1, ..., \Gamma_k \Rightarrow_{n-1} A_k}{\Pi \Rightarrow_{n-1} C}}\ {\scriptstyle Sub_n^c}
$$

These rules allow us to substitute premise/conclusion metainferences for *equivalent* metainferences: that is why each *Sub* rule has as a premise a double-line metainference, whose role is to guarantee that the substitute and the substitutee are indeed equivalent. These substitution rules are clearly derivable from $Tran_n^p$, $Tran_n^c$, but they are weaker than the transitivity rules. For to apply such a rule, we need a premise in the form of a double-line metainference.

Now, the substitution rules will be needed in the absence of the transitivity rules. In effect, each logic L_m in the BPS hierarchy (when $m \geq 2$) becomes non-transitive at level $m - 1$, and is thus governed at level n (for all $n \in \mathbb{N} \cup \{0\}$) by the following sequent rules and axioms:

$$
L_m^n = \begin{cases} CL^n & 0 \leq n \leq m-2 \\ L_m^{n-1} \cup Weak_n \cup Ref_n \cup Aux_n \cup Sub_n & n = m-1 \\ L_m^{n-1} \cup Weak_n \cup Ref_n \cup Aux_n \cup Tran_n & n > m-1. \end{cases}
$$

As before, let us define $L_m^\infty = \bigcup_{n=1}^{\infty} L_m^n$. Observe that for each m, we have

$$
L_m^\infty = CL^{m-2} \cup Weak_{m-1} \cup Ref_{m-1} \cup Aux_{m-1} \cup Sub_{m-1} \cup \bigcup_{n=m}^{\infty} (Weak_n \cup Ref_n \cup Aux_n \cup Tran_n).
$$

And indeed, all those systems prove to be sound and complete:

Theorem 9 *For all* $m \in \mathbb{N} \cup \{0\}$, *for all* $n \in \mathbb{N} \cup \{0\}$, *and for every metainference* $\Gamma \Rightarrow_n A$: $\vdash_{L_m^\infty} \Gamma \Rightarrow_n A$ *iff* $\models_{L_m} \Gamma \Rightarrow_n A$.

So much for my previous work. I now turn to make some novel observations based on the above results. As a point of departure, recall that BPS conceive of their hierarchy as if its logics are progressively classical. Strictly speaking, this is inaccurate. To see why, consider again TS/ST,[16] which is supposed to be "more classical" than ST as it presumably agrees with CL not only on inferences, but also on metainferences of the first level (for one thing, unlike ST, TS/ST has Cut). But how about higher levels? From level 3 upwards, ST is as classical as TS/ST, as both logics admit all classical

[16] I take TS/ST to be my case in point throughout this paper. But there is nothing special about TS/ST here. Every claim I'm about to make clearly extends to the logics above TS/ST in the hierarchy.

rules and axioms at those levels. At level 2, though, ST turns out to be more classical than TS/ST. For, unlike TS/ST, ST admits the transitivity rules $Tran_1^p$, $Tran_1^c$.[17] Hence, even though TS/ST appears to be more classical than ST at level 1, it is less classical than ST at level 2, and at higher levels the two logics are in full (classical) agreement. Therefore, I see no reason to believe that TS/ST is more classical than ST. For those who are not yet convinced by this line of thought, I shall now put forward a much stronger argument for the non-classicality of TS/ST, based on its lack of the $Tran_1$ rules.

Up until now, I've only hinted at TS/ST's lack of transitivity rules for first-level metainferences. But now I wish to elaborate on this point, which is crucial for several reasons. To get a better understanding of what is going on with TS/ST at the first metainferential level, consider the following valuation v, which has: $v(A) = 1$, $v(B) = 1$, $v(C) = 1$, $v(D) = \frac{1}{2}$, $v(E) = 1$, and $v(F) = 0$. It is not difficult to verify that $v \models_{TS} A \Rightarrow B$, $v \not\models_{TS} C \Rightarrow D$, $v \models_{ST} C \Rightarrow D$, and $v \not\models_{ST} E \Rightarrow F$. Therefore:

$$v \not\models_{TS/ST} \frac{\frac{C\Rightarrow D}{A\Rightarrow B} \quad \frac{E\Rightarrow F}{C\Rightarrow D}}{\frac{E\Rightarrow F}{A\Rightarrow B}} \tag{4}$$

Needless to say, (4) is clearly ST-valid, as well as CL-valid.

This is a big deal. The absence of the $Tran_1$ rules makes it unsafe to apply any "regular" sequent rule in TS/ST, thereby reducing the prospects of constructing sequent-to-sequent derivations in this logic. After all, each and every "regular" proof tree—namely, proof tree of sequents—makes implicit use of those transitivity rules. For each sequent in such a proof tree (except for the root) is first inferred as a *conclusion*, and then serves as a *premise* in yet another step in the derivation. Unfortunately, it is exactly this role reversal that cannot be automatically guaranteed in TS/ST due to the absence of the $Tran_1$ rules. Thus, TS/ST allows us to derive sequent-axioms (from no premises), but provides no guarantee that those axioms can serve as starting points of derivations of length greater than 1.

Here is another way to think about this aspect of TS/ST. In this logic, any rule-involving derivation like

$$\frac{\dfrac{A \Rightarrow B}{C \Rightarrow D}}{E \Rightarrow F} \tag{5}$$

can no longer be automatically carried out. Rather, to derive $E \Rightarrow F$ from $A \Rightarrow B$ in TS/ST, one first needs to make sure that the sequent $C \Rightarrow D$ can play the dual role

[17] Several authors (e.g., [1, 19]) seem inclined to believe that ST remains non-classical at the second metainferential level (like TS/ST) due to the fact that it doesn't have meta-Cut. However, TS/ST doesn't have meta-Cut either, and hence TS/ST is not "more classical" than ST at level 2. Moreover, recall derivation (3): the reason why ST doesn't have meta-Cut is because we need Cut as an axiom to derive meta-Cut. Thus, ST's inability to admit meta-Cut is really caused by its first-level non-classical behavior, rather than its second-level behavior.

of both premise and conclusion, as such duality can no longer be taken for granted.[18] In other words, to guarantee the derivation of $E \Rightarrow F$ from $A \Rightarrow B$ in TS/ST, one actually needs to make the following metainferential derivation:

$$\frac{\dfrac{A\Rightarrow B}{C\Rightarrow D} \quad \dfrac{C\Rightarrow D}{E\Rightarrow F}}{\dfrac{A\Rightarrow B}{E\Rightarrow F}} \tag{6}$$

That is to say, if we dispense with the transitivity rules for first-level metainferences, we cannot automatically carry out each sequent-to-sequent derivation such as (5). Rather, we need to ground it in yet another derivation: a metainferential derivation of level 1 such as (6). The need for such grounding results from that fact that in dropping the $Tran_1$ rules, TS/ST goes substructural not about the sequent arrow $\Rightarrow$ but about the bar —— between sequents in proof trees like (5). Ultimately, that is why metainferences matter from a proof-theoretic perspective: by introducing higher-order rules, our presuppositions about the structural properties of sequent-to-sequent derivations (as well as higher-order derivations) are brought to light, and no longer taken for granted. Hence, it becomes possible to go substructural about the bar between sequents, as well as higher-order bars between metainferences of any arbitrary level.

Let me dwell a bit further on this issue. It is well known that in sequent calculi, as opposed to Hilbert-style systems, the level where rules are applied and derivations genuinely take place is that of metainferences. As Dicher and Paoli explain:

> "Notice... that in a sequent calculus all of the action takes place at the level of sequent-to-sequent rules, whereby from one or more sequents (intuitively understood as 'inferences') we derive more sequents (i.e., more 'inferences'). Which is to say, the action takes place at the level of metainferences. It is therefore only natural to account for metainferences as syntactic objects of the system under consideration." [9, p. 8]

Gentzen's ingenious insight, one might say, was exactly this: to transform what would otherwise (in a Hilbert-style system) be a proof consisting of a series of inferences between sentences (or sets thereof) into a proof tree where such inferences are themselves being derived from one another, whereby our presuppositions about the structural properties of derivations are brought to light. By doing so, Gentzen opened the door to the possibility of going substructural in various ways. In TS/ST, this Gentzenian move is literally taken to a higher degree, so that the "real action" takes place not at the first metainferential level, but at the second.[19] Likewise, the higher we climb up the BPS hierarchy, the higher the level at which "real action"

[18] I focus above on the dual role of $C \Rightarrow D$, but only for the sake of clarity. The same is true also for $A \Rightarrow B$. If $A \Rightarrow B$ is an axiom, then it is actually derived from no premises ($\frac{\emptyset}{A\Rightarrow B}$), and so there is no guarantee in TS/ST that $A \Rightarrow B$ can serve as a premise from which $C \Rightarrow D$ can be derived (by some rule), due to the absence of the $Tran_1$ rules.

[19] Strictly speaking, such activity can take place in TS/ST at all levels$\geq$ 2. Yet, my point is that first-level action is no longer possible, so that sequent-to-sequent derivations need to be grounded in metainferential derivations, in the same way that inferences between sentences in Hilbert-style systems are transformed into inference between sequents in sequent calculi.

takes place, since the lack of transitivity rules at some level results in the need to ground lower-level derivations—except for one-step derivations of axioms— in corresponding higher-level derivations.

On top of that, notice that each such logic stops being "classical" exactly at the level where the action takes place: ST stops being classical at the first metainferential level, TS/ST at the second metainferential level, and so on. Ultimately, that is why I have doubts about the BPS claim that the logics in the hierarchy progressively agree with CL. For at the level where there is such "agreement," real inferential action—except for one-step derivations of axioms—cannot be carried out, and needs to be grounded in higher-order derivations. Thus, TS/ST doesn't really "agree" with CL on inferences and metainferences, nor does any logic higher up in the hierarchy agree with CL at further metainferential levels. Whereas CL presupposes all the structural properties of derivations and thereby allows real inferential action to take place at any inferential level, each logic in the hierarchy goes substructural at some metainferential level, which results in the need to ground lower-level derivations in corresponding higher-level derivations.

In light of the above, one may well wonder where the illusion that TS/ST is classical up to the second metainferential level comes from. I believe that the illusion is maintained by the fact that TS/ST has something to offer instead of the $Tran_1$ rules, namely, the Sub_1 rules. The latter rules do not make us free of the need to ground sequent-to-sequent derivations in corresponding metainferential derivations. Yet, the Sub_1 rules do guarantee that any sequent-to-sequent derivation with double-line bars at each step can *in principle* be grounded in this way. For example, a derivation where each step is invertible like

$$\dfrac{\dfrac{A \Rightarrow B}{C \Rightarrow D}}{E \Rightarrow F} \tag{7}$$

can always be grounded in TS/ST, because

$$\dfrac{\dfrac{\dfrac{A \Rightarrow B}{C \Rightarrow D} \quad \dfrac{C \Rightarrow D}{E \Rightarrow F}}{A \Rightarrow B}}{E \Rightarrow F} \tag{8}$$

is an instance of Sub^c_1. Fortunately, our operational rules are formulated as double-line rules,[20] which gives rise to the hope that many classical sequent-to-sequent derivations can be grounded in TS/ST.

Actually, there is some really good news on this front: at least for the logical fragment of the language, we don't need more than double-line derivations because the $Tran_1$ rules are *admissible* in TS/ST. That is to say, if some metainference can be proven in TS/ST with the addition of the $Tran_1$ rules, then there is a corresponding

[20]See footnote 5. One may wonder whether the double line between $A \Rightarrow B$ and $C \Rightarrow D$ in derivation (7) is necessary. It is, because as noted in footnote 18, $A \Rightarrow B$ is itself derived from no premises, and so there is a need to make sure that it can serve as a premise in yet another step in the derivation. As I said in footnote 18, I focus above on the dual role of $C \Rightarrow D$, but only for the sake of clarity.

TS/ST derivation of the same result that makes no use of the $Tran_1$ rules. As a result, we get a guarantee that a sequent-to-sequent derivation can always be grounded in a corresponding metainferential derivation, to the extent that it involves only logical axioms and rules. To go back to our previous example, if derivation (5) involves only logical rules and axioms, then we are guaranteed that there is an alternative to derivation (6) in TS/ST—with the same premises (with optionally more leafs with axioms) and the same conclusion— that makes no use of the $Tran_1$ rules. Analogous results hold for each logic in the BPS hierarchy, as confirmed by the following theorem.

Theorem 10 *For all $n \geq 1$, $Tran_{n-1}$ is admissible in L_n^∞; namely, the set of metainferences of level $n-1$ derivable in L_n^∞ doesn't change if $Tran_{n-1}$ is added to its rules. Put differently, given some metainference $\Gamma \Rightarrow_{n-1} A$ of level $n-1$: if $\vdash_{L_n^\infty \cup Tran_{n-1}} \Gamma \Rightarrow_{n-1} A$ then $\vdash_{L_n^\infty} \Gamma \Rightarrow_{n-1} A$.*

Proof By induction. The base case is $n = 1$, namely, ST^∞. But we know from Gentzen that $Tran_0$, i.e., Cut, is eliminable in ST^∞. To proceed with the proof, we first need to recursively define the translation function $lower : \bigcup_{n<\omega} SEQ^n(\mathcal{L}) \rightarrow \mathcal{L} \cup \bigcup_{n<\omega} SEQ^n(\mathcal{L})$, as

$$lower(\Gamma \Rightarrow \Delta) = ((\wedge\Gamma) \supset (\vee\Delta))$$
$$lower(\Gamma \Rightarrow_{n+1} A) = \{lower(\gamma) \mid \gamma \in \Gamma\} \Rightarrow_n lower(A).$$

Notice that the output of $lower$, when applied to a metainference of level $n+1$ (for any n), is a metainference of level n. Importantly, it is proven in [1, 14] that

(i) for all n: $\vdash_{L_{n+1}^\infty} \Gamma \Rightarrow_n A$ iff $\vdash_{L_n^\infty} lower(\Gamma \Rightarrow_n A)$, and

(ii) $\vdash_{CL^\infty} \Gamma \Rightarrow_n A$ iff $\vdash_{CL^\infty} lower(\Gamma \Rightarrow_n A)$.

With properties (i) and (ii) in mind, we can proceed to the inductive step. Let us therefore assume that the claim holds for some n, namely, that the $Tran_{n-1}$ rules are eliminable in L_n. Let $\Gamma \Rightarrow_n A$ be some metainference of level n, and assume that $\vdash_{L_{n+1}^\infty \cup Tran_n} \Gamma \Rightarrow_n A$. Notice that, actually, $L_{n+1}^\infty \cup Tran_n = CL^\infty$, and so $\vdash_{CL^\infty} \Gamma \Rightarrow_n A$. By property (ii), this is the case iff $\vdash_{CL^\infty} lower(\Gamma \Rightarrow_n A)$, and since $CL^\infty = L_n \cup Tran_{n-1}$, we get that $\vdash_{L_{n+1}^\infty \cup Tran_n} \Gamma \Rightarrow_n A$ iff $\vdash_{L_n^\infty \cup Tran_{n-1}} lower(\Gamma \Rightarrow_n A)$. By the inductive hypothesis, $Tran_{n-1}$ is eliminable in L_n^∞, which means that $\vdash_{L_n^\infty \cup Tran_{n-1}} lower(\Gamma \Rightarrow_n A)$ iff $\vdash_{L_n^\infty} lower(\Gamma \Rightarrow_n A)$. By property (i), $\vdash_{L_n^\infty} lower(\Gamma \Rightarrow_n A)$ iff $\vdash_{L_{n+1}^\infty} \Gamma \Rightarrow_n A$. Summing up, we get that $\vdash_{L_{n+1}^\infty \cup Tran_n} \Gamma \Rightarrow_n A$ iff $\vdash_{L_{n+1}^\infty} \Gamma \Rightarrow_n A$, as required. □

This result is of paramount importance. Later on, it will help us solve the mystery around CM_ω. Right now, though, I wish to point out that as far as classicality is concerned, Theorem 10 offers nothing more than an illusion. For the $Tran_n$ rules may not be admissible outside the logical fragment of the language, and so sequent-to-sequent derivations involving, say, Peano axioms, may be impossible to ground in the logics of the BPS hierarchy. In effect, those logics block liar derivations like (1) exactly due to failures of transitivity that occur outside the logical fragment.

Let me demonstrate the last point with TS/ST. Assume we add Peano arithmetic or simply enrich the language with a truth predicate along with the TL and TR rules.

For the sake of the argument, assume further that we've somehow managed to derive both $\lambda \Rightarrow$ and $\Rightarrow \lambda$ (where λ is a liar sentence), from which point we should proceed to the last step of derivation (1), cutting on these sequents so as to derive $\Rightarrow$. Yet, even though TS/ST proves the relevant instance of Cut—after all, $\vdash_{TS/ST} \frac{\lambda\Rightarrow \quad \Rightarrow\lambda}{\Rightarrow}$—this logic does not allow us to actually cut on $\lambda \Rightarrow, \Rightarrow \lambda$ because as derived sequents they cannot play the role of premises.[21] We rather need to ground the last step of derivation (1) in a corresponding metainferential derivation. However, due to the absence of $Tran_1$:

$$\nvdash_{TS/ST} \quad (2)\,\frac{(1)\,\frac{\emptyset}{\lambda\Rightarrow} \qquad (1)\,\frac{\emptyset}{\Rightarrow\lambda} \qquad (1)\,\frac{\lambda\Rightarrow \quad \Rightarrow\lambda}{\Rightarrow}}{(1)\,\frac{\emptyset}{\Rightarrow}} \tag{9}$$

In this way, we get a straightforward proof that TS/ST is safe from paradox: no axiom of TS/ST is empty, and no rule of TS/ST except for Cut can go from non-empty premises to an empty conclusion. Yet, in TS/ST even Cut cannot do so by itself; it needs the underlying support of the $Tran_1$ rules, which it does not have. What's more, notice that if $Tran_1^c$ were admissible in this extension of TS/ST, the conclusion metainference of derivation (9) would be derivable, in which case the consequence relation of TS/ST as well its notion of first-level metainferential validity—designated by the bar between sequents—would both be trivialized.[22] Thus, we also get a proof that the transitivity rules are not admissible in the extension formed by adding Peano axioms or the truth predicate to TS/ST.[23] To sum up, going substructural about the bar between sequents makes TS/ST immune to the paradox even in the presence of Cut; the only price TS/ST pays for this is that the $Tran_1$ rules stop being admissible, and so there is no guarantee that we can ground classical sequent-to-sequent derivations in this logic outside the logical fragment of the language.[24]

In general, each logic L_n in the hierarchy does not admit the $Tran_{n-1}$ rules, and thereby goes substructural about the bar between metainferences of level $n-1$. By the same reasoning, each such logic successfully accommodates transparent truth. It's just that the $Tran_{n-1}$ rules stop being admissible outside the logical fragment of

[21] Ripley [22] makes a similar distinction between *containing* a rule and *obeying* it. In his language, TS/ST *contains* Cut, but doesn't *obey* it. I find this terminology to be not entirely convincing. The problem at hand is not whether TS/ST obeys Cut, but what it means to obey Cut in a logic that goes substructural about the bar between sequents.

[22] If the conclusion metainference of derivation (9) were derivable in TS/ST, then one could also derive $\Rightarrow\bot$ from $\frac{\emptyset}{\Rightarrow\bot}$ by reasoning in the metalanguage (given that the metalanguage is transitive). Worse, the conclusion metainference of derivation (9) is not derivable in CL^1 and so, in a setting like TS/ST that is classical up to the first metainferential level we reach a contradiction.

[23] As Girard writes, "The *Hauptsatz* fails for systems with proper axioms," [12, p. 125] meaning that once we add Peano axioms to the sequent rules and axioms of CL Cut is no longer admissible. It is thus no wonder that higher order transitivity principles stop being admissible just as well once Peano axioms or the like is added. It is worthwhile to mention that the admissibility of Cut can be recovered by adding Peano arithmetic formulated as a set of non-logical rules rather than as axioms, but then the subformula property is lost, and so consistency remains a problem. See [18] for a discussion.

[24] From a model-theoretic perspective, we note that TS/ST treats liar sentences in the same way as ST: it assigns the value $\frac{1}{2}$ to λ in every model, which makes both sequents $\lambda \Rightarrow, \Rightarrow \lambda$ TS/ST-valid, since they are ST-valid. However, these sequents are *not* both TS-valid, which indicates that it is impossible to actually cut on them.

the language once the truth predicate is added, and so liar derivations like (1) cannot be grounded in L_n by a corresponding metainferential derivation of level $n-1$. Thus, even though the logics in the hierarchy all admit the rules and axioms involved in liar derivations like (1), they undermine the structural properties of such derivations in such a way that these rules and axioms cannot be actually applied.

In conclusion, the logics in the hierarchy go progressively substructural. As a result, they each block liar derivations like (1) due to the lack of transitivity at the relevant level. In this way, all these logics turn out to be immune to the paradox. But how about CM_ω? In the next section I show that CM_ω goes substructural at *all* levels, and that, as a result, there is no way to ultimately ground derivations in CM_ω. That is to say, there is no room for real inferential action in this logic. At the end of the day, this is why I believe that CM_ω is not only not identifiable with CL but—in fact—it can hardly be regarded as "logic," and so its solution to the paradoxes is dubious.

5 CM_ω, with and without the Truth Predicate

How can we go about providing a proof system for CM_ω? The problem is that we need something that looks like CL at all levels, but doesn't behave like it. In particular, CM_ω should give us the illusion that even after the truth predicate is added it endorses each and every classical principle—rules and axioms alike—while blocking liar derivations like (1) at the same time.

Let us begin with what is already known about how such a system should behave. Undoubtedly, it should admit all the operational rules, as well as all the structural rules other than the transitivity ones. That is, CM_ω should contain all of ST^0, along with $Weak_n$, Ref_n, Aux_n, and Sub_n, at all levels. It remains to consider how it deals with transitivity at all levels, starting with Cut. After all, all transitivity principles come out as CM_ω-valid, as each one of them is satisfied by at least one logic in the hierarchy. That is, for all $r \in \bigcup_{n=0}^{\infty} Tran_n$, we have $\models_{CM_\omega} r$ and so, for completeness to hold we must have $\vdash_{CM_\omega} r$, regardless of whether the truth predicate is added to the language.

The dilemma is clear: CM_ω must be able to prove all transitivity principles—admissibility is not enough—and yet block them all from being used in liar derivations like (1). With the proof-theoretic account of the BPS hierarchy in mind, the solution is at hand. Recall that in the account laid out in the previous section each rule has a dual role to play: primarily as a rule, but secondarily as an axiom in higher-order derivations. Consider, for example, the role of Cut in CL^∞. On the one hand, Cut plays the role of a structural rule in classical sequent-to-sequent derivations. On the other hand, Cut may occur as an axiom in a derivation of metainferences of the first level as it does, e.g., in derivation (3). It is due to the latter role that $\vdash_{CL^\infty} \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}$.

So here is the solution: to provide a proof system for CM_ω we have to abandon the duality of roles in the case of the transitivity principles. The result is a system where *all* transitivity principles are admitted, but only as *axioms*, not as *rules*. Thus, Cut may only occur as an axiom in proof trees of metainferences, as it does in derivation

(3), but not as a rule in sequent-to-sequent derivations. In general, each $r \in Tran_n$ (for some n) may occur as an axiom in proof trees of metainferences of level $n+1$, but not as a rule in proof trees of metainferences of level n. If we denote by r^{axiom} a schemata of metainferences (or a set thereof) whose instances can only play the role of axioms, then the proof system we were looking for can be defined as

$$CM_\omega = ST^0 \cup \{Cut^{axiom}\} \cup \bigcup_{n=1}^{\infty} (Weak_n \cup Ref_n \cup Aux_n \cup Sub_n \cup Tran_n^{axiom})$$

I shall now suggest the following lemma (with two corollaries) regarding this proof system:

Lemma 11 *For all $n \geq 0$ and every metainference $\Gamma \Rightarrow_n A$: $\vdash_{CM_\omega} \Gamma \Rightarrow_n A$ iff $\vdash_{CL^\infty} \Gamma \Rightarrow_n A$.*

Proof The left-to-right direction is trivial. On the other hand, assume that $\vdash_{CL^\infty} \Gamma \Rightarrow_n A$. It is enough to show that all transitivity rules, at all levels, are eliminable in CL^∞. If so, then there is a CL^∞-proof of $\Gamma \Rightarrow_n A$ that doesn't make use of any transitivity rule, and this proof is clearly available in CM_ω, so $\vdash_{CM_\omega} \Gamma \Rightarrow_n A$.

Let $\Gamma \Rightarrow_k A$ be some metainference of level k (for some k) such that $\vdash_{CL^\infty} \Gamma \Rightarrow_k A$. In each step in the proof we derive some metainference of level k by applying a rule in the form of a metainference of level $k+1$, including $Tran_k$. By Theorem 4, $\vdash_{CL^\infty} \Gamma \Rightarrow_k A$ iff $\vdash_{L^\infty_{k+1}} \Gamma \Rightarrow_k A$,[25] but L^∞_{k+1} doesn't have $Tran_k$, and so there is a L^∞_{k+1}-proof of $\Gamma \Rightarrow_k A$ that doesn't make any use of $Tran_k$, and that proof is clearly available in CL^∞. We didn't make any assumption about k, and so we may conclude that all transitivity rules are eliminable in CL^∞, as required. □

Corollary 12 *For all $m \geq 1$, $n \geq 0$, and every metainference $\Gamma \Rightarrow_n A$: if $\vdash_{L^\infty_m} \Gamma \Rightarrow_n A$ then $\vdash_{CM_\omega} \Gamma \Rightarrow_n A$.*

Proof If $\vdash_{L^\infty_m} \Gamma \Rightarrow_n A$ then clearly $\vdash_{CL^\infty} \Gamma \Rightarrow_n A$ since CL^∞ simply has more rules and axioms than L^∞_m. By Lemma 11, $\vdash_{CM_\omega} \Gamma \Rightarrow_n A$. □

Corollary 13 *For all n and every metainference $\Gamma \Rightarrow_n A$: if $\vdash_{CM_\omega} \Gamma \Rightarrow_n A$ then there is some m such that $\vdash_{L^\infty_m} \Gamma \Rightarrow_n A$.*

Proof Consider L^∞_{n+1}. By Theorem 4,[26] $\vdash_{L^\infty_{n+1}} \Gamma \Rightarrow_n A$ iff $\vdash_{CL^\infty} \Gamma \Rightarrow_n A$ and, by Lemma 11, that's the case iff $\vdash_{CM_\omega} \Gamma \Rightarrow_n A$. □

Based on the above results, we can quickly establish soundness and completeness:

[25] Theorem 4 actually states the model-theoretic version of that biconditional. But the proof-theoretic version follows straightforwardly, given the soundness and completeness of CL^∞ and L^∞_{k+1}.

[26] See the previous footnote.

Theorem 14 *For all n and every metainference* $\Gamma \Rightarrow_n A$: $\models_{CM_\omega} \Gamma \Rightarrow_n A$ *iff* $\vdash_{CM_\omega} \Gamma \Rightarrow_n A$.

Proof We begin with soundness. Assume that $\vdash_{CM_\omega} \Gamma \Rightarrow_n A$. By Corollary 13, there is some m such that $\vdash_{L_m^\infty} \Gamma \Rightarrow_n A$. By Theorem 9, $\models_{L_m^\infty} \Gamma \Rightarrow_n A$. So by definition, $\models_{CM_\omega} \Gamma \Rightarrow_n A$.

On the other hand, let $\Gamma \Rightarrow_n A$ be some metainference of level n (for some n), and assume that $\models_{CM_\omega} \Gamma \Rightarrow_n A$. By definition, it follows that there is some m such that $\models_{L_m} \Gamma \Rightarrow_n A$. By Theorem 9, $\vdash_{L_m^\infty} \Gamma \Rightarrow_n A$. By Corollary 12, $\vdash_{CM_\omega} \Gamma \Rightarrow_n A$. □

With this proof system at hand, it becomes clear how CM_ω blocks liar derivations like (1): it does so simply by not granting any transitivity principle the status of a rule, even though it admits it as an axiom. Let me demonstrate this point with derivation (1) itself. Suppose that we've somehow managed to derive $\lambda \Rightarrow$ and $\Rightarrow \lambda$ in CM_ω. To derive the empty sequent from these sequents, we need to cut on them. Yet, CM_ω only admits Cut as an axiom, not as a rule. Thus, derivation (1) needs to be grounded in CM_ω by derivation (9). Now, derivation (9) was blocked by TS/ST because it doesn't admit $Tran_1^c$. Unlike TS/ST, CM_ω does admit $Tran_1^c$, but only as a an axiom and not as a rule. As a result, derivation (9) is itself in need of grounding in CM_ω by a corresponding derivation of metametainferences, which will itself be in need of further grounding, and so on ad infinitum. In this way, CM_ω admits merely as an axiom each and every principle that would otherwise let us ground liar derivations like (1). Consequently, this system has no problem accommodating a transparent truth predicate.

6 Philosophical Discussion

The last section ended with the illusion that CM_ω provides an irreproachable solution to the paradoxes, as it doesn't give up any classical principle while accommodating transparent truth. But too-good-to-be-true solutions are just that. Admittedly, this system blocks the paradox, but it does so at a grave, and even unbearable cost, to be discussed in this section.

As a point of departure, notice that unlike the rest of the logics in the hierarchy, CM_ω pays a double price for blocking the paradox. First, once the truth predicate is added to the language the transitivity principles lose their status as admissible. Indeed, CM_ω could be said to maintain an illusion of classicality due to Lemma 11 (according to which a metainference is CM_ω-valid iff it is classically valid), but as in the case of TS/ST and the rest of the logics in the hierarchy, this illusion cannot be maintained outside the logical fragment of the language. When we deal with, say, Peano arithmetic in CM_ω, or when we simply introduce a truth predicate into the language, the transitivity rules stop being admissible, and the classicality illusion fades away.

Some may find this first price bearable, as it was with the rest of the logics in the hierarchy. But in the specific case of CM_ω there is a second price to pay, which, I believe, is unbearable. For in its refusal to grant any transitivity principle the status of a rule, CM_ω goes substructural at all inferential levels, and so there is no room for real inferential action in this system. Recall the previous analysis of the logics in the BPS hierarchy. It turned out that the higher we climb up the hierarchy, the higher the level where real inferential action takes place and lower-level derivations may be grounded: in ST the action takes place at the level of metainferences, in TS/ST at the level of metametainferences, and so on. Below such an action level, derivations of length>1 cannot be grounded. That goes even more so for CM_ω. As CM_ω doesn't grant any transitivity principle the status of a rule, it goes substructural about the bar between sequents, and the bar between first-level metainferences, and so on ad infinitum. Thus, as in the case of derivation (1) (discussed toward the end of Section 5), CM_ω has no way to ground derivations of any level: the lack of transitivity rules results in a constant need to ground any derivation of length>1—including those that involve only logical rules and axioms—in a corresponding higher-level derivation, which will itself be in need of further grounding, ad infinitum. Therefore, derivations of length>1 cannot get off the ground in CM_ω. In essence, then, all this system has is a bunch of axioms in the form of metainferences of various levels. I would like to argue that it is this second price that makes CM_ω a very poor candidate for a theory of truth.

As a point of departure for my argument, notice that paradoxes like the liar can be said to pose a threat to the notion of truth only in a context where genuine reasoning—however it is construed—can in principle be carried out. If there is no genuine reasoning, there is no way to derive the paradox and hence, one could argue, no paradox at all.[27] Ultimately, that is why all attempts to solve the paradox aim at accommodating truth—preferably, transparent truth—within a *workable* proof system, i.e., system in which genuine reasoning can in principle be carried out.[28]

At this point, one may ask what genuine reasoning actually is, and whether such "workability" (of proof systems) can be accounted for in more concrete terms. I am not going to address these questions in detail, given the scope of the present paper. Suffice it to point out a necessary condition for genuine reasoning, which will in turn give rise to a specific criterion for workability. The idea is that genuine reasoning must be non-trivial, i.e., its starting-point and endpoint ("premises" and "conclusion") cannot be the same; otherwise, it is simply unclear what reasoning is for. Technically, this idea gives rise to the following criterion for workability: a workable proof system has to allow us to carry out derivations of length>1, which is a condition for the possibility of having derivations with non-identical premises and conclusions.

Let me stress this point. Consider the following degenerate proof system: there are no inference rules, but each classically valid sequent is taken to be an axiom.

[27] One might still wonder how to interpret sentences like "this sentence is false" in such degenerate contexts, but that would be more of a riddle than a paradox. It is widely agreed that what's really bothering about the liar and other paradoxes is that they pose a threat to the coherence of our very idea of rationality. Evidently, such a threat can only be posed in a framework where inferences can in principle be carried out.

[28] In effect, there is nothing special about truth here. Similar points can clearly be made about other semantic notions such as validity.

There is clearly no problem in accommodating transparent truth in this degenerate system: it is impossible to derive the paradox simply because it is impossible to derive anything. So on the face of it, we achieve both full agreement with CL and transparent truth. But that is definitely a cheating solution. For, such a system does not allow us to carry genuine reasoning, as derivations of length>1 cannot be carried out in it. Indeed, such a degenerate system can hardly be called a logic.[29] Notice further, that this proof system counts as degenerate even though there is a sense in which it does contain "inferences," as it contains sequents. Yet, those sequents can be hardly interpreted as genuine inferences, precisely because they all designate mere starting-points of derivations, rather than substantial moves between non-identical premises and conclusions.

I would like to argue that the same is true of CM_ω. As was already hinted at by Scambler in [24], the situation with CM_ω resonates with Lewis Carroll's famous regress arguments [5]. Under the proof-theoretic account of CM_ω, we already saw why this is so: there is no way to ground derivations of any level in CM_ω, since each derivation is in constant need of further grounding, due to CM_ω's lack of transitivity rules at all levels. As a result (and as we already saw), all CM_ω has (outside the logical fragment of the language) is a bunch of axioms in the form of metainferences of various levels. So outside the logical fragment of the language, there is no way to carry out derivations of length>1, and thus no way to carry out genuine reasoning.

Now, Carroll's paradox is customarily taken to show that a workable proof system must have rules in addition to axioms. Robert Brandom, to mention just one prominent philosopher, makes the point thus:

> "[I]n a formal logical system, statements are inferentially inert. Even conditionals, whose expressive job it is to make inferential relations explicit as the contents of claims, license inferential transitions from premises to conclusions only in the context of rules permitting detachment. Rules are needed to give claims, even conditional claims, a normative significance for action. Rules specify how conditionals are to be used-how it would be correct to use them. It is the rules that fix the inference-licensing role of conditionals, and so their significance for what it is correct to do (infer, assert). Although particular rules can be traded in for axioms (in the form of conditional claims), one cannot in principle trade in all rules for axioms. So one cannot express all of the rules that govern inferences in a logical system in the form of propositionally explicit postulates within that system." [4, p. 22]

Brandom's distinction is made here with respect to conditionals and rules, but the Brandomian point applies more generally: it is about the necessity of *rules* for launching derivations with non-identical premises and conclusions, rather than about the

[29] One might object that the real culprit in such a degenerate system is undecidability: there is no effective procedure for deciding whether a given sequent is an axiom (in cases where the language includes equality and at least one other predicate with two or more arguments). But undecidability is itself denied on the grounds that it is unworkable for us, finite creatures. The same is true of any system with no workable proof system.

devices used to express that such rules are valid. CM_ω has a unique device for expressing the validity of all transitivity principles: axioms in the form of metainferences of any arbitrary level. But the same moral from the above degenerate system applies here as well: all these axioms are no less “inert,” i.e., the cannot be interpreted as genuine “inferences,” as they all fall short of launching any real inferential action. It is for this reason that CM_ω can be hardly called a logic, and that its solution to the paradox is dubious.

7 Conclusion

My primary goal in this paper was to solve an open problem in the literature: to provide a sound and complete proof system for CM_ω, and explain how it combines transparent truth with classicality at all inferential levels. Having achieved that goal, I also made certain philosophical observations about CM_ω and its solution to the paradoxes. First and foremost, a system where real inferential action doesn’t take place can hardly be regarded as a “logic.” After all, we can never launch any genuine derivation in CM_ω, and so it’s not a workable proof system. I believe that this is a sufficient reason for rejecting both the identification of CM_ω with CL, and its solution to the paradoxes.[30]

How about the other logics in the hierarchy? We saw that the higher a logic is situated in the hierarchy, the higher the level in which inferential action takes place, and lower-level derivations may be grounded. Therefore, it is inaccurate to conceive of such logics as progressively classical. Rather, classicality is achieved by each such logic only at levels where inferential action does not take place. Consequently, I see no reason why the logics in the hierarchy are more classical than, and hence preferable to, ST. Quite the contrary, there is at least one aspect in which ST emerges as more classical than all the logics in the hierarchy: it has a “regular” proof system, i.e., a sequent calculus where, as is usually the case, the inferential action takes place at the first metainferential level.

It is worthwhile to mention that proponents of ST sometimes vindicate their appeal to this logic on independent grounds. In particular, starting with [21], Ripley has been constantly advocating a pragmatic, strict-tolerant interpretation according to which ST models the notion of coherence of discursive positions. Although I do not purport to make up my mind here on whether we should choose ST to be our favorite theory of truth, it is quite clear in light of the above results that this pragmatic story, rather than similarity to CL, makes ST an attractive candidate for dealing with the paradoxes.

[30]One may suggest that CM_ω be viewed as offering a purely model-theoretic approach to the paradoxes. Indeed, especially in the case of semantic paradoxes, it’s not unusual to work only (or mainly) in a model-theoretic framework (think, for example, about the works of Kripke and Field). This is undoubtedly true, but notice that a purely model-theoretic approach would offer no explanation as to how CM_ω blocks the paradox, as opposed to Kripke and Field who, among others, explicitly reject classically-valid principles such as the law of excluded middle. Hence, a purely model-theoretic approach based on CM_ω is at the very least explanatorily incomplete.

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