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# Awareness as Potential for Knowledge

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**Abstract**

With the idea of analyzing awareness as potential for knowledge, we propose a novel semantics for awareness logic. Expressivities of languages with different combinations of modalities for this semantics are investigated. We explore the properties of our semantics and compare our model with Fagin & Halpern model and the model by Heifetz et al. in partitional settings. A series of equivalence results are established from the comparison. Finally, we provide two axiomatizations for implicit knowledge and explicit knowledge, respectively, and prove soundness and completeness for them.

**Keywords** Awareness · Knowledge · Epistemic logic · Partition

## 1 Introduction

In the area of epistemic logic, formalizing awareness is an important research topic that can be applied in many areas. For example, it plays an important role in capturing explicit knowledge and implicit knowledge [11], reasoning about bounded rationality [23], and dealing with the well-known logical omniscience problem [10].

Works treating awareness either follow the *semantic* approach [3, 8, 14, 19], where awareness is generated by atomic propositions, or the *syntactic* approach [1, 3, 5, 22], where the set of formulas of which an agent is aware can be simply any given set of formulas. Fagin & Halpern [3] lay the foundation of both the two approaches

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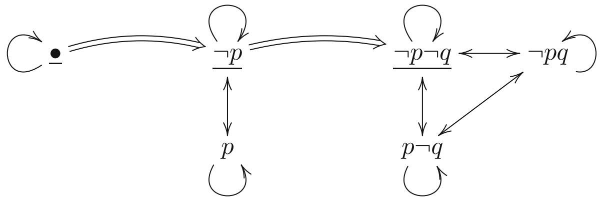


**Fig. 1** Three single-agent models showing Hans's changing epistemic status

and inspire the researchers along their chosen paths. This paper falls into the semantic corner and talks about awareness of atomic propositions, which is also called propositional awareness.

In the semantic approach, awareness is introduced as a complement to uncertainty in models for knowledge and rational interaction. In short, while being uncertain about the value of atomic propositions, agents may also be unaware of these propositions. Thus, we differentiate incompleteness (i.e., unawareness) from uncertainty in epistemic models. An honored principle is that incompleteness precedes uncertainty. Consider the scenario in [20].

> Hans wakes up in the morning with his mind empty. Thus, he is unaware of his surroundings.[1] Subsequently, Hans realizes what is lacking: coffee. He starts to wonder if coffee would already be served in the restaurant below. On his way to the lower floor, someone in the elevator mentions that you can't have both coffee and orange juice for breakfast. This makes Hans aware that orange juice is an issue.

In Fig. 1, three single-agent models are presented to illustrate the epistemic process in this scenario. The single-lined arrows are accessibility relations for Hans. The left model is a singleton with an empty set of atomic propositions, which describes the status when Hans just wakes up. The model in the middle formalizes the condition that Hans becomes aware of coffee, where $p$ indicates "coffee is served in the restaurant below". The rightmost model corresponds to the situation that Hans is informed by someone, where $q$ stands for "orange juice is served in the restaurant below". The underlined states are the actual states, and the double-lined arrows denote the direction of epistemic update.

Models in Fig. 1 are specified for propositions within awareness and provide a subjective view. It complies with the principle that incompleteness precedes uncertainty. However, if we ask the question whether Hans is certain about $p$ when he wakes up, apparently the answer should be negative, and it is not shown in the model of singleton. As such, we would like to introduce an objective view for this occasion, and present it in Fig. 2.

Three single-agent models in Fig. 2 are each with an infinite amount of states and an infinite set of atomic propositions. The solid ellipses denote the agent's information cells, and the dashed ellipses encircle equivalent classes of the agent's upcoming

[1] Slightly different with the story in [20], an agent becoming aware of himself is not an issue here, as we only consider awareness of propositions.

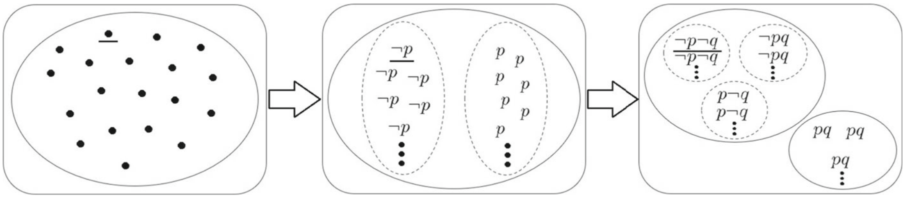


**Fig. 2** Three single-agent models showing Hans's changing epistemic status

knowledge, the knowledge of primitive propositions an agent knows that he is going to know by certain means. The three models correspond to the three circumstances of the previous scenario, respectively, where the meanings of $p$ and $q$ remain the same, and the underlined states represent the actual states. We assume that states of all combinations of consistent valuations can be found in every encircled subset. For example, for every combination of truth values of all variables, we can find a state in the left model.

When Hans wakes up, his accessible states incorporate every valuation of the infinite variables, so he doesn't hold any information. Then, he becomes aware of coffee and wonders if coffee is served. Thus, his accessibility remains unchanged, but he knows that his information cell can be divided into two parts, one with $p$ and another with $\neg p$. We call such parts the equivalent classes of upcoming knowledge. At the moment that he is informed by someone in the elevator, the set of accessible states varies and he knows his information cell can be divided into three parts, but he doesn't know into which part the actual state falls.

The existed formalizations of the semantic approach can be regarded as realizations of the subjective view, including the awareness structure by Fagin & Halpern (FH structure) [3] and the lattice structure by Heifetz et al. (HMS structure) [8, 9]. They use either awareness functions or states with partial valuations to capture incompleteness. Compared with these works, the objective view displays different aspects of awareness.[2] Firstly, as is shown in Fig. 2, uncertainty of variables out of awareness is depicted in the model. Secondly, it explains the phenomenon of becoming aware from gaining knowledge. To show this point, let us consider the following scenario.

> Bob arrives at a restaurant without any information of the service. Then, he notices that there is a wine glass on the table. This makes Bob become aware that wine is an issue. But he is not sure whether wine is served, since it is possible that the glass is left over by yesterday's party and wine is not available today. Let $p$ indicate

[2]This differentiation is first implemented on "states" by Halpern & Rêgo [7]. According to their discussion, HMS view states as subjective, while states in FH structure are objective. By contrast, our differentiation is applied to model level. Concretely speaking, our semantics is objective because the model is with full range of valuations of all atoms, where knowledge and awareness are formalized. In contrast, FH semantics and HMS semantics identify awareness set of each agent, then generate subjective models for them. Someone might argue that FH semantics does not build subjective models for each agent. However, with the restriction of the awareness function, we can simply view the atoms not belonging to the awareness set as irrelevant and abstract a subjective model for each agent.

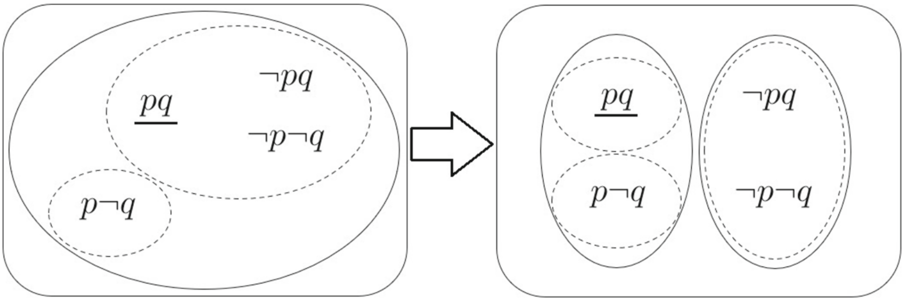


**Fig. 3** Two single-agent models showing Bob's changing epistemic status

"there is a wine glass on the table", $q$ indicate "wine is served". Initially, Bob does not know whether $p$ or $q$ and is not aware of either of them. After he notices $p$, he becomes aware of $q$. However, he is still ignorant about $q$.

As is shown in Fig. 3, Bob becomes aware of $q$ by knowing $p$.[3] Such a phenomenon shows that, there are certain connections between different atomic propositions (within the states where $p$ is true, the value of $q$ is separated by the dashed ellipses), and agents can acquire awareness of new atomic propositions based on these connections.

Therefore, we propose a novel structure based on the objective view. Intuitively, we are convinced that "agent $i$ is aware of $p$" can be conceived as "agent $i$ knows that he is going to know whether $p$ somehow". As in the previous scenario, after Bob notices the wine glass, he knows that he is going to know whether wine is served by asking the waitress. Following this idea, we provide two binary relations for each agent in our model. One represents the accessibility relation induced by *current knowledge*, the other by *upcoming knowledge*. And awareness about atomic propositions is generated from the two relations.

In general, this paper proposes a novel semantics for propositional awareness with the following features:

- Knowledge and a non-trivial notion of awareness can be formalized by a two-layer partition structure in a two-value setting;
- Being aware of a formula is analyzed as a condition satisfied by every accessible state, showing that awareness is a new kind of knowledge.

Our work is grounded on the tradition of epistemic logic [10] and in particular multi-agent epistemic logic [4, 13]. The review of awareness logic [15] inspires us to build our semantics. We prove completeness by model equivalences, a technique well presented by Lorini [12]. Works by van Ditmarsch et al. [18, 20, 21] provide a novel foundation for knowledge and awareness, by which we can explain some results of our theory and find some directions for future work.

[3] For convenience, we only show relevant atomic propositions in the figures from now on.

The remainder of this paper is organized as follows. In Section 2 we define the language for implicit knowledge, explicit knowledge and awareness. Following that, we present the novel semantics, recall FH semantics and HMS semantics. Section 3 compares the expressivities of languages with different combinations of modalities and identifies an expressivity hierarchy. Section 4 focuses on proving some important properties of our structure. Then, based on these properties, we demonstrate some equivalence results in partitional settings, in order to establish connections between our framework and FH structure, and between our framework and HMS structure. Section 5 offers two axiomatics and proves soundness and completeness for them. In Section 6, we focus on some interesting discussions about the philosophy of this novel semantics. Section 7 concludes the paper and points out future works.

## 2 Language and Semantics

In this section, we define the language of our logic LAPK. The latter is short for Logic of Awareness as Potential for Knowledge. After that, we give three semantics for it, one is the novel semantics, the other two are the Fagin & Halpern semantics and the semantics by Heifetz et al.

### 2.1 Language

Throughout the contribution, we assume a countably infinite set of atomic propositions $Atm$ and a finite non-empty set of agents $Agt$. Let us define the language for LAPK.

**Definition 1** The language $\mathcal{L}^{XKA}(Atm, Agt)$ is defined as follows:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid X_i\varphi \mid K_i\varphi \mid A_i\varphi,$$

where $p$ ranges over $Atm$ and $i$ ranges over $Agt$.

$\mathcal{L}^{XKA}(Q, Agt)$ is the language with a vocabulary restricted to $Q \subseteq Atm$. And we write $\mathcal{L}^{XKA}$ for $\mathcal{L}^{XKA}(Atm, Agt)$.

The language has three modalities for an arbitrary agent $i$, which are $X_i$, $K_i$ and $A_i$. We can delete some of the modalities and get a fragment of $\mathcal{L}^{XKA}$. For example, $\mathcal{L}^{XA}$ and $\mathcal{L}^{KA}$ are languages of $\{X_i, A_i\}_{i\in Agt}$ and $\{K_i, A_i\}_{i\in Agt}$, respectively. We write $\mathcal{L}(Q)$ to indicate the language restricted in $Q \subseteq Atm$ with no modality, i.e., the language of propositional logic. Let $\mathcal{L}$ denote $\mathcal{L}(Atm)$. The formulas $X_i\varphi$ and $K_i\varphi$ are read as "agent $i$ explicitly knows $\varphi$" and "agent $i$ implicitly knows $\varphi$", respectively, and the formula $A_i\varphi$ stands for "agent $i$ is aware of $\varphi$". We define other boolean connectives such as disjunction $\vee$, implication $\rightarrow$ and biimplication $\leftrightarrow$ as usual. And $\top$ stands for truth.

The (uniform) substitution of $p$ by $\varphi$ in a formula $\psi$, notation $\psi[\varphi\backslash p]$ is inductively defined by replacing all occurrences of $p$ in $\psi$ by $\varphi$.

## 2.2 Semantics of Awareness as Potential for Knowledge

In this part we present the novel structure for awareness logic. Before defining the models, let us illustrate the idea through some examples.

In Fig. 4, solid dots denote the states at which $p$ is true, and the hollow dots indicate the states with $\neg p$ being true. The solid lines and the dashed lines represent the current knowledge relation and upcoming knowledge relation, respectively. The only difference between $M$ and $M'$ is that $p$ is true at $u$ but false at $u'$.

Figure 4 gives a simple representation for our awareness semantics. In the state $s$, the agent is aware of $p$ because $p$ stays the same value (being true in this example) on the upcoming knowledge alternatives (states $u$ and $v$) of his each current knowledge alternative (state $t$). While in the state $s'$, the agent is unaware of $p$ as the values of $p$ are different on $u'$ and $v'$. Informally, awareness is defined as follows.

> In a state $s$, an agent $i$ is aware of a formula $\varphi$ if for any atomic proposition $p$ occurring in $\varphi$ and any current knowledge alternative $t$, $p$ has a uniform value in all the upcoming knowledge alternatives of $t$.

Complicated as it seems, the idea is very simple: agent $i$ is aware of $\varphi$ means that, for every atomic proposition $p$ occurring in $\varphi$, agent $i$ knows that he is going to know whether $p$ somehow.

We now put forward a more realistic example based on equivalent relations, which appeals to our intuitions about knowledge and awareness.

*Example 1* Suppose that my 5-year-old nephew and I find a mushroom in a forest. Both of us do not know whether it is poisonous or not; accordingly we lack knowledge about its poisonousness. Nevertheless, I know that the mushroom could be poisonous, which means knowing that I'm going to know its poisonousness by certain means (for instance, take a picture of the mushroom and send it to an expert). By contrast, my nephew is so young that he does not have a conception about poisonousness. In other words, he is unaware of that a mushroom could be poisonous or not, which means that he doesn't know that he is going to know whether it's poisonous by any means. The two different epistemic structures for capturing our knowledge and awareness can be essentially depicted as in Fig. 5.

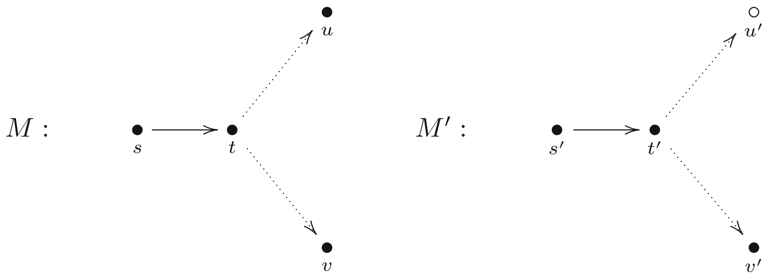


**Fig. 4** Two single-agent models for agent $i$ with only one atomic proposition


**Fig. 5** Two single-agent models with equivalent relations of current knowledge and upcoming knowledge. The left one indicates my nephew's epistemic status, while the right one formalizes mine

In Fig. 5, the solid dot denotes the state at which $p$ is true, where $p$ stands for "the mushroom is poisonous". And the hollow dot indicates the state at which $\neg p$ is true. The solid lines indicate current knowledge equivalent classes, while the dashed line represents upcoming knowledge equivalent classes. Note that there should be reflexive arrows for every dot with respect to the two kinds of relations, and we do not show them in the figure.

$M$ of Fig. 5 illustrates my nephew's epistemic status. The two states are both included in a single current knowledge partition and upcoming knowledge partition. The structure underlines that my nephew does not know whether the mushroom is poisonous or not and is not aware of it, the latter is due to that he does not know that the value of $p$ can be distinguished by his upcoming knowledge. Similarly, the current knowledge partition in $M'$ of Fig. 5, my epistemic structure, includes the two states, showing that I don't know whether the mushroom is poisonous either. However, there are two upcoming knowledge partitions in my epistemic structure that distinguish the values of $p$. As such, it indicates that I am aware of $p$, which means that I know that I'm going to know whether $p$ somehow. By equivalent relations, we construct a *two-layer partition awareness structure*.

Now we present the novel structure for our logic. A potential knowledge awareness model is a multi-agent Kripke model enriched with a relation of upcoming knowledge for every agent.

**Definition 2** A potential knowledge awareness model (PKAM) is a tuple $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ where

- $S$ is a nonempty set of states,
- $\mathcal{R} : Agt \times S \longrightarrow 2^S$ indicates the accessibility relations for current knowledge,
- $\mathcal{T} : Agt \times S \longrightarrow 2^S$ indicates the accessibility relations for upcoming knowledge,
- $\mathcal{V} : Atm \longrightarrow 2^S$ is the valuation function.

And it satisfies the following condition:

(C1) $\mathcal{T}(i, s) \subseteq \mathcal{R}(i, s)$ for all $i \in Agt$ and all $s \in S$.

For simplicity, we write $s\mathcal{R}_i t$ for $t \in \mathcal{R}(i, s)$. Let $\mathcal{R}_i$ be $\{(s, t) : t \in \mathcal{R}(i, s)\}$, $\mathcal{T}_i$ be $\{(s, t) : t \in \mathcal{T}(i, s)\}$. For $s \in S$, a pair $(M, s)$ is called a pointed PKAM.

A few points need to be clarified. Condition (C1) stipulates that an upcoming knowledge alternative should be his current knowledge alternative, while the opposite direction may fail. It coincides with our intuition that the knowledge of an agent would increase by receiving information, resulting in the epistemic alternatives becoming less, and that knowledge increasing does not cause knowledge change, which means the accessible states should be accessible before. Another issue is whether there are certain connections of properties of the two relations. As we all know, epistemic properties corresponds to certain properties of the accessibility relations. We focus on three properties: reflexive, transitive and Euclidean, which are defined for every $i \in Agt$ and for every $s, t, u \in S$ as follows:

reflexive: $s\mathcal{R}_i s$,

transitive: if $s\mathcal{R}_i t$ and $t\mathcal{R}_i u$, then $s\mathcal{R}_i u$,

Euclidean: if $s\mathcal{R}_i t$ and $s\mathcal{R}_i u$, then $t\mathcal{R}_i u$.

One might assume another condition:

(C2) $\mathcal{T}_i$ preserves all the properties of $\mathcal{R}_i$ for all $i \in Agt$.

It suggests that an agent would not lose his epistemic properties after receiving information. For example, if $\mathcal{R}_i$ is transitive, then $\mathcal{T}_i$ is transitive. Such an assumption might be reasonable, but with some defects. It excludes the situation that properties of $\mathcal{R}_i$ might not be inherent. For instance, $\mathcal{R}_i$ is accidentally transitive in a non-transitive setting. This does not require $\mathcal{T}_i$ to be transitive.[4] However, if we study certain classes of models, we suggest that (C2) holds for the properties defining the classes. For example, $\mathcal{T}_i$ is reflexive for all $i \in Agt$ when we study models with $\mathcal{R}_i$ being inherently reflexive for all $i \in Agt$. This assumption is coherent with the following setup.

We are specifically concerned about the partitional setting, i.e., $\mathcal{R}_i$ is inherently an equivalent relation (being both reflexive and Euclidean) for every $i \in Agt$, and generates partitions on $S$. Then, $\mathcal{T}_i$ is an equivalent relation for every $i \in Agt$ and generates partitions on $S$ as well. Interestingly, the partitions generated by $\mathcal{T}_i$ turn out to be a refinement of the partitions generated by $\mathcal{R}_i$. In this case, we construct models with two-layer partitions [16]. The following definition specifies such models.

**Definition 3** A two-layer partition awareness model (TPAM) is a PKAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ that satisfies the following condition:

(C1*) for all $i \in Agt$ and all $s \in S$, $\mathcal{T}(i, s)$ and $\mathcal{R}(i, s)$ are equivalent relations, and $\mathcal{T}(i, s) \subseteq \mathcal{R}(i, s)$.

**PKAM** denotes the class of all PKAMs. We write **TPAM** for the class of all TPAMs.

For the reason that we are talking about propositional awareness, we need the following function to indicate the set of atomic propositions occurring in a formula $\varphi$.

**Definition 4** The set of atomic propositions occurring in $\varphi$, denoted by $Atm(\varphi)$, is defined inductively as follows:

$$\begin{aligned}
Atm(\top) &= \emptyset, \\
Atm(p) &= \{p\}, \text{ for all } p \in Atm, \\
Atm(\neg\varphi) &= Atm(\varphi), \\
Atm(\varphi \wedge \psi) &= Atm(\varphi) \cup Atm(\psi), \\
Atm(Y_i\varphi) &= Atm(\varphi) \text{ for } Y \in \{X, K, A\}.
\end{aligned}$$

[4] We are very grateful to the reviewer for pointing out this problem.

For $\Sigma \subseteq \mathcal{L}^{XKA}$, we slightly abuse the notion and let $Atm(\Sigma) = \bigcup_{\varphi \in \Sigma} Atm(\varphi)$. The following definition provides interpretations of $\mathcal{L}^{XKA}$ formulas with respect to PKAMs.

**Definition 5** Let $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ be a PKAM. Then,

$$
\begin{aligned}
(M,s) &\models p \iff s \in \mathcal{V}(p),\\
(M,s) &\models \neg\varphi \iff (M,s) \not\models \varphi,\\
(M,s) &\models \varphi \wedge \psi \iff (M,s) \models \varphi \text{ and } (M,s) \models \psi,\\
(M,s) &\models A_i\varphi \iff \text{ for all } p \in Atm(\varphi) \text{ and all } t, u, v \text{ with } s\mathcal{R}_i t, t\mathcal{T}_i u, t\mathcal{T}_i v\\
&\qquad\qquad\qquad (M,u) \models p \text{ iff } (M,v) \models p,\\
(M,s) &\models K_i\varphi \iff (M,t) \models \varphi \text{ for all } t \text{ with } s\mathcal{R}_i t.
\end{aligned}
$$

Let $X_i\varphi$ be an abbreviation of $K_i\varphi \wedge A_i\varphi$. We stipulate that $(M, s) \models \top$ for every $s \in S$. As usual, explicit knowledge emerges from implicit knowledge and awareness. Compared with the work by Fagin & Halpern, the only difference of our semantics is the way of interpreting awareness modalities.

Formula $\varphi$ is valid on a PKAM $M$, denoted as $M \models \varphi$, if and only if $(M, s) \models \varphi$ for all $s \in S$. Formula $\varphi$ is valid for a class $\mathbf{C}$ of PKAMs, denoted as $\mathbf{C} \models \varphi$, if and only if $M \models \varphi$ for all $M \in \mathbf{C}$. A logic is the set of all the valid formulas in a certain language with respect to a certain class of PKAMs. Formula $\varphi$ is satisfiable for a class of PKAMs if and only if there exists PKAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ belonging to the class and there exists $s \in S$ with $(M, s) \models \varphi$.

### 2.3 FH Semantics

We recall the semantics of Logic of General Awareness by Fagin & Halpern [3] in this segment.

**Definition 6** A General Awareness Model (GAM) is a tuple $M = (\Omega, \Rightarrow, \rho, \pi)$, where

- $\Omega$ is a nonempty set of states,
- $\Rightarrow: Agt \times \Omega \longrightarrow 2^{\Omega}$ indicates the accessible states for each agent and each state,
- $\rho : Agt \times \Omega \longrightarrow 2^{\mathcal{L}^{XKA}}$ indicates the awareness set for each agent and each state,
- $\pi : Atm \longrightarrow 2^{\Omega}$ is the valuation function.

Similarly with PKAMs, we write $s \Rightarrow_i t$ for $t \in \Rightarrow(i, s)$, and $\Rightarrow_i$ for $\{(s,t) : t \in \Rightarrow (i, s)\}$. A pair $(M, s)$ is called a pointed GAM. The following two properties of GAM are necessary for the rest of the paper:

- A GAM $M = (\Omega, \Rightarrow, \rho, \pi)$ is image-finite iff $\Rightarrow(i, s)$ is finite for every $i \in Agt$ and every $s \in \Omega$,
- A GAM $M = (\Omega, \Rightarrow, \rho, \pi)$ is awareness-finite iff $Atm(\rho(i, s))$ is finite for every $i \in Agt$ and every $s \in \Omega$.

The following definition gives the interpretations of $\mathcal{L}^{XKA}$ formulas with respect to GAMs.

**Definition 7** Let $M = (\Omega, \Rightarrow, \rho, \pi)$ be a GAM. Then,

$$
\begin{aligned}
&(M, s) \models p \Longleftrightarrow s \in \pi(p),\\
&(M, s) \models \neg\varphi \Longleftrightarrow (M, s) \not\models \varphi,\\
&(M, s) \models \varphi \wedge \psi \Longleftrightarrow (M, s) \models \varphi \text{ and } (M, s) \models \psi,\\
&(M, s) \models A_i\varphi \Longleftrightarrow \varphi \in \rho(i, s),\\
&(M, s) \models K_i\varphi \Longleftrightarrow (M, t) \models \varphi \text{ for all } t \text{ with } s \Rightarrow_i t.
\end{aligned}
$$

$X_i\varphi$ is the abbreviation of $K_i\varphi \wedge A_i\varphi$ as usual. We stipulate that $(M, s) \models \top \wedge A_i \top$ for every $s \in \Omega$ and every $i \in Agt$. Formula $\varphi$ is valid on a GAM $M$, denoted as $M \models \varphi$, if and only if $(M, s) \models \varphi$ for all $s \in \Omega$. Formula $\varphi$ is valid for a class of GAMs if and only if $M \models \varphi$ for all $M$ belonging to the class. A logic is the set of all the valid formulas in a certain language with respect to a certain class of GAMs. Formula $\varphi$ is satisfiable for a class of GAMs if and only if there exists GAM $M = (\Omega, \Rightarrow, \rho, \pi)$ belonging to the class and there exists $s \in \Omega$ with $(M, s) \models \varphi$. We will define some useful classes of GAMs in the rest of the paper. The first one is given in the next definition.

Actually, a GAM is a model for Logic of General Awareness, which plays the central role in the syntactic approach. To make it a model for Logic of Propositional Awareness, additional conditions are needed.

**Definition 8** A Propositionally Determined Awareness Model (PDAM) is a GAM $M = (\Omega, \Rightarrow, \rho, \pi)$ satisfying the following conditions [6].

(i) Awareness is generated by primitive propositions if for any $i \in Agt$, any $s \in \Omega$, and any $\varphi \in \mathcal{L}^{XKA}$, we have $\varphi \in \rho_i(s)$ iff $Atm(\varphi) \subseteq \rho_i(s)$,
(ii) Agents know what they are aware of if for any $i \in Agt$ and any $s, s' \in \Omega$, $s \Rightarrow_i s'$ implies $\rho_i(s') = \rho_i(s)$.

It is worth mentioning that, Halpern and Rêgo [7] prove an equivalence result between PDAMs and HMS structure [8, 9].

### 2.4 HMS Semantics

Here we briefly review the lattice structure by Heifetz et al. Since HMS structure is introduced as a syntax-free approach, we first define an HMS frame, then we combine it with a valuation function to create an HMS model.

**Definition 9** An HMS frame is a tuple $F = (\Theta, \preceq, R, \Pi)$, where

- $(\Theta, \preceq)$ is a complete lattice with $\Theta = \{S, S', \dots\}$ a set of disjoint, non-empty state-spaces $S = \{s, s', \dots\}$ satisfying that $S \preceq S'$ implies $|S| \leq |S'|$, and let $S_\Theta = \bigcup_{S \in \Theta} S$ be the disjoint union of state-spaces in $\Theta$,

- $R = \{r_S^{S'} : S, S' \in \Theta, S \preceq S'\}$ is a family of surjective projections $r_S^{S'} : S' \longrightarrow S$, where $r_S^S$ is the identity, $S \preceq S' \preceq S''$ implies commutativity: $r_S^{S''} = r_{S'}^{S''} \circ r_S^{S'}$, and $D^{\uparrow} = \bigcup_{S' \succeq S} (r_S^{S'})^{-1}(D)$ is the upward closure of $D \subseteq S \in \Theta$[5],
- $\Pi : Agt \times S_\Theta \longrightarrow 2^{S_\Theta}$ assigns each agent $i \in Agt$ a possibility correspondence satisfying (we write $\Pi_i(s)$ instead of $\Pi(i, s)$):
  - (0) Confinedness: if $s \in S'$, then $\Pi_i(s) \subseteq S$ for some $S \preceq S'$,
  - (1) Generalized Reflexivity: $s \in (\Pi_i(s))^{\uparrow}$ for every $s \in S_\Theta$,
  - (2) Stationarity: $s' \in \Pi_i(s)$ implies $\Pi_i(s') = \Pi_i(s)$,
  - (3) Projections Preserve Ignorance: if $s \in S'$ and $S \preceq S'$, then $(\Pi_i(s))^{\uparrow} \subseteq (\Pi_i(r_S^{S'}(s)))^{\uparrow}$,
  - (4) Projections Preserve Knowledge: if $S \preceq S' \preceq S''$, $s \in S''$ and $\Pi_i(s) \subseteq S'$, then $r_S^{S'}(\Pi_i(s)) = \Pi_i(r_S^{S''}(s))$.

**Definition 10** An HMS model (HMSM) is a tuple $M = (F, \pi)$ with $F = (\Theta, \preceq, R, \Pi)$ an HMS frame, where

- $\Theta = \{S_\Phi, S_\Psi, \dots\}$ with $\Phi, \Psi \subseteq Atm$, satisfying that $\Phi \subseteq \Psi$ implies $S_\Phi \preceq S_\Psi$,
- $\pi : S_\Theta \times Atm \longrightarrow \{0, 1, 1/2\}$ is a valuation function such that for $s \in S_\Psi$, $\pi(s, p) \neq 1/2$ iff $p \in \Psi$, and that if $s = r_{S_\Psi}^{S_\Phi}(s')$, then $\pi(s, p) = \pi(s', p)$ for every $p \in \Psi$.

A pair $(M, s)$ is called a pointed HMSM as usual. Distinct from previous semantics, in HMS semantics, implicit knowledge modalities are not involved, and awareness operators are defined as

$$A_i\varphi := X_i\varphi \vee X_i\neg X_i\varphi.$$

Thus, we need only care about the interpretations of formulas in $\mathcal{L}^X$. The following definition gives the interpretations of $\mathcal{L}^X$ formulas with respect to HMSMs.

**Definition 11** Let $M = (\Theta, \preceq, R, \Pi, \pi)$ be an HMSM. Then,

$$(M, s) \models p \Longleftrightarrow \pi(s, p) = 1,$$
$$(M, s) \models \neg\varphi \Longleftrightarrow s \in S_\Phi, Atm(\varphi) \subseteq \Phi \text{ and } (M, s) \not\models \varphi,$$
$$(M, s) \models \varphi \wedge \psi \Longleftrightarrow (M, s) \models \varphi \text{ and } (M, s) \models \psi,$$
$$(M, s) \models X_i\varphi \Longleftrightarrow (M, t) \models \varphi \text{ for all } t \in \Pi_i(s).$$

Same with former semantics, we stipulate that $(M, s) \models \top$ for every $s \in S_\Theta$.

## 3 Expressivity

As languages with different modalities can be interpreted in PKAMs, we investigate the expressivities of the languages with respect to PKAMs in this section. We employ

[5] Note that we slightly abuse the notion and let $r_S^{S'}(D)$ be $\{r_S^{S'}(s) : s \in D\}$ when $D \subseteq S'$ and $S \preceq S'$.

the definition of expressivity in [17], by which we obtain an expressivity hierarchy in the end of this section.

**Definition 12** Given two logical languages $\mathcal{L}_1$ and $\mathcal{L}_2$ that are interpreted in the same class of models,

- $\mathcal{L}_2$ is at least as expressive as $\mathcal{L}_1$, denoted by $\mathcal{L}_1 \preceq \mathcal{L}_2$, iff for every formula $\varphi_1 \in \mathcal{L}_1$ there is a formula $\varphi_2 \in \mathcal{L}_2$ such that $\varphi_1 \leftrightarrow \varphi_2$ is valid,
- $\mathcal{L}_1$ and $\mathcal{L}_2$ are equally expressive, denoted by $\mathcal{L}_1 \equiv \mathcal{L}_2$, iff $\mathcal{L}_1 \preceq \mathcal{L}_2$ and $\mathcal{L}_2 \preceq \mathcal{L}_1$,
- $\mathcal{L}_1$ is less expressive than $\mathcal{L}_2$, denoted by $\mathcal{L}_1 \prec \mathcal{L}_2$, iff $\mathcal{L}_1 \preceq \mathcal{L}_2$ and $\mathcal{L}_2 \not\preceq \mathcal{L}_1$,
- $\mathcal{L}_1$ and $\mathcal{L}_2$ are incomparable, denoted by $\mathcal{L}_1 \asymp \mathcal{L}_2$, iff $\mathcal{L}_1 \not\preceq \mathcal{L}_2$ and $\mathcal{L}_2 \not\preceq \mathcal{L}_1$.

For convenience, we give the following definition for the proof of expressivity.

**Definition 13** Two pointed PKAMs $(M, s)$ and $(M', s')$ cannot be distinguished by a certain language $\mathcal{L}_0$ iff for every $\varphi \in \mathcal{L}_0$,

$$(M, s) \models \varphi \text{ iff } (M', s') \models \varphi.$$

According to Definition 13, if there exists $\varphi \in \mathcal{L}_0$ such that $(M, s) \models \varphi$ and $(M', s') \models \neg\varphi$, we say the two pointed PKAMs can be distinguished by $\mathcal{L}_0$.

**Lemma 1** *Given two pointed PKAMs $(M, s)$ and $(M', s')$ belonging to a certain class* **C** *of PKAMs, and two language $\mathcal{L}_1$ and $\mathcal{L}_2$, if the two pointed PKAMs can be distinguished by $\mathcal{L}_1$ but not by $\mathcal{L}_2$, then $\mathcal{L}_1 \not\preceq \mathcal{L}_2$ for* **C**.

*Proof* Let us prove by contradiction. Suppose $\mathcal{L}_1 \preceq \mathcal{L}_2$ for the class **C**. Suppose $\varphi \in \mathcal{L}_1$ distinguishes the two pointed PKAMs $(M, s)$ and $(M', s')$. By Definition 12, we can find $\varphi' \in \mathcal{L}_2$ such that $\mathbf{C} \models \varphi \leftrightarrow \varphi'$. Then the two pointed PKAMs should also be distinguished by $\varphi'$, which contradicts the precondition. □

**Proposition 2** *$\mathcal{L}^{KA} \equiv \mathcal{L}^{XK}$, $\mathcal{L}^{X} \equiv \mathcal{L}^{XA}$, $\mathcal{L}^{XA} \prec \mathcal{L}^{KA}$ with respect to* **PKAM**.

*Proof* Since $\mathbf{PKAM} \models A_i\varphi \leftrightarrow X_i(\varphi \vee \neg\varphi)$ and $\mathbf{PKAM} \models X_i\varphi \leftrightarrow K_i\varphi \wedge A_i\varphi$, we have $\mathcal{L}^{KA} \equiv \mathcal{L}^{XK}$. And $\mathbf{PKAM} \models A_i\varphi \leftrightarrow X_i(\varphi \vee \neg\varphi)$ gives $\mathcal{L}^{X} \equiv \mathcal{L}^{XA}$.

As $\mathbf{PKAM} \models X_i\varphi \leftrightarrow K_i\varphi \wedge A_i\varphi$, we have $\mathcal{L}^{XA} \preceq \mathcal{L}^{KA}$. In Fig. 6, the two pointed PKAMs $(M, t)$ and $(M', t')$ cannot be distinguished by $\mathcal{L}^{XA}$ formulas. But it is apparent that $(M, t) \models K_i p$ and $(M', t') \not\models K_i p$. So $(M, t)$ and $(M', t')$ can be distinguished by $\mathcal{L}^{KA}$ formulas. Hence, $\mathcal{L}^{KA} \not\preceq \mathcal{L}^{XA}$. By Definition 12, we have $\mathcal{L}^{XA} \prec \mathcal{L}^{KA}$. □

**Proposition 3** *$\mathcal{L}^{A} \asymp \mathcal{L}^{K}$, $\mathcal{L}^{A} \prec \mathcal{L}^{XA}$, $\mathcal{L}^{K} \asymp \mathcal{L}^{XA}$, $\mathcal{L}^{K} \prec \mathcal{L}^{KA}$ with respect to* **PKAM**.

*Proof* Let us prove the first assertion. To do this, consider the models in Fig. 7. It is clear that $(M, s)$ and $(M', s')$ cannot be distinguished by $\mathcal{L}^{K}$ formulas. But we

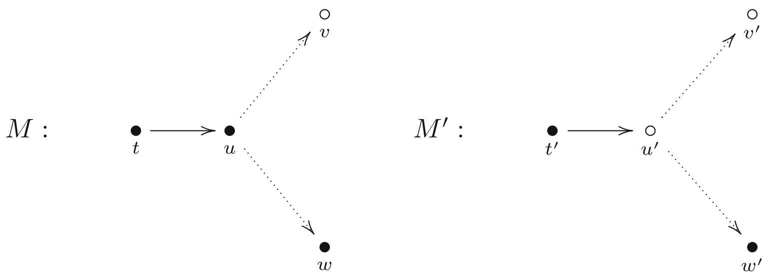


**Fig. 6** The meanings of the elements in this figure are the same as those in Fig. 4

have $(M, s) \models A_i p$ and $(M', s') \not\models A_i p$. Hence, $\mathcal{L}^A \not\preceq \mathcal{L}^K$ by Lemma 1. For the other direction, it follows by the models in Fig. 8 that $(M, s)$ and $(M', s')$ cannot be distinguished by $\mathcal{L}^A$ formulas. But we have $(M, s) \models K_i p$ and $(M', s') \not\models K_i p$. Hence, $\mathcal{L}^K \not\preceq \mathcal{L}^A$ by Lemma 1. So we have $\mathcal{L}^A \asymp \mathcal{L}^K$ by Definition 12.

The proof of the second assertion is based on Fig. 8. In this figure, we can see that $(M, s)$ and $(M', s')$ cannot be distinguished by $\mathcal{L}^A$ formulas. But we have $(M, s) \models X_i p$ and $(M', s') \not\models X_i p$. So we have $\mathcal{L}^A \prec \mathcal{L}^{XA}$ by Definition 12.

Finally, we can check that the other two assertions hold by the above assertions and Proposition 2. □

It is apparent that $\mathcal{L}$ is less expressive than $\mathcal{L}^A$ and $\mathcal{L}^K$ with respect to **PKAM**.

In summary, we establish an expressivity hierarchy of the languages in Fig. 9. Note that the $\prec$ relation is transitive.

We wonder if the expressivity result with respect to **TPAM** stays the same with what is presented in Fig. 9. It is straightforward that all $\preceq$ relations for **PKAM** can be inherited by **TPAM** since **TPAM** is a subclass of **PKAM**. So we only need to verify the $\not\preceq$ relations.

**Proposition 4** *$\mathcal{L}^{XA} \not\preceq \mathcal{L}^A$, $\mathcal{L}^A \not\preceq \mathcal{L}^K$, $\mathcal{L}^K \not\preceq \mathcal{L}^A$, $\mathcal{L}^{KA} \not\preceq \mathcal{L}^{XA}$, $\mathcal{L}^{KA} \not\preceq \mathcal{L}^K$, $\mathcal{L}^K \not\preceq \mathcal{L}^{XA}$, $\mathcal{L}^{XA} \not\preceq \mathcal{L}^K$ with respect to* **TPAM**.

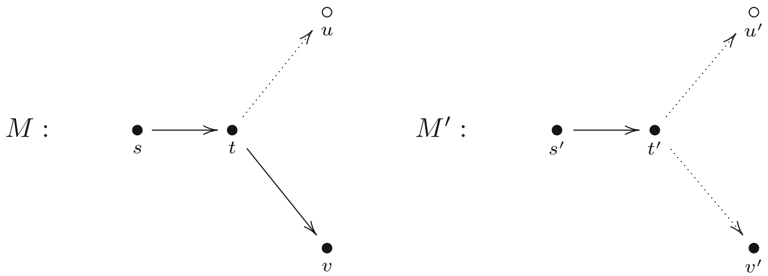


**Fig. 7** The meanings of the elements in this figure are the same as those in Fig. 4

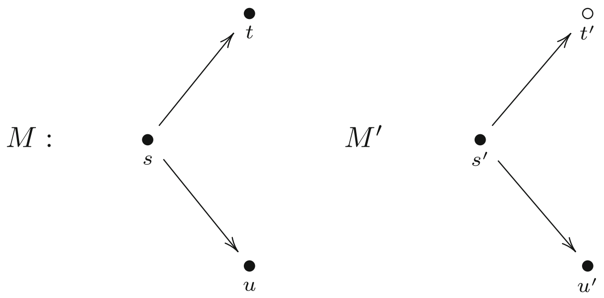


**Fig. 8** The explanation of this figure is the same as Fig. 4

*Proof* Let us prove the first assertion. Consider the models in Fig. 10, it is clear that $(M, s)$ and $(M', s')$ cannot be distinguished by $\mathcal{L}^A$ formulas. But we have $(M, s) \models X_i p$ and $(M', s') \not\models X_i p$. Hence, $\mathcal{L}^{XA} \not\preceq \mathcal{L}^A$ by Lemma 1.

The proof of the second assertion is based on Fig. 5. Suppose the two models are both for a single agent $i$. It is apparent that $(M, s)$ and $(M', s')$ cannot be distinguished by $\mathcal{L}^K$ formulas, while $(M, s) \not\models A_i p$ and $(M', s') \models A_i p$. It follows by Lemma 1 that $\mathcal{L}^A \not\preceq \mathcal{L}^K$.

The third assertion is proved by Fig. 10 similarly with the first assertion.

For the fourth assertion, let us consider the models in Fig. 11. $(M, s)$ and $(M', s')$ cannot be distinguished by $\mathcal{L}^{XA}$ formulas, because the agent is aware of nothing. But we have $(M, s) \models K_i(p \leftrightarrow q)$ and $(M', s') \not\models K_i(p \leftrightarrow q)$. So $\mathcal{L}^{KA} \not\preceq \mathcal{L}^{XA}$ by Lemma 1.

The last three assertions follow from the previous proofs. □

It is apparent that $\mathcal{L}$ is less expressive than $\mathcal{L}^A$ and $\mathcal{L}^K$ with respect to **TPAM**.

From Proposition 4, we can infer that the hierarchy in Fig. 9 applies to expressivity relations with respect to **TPAM**.

The significance of this section lies in that, whether in partitional structure or non-partitional structure of PKAMs, $\mathcal{L}^X$ is equally expressive with $\mathcal{L}^{XA}$, which means the $A_i$ operator can be defined by $X_i$. Such a result complies with HMS semantics. However, Halpern [6] has proved that, only in partitional GAMs, propositional awareness can be defined by explicit knowledge. This contrast inspires us to explore the relation between TPAMs and partitional PDAMs in the next section.

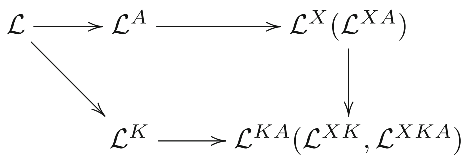


**Fig. 9** This graph shows expressivity relations with respect to **PKAM** or **TPAM**. An arrow represents a $\prec$ relation and points from a less expressive language to a more expressive language. Note that the arrows that follows from the transitivity of $\prec$ relations are omitted. Absence of an arrow represents incomparability. The languages are equally expressive with the languages in the bracket following them


**Fig. 10** The explanation of this figure is the same as Fig. 5 except that the two models here are all for a single agent $i$

# 4 Equivalences of Partitional Models

Although we take into consideration the class of all PKAMs, what concerns us most is the class of **TPAM**. It is due to the fact that partitional structure is the well-accepted structure in epistemic logic [4]. This section explores the relation between TPAMs and partitional PDAMs. Before the comparison, we shall investigate the properties of PKAMs.

## 4.1 Properties of PKAMs

In this part, some valid formulas that are quite different with those in other works of awareness logic are found out.

**Definition 14** Given a PKAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$, $\varphi$ is an awareness consequence of $\psi$ at the state $s$ for agent $i$ ($\varphi$ is an $i$-$s$ awareness consequence of $\psi$) if for any PKAM $M' = (S, \mathcal{R}, \mathcal{T}', \mathcal{V})$[6], $(M', s) \models A_i\psi \to A_i\varphi$.

Informally, a formula being an $i$-$s$ awareness consequence of another formula means that, no matter how we vary his upcoming knowledge, if $i$ is aware of the latter on $s$, then $i$ is aware of the former. The following two propositions are apparent results from Definition 14.

**Proposition 5** *Given a PKAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$, $\varphi$ is not an $i$-$s$ awareness consequence of $\psi$ if and only if there exists a PKAM $M' = (S, \mathcal{R}, \mathcal{T}', \mathcal{V})$ such that $(\mathcal{M}', s) \models A_i\psi \wedge \neg A_i\varphi$.*

**Proposition 6** *Given a PKAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$, if $Atm(\varphi) \subseteq Atm(\psi)$, then $\varphi$ is an $i$-$s$ awareness consequence of $\psi$.*

The following propositions are aimed at demonstrating a relation between implicit knowledge and awareness.

**Proposition 7** *Given a PKAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$, we have $M \models K_i(K_ip \vee K_i\neg p) \to A_ip$.*

*Proof* Let $s \in S$ be an arbitrary state. Suppose $(M, s) \models K_i(K_ip \vee K_i\neg p)$. By Definition 5, for any set $\{t, u, v\} \subseteq S$ such that $s\mathcal{R}_it, t\mathcal{R}_iu, t\mathcal{R}_iv$, we have that

[6] $\mathcal{T}$ is replaced by $\mathcal{T}'$ without changing anything else of $M$. Similarly hereinafter.


**Fig. 11** The explanation of this figure is the same as Fig. 10 except that there are two propositions $p$ and $q$ in each of the models. In every state, there are two dots, where the left dot denotes the truth value of $p$, and the right dot denotes the truth value of $q$

$(M, u) \models p$ if and only if $(M, v) \models p$. As $\mathcal{T}(i, t) \subseteq \mathcal{R}(i, t)$, we have that $(M, u') \models p$ if and only if $(M, v') \models p$ for any $\{u', v'\} \subseteq \mathcal{T}(i, t)$. Thus, by Definition 5, we have $(M, s) \models A_i p$. Since $s$ is arbitrary, we have $M \models K_i(K_i p \vee K_i \neg p) \to A_i p$[7]. □

Informally, Proposition 7 says that if one knows that he knows (implicitly) whether $p$, then he is aware of $p$. So awareness of atomic propositions can be generated by implicit knowledge introspection.

**Corollary 8** *Given a PKAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ and a state $s \in S$, if $(M, s) \models K_i(K_i p \vee K_i \neg p)$, then $p$ is an $i$-$s$ awareness consequence of any formula.*

*Proof* It is obvious that $(M', s) \models K_i(K_i p \vee K_i \neg p)$ for any PKAM $M' = (S, \mathcal{R}, \mathcal{T}', \mathcal{V})$. By Proposition 7, we have $(M', s) \models A_i p$. Hence $(M', s) \models A_i \varphi \to A_i p$ for any $\varphi \in \mathcal{L}^{XKA}$. □

By the previous result of this section, we know that, given a state of a PKAM and an agent, awareness of an atomic proposition may not be independent of awareness of $\varphi$ where $p \notin Atm(\varphi)$. So we are wondering if there exists certain circumstance such that being aware of any formula is independent of awareness of any other formula that is formed by at least one different atomic proposition. The following definition gives a concept about it.

**Definition 15** A PKAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ is *i-s* awareness free if for any formulas $\varphi$ and $\psi$ with $Atm(\varphi) \not\subseteq Atm(\psi)$, $\varphi$ is not an *i-s* awareness consequence of $\psi$.

**Proposition 9** *If a PKAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ is i-s awareness free, then there does not exist a primitive proposition $p$ with $(M, s) \models K_i(K_i p \vee K_i \neg p)$.*

*Proof* It is an immediate consequence of Corollary 8. □

**Proposition 10** *A model $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ is i-s awareness free if and only if for any atomic proposition $p$, $p$ is not an i-s awareness consequence of $\varphi$ with $p \notin Atm(\varphi)$.*

*Proof* The direction from left to right is immediate. To prove the other direction, assume that for any atomic proposition $p$, $p$ is not an *i-s* awareness consequence

[7] Any two elements of the set $\{t, u, v\}$ are possibly identical. Similarly hereinafter.

of an arbitrary formula $\varphi$, where $p \notin Atm(\varphi)$. Suppose again that $\psi$ and $\eta$ satisfy $Atm(\psi) \nsubseteq Atm(\eta)$. Then there exists an atom $q$ satisfying $q \in Atm(\eta)$ and $q \notin Atm(\psi)$. By the assumption and Proposition 5, there is a PKAM $M' = (S, \mathcal{R}, \mathcal{T}', \mathcal{V})$ satisfying $(M', s) \models \neg A_i q \wedge A_i \eta$. By $q \in Atm(\psi)$ and $(M', s) \models \neg A_i q$, we get $(M', s) \models \neg A_i \psi \wedge A_i \eta$. So $\psi$ is not an $i$-$s$ awareness consequence of $\eta$. By Definition 15, $M$ is $i$-$s$ awareness free. □

Proposition 10 provides a simple way to check whether a model is $i$-$s$ awareness free.

Now we generalize Proposition 7 and present a relation between implicit knowledge introspection and awareness. Before doing that, we need to prepare a notion. Given a finite set $Q \subseteq Atm$, let

$$C(Q) = \left\{ \bigwedge_{p \in k} p \wedge \bigwedge_{q \in Q \setminus k} \neg q : k \in 2^Q \right\},$$

$$D(Q) = \left\{ \bigvee_{\varphi \in k} \varphi : k \in 2^{C(Q)} \right\}.$$

As $Q$ is finite we have that $2^Q$ is finite. If we neglect the order of the conjuncts in each conjunction, it is easy to see that $C(Q)$ is finite. For $D(Q)$, if we neglect the order of the disjuncts in each disjunction, $D(Q)$ is also finite. Intuitively, $C(Q)$ includes all valuations for propositions in $Q$, and $D(Q)$ includes all combinations of the valuations. Apparently, for every $\varphi \in \mathcal{L}(Q)$, we can find $\psi \in D(Q)$ such that $\varphi \leftrightarrow \psi$ is a propositional tautology.

**Proposition 11** *Given a PKAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$, an atomic proposition $p$ and a finite set $Q \subseteq Atm$, the following assertions hold.*

*(a)* $M \models K_i \left( \bigvee_{\varphi \in D(Q)} K_i(p \leftrightarrow \varphi) \right) \to \left( \bigwedge_{q \in Q} A_i q \to A_i p \right)$,

*(b) if $p$ is an $i$-$s$ awareness consequence of $\varphi$, then*

$$(M, s) \models K_i \left( \bigvee_{\psi \in D(Atm(\varphi))} K_i(p \leftrightarrow \psi) \right).$$

*Proof* Consider the case that $Q$ is empty, then $\varphi$ has to be $\top$ or $\neg\top$. By Proposition 7, the formula of (a) is established. Regard the other case that $Q$ is non-empty. For an arbitrary $s \in S$, suppose that

$$(M, s) \models K_i \left( \bigvee_{\varphi \in D(Q)} K_i(p \leftrightarrow \varphi) \right) \wedge \bigwedge_{q \in Q} A_i q. \qquad (1)$$

By Definition 5, for all $q \in Q$ and for any set $\{t, u, v\} \subseteq S$ with $s\mathcal{R}_i t$, $t\mathcal{T}_i u$, $t\mathcal{T}_i v$, we have $(M, u) \models q$ if and only if $(M, v) \models q$. By (1) and Definition 5, there exists

$\varphi \in D(Q)$ such that $(M, u) \models p \leftrightarrow \varphi$ and $(M, v) \models p \leftrightarrow \varphi$. Since $Atm(\varphi) \subseteq Q$, we have $(M, u) \models \varphi$ if and only if $(M, v) \models \varphi$, which yields $(M, u) \models p$ if and only if $(M, v) \models p$. By Definition 5, we thus get $(M, s) \models A_i p$. This completes the proof of (a).

We now turn to prove the second part. Case 1. Suppose that $\varphi$ is $\top$ or $\neg\top$. Then for any $M' = (S, \mathcal{R}, \mathcal{T}', \mathcal{V})$, we have $(M', s) \models A_i p$. Hence, it is impossible to find $\{t, u, v\} \subseteq S$ with $s\mathcal{R}_i t$, $t\mathcal{R}_i u$, $t\mathcal{R}_i v$, satisfying $(M', u) \models p$ and $(M', v) \models \neg p$. So we have $(M, s) \models K_i(K_i p \vee K_i \neg p)$, and (b) is proved in this case.

Case 2. Assume that $\varphi$ is not $\top$ and not $\neg\top$. Suppose $s\mathcal{R}_i t$ and $T_p = \{u \in \mathcal{R}(i, t) \mid (M, u) \models p\}$. Let $P_{T_p}^{Atm(\varphi)} \subseteq 2^{T_p}$ such that,

- for any $\{x, y\} \subseteq P_{T_p}^{Atm(\varphi)}$, $x \cap y = \emptyset$,
- $\bigcup_{x \in P_{T_p}^{Atm(\varphi)}} = T_p$,
- for any $x \in P_{T_p}^{Atm(\varphi)}$, if $\{u', v'\} \subseteq x$, then $(M, u') \models q$ iff $(M, v') \models q$ for any $q \in Atm(\varphi)$,
- for any $x, y \in P_{T_p}^{Atm(\varphi)}$, if $u' \in x$, $v' \in y$, then there exists $q \in Atm(\varphi)$ such that $(M, u') \models q$ iff $(M, v') \models \neg q$.

Intuitively, we make partitions on $T_p$ to classify the values of all atoms in $Atm(\varphi)$. It is obvious that $P_{T_p}^{Atm(\varphi)}$ is finite. We construct $\psi$ as follows,

$$\psi = \bigvee_{x \in P_{T_p}^{Atm(\varphi)}} \left( \bigwedge_{q \in Atm(\varphi), x \subseteq \mathcal{V}(q)} q \wedge \bigwedge_{q \in Atm(\varphi), x \cap \mathcal{V}(q) = \emptyset} \neg q \right).$$

Since $p$ is an *i-s* awareness consequence of $\varphi$, it is impossible to satisfy the following conditions all:

- there exists $\{t, u, v\} \subseteq S$ with $s\mathcal{R}_i t$, $t\mathcal{R}_i u$, $t\mathcal{R}_i v$,
- $(M, u) \models q$ if and only if $(M, v) \models q$ for all $q \in Atm(\varphi)$,
- $(M, u) \models p$ iff $(M, v) \models \neg p$.

So it has to be that there exists $q \in Atm(\varphi)$ such that $(M, u) \models q$ iff $(M, v) \models \neg q$, with the first and the third condition being true.

Let $F_p = \{u \in \mathcal{R}(i, t) \mid (M, u) \models \neg p\}$. It follows that, for any $u' \in T_p$ and $v' \in F_p$, there exists $q \in Atm(\varphi)$ such that $(M, u') \models q$ iff $(M, v') \models \neg q$. By construction of $\psi$, we have that $(M, v') \models \neg\psi$ for any $v' \in F_p$. As a result, $(M, u') \models p \leftrightarrow \psi$ for any $u' \in \mathcal{R}(i, t)$. Hence,

$$(M, t) \models K_i(p \leftrightarrow \psi).$$

By the construction of $\psi$, it is easy to see that $\psi \in D(Atm(\varphi))$. So we have

$$(M, t) \models \bigvee_{\psi \in D(Atm(\varphi))} K_i(p \leftrightarrow \psi), \ \forall t \in \mathcal{R}_i(s).$$

Thus, we obtain

$$(M,s) \models K_i\left(\bigvee_{\psi\in D(Atm(\varphi))} K_i(p \leftrightarrow \psi)\right).$$

□

With Proposition 11, we can further establish the following corollary that finds the equivalent condition for $i$-$s$ awareness free.

**Corollary 12** *A PKAM* $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ *is i-s awareness free if and only if* $(M,s) \not\models K_i\left(\bigvee_{\varphi\in D(Atm\setminus\{p\})} K_i(p \leftrightarrow \varphi)\right)$ *for all* $p \in Atm$.

*Proof* The direction from right to left is a consequence of Proposition 10 and the converse-negative of part (b) of Proposition 11. The other direction can be deducted from Proposition 10 and the converse-negative of part (a) of Proposition 11. □

Although Proposition 11 explores a relation between awareness and implicit knowledge, it seems quite complex and far away from our intuition. What concerns us more is how it fits in the two-layer partition structure, with the accessibility relations being equivalent. The following proposition and corollary are analogous to Proposition 11 and Corollary 12, respectively.

**Proposition 13** *Given a two-layer partition awareness model (TPAM)* $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$*, we have*

(a) $M \models K_i(p \leftrightarrow \varphi) \rightarrow (A_i\varphi \rightarrow A_i p)$ *where* $\varphi \in \mathcal{L}$,

*(b) if p is an i-s awareness consequence of* $\varphi$*, then there exists* $\psi \in \mathcal{L}$ *such that* $Atm(\psi) \subseteq Atm(\varphi)$ *and* $(M,s) \models K_i(p \leftrightarrow \psi)$.

*Proof* Part (a) is a special case of Part (a) of Proposition 11. Part (b) is a special case of Part (b) of Proposition 11. □

**Corollary 14** *A TPAM* $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ *is i-s awareness free if and only if* $(M,s) \not\models K_i(p \leftrightarrow \varphi))$ *for all* $p \in Atm$ *and* $\varphi \in \mathcal{L}(Atm\setminus\{p\})$.

*Proof* It is a special case of Corollary 12. □

### 4.2 A Comparison with FH Structure

This part proves several equivalence results for different partitional structures. The equivalences are established with respect to the language $\mathcal{L}^{XKA}$ and the language $\mathcal{L}^{XA}$. In the end, two figures about the relations of the models are presented. With the help of Proposition 13, we define a special class of PDAMs in the first place.

**Definition 16** A PDAM $M = (\Omega, \Rightarrow, \rho, \pi)$ is called a partitional PDAM (PPDAM) if it satisfies (i) of the following conditions, and a PPDAM is called a restricted partitional PDAM (RPPDAM) if it satisfies (ii) of the following conditions.

(i) $\Rightarrow_i$ is equivalent relation for all $i \in Agt$,
(ii) $M \models K_i(p \leftrightarrow \varphi) \rightarrow (A_i\varphi \rightarrow A_i p)$, where $\varphi \in \mathcal{L}$.

Based on the conditions in Definition 16, we are able to establish the equivalence between TPAMs and RPPDAMs.

**Theorem 15** *(a) Given a TPAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$, there is an RPPDAM $M' = (\Omega, \Rightarrow, \rho, \pi)$ with $\Omega = S$ such that for all formulas $\varphi \in \mathcal{L}^{XKA}$ and all $s \in S$, $(M, s) \models \varphi$ if and only if $(M', s) \models \varphi$,*
*(b) Given an RPPDAM $M = (\Omega, \Rightarrow, \rho, \pi)$, if $M$ is image-finite or awareness-finite, then there is a TPAM $M' = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ with $S = \Omega$ such that for all formulas $\varphi \in \mathcal{L}^{XKA}$ and all $s \in \Omega$,*

$$(M, s) \models \varphi \text{ if and only if } (M', s) \models \varphi.$$

*Proof* We first prove part (a). Given a TPAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$, for each state $s \in S$ and each $i \in Agt$, we define $\rho(i, s) = \mathcal{L}^{XKA}(\{p \mid (M, s) \models A_i p\}, Agt)$. Let $\Rightarrow = \mathcal{R}$ and $\pi = \mathcal{V}$. Let $M' = (\Omega, \Rightarrow, \rho, \pi)$. It is clear that $M'$ is a PPDAM. By Proposition 13, $M'$ also satisfies (ii) in Definition 16. Thus, $M'$ is an RPPDAM. Then we prove by induction on the structure of formulas.

**Base case** If $\varphi$ is an atomic proposition $p$, then it is immediate from the definition of $M$ and $M'$.

**Induction hypothesis** Suppose that it holds for $\varphi_1$ and $\varphi_2$.

**Induction step**

Case 1. $\varphi$ is a form $\varphi_1 \wedge \varphi_2$ or $\neg\varphi_1$. It follows immediately from the induction hypothesis and the semantics.

Case 2. $\varphi$ is a form $A_i\varphi_1$. ($\Rightarrow$) For an arbitrary state $s \in S$, suppose $(M, s) \models A_i\varphi_1$. By the semantics, it follows that, for all $p \in Atm(\varphi_1)$, $(M, s) \models A_i p$. By the definition of $\rho(i, s)$, the latter leads to $\varphi_1 \in \rho(i, s)$, which means $(M', s) \models A_i\varphi_1$.

($\Leftarrow$) By following the proof of ($\Rightarrow$) in reverse, this direction is easily proved.

Case 3. $\varphi$ is a form $K_i\varphi_1$. It follows immediately from the hypothesis and the definition of $M$ and $M'$.

Case 4. $\varphi$ is a form $X_i\varphi_1$. Since $X_i\varphi_1$ is equivalent to $K_i\varphi_1 \wedge A_i\varphi_1$, it follows from the previous cases.

We move on to part (b). Given an RPPDAM $M = (\Omega, \Rightarrow, \rho, \pi)$, let $\mathcal{R} = \Rightarrow$ and $\mathcal{V} = \pi$.

For TPAM $M' = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$, for each $s \in S$ and each $i \in Agt$, within the set $\mathcal{R}(i, s)$, we establish partitions by filtrating using $Atm(\rho(i, s))$, which generates the set of partitions $\Delta = \{\delta_1, \ldots, \delta_n\}$ satisfying the following conditions:

- for any $\delta_k \in \Delta$, any $u, v \in \delta_k$ and any $p \in Atm(\rho(i, s))$, we have $(M', u) \models p$ iff $(M', v) \models p$,

- for any $\delta_k, \delta_m \in \Delta$, there exists $p \in Atm(\rho(i,s))$ such that $\delta_k \subseteq \mathcal{V}(p)$ iff $\delta_m \cap \mathcal{V}(p) = \emptyset$.

When $M$ is awareness-finite, for the reason that there are at most $2^{|Atm(\rho(i,s))|}$ valuations for $Atm(\rho(i,s))$, such partitions exist and $\Delta$ is finite. When $M$ is image-finite, such partitions also exist with $\Delta$ being finite, as we can imagine the extreme case that each partition is a singleton.

For every $i \in Agt$, we make partitions in this way for each current knowledge partition of him. Let $\mathcal{T}_i$ be the equivalent relation generating the partitions we make for every $i \in Agt$. Thus, we have defined the TPAM $M' = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$. Then we prove by induction on the structure of formulas.

The base case and the induction hypothesis stay the same with the proof of part (a). The boolean cases and the cases of $K_i\varphi_1$ and $X_i\varphi_1$ are immediate. Here we only prove the case that $\varphi$ is a form $A_i\varphi_1$.

($\Rightarrow$) For an arbitrary state $s \in \Omega$, suppose $(M, s) \models A_i\varphi_1$. By the semantics, for all $p \in Atm(\varphi_1)$, $p \in \rho(i,s)$. By the previous construction, the partitions generated by $\mathcal{T}_i$ within $\mathcal{R}(i,s)$ classify the values of all $p \in Atm(\varphi_1)$. Then by the semantics, we have $(M', s) \models A_i\varphi_1$.

($\Leftarrow$) We prove it by contradiction. For an arbitrary state $s \in S$, suppose $(M', s) \models A_i\varphi_1$ and $(M, s) \models \neg A_i\varphi_1$. Then there exists $p \in Atm(\varphi_1)$ such that $(M', s) \models A_i p$ and $(M, s) \models \neg A_i p$. By the semantics, from $(M, s) \models \neg A_i p$ we have $p \notin \rho(i,s)$. By the previous construction, the partitions generated by $\mathcal{T}_i$ within $\mathcal{R}(i,s)$ classify the values of all $q \in \rho(i,s)$. By the semantics, from $(M', s) \models A_i p$ it follows that the partitions also classify the value of $p$. Now we prove that there exists $\psi \in \rho(i,s)$ such that $(M, s) \models K_i(p \leftrightarrow \psi)$. If $\psi$ is $\top$ or $\neg\top$, then from $(M, s) \models K_i(p \leftrightarrow \psi) \to (A_i\psi \to A_i p)$ we get $(M, s) \models A_i p$, which contradicts $(M, s) \models \neg A_i p$. Then we suppose that there exist $u, v \in \Rightarrow (i,s)$ such that $(M, u) \models p$ and $(M, v) \models \neg p$.

Firstly, we consider that $M$ is awareness-finite. Let $\Delta_p = \{\delta \in \Delta \mid \text{for any } u \in \delta, (M', u) \models p\}$. Let

$$\psi = \bigvee_{\delta \in \Delta_p} \left( \bigwedge_{q \in \rho(i,s), \delta \subseteq \mathcal{V}(q)} q \wedge \bigwedge_{q \in \rho(i,s), \delta \cap \mathcal{V}(q) = \emptyset} \neg q \right).$$

By the construction of $\psi$ and the construction of $\Delta$, we have that, for every $t \in \mathcal{R}(i,s)$, $(M', t) \models p \leftrightarrow \psi$. Thus, $(M', s) \models K_i(p \leftrightarrow \psi)$. By the other cases, it follows that $(M, s) \models K_i(p \leftrightarrow \psi)$. Then from $(M, s) \models K_i(p \leftrightarrow \psi) \to (A_i\psi \to A_i p)$, we have $(M, s) \models A_i p$, which contradicts $(M, s) \models \neg A_i p$.

Secondly, we consider that $M$ is image-finite, which means $M'$ is also image-finite. Let $P(i,s) \subseteq Atm(\rho(i,s))$ be finite and satisfy that, for any $\delta_k, \delta_m \in \Delta$, there exists $p \in P(i,s)$ such that $\delta_k \subseteq \mathcal{V}(p)$ iff $\delta_m \cap \mathcal{V}(p) = \emptyset$. By the construction of $\Delta$ and the fact that $\mathcal{R}(i,s)$ is finite, it is apparent that such $P(i,s)$ exists. Let

$$\psi = \bigvee_{\delta \in \Delta_p} \left( \bigwedge_{q \in P(i,s), \delta \subseteq \mathcal{V}(q)} q \wedge \bigwedge_{q \in P(i,s), \delta \cap \mathcal{V}(q) = \emptyset} \neg q \right).$$

By the construction of $\psi$ and the construction of $\Delta$, we have that, for every $t \in \mathcal{R}(i, s)$, $(M', t) \models p \leftrightarrow \psi$. For the same reason as before, it causes contradiction. $\square$

Theorem 15 provides some insights about the relation between TPAMs and RPPDAMs. For any TPAM, there is an RPPDAM equivalent to it. However, for the opposite direction, the establishment of the equivalence requires additional conditions, i.e. for any image-finite or awareness-finite RPPDAM, there is a TPAM equivalent to it. Thus, in order to establish an equivalence between TPAMs and RPPDAMs, we need to transform RPPDAMs to image-finite or awareness-finite RPPDAMs. For convenience, here we only adopt awareness-finite RPPDAMs.

**Lemma 16** *Let $\Sigma \subseteq \mathcal{L}^{XKA}$ be finite. Given an RPPDAM $M = (\Omega, \Rightarrow, \rho, \pi)$, there is an awareness-finite RPPDAM $M' = (\Omega, \Rightarrow, \rho', \pi)$ such that for every $\varphi \in \Sigma$ and every $s \in \Omega$,*

$$(M, s) \models \varphi \text{ if and only if } (M', s) \models \varphi.$$

*Proof* For every $s \in \Omega$ and every $i \in Agt$, let $P(i, s) = Atm(\rho(i, s)) \cap Atm(\Sigma)$, and let $\rho'(i, s) = \mathcal{L}^{XKA}(P(i, s), Agt)$. It is clear that $M'$ so defined is awareness-finite. Then we prove the equivalence by induction on the structure of $\varphi$. The proof is a routine procedure and omitted here. $\square$

Let **RPPDAM** denote the class of all RPPDAMs, let af-**RPPDAM** denote the class of all awareness-finite RPPDAMs.

**Theorem 17** *Let $\varphi \in \mathcal{L}^{XKA}$. Then, the following three statements are equivalent:*

- *$\varphi$ is satisfiable for the class* **TPAM***,*
- *$\varphi$ is satisfiable for the class* af-**RPPDAM***,*
- *$\varphi$ is satisfiable for the class* **RPPDAM**.

*Proof* The theorem is a direct consequence of Theorem 15 and Lemma 16. The relations of the three classes are shown in Fig. 12. $\square$

Till now, we have explored partitional structures of PKAMs and PDAMs for $\mathcal{L}^{XKA}$, and established the relations of them. Not only that, we are also concerned about the language $\mathcal{L}^{XA}$, and incline to study the relations of the partitional structures

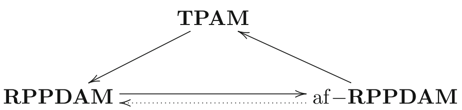


**Fig. 12** Relations between our semantics and FH semantics for the language $\mathcal{L}^{XKA}$. An arrow means that satisfiability relative to the first class of models implies satisfiability relative to the second class of models. Full arrows correspond to the results stated in Theorems 15 and Lemma 16. Dotted arrows denote relations that follow straightforwardly given the inclusion between classes of models

with respect to $\mathcal{L}^{XA}$. There are some reasons for us to skip implicit knowledge and investigate explicit knowledge directly. Firstly, as mentioned before, in HMS semantics there is no such distinction between explicit knowledge and implicit knowledge. In fact, what they call knowledge is explicit knowledge. Secondly, Halpern [6] considers the properties of explicit knowledge by establishing axiomatics for $\mathcal{L}^{XA}$, without taking the indirect route though implicit knowledge, for the reason as he says "the interplay between assumptions about awareness and axioms for explicit knowledge is far more significant here than for the language $\mathcal{L}^{XKA}$". Thirdly, in our logic, agent $i$ cannot implicitly know an atom $p$ if $i$ is not aware of $p$, which makes sense, and corrects an arguable defect in FH structure. However, agent $i$ can implicitly know $p \leftrightarrow q$ without being aware of $p$ or $q$. What does such implicit knowledge mean?[8] It seems that implicit knowledge lacks an ontological justification in our logic. Last but not least, compared with implicit knowledge, we have a better option. Van Ditmarsch et al. [18] suggest to regard the value of $p$ as 'don't care', if $i$ is unaware of $p$. As such, they propose a novel notion called *speculative knowledge* [20, 21], which is knowledge modulo speculation over unaware variables. In fact, [20] suggests that speculative knowledge should really be called implicit knowledge, and $K_i$ in our language should be delegated to the level of a mere technical primitive, or to something called 'latent' knowledge. Levesque [11], the work predating Fagin & Halpern [3], supports this view. In Levesque's work, variables in the unaware set can be speculated over and that is exactly called implicit knowledge in his writings. We will apply this idea in our future work. In this contribution we focus on setting up a framework as a theoretical foundation.

**Theorem 18** *Given a PPDAM $M = (\Omega, \Rightarrow, \rho, \pi)$, there is an RPPDAM $M' = (\Omega', \Rightarrow', \rho', \pi')$ with $\Omega' \supseteq \Omega$ such that for all formulas $\varphi \in \mathcal{L}^{XA}$ and all $s \in \Omega$,*

$$(M, s) \models \varphi \text{ if and only if } (M', s) \models \varphi.$$

*Proof* The only difference between a PPDAM $M$ and an RPPDAM $M'$ is that, $M' \models K_i(p \leftrightarrow \psi) \rightarrow (A_i\psi \rightarrow A_i p)$ where $\psi \in \mathcal{L}$, while such a formula may be falsified in $M$. So we are going to transform $M$ to an RPPDAM without changing the satisfiability of formulas in $\mathcal{L}^{XA}$.

For an arbitrary agent $i \in Agt$, let $\Delta = \{\delta_1, \delta_2, \dots\}$ be the set of all partitions of $i$. For $\delta_1 \in \Delta$, let $U = \{p \in Atm \mid \text{for every } s \in \delta_1, p \notin \rho(i, s)\}$. Then we interpolate a new state $t$ into $\delta_1$ to make a new model $M^1 = (\Omega^1, \Rightarrow^1, \rho^1, \pi^1)$ with the following conditions:

- $\Omega^1 = \Omega \cup \{t\}$,
- $\Rightarrow_i^1 = \Rightarrow_i \cup \{(u, t), (t, u) \mid u \in \delta_1\} \cup \{(t, t)\}$,
  for every $j \in Agt \setminus \{i\}$, $\Rightarrow_j^1 = \Rightarrow_j \cup \{(t, t)\}$,
- for every $u \in \Omega$ and for every $q \in Atm$, $u \in \pi(q)$ iff $u \in \pi^1(q)$,
  there exists $v \in \delta_1$ such that, for every $p \in U$, $v \in \pi^1(p)$ iff $t \notin \pi^1(p)$, and for every $q \in Atm \setminus U$, $v \in \pi^1(q)$ iff $t \in \pi^1(q)$,

[8]This question is raised by the reviewer.

- for every $j \in Agt$ and every $u \in \Omega$, $\rho^1(j, u) = \rho(j, u)$,
  for every $j \in Agt \setminus \{i\}$, $\rho^1(j, t) = \mathcal{L}^{XA}$,
  $\rho^1(i, t) = \rho(i, v)$ where $v \in \delta_1$.

It is apparent that $M^1$ so defined is a PPDAM.

Intuitively, we interpolate a new state into $\delta_1$ from the counterpart $v \in \delta_1$. The new state changes the value of all atoms that $i$ is unaware of, and keeps the other atoms' value. For agents other than $i$, the state is a singleton partition. And on this new state, $i$'s awareness does not change, and agents other than $i$ are aware of everything.

Now we show by contradiction that, after the interpolation, $(M^1, s) \models K_i(p \leftrightarrow \psi) \rightarrow (A_i\psi \rightarrow A_i p)$ where $\psi \in \mathcal{L}$ and $s \in \delta_1$. Suppose $(M, s) \models K_i(p \leftrightarrow \psi) \wedge A_i\psi \wedge \neg A_i p$. Then $p \notin \rho(i, s)$, $\psi \in \rho(i, s)$ and $(M, v) \models p \leftrightarrow \psi$. By the construction, $(M^1, t) \models p$ iff $(M^1, t) \models \neg\psi$, which means $(M^1, t) \models \neg(p \leftrightarrow \psi)$. Thus, $(M^1, s) \models \neg K_i(p \leftrightarrow \psi)$, which leads to a contradiction.

We repeat the interpolation for every $\delta_k \in \Delta$. After that, we repeat it for every $i \in Agt$. During the procedure, we get the models $M^1, M^2, \ldots$ and finally get the model $M' = (\Omega', \Rightarrow', \rho', \pi')$. Apparently, $M'$ is an RPPDAM.

Finally, we prove by induction on the structure of $\varphi$ that $(M, s) \models \varphi$ iff $(M', s) \models \varphi$ for every $s \in \Omega$ and every $\varphi \in \mathcal{L}^{XA}$. It holds for the reason that, in the construction of $M'$,

- for every $p \in Atm$ and every $s \in \Omega$, we do not change value of $p$, which means every propositional formula stays the same value,
- for every $i \in Agt$ and every $s \in \Omega$, we do not change the awareness set $\rho(i, s)$, which means the formulas of the form $A_i\psi$ stays the same value,
- for every partition of agent $i$, we interpolate a state, on which every other agent being fully aware, and on which only the atoms that $i$ is unaware of change their values, such that the interpolation does not change the value of the formulas of the form $X_i\psi$.

We omit the details of the induction and conclude the proof. □

Theorem 18 demonstrates an unexpected result. Rephrased informally, PPDAMs and RPPDAMs are equivalent with respect to $\mathcal{L}^{XA}$, or we can say RPPDAMs are as rich as PPDAMs if we only consider explicit knowledge and awareness. Since RPPDAMs are equivalent to TPAMs, such a principle provides a strong justification for TPAMs.

Let **PPDAM** denote the class of all PPDAMs. We have the following theorem analogous to Theorem 17.

**Theorem 19** *Let $\varphi \in \mathcal{L}^{XA}$. Then, the following three statements are equivalent:*

- *$\varphi$ is satisfiable for the class* **TPAM**,
- *$\varphi$ is satisfiable for the class* af-**RPPDAM**,
- *$\varphi$ is satisfiable for the class* **RPPDAM**,
- *$\varphi$ is satisfiable for the class* **PPDAM**.

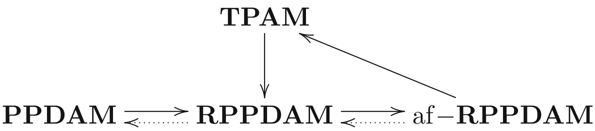


**Fig. 13** Relations between our semantics and FH semantics for the language $\mathcal{L}^{XA}$. The arrows are explained in the same way as in Fig. 12, where full arrows correspond to the results stated in Lemma 16 and Theorems 15, 18

*Proof* Given the fact that $\mathcal{L}^{XA}$ is a sublanguage of $\mathcal{L}^{XKA}$, this theorem follows directly from Theorem 15, Lemma 16 and Theorem 18. The relations of the three classes are showen in Fig. 13. □

### 4.3 A Comparison with HMS Structure

This segment establishes the relationship between our semantics and HMS semantics. It has to be relative to $\mathcal{L}^{XA}$, because HMS semantics does not involve implicit knowledge. Analogous to the previous part, a figure emerges in the end showing how such an equivalence is formed. The following lemma is necessary for the proof of the theorem immediately after.

**Lemma 20** *Let* $M = (\Theta, \preceq, R, \Pi, \pi)$ *be an HMSM. Given* $\Psi \subseteq \Phi$, $s' \in S_\Phi$, $s = r^{S_\Phi}_{S_\Psi}(s')$ *and* $\varphi \in \mathcal{L}^{XA}$, *we have that*

$$\text{if } (M, s) \models \varphi \text{ then } (M, s') \models \varphi.$$

*Proof* This lemma is easily proved by induction on the structure of formulas using the property of Projections Preserve Ignorance. □

**Theorem 21** *(a)* *Given a TPAM* $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$, *there is an HMSM* $M' = (\Theta, \preceq, R, \Pi, \pi)$ *with* $\Theta = S \times 2^{Atm}$, $S_\Psi = S \times \{\Psi\}$ *for all* $\Psi \subseteq Atm$, *such that for all formulas* $\varphi \in \mathcal{L}^{XA}$ *and all* $s \in S$, *if* $Atm(\varphi) \subseteq \Psi$, *then*

$$(M, s) \models \varphi \text{ if and only if } (M', (s, \Psi)) \models \varphi,$$

*(b)* *Let* $\Sigma \subseteq \mathcal{L}^{XA}$ *be finite. Given an HMSM* $M = (\Theta, \preceq, R, \Pi, \pi)$, *then there is a TPAM* $M' = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ *with* $S \supseteq S_\Theta$ *such that for all formulas* $\varphi \in \Sigma$ *and all* $s \in S_\Theta$, *if* $s \in S_\Psi$ *and* $Atm(\varphi) \in \Psi$, *then*

$$(M, s) \models \varphi \text{ if and only if } (M', s) \models \varphi.$$

*Proof* For part (a), given a TPAM $M = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$, let $M' = (\Theta, \preceq, R, \Pi, \pi)$ be an HMSM such that

- $\Theta = S \times 2^{Atm}$,
- $S_\Psi = S \times \{\Psi\}$ for all $\Psi \subseteq Atm$,
- $R = \{r^{S_\Phi}_{S_\Psi} : \Psi \subseteq \Phi \subseteq Atm\}$, where $r^{S_\Phi}_{S_\Psi}((s, \Phi)) = ((s, \Psi))$,

- for every $s \in S$, $i \in Agt$ and $\Psi \subseteq Atm$,
  $\Pi_i((s, \Psi)) = \{(t, \Psi \cap \mathcal{A}_i(s)) : s\mathcal{R}_i t\}$, where $\mathcal{A}_i(s) = \{p \in Atm : (M, s) \models A_i p\}$,
- for every $p \in Atm$, $s \in S$ and $\Psi \subseteq Atm$,
  $\pi((s, \Psi), p) = 1$ if $p \in \Psi$ and $s \in \mathcal{V}(p)$,
  $\pi((s, \Psi), p) = 0$ if $p \in \Psi$ and $s \notin \mathcal{V}(p)$,
  $\pi((s, \Psi), p) = 1/2$ otherwise.

Then, we show that $M'$ satisfies the properties (0)-(4) in Definition 9.

Confinedness follows since $\Pi_i((s, \Psi)) \subseteq S_{\Psi \cap \mathcal{A}_i(s)}$.

Since $M$ is a TPAM, we have $(s, \Psi \cap \mathcal{A}_i(s)) \in \Pi_i((s, \Psi))$. So $(s, \Psi) \in (\Pi_i((s, \Psi)))^{\uparrow}$, which means $M'$ satisfies Generalized Reflexivity.

Given $\Pi_i((s, \Psi)) = \{(t, \Psi \cap \mathcal{A}_i(s)) : s\mathcal{R}_i t\}$, we have $\Pi_i((t, \Psi \cap \mathcal{A}_i(s))) = \{(u, \Psi \cap \mathcal{A}_i(s) \cap \mathcal{A}_i(t)) : t\mathcal{R}_i u\}$ for $t \in \mathcal{R}(i, s)$. Since $M$ is a TPAM, $s\mathcal{R}_i t$ implies $\mathcal{A}_i(s) = \mathcal{A}_i(t)$ and $\mathcal{R}(i, s) = \mathcal{R}(i, t)$. As a result, $\Pi_i((t, \Psi \cap \mathcal{A}_i(s))) = \Pi_i((s, \Psi))$, which means $M'$ satisfies Stationarity.

To prove that Projections Preserve Ignorance, note that $(\Pi_i((s, \Psi')))^{\uparrow} = \{(t, \Psi'') : s\mathcal{R}_i t \text{ and } \Psi' \cap \mathcal{A}_i(s) \subseteq \Psi''\}$ and $(\Pi_i(r^{S_{\Psi'}}_{S_\Psi}((s, \Psi'))))^{\uparrow} = \{(t, \Psi'') : s\mathcal{R}_i t \text{ and } \Psi \cap \mathcal{A}_i(s) \subseteq \Psi''\}$. Since $\Psi \subseteq \Psi'$, it follows that $(\Pi_i((s, \Psi')))^{\uparrow} \subseteq (\Pi_i(r^{S_{\Psi'}}_{S_\Psi}((s, \Psi'))))^{\uparrow}$. Therefore, Projections Preserve Ignorance.

To prove Projections Preserve Knowledge, suppose that $\Psi_1 \subseteq \Psi_2 \subseteq \Psi_3$ and $\Pi_i((s, \Psi_3)) \subseteq S_{\Psi_2}$. Then we have $\Psi_2 = \Psi_3 \cap \mathcal{A}_i(s)$ and $\Psi_1 = \Psi_1 \cap \mathcal{A}_i(s)$. Thus, $r^{S_{\Psi_2}}_{S_{\Psi_1}}(\Pi_i((s, \Psi_3))) = \Pi_i(r^{S_{\Psi_3}}_{S_{\Psi_1}}((s, \Psi_3))) = \{(t, \Psi_1) : s\mathcal{R}_i t\}$. Therefore, Projections Preserve Knowledge.

As such, we have proved that $M'$ is an HMSM. We now show by induction on the structure of $\varphi$ that if $Atm(\varphi) \subseteq \Psi$, then $(M, s) \models \varphi$ iff $(M', (s, \Psi)) \models \varphi$. For the boolean cases, the result is obvious either from the definition of $\pi$ or from the induction hypothesis. Now we prove the case $\varphi = X_i\psi$.

($\Rightarrow$) For an arbitrary state $s \in S$, suppose $(M, s) \models X_i\psi$ and $Atm(\psi) \subseteq \Psi$. Then $(M, s) \models K_i\psi$ and $Atm(\psi) \subseteq \mathcal{A}_i(s)$. Hence, $Atm(\psi) \subseteq (\Psi \cap \mathcal{A}_i(s))$. $(M, s) \models K_i\psi$ implies that $(M, t) \models \psi$ for all $t \in \mathcal{R}(i, s)$. By the induction hypothesis, since $Atm(\psi) \subseteq (\Psi \cap \mathcal{A}_i(s))$, $(M', (t, \Psi \cap \mathcal{A}_i(s))) \models \psi$ for all $t \in \mathcal{R}(i, s)$. By the definition of $\Pi$, $(M', (t, \Psi \cap \mathcal{A}_i(s))) \models \psi$ for all $(t, \Psi \cap \mathcal{A}_i(s)) \in \Pi_i((s, \Psi))$. Thus, $(M', (s, \Psi)) \models X_i\psi$.

($\Leftarrow$) For an arbitrary state $s \in S$ and $\Psi \subseteq Atm$, suppose $(M', (s, \Psi)) \models X_i\psi$. Then for all $(t, \Psi \cap \mathcal{A}_i(s)) \in \Pi_i((s, \Psi))$, $(M', (t, \Psi \cap \mathcal{A}_i(s))) \models \psi$. By the induction hypothesis and the definition of $\Pi$, it follows that for all $t \in \mathcal{R}(i, s)$, $(M, t) \models \psi$, so $(M, s) \models K_i\psi$. Also note that if $(M', (t, \Psi \cap \mathcal{A}_i(s))) \models \psi$, then $\psi$ is defined at all states in $S_{\Psi \cap \mathcal{A}_i(s)}$. Hence, $Atm(\psi) \subseteq \Psi \cap \mathcal{A}_i(s) \subseteq \mathcal{A}_i(s)$. Thus, $(M, s) \models A_i\psi$, which implies $(M, s) \models X_i\psi$.

As $A_i\varphi$ is defined as an abbreviation of $X_i\varphi \vee X_i\neg X_i\varphi$, we have proved part (a).

For part (b), we first prove that, given an HMSM $M = (\Theta, \preceq, R, \Pi, \pi)$, there is a PPDAM $M^p = (\Omega, \Rightarrow, \rho, \pi_0)$ with $\Omega = S_\Theta$ such that for all formulas $\varphi \in \mathcal{L}^{XA}$ and all $s \in S_\Theta$, if $s \in S_\Psi$ and $Atm(\varphi) \in \Psi$, then[9]

$$(M, s) \models \varphi \text{ if and only if } (M^p, s) \models \varphi. \tag{1}$$

Given an HMSM $M = (\Theta, \preceq, R, \Pi, \pi)$, let $M^p = (\Omega, \Rightarrow, \rho, \pi_0)$ be a GAM such that

- $\Omega = S_\Theta$,
- for every $i \in Agt$, let $\Pi_i = \{(s, t) : t \in \Pi_i(s)\}$,
  $\Rightarrow_i = \bigcap\{E : E$ is an equivalent relation on $S_\Theta$ and $\Pi_i \subseteq E\}$,
- for every $i \in Agt$ and $s \in S_\Theta$,
  if $\Pi_i(s) \subseteq S_\Psi$ then $\rho(i, s) = \mathcal{L}^{XA}(\Psi, Agt)$,
- for every $p \in Atm$,
  $\pi_0(p) = \{s \in S_\Theta : \pi(s, p) = 1\}$.

By the construction and Stationarity, it is apparent that $M^p$ is a partitional model and satisfies (i)[10] & (ii) of Definition 8. So $M^p$ is a PPDAM.

We complete the proof of part (b) by proving, by induction on the structure of $\varphi$, that if $s \in S_\Psi$ and $Atm(\varphi) \subseteq \Psi$, then $(M, s) \models \varphi$ iff $(M^p, s) \models \varphi$. For the boolean cases, the result is obvious either from the definition of $\pi_0$ or from the induction hypothesis. Now we prove the case $\varphi = X_i\psi$.

($\Rightarrow$) For an arbitrary state $s \in S_\Psi$ and $\Psi \subseteq Atm$, suppose $(M, s) \models X_i\psi$ and $Atm(\psi) \subseteq \Psi$. Hence, $(M, t) \models \psi$ for all $t \in \Pi_i(s)$. By Stationarity, $\Pi_i(t) = \Pi_i(s)$ for every $t \in \Pi_i(s)$. By Stationarity again, for any $s' \in S_\Theta$, if $\exists t \in \Pi_i(s)$ such that $t \in \Pi_i(s')$, then $\Pi_i(s') = \Pi_i(t) = \Pi_i(s)$. Then, by the definition of $\Rightarrow$, $\Rightarrow(i, s) = \{s' \in S_\Theta : \Pi_i(s') = \Pi_i(s)\}$. By Generalized Reflexivity, for any $s' \in\Rightarrow(i, s)$, we have $s' \in (\Pi_i(s))^\uparrow$, which implies $(M, s') \models \psi$ by Lemma 20. Hence, by the induction hypothesis and the definition of $\rho$, we have $Atm(\psi) \in \rho(i, s)$, and for all $s' \in\Rightarrow(i, s)$, $(M^p, s') \models \psi$, which implies $(M^p, s) \models X_i\psi$.

($\Leftarrow$) For an arbitrary state $s \in S_\Psi$ and $\Psi \subseteq Atm$, suppose $(M^p, s) \models X_i\psi$ and $Atm(\psi) \subseteq \Psi$. Hence, $(M^p, s') \models \psi$ for all $s' \in\Rightarrow(i, s)$. By the induction hypothesis and the definition of $\Rightarrow$, it follows that for all $t \in \Pi_i(s)$, $(M, t) \models \psi$, which implies $(M, s) \models X_i\psi$.

As such, given that $A_i\varphi$ is defined as $X_i\varphi \vee X_i\neg X_i\varphi$, (1) is demonstrated.

[9]In fact, this result is covered in Theorem 3.2 of [7]. However, we believe that there is a mistake in Halpern & Rêgo's construction when it comes to partitional structures, where the resulting awareness structure might not be partitional given the build-from HMS structure being partitional. So we prove it again by using equivalent closure to construct the accessibility relations in $M^p$.

[10]Compared with (i) of Definition 8, here we only consider formulas in $\mathcal{L}^{XA}$.

By Theorem 18, there is an RPPDAM $M^r = (\Omega', \Rightarrow', \rho', \pi')$ with $\Omega' \supseteq \Omega$ such that for all formulas $\varphi \in \mathcal{L}^{XA}$ and all $s \in \Omega$,

$$(M^p, s) \models \varphi \text{ if and only if } (M^r, s) \models \varphi. \tag{2}$$

By Lemma 16, there is an awareness-finite RPPDAM $M^a = (\Omega', \Rightarrow', \rho'', \pi')$ such that for every $\varphi \in \Sigma$ and every $s \in \Omega'$,

$$(M^r, s) \models \varphi \text{ if and only if } (M^a, s) \models \varphi. \tag{3}$$

By Theorem 15, there is a TPAM $M' = (S, \mathcal{R}, \mathcal{T}, \mathcal{V})$ with $S = \Omega'$ such that for all formulas $\varphi \in \Sigma$ and all $s \in \Omega'$,

$$(M^a, s) \models \varphi \text{ if and only if } (M', s) \models \varphi. \tag{4}$$

By (1)–(4), we conclude that part (b) holds. Figure 14 provides an overview of this theorem. □

## 5 Axiomatics

In this section, we define two variants of the LAPK logics and prove their soundness and completeness for the class **TPAM**. The two logics, notation $\mathsf{LAPK}^K$ and $\mathsf{LAPK}^X$, are LAPK for implicit knowledge and LAPK for explicit knowledge, respectively. What calls for special attention is that the proofs in this section are not self-contained and rely heavily on Section 4 and results of [6] and [2].

For language $\mathcal{L}^{XKA}$, we define the logic $\mathsf{LAPK}^K$ to be the extension of all propositional tautologies including $\top$, which is given by the following axioms and rule of inference:

$$A_i \top \tag{AT}$$

$$A_i\varphi \leftrightarrow A_i\neg\varphi \tag{AS}$$

$$A_i(\varphi \wedge \psi) \leftrightarrow A_i\varphi \wedge A_i\psi \tag{AC}$$

$$A_i\varphi \leftrightarrow A_i K_j \varphi \tag{AKR}$$


**Fig. 14** Relations between our semantics and HMS semantics for the language $\mathcal{L}^{XA}$. The arrows are explained in the same way as in Fig. 12, where full arrows correspond to the results stated in Lemma 16 and Theorems 15, 18, 21

$$A_i\varphi \leftrightarrow A_i A_j \varphi \tag{AR}$$

$$A_i\varphi \to K_i A_i \varphi \tag{AI}$$

$$\neg A_i\varphi \to K_i \neg A_i \varphi \tag{UI}$$

$$K_i(p \leftrightarrow \varphi) \to (A_i\varphi \to A_i p) \text{ where } \varphi \in \mathcal{L} \tag{ARKE}$$

$$X_i\varphi \leftrightarrow K_i\varphi \wedge A_i\varphi \tag{XK}$$

$$(K_i\varphi \wedge K_i(\varphi \to \psi)) \to K_i\psi \tag{K}$$

$$K_i\varphi \to \varphi \tag{T}$$

$$K_i\varphi \to K_i K_i \varphi \tag{4}$$

$$\neg K_i\varphi \to K_i \neg K_i \varphi \tag{5}$$

$$\frac{\varphi, \varphi \to \psi}{\psi} \tag{MP}$$

$$\frac{\varphi}{K_i\varphi} \tag{NEC}$$

For every $\varphi \in \mathcal{L}^{XKA}$, we write $\vdash_{\mathsf{LAPK}^K} \varphi$ to mean that $\varphi$ is deducible in $\mathsf{LAPK}^K$, that is, there is a sequence of formulas $(\varphi_1, \ldots, \varphi_m)$ such that:

- $\varphi_m = \varphi$, and
- for every $1 \leq k \leq m$, either $\varphi_k$ is an instance of one of the axiom schema[11] of $\mathsf{LAPK}^K$ or there are formulas $\varphi_{k_1}, \ldots, \varphi_{k_t}$ such that $k_1, \ldots, k_t < k$ and $\frac{\varphi_{k_1}, \ldots, \varphi_{k_t}}{\varphi_k}$ is an instance of some inference rule of $\mathsf{LAPK}^K$.

We say that the set of formulas $\Gamma \subseteq \mathcal{L}^{XKA}$ is $\mathsf{LAPK}^K$-consistent if there are no formulas $\varphi_1, \ldots, \varphi_m \in \Gamma$ such that $\vdash_{\mathsf{LAPK}^K} (\varphi_1 \wedge \ldots \wedge \varphi_m) \to \neg\top$. In particular, $\varphi$ is $\mathsf{LAPK}^K$-consistent if $\{\varphi\}$ is $\mathsf{LAPK}^K$-consistent.

Note that the four axioms (AS), (AC), (AKR) and (AR) together are equivalent to the following axiom:

$$A_i\varphi \leftrightarrow \bigwedge_{p \in Atm(\varphi)} A_i p \tag{AGPP}$$

which means being of a formula is equivalent to being aware of every atomic proposition occurring in the formula.

[11]To form an instance of axiom (ARKE), we can only replace $p$ with an atomic proposition, and replace $\varphi$ with a propositional formula.

**Theorem 22** *The logic* $\mathsf{LAPK}^K$ *is sound and complete for the class* **RPPDAM**.

*Proof* For soundness, we just verify that every axiom of $\mathsf{LAPK}^K$ is valid in **RPPDAM** and every inference rule of $\mathsf{LAPK}^K$ preserves validity in **RPPDAM**. Note that the validity of Eq. (ARKE) is included in the definition of RPPDAMs. We omit the details of the verifying procedure. For completeness, this is a straightforward modification of the proof of **S5** completeness (cf. Theorem 4.29 in [2].) We omit details here. □

**Theorem 23** *The logic* $\mathsf{LAPK}^K$ *is sound and complete for the class* **TPAM**.

*Proof* For soundness, we just verify that every axiom of $\mathsf{LAPK}^K$ is valid in **TPAM** and every inference rule of $\mathsf{LAPK}^K$ preserves validity in **TPAM**. Note that the validity of Eq. (ARKE) is proved by Proposition 13. We omit the details of the verifying procedure. Then we prove completeness. By Theorem 22, for every $\varphi \in \mathcal{L}^{XKA}$, if $\varphi$ is $\mathsf{LAPK}^K$-consistent, then $\varphi$ is satisfiable for the class **RPPDAM**. Then, by Theorem 17, $\varphi$ is also satisfiable for the class **TPAM**. □

For language $\mathcal{L}^{XA}$, we define the logic $\mathsf{LAPK}^X$ to be the extension of all propositional tautologies including $\top$, which is given by the following axioms and rule of inference[12]:

$$A_i \top \tag{AT}$$

$$A_i\varphi \leftrightarrow A_i\neg\varphi \tag{AS}$$

$$A_i(\varphi \wedge \psi) \leftrightarrow A_i\varphi \wedge A_i\psi \tag{AC}$$

$$A_i\varphi \leftrightarrow A_i X_j\varphi \tag{AXR}$$

$$A_i\varphi \leftrightarrow A_i A_j\varphi \tag{AR}$$

$$A_i\varphi \rightarrow X_i A_i\varphi \tag{AI$_X$}$$

$$X_i\varphi \rightarrow A_i\varphi \tag{KA}$$

$$(X_i\varphi \wedge X_i(\varphi \rightarrow \psi)) \rightarrow X_i\psi \tag{K$_X$}$$

$$X_i\varphi \rightarrow \varphi \tag{T$_X$}$$

[12] By Theorem 5.3 in [6], the inference rule (Irr) is necessary for the completeness proof of $\mathcal{L}^{XA}$ axiomatizations with respect to **PPDAM**.

$$X_i\varphi \to X_iX_i\varphi \tag{$4_X$}$$

$$\neg X_i\varphi \wedge A_i\varphi \to X_i\neg X_i\varphi \tag{$5_X$}$$

$$\frac{\varphi, \varphi \to \psi}{\psi} \tag{MP}$$

$$\frac{\varphi}{A_i\varphi \to X_i\varphi} \tag{$\mathrm{NEC}_X$}$$

$$\text{if } Atm(\varphi) \cap Atm(\psi) = \emptyset, \text{ then } \frac{\neg A_i\varphi \to \psi}{\psi} \tag{Irr}$$

For every $\varphi \in \mathcal{L}^{XA}$, we write $\vdash_{\mathsf{LAPK}^X} \varphi$ to mean that $\varphi$ is deducible in $\mathsf{LAPK}^X$, which is defined in the usual way. We say that the set of formulas $\Gamma \subseteq \mathcal{L}^{XA}$ is $\mathsf{LAPK}^X$-consistent if there are no formulas $\varphi_1, \ldots, \varphi_m \in \Gamma$ such that $\vdash_{\mathsf{LAPK}^X} (\varphi_1 \wedge \ldots \wedge \varphi_m) \to \neg\top$. In particular, $\varphi$ is $\mathsf{LAPK}^X$-consistent if $\{\varphi\}$ is $\mathsf{LAPK}^X$-consistent.

**Theorem 24** *The logic* $\mathsf{LAPK}^X$ *is sound and complete for the class* **PPDAM**.

*Proof* For soundness, we just verify that every axiom of $\mathsf{LAPK}^X$ is valid in **PPDAM** and every inference rule of $\mathsf{LAPK}^X$ preserves validity in **PPDAM**. We omit the details of the verifying procedure. The completeness of the single-agent version of $\mathsf{LAPK}^X$ with respect to **PPDAM** is proved by Halpern (cf. Theorem 5.3 in [6]), which can be easily modified to be the completeness proof for $\mathsf{LAPK}^X$. We omit the details here. □

**Theorem 25** *The logic* $\mathsf{LAPK}^X$ *is sound and complete for the class* **TPAM**.

*Proof* For soundness, we just verify that every axiom of $\mathsf{LAPK}^X$ is valid in **TPAM** and every inference rule of $\mathsf{LAPK}^X$ preserves validity in **TPAM**. We omit the details of the verifying procedure. Then we prove completeness. By Theorem 24, for every $\varphi \in \mathcal{L}^{XA}$, if $\varphi$ is $\mathsf{LAPK}^X$-consistent, then $\varphi$ is satisfiable for the class **PPDAM**. Then, by Theorem 19, $\varphi$ is also satisfiable for the class **TPAM**. □

Note that there is no further restriction on every axiom schema of $\mathsf{LAPK}^X$, so it supports the rule of substitution.

$$\frac{\varphi}{\varphi[\psi \backslash p]} \tag{US}$$

## 6 Discussion

A few points need to be clarified about this novel semantics. Firstly, "an agent knows that he is going to know whether $p$ by certain means" does not imply the means exists in reality. Absurd as it seems, it reveals some insights about awareness. One example comes from the fancy concept "unicorn". Let agent $i$ be aware of $p$, where $p$ stands for "unicorns can fly", and let agent $j$ be not aware of $p$. Thus, $i$ knows that he is going to know whether $p$ by traveling to a fairy tale world, while $j$ does not. It makes sense because the means does disclose the nature of the concept "unicorn", i.e., $i$ knows that unicorn is an imaginary animal while $j$ does not. So it is reasonable to link propositional awareness to the means of truth inquiry. Secondly, understandings of the means of truth inquiry implicate understandings of the concept. Let's think about mathematical hypotheses. Let $p$ denote "Riemann hypothesis is true". Then, what does it mean by being aware of $p$? If an elementary school student $i$ mentions $p$, shall we deem it as that $i$ is aware of $p$? At the same time, a graduate student $j$ of mathematics with a good understanding of complex variables functions utters that he is aware of $p$. Is it more reasonable for $j$ being aware of $p$ than $i$ being aware of $p$? The answer is yes, since $j$ has more understandings of proving $p$ even though he can not tell how to do it. The crucial point is that, being aware of a proposition is a lot more than the ability to put it into words.[13] This principle applies to other more abstract examples such as art. Perhaps we will never be able to show a definite path to art, but we can definitely tell that it is more reasonable for someone to be considered as being aware of art than someone else, when the former knows more about how to approach art.

Another question emerges when we think in the other way around. Is it possible that an agent knows how to inquire the truth of a proposition while not aware of it? Let's consider a simple scenario: agent $i$ is not aware of what is behind the door, but he knows that after opening it he will know what is behind the door. Should that mean $i$ is aware of it? Assume that there does exist a teddy bear behind the door. Let $p$ denote "there is something behind the door", let $q$ denote "there is a teddy bear behind the door". Apparently, $p$ and $q$ are different propositions. Thus, this scenario is similar with the Bob story in Introduction and shows the connections between different atoms. The process is presented in Fig. 15.

By the semantics, originally, $i$ does not know $p$ or $q$, but he is aware of $p$. The latter is due to that he knows he is going to know whether $p$ by opening the door. Note that $i$ is not aware of $q$. It may be controversial, since apparently $i$ knows he is going to know whether $q$ by opening the door. However, it is meaningless for that we can come up with millions of similar propositions, such as "there is a Barbie doll behind the door". An agent need not consider so many propositions in this particular

[13] As pointed out by the reviewer, this scenario inspires another explanation for our semantics: "agent $i$ is aware of $p$" can be understood as "agent $i$ can imagine knowing whether $p$ is true". Thus, the converse notion: "agent $i$ is unaware of $p$ if he cannot imagine knowing whether $p$" is also interesting and justified by this scenario, where a school student may have heard the name of the problem, but is arguably unaware of the hypothesis itself, as he cannot conceptualize the nature of its proof or refutation. This notion of awareness is justifiable in an ontological sense and is an interesting addition to epistemology. However, in a technical sense, the notion of "imagine" seems not easy to be integrated into our semantics. The temporal connotation in this paper is consistent with the traditional way of formalizing knowledge, as it interprets awareness as a property satisfied by every $i$-accessible state. If we bring in the notion of "imagine", it seems that an in-depth connection between knowledge and imagination needs to be clarified.

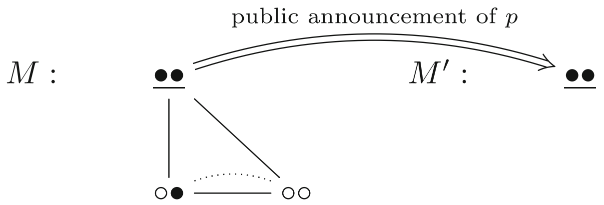


**Fig. 15** Two single-agent TPAMs for agent $i$. The explanation of elements in the figure is the same with that of Fig. 11. The underlined states indicate the current states. By the public announcement of $p$, $M$ is transformed to $M'$

circumstance. Instead, $i$ should first be concerned about whether $p$. In this example, we consider opening the door as public announcement of $p$. After the announcement, $i$ becomes aware of $q$ and knowing $q$. It happens through the objective connection between $p$ and $q$. Once again, the unique feature of our semantics is revealed by dynamic processes.

## 7 Conclusions and Future Work

We have brought up a novel semantics of awareness, called Semantics of Awareness as Potential for Knowledge, for the semantic approach of awareness logic. In our structure, two kinds of accessibility relations are attached to each agent, and from the relations awareness, implicit knowledge and explicit knowledge are generated. Expressivities of languages with different modalities are compared in the proposed semantics. We have connected our semantics to the existed works in two directions. Firstly, we have investigated the relation between our semantics and the semantics by Fagin & Halpern in partitional settings, which results in two equivalence patterns, one for language of awareness and implicit knowledge, the other for language of awareness and explicit knowledge. Secondly, likewise in partitional settings, we have proved that our structure is equivalent with the lattice structure by Heifetz et al. with respect to language of awareness and explicit knowledge. Finally, we have constructed two axiomatics for partitional models, one for logic of awareness and implicit knowledge, the other for logic of awareness and explicit knowledge.

Future work will be devoted to dynamic extensions of our logic, where public announcements and action models of our structure will be studied. As the Bob scenario in Introduction shows, getting knowledge would possibly cause an agent to become aware of an atomic proposition that is not part of the knowledge. Figures 3 and 15 can both be regarded as dynamic processes of this kind, where by public announcement of $p$, agent $i$ becomes aware of $q$. Moreover, dynamics of TPAMs not only involves current knowledge update, but also comprises upcoming knowledge change, revealing distinct features of TPAMs and modeling phenomena of science discovery. We also plan to introduce the notion of speculative knowledge into our logic. As what we have discussed in Section 4.2, speculative knowledge is necessary for our theory in order to remove the defect of implicit knowledge.

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