Journal of Philosophical Logic (2023) 52:415–493
https://doi.org/10.1007/s10992-022-09676-2

A Semantics for the Impure Logic of Ground

Louis deRosset[1] · **Kit Fine**[2]

Received: 23 August 2021 / Accepted: 15 July 2022 / Published online: 10 September 2022

Abstract
This paper establishes a sound and complete semantics for the impure logic of ground. Fine (*Review of Symbolic Logic*, 5(1), 1–25, 2012a) sets out a system for the pure logic of ground, one in which the formulas between which ground-theoretic claims hold have no internal logical complexity; and it provides a sound and complete semantics for the system. Fine (2012b) [§§6-8] sets out a system for an impure logic of ground, one that extends the rules of the original pure system with rules for the truth-functional connectives, the first-order quantifiers, and λ-abstraction. However, no semantics has yet been provided for this system. The present paper partly fills this lacuna by providing a sound and complete semantics for a system GG containing the truth-functional operators that is closely related to the truth-functional part of the system of Fine (2012b).

Keywords Impure logic of ground · Truthmaker semantics · Logic of ground · Ground

1 Introduction

This paper establishes a sound and complete semantics for the impure logic of ground. Fine [6] sets out a system for the pure logic of ground, one in which the formulas between which ground-theoretic claims hold have no internal logical complexity; and it provides a sound and complete semantics for the system. Fine [7][§§6-8] sets out a system for an impure logic of ground, one that extends the rules of the original pure system with rules for the truth-functional connectives, the first-order quantifiers, and λ-abstraction. However, it does not provide a semantics for this system. The present paper partly fills this lacuna by providing a sound and

✉ Louis deRosset
Louis.deRosset@uvm.edu

[1] University of Vermont, Burlington, Burlington, VT, USA

[2] New York University, New York, NY, USA

complete semantics for a system GG containing the truth-functional operators that is closely related to the truth-functional part of the system of [7].[1]

The present section provides an informal introduction to the leading ideas behind the paper. In the rest of the paper, we describe the target system GG and the proposed semantics and provide a proof that the system is sound and complete for the proposed semantics. Sections 2 and 3 provide the formal specifications of the system GG and its semantics, and establish soundness and consistency for the system. The ensuing proof of completeness is Henkin-style. In Section 4, we define the canonical model for a given set of grounding claims S, and discuss its principal features. Sections 5–7 establish the adequacy of the construction. Section 8 proves completeness, and Section 9 concludes with a sketch of directions for further work.

The reader may find it helpful to have the above two papers at hand, but let us remind her of some key features of the earlier systems. A distinction is drawn between *weak* and *strict* ground. Intuitively, we might think of a strict ground as being on a lower explanatory level than what it grounds, while a weak ground can also be at the same explanatory level. Thus A will always be a weak ground for itself though never a strict ground. We also introduce the notion of a *partial*, as opposed to a *full*, ground. A *weak partial* ground is a part of a weak full ground, while a *strict partial* ground is a weak partial ground which cannot be reversed. Thus A, B, together, will be a strict full ground for $A \wedge B$, while A or B on their own will be strict partial grounds for $A \wedge B$ though not, in general, strict full grounds; and if it is granted that, for distinct bodies x, y and z, x being of the same mass as y and y the same mass as z weakly fully grounds x being of the same mass as z, then x being of the same mass as y will be a weak partial ground for x being of the same mass as z without being either a strict partial ground or a weak full ground. We are thereby led to a fourfold classification of ground - strict full, weak full, strict partial, and weak partial - for which we use the respective symbols $<$, $\leq$, $\prec$, and $\preceq$ and the systems we consider will treat each of these four types of ground as syntactic primitives.

In the impure system, there are two principal sets of rules concerning the interaction between ground and the truth-functional connectives. There are, first of all, the introduction rules, which specify the grounds for a truth-functionally complex statement of a given form in terms of simpler statements. Thus the fact that A, B strictly grounds $A \wedge B$ serves as an introduction rule for conjunction. There are, in the second place, the elimination rules, which tell us how an arbitrary ground for a truth-functionally complex statement of a given form must be related to the grounds for simpler statements. Thus in the case of conjunction, the elimination rule will tell us that when a set of statements Δ strictly grounds $A \wedge B$, it must be possible to split Δ into two (perhaps overlapping) parts, one of which weakly grounds A and the other of which weakly grounds B.

The development of a semantics for the logic of ground faces two main tasks: it must provide an account of the content of the statements that go to make up a

[1]The main differences between the two systems are that we now only allow finitely many formulas to occur to the left of the ground-theoretic operator and that we have added the Irreversibility Rule, which should have been part of the original system.

grounding claim; and it must provide an account of the ground-theoretic connection that should hold among the contents of those statements when the claim is true. The two tasks go hand in hand, since the account of content should be precisely what is needed to provide the resources by which a suitable account of the ground-theoretic connections might be given.

In dealing with these two tasks, we have found it convenient to adopt a form of truthmaker semantics.[2] The main idea behind such a semantics is that truthmaking should be exact, *i.e.,* a truthmaker should bear as a whole upon the statement that it makes true. Since ground is also exact, which is to say that the grounds should bear as a whole upon what is grounded, it is perhaps no surprise that a semantics for the logic of ground should also be exact. The exactitude of ground will be mirrored in the exactitude of the truth-makers.[3]

Another feature of truthmaker semantics – at least within the setting of classical logic – is that it is bilateral. The full content, or meaning, of a statement is not simply given by its truth-makers but also by its falsity-makers. Thus we may take the truth-condition (sometimes called the *positive content*) of a statement to be given by the set of its truth-makers, the falsity-condition (or *negative content*) to be given by the set of its falsity-makers, and its content (or *full content*) to be given by the ordered pair consisting of its truth-condition, or positive content, and its falsity-condition, or negative content.

Our semantics for the impure logic of ground will take over these features; it will be both exact and bilateral. However, the standard "flat" form of truthmaker semantics, described in [9], will not serve our purpose, since it does not provide us with a sufficiently fine-grained conception of content. The problem is that our impure logic of ground is highly hyper-intensional; it distinguishes in a very radical way between logically equivalent statements. Thus: even though $A \wedge B$ is logically equivalent to $B \wedge A$, $A \wedge B$ will be a weak ground for $A \wedge B$ but not generally for $B \wedge A$; even though $A \vee B$ is logically equivalent to $B \vee A$, $A \vee B$ is a weak ground for $A \vee B$ but not generally for $B \vee A$; and, even though A is logically equivalent to $\neg\neg A$, $\neg\neg A$ will be a weak ground for $\neg\neg A$ but not for A.

Now the standard truthmaker semantics is indeed hyper-intensional; it will distinguish, for example, between the truthmakers for A and for $A \vee (A \wedge B)$, since the fusion of a truth-maker for A and for B will be a truth-maker for $A \wedge B$ and hence for $A \vee (A \wedge B)$, yet not in general for A. However, it is not hyper-intensional enough. For under the standard semantics, the truth- and falsity-makers of $A \wedge B$ and $B \wedge A$, and of $A \vee B$ and $B \vee A$, and of A and $\neg\neg A$ will be the same. We therefore require a more fine-grained conception of content and a more refined conception of truth- and falsity-making by which it might be defined.

To this end, it will be helpful to see how this more refined conception of truth-making of our semantics might have evolved, through successive differentiation, from the original, less refined, notion of truthmaking of the standard semantics. (This reflects the actual development of our semantics). Consider first the relationship

[2]A survey of this style of semantics can be found in [9].

[3]This connection between ground and truthmaking is further discussed in [11].

between $A \wedge B$ and $B \wedge A$. Under the standard semantics, the truthmakers for $A \wedge B$ are the fusions of the form $(a \sqcup b)$ and the truthmakers for $B \wedge A$ are the fusions of the form $(b \sqcup a)$, for a a truthmaker for A and b a truthmaker for B, and, since $(a \sqcup b)$ is assumed to be the same as $(b \sqcup a)$, the truthmakers for $A \wedge B$ and for $B \wedge A$ will be the same. It turns out that the falsitymakers for $A \wedge B$ and for $B \wedge A$ are also the same; and so the standard semantics will be incapable of distinguishing, as it should, between the contents of $A \wedge B$ and $B \wedge A$.

We may overcome this problem by adopting a more fine-grained conception of fusion, which we now call *combination* and which is not subject to the usual "leveling" constraints, such as associativity, commutativity and idempotence. The combination $[a.b]$ of a and b, for example, need not be taken to be the same as the combination $[b.a]$ of b and a. We also allow, in the spirit of generality, for combination to apply to any finite number of elements (or to an infinite number of elements in some further applications we will consider). In the special case in which combination applies to zero items, we will get a null item, which corresponds to the fusion of zero states in the standard semantics; and, in the special case of the unit combination of a single item a, combination will take us up a level to a "raised" version $[a]$ of the item, which, in contrast to the unit fusion, is never the same as the item itself. The semantics for conjunction is now explained in terms of combination rather than fusion and, since the combination $[a.b]$ of a and b need not be the same as the combination $[b.a]$ of b and a, the previous difficulty is avoided.

Similar problems beset the relationship between $(A \vee B)$ and $(B \vee A)$. Under the standard semantics, the truthmakers for $(A \vee B)$ are the truthmakers for A and for B (and possibly also for $A \wedge B$) and so will be the same as the truthmakers for $(B \vee A)$. It turns out that the falsity-makers for $(A \vee B)$ and for $(B \vee A)$ are also the same; and so the standard semantics will be incapable of distinguishing, as it should, between the contents of $(A \vee B)$ and $(B \vee A)$.

We overcome this problem by supposing that, in addition to the operation of combination, there is an operation of *choice* which applies to any finite number (or, more generally, to any number) of items and which is, again, not subject to leveling. The choice $[a + b]$ between a and b, for example, need not be the same as the choice $[b + a]$ between b and a. Choices are in general different from combinations but, in the special case of a single element a, we shall find no need to distinguish between the unit choice of a and the unit combination $[a]$. The semantics for disjunction is now explained in terms of choice and, since the choice $[a + b]$ between a and b need not be the same as the choice $[b + a]$ between b and a, we will be in a position to distinguish between the contents of $(A \vee B)$ and $(B \vee A)$.

This change to the standard semantics brings a more sweeping change in its wake. Before, we could identify the truth-condition of a statement with the *set* of its truthmakers and the falsity-condition of the statement with the *set* of its falsity-makers and we were able, moreover, to provide a recursive specification of the truth-and falsity-makers of a conjunction or disjunction in terms of the truth- and falsity-makers of their immediate components (and their fusions). We could therefore take as our semantic primitives the notions of a state being a truth-maker for a given statement and of a state being a falsity-maker for a given statement. This is no longer possible, for the difference between $(A \vee B)$ and $(B \vee A)$, for example, will lie not in the truth-makers for their components, which are the same, but in the order in which they are given. Thus the semantics must proceed by providing a recursive specification of the truth- and falsity-*conditions*, rather than the truth- and falsity-*makers*, and combination and choice must be regarded (at least for now) as operations on truth- and falsity-conditions without there necessarily being any explanation of the operations solely in terms of the truth- and falsity-makers by which the conditions are constituted.

Negation introduces a further complication. Under the standard semantics, the truth-condition for $\neg A$ is the falsity-condition for A and the falsity-condition for $\neg A$ is the truth-condition for A. This means that A and $\neg\neg A$ will have the same truth-condition and the same falsity-condition; and so the semantics will be incapable of distinguishing, as it should, between the contents of A and $\neg\neg A$.

Let us grant that the falsity-condition for A is indeed the truth-condition for $\neg A$. (Indeed, we might even take the falsity-condition for A to be, by definition, the truth-condition for $\neg A$.) Is it then so clear that the falsity-condition for $\neg A$ will be the truth-condition for A? For the falsity-condition for $\neg A$, we have already assumed, is the truth-condition for $\neg\neg A$. But the direct truth-condition a for A is only an indirect truth-condition for $\neg\neg A$; it makes $\neg\neg A$ true through first making A true. And we may mark this difference by making the direct truth-condition for $\neg\neg A$ to be, not a, but the unit combination $[a]$ (cf. [13, 10-12]). Thus in providing a semantics for $\neg A$, there is not simply a reversal of the truth- and falsity-conditions but a *raised* reversal, in which the truth-condition a for A is converted into a raised falsity-condition $[a]$ for $\neg A$. We can then distinguish between A and $\neg\neg A$ since, when a is the truth-condition for A, it is $[a]$ rather than a that will be the truth-condition for $\neg\neg A$ and, in general, when (a, a') is the content of A then $([a], [a'])$ will be the content of $\neg\neg A$.

We are not yet done. We have so far assumed that the truth-condition for $A \wedge B$ is the combination of the truth-conditions for A and B respectively and that the truth-condition for $A \vee B$ is the choice between the truth-conditions for A and B; and similarly for the other cases. But this leads to unwanted results. For suppose that (a, a') is the content of A and (b, b') the content of B. Then the content of $A \vee B$ is $([a + b], [a'.b'])$, so the content of $\neg(A \vee B)$ is $([a'.b'], [[a + b]])$, and so the content of $\neg\neg\neg(A \vee B)$ is $([[a'.b']], [[[a + b]]])$; and the respective contents of $\neg A$ and $\neg B$ are $(a', [a])$ and $(b', [b])$, so the content of $(\neg A \wedge \neg B)$ is $([a'.b'], [[a] + [b]])$, and so the content of $\neg\neg(\neg A \wedge \neg B)$ is $([[a'.b']], [[[a] + [b]]])$.

Now $\neg(A \vee B)$ is a strict full ground for $\neg\neg\neg(A \vee B)$ and so we will want an appropriate ground-theoretic connection to hold between the content $([a'.b'], [[a + b]])$ of $\neg(A \vee B)$ and the content $([[a'.b']], [[[a + b]]])$ of $\neg\neg\neg(A \vee B)$. But in the semantics, we will want the grounding connection between the contents of some grounds and a grounded statement to depend only upon the positive content of the grounded statement (we might call this 'positive bias', since only the positive content of the grounded statement is taken into account).

Such a view might plausibly be taken to be built into our conception of positive content, which concerns the ways in which a proposition might be true, but not the ways in which it might be false, i.e. its negative content. (It will also receive some support from the idea, developed below, of contents as bi-lateral menus.) But this means, in the particular case above, that it is only the positive content $[[a'.b']]$

of $\neg\neg\neg(A \vee B)$ that it is relevant to $\neg(A \vee B)$ grounding $\neg\neg\neg(A \vee B)$. So the same ground-theoretic connection should hold between the content $([[a'.b']], [[[a + b]]])$ of $\neg(A \vee B)$ and the content $([[a'.b']], [[[a] + [b]]])$ of $\neg\neg(\neg A \wedge \neg B)$ and, consequently, $\neg(A \vee B)$ should also be a strict ground for $\neg\neg(\neg A \wedge \neg B)$. But our system leaves open whether this is so.

We solve this problem by supposing that combination and choice are operations, not on conditions, but on contents. Thus the truth-condition for $(A \wedge B)$ will be the combination of the respective *contents* (not truth-*conditions*) of A and B, the truth-condition for $(A \vee B)$ will be the choice of the respective contents of A and B, the falsity-condition for $\neg A$ will be the unit combination of the content of A, and similarly for the other cases. There is thus an interplay between conditions and contents, with contents formed through the pairing of conditions and conditions formed through the combination and choice of contents. The previous problem will not then arise since $\neg\neg\neg(A \vee B)$ and $\neg\neg(\neg A \wedge \neg B)$ will end up having different truth-conditions.

From an intuitive point of view, we should think of contents as bilateral or two-sided; what matters to them is when they are true and when they are false. We should, by contrast, think of conditions as unilateral or one-sided; all that matters to them is when they obtain. A content will then be constituted by a truth-condition, being true when the condition obtains, and by a falsity-condition, being false when this condition obtains. It remains to explain why, from an intuitive point of view, we should take conjunctive conditions to be combinations of contents rather than conditions. In other words, given two propositions (a, a') and (b, b'), why should we take the truth-condition c for their conjunction to be the combination of the two propositions themselves rather than the combination of their truth-conditions a and b? (A similar problem also arises for disjunction). The reason is that we adopt a representational conception of the truth-conditions. It matters to the identity of a conjunctive truth-condition, or combination, c not only what the component truth-conditions a and b are but also how they get "carried" into the combination via the respective propositions (a, a') and (b, b').

The proposed semantics reveals then an interesting feature of the truthmaker approach. Truthmaker semantics is generally contrasted with more structural approaches to propositional identity. For instance, a structural approach might draw a distinction between A and $(A \vee A)$, whereas standard truthmaker semantics does not. As we have seen, our target logic GG requires distinctions of this sort. The soundness and completeness of that logic under the present approach therefore reveals that there are natural modifications of standard truthmaker semantics that accommodate these more finely-grained distinctions, thereby achieving a kind of semantic commonality between the more coarse-grained approaches to ground based on standard truthmaker semantics [2, 7][§1.10] and the more fine-grained approaches [7] [§§1.7-1.9] treated under the present modification.

We turn to the second task, of providing an account of ground-theoretic connection. We here appeal to the abstract theory of menus gestured at in §4 of [10]. A menu provides a vehicle for selection. Thus from the two-item menu listing eggs-and-bacon and porridge, one can select either eggs-and-bacon or porridge and, from the one-item menu listing eggs-and-bacon, one can select the two component items, eggs and bacon, and consequently, from the original two-item menu, one can select either eggs and bacon or porridge.

The theory of menus provides a general abstract account of selection. Within such a theory, we take the domain of items to be closed under combination and choice. Thus, given any finite number of items $a_1, a_2, \ldots$ of the domain, the choice $[a_1 + a_2 + \cdots]$ and the combination $[a_1.a_2. \cdots]$ of those items will also be items of the domain. Menus are either combinations or choices and so may themselves figure as items on a menu. So, in the example above, the breakfast menu will be of the form $b = [[a_1.a_2] + a_3]$, where a_1 is eggs, a_2 is bacon and a_3 is porridge. This menu may then be part of another, meta-menu $[b + l]$, which provides a choice between the breakfast menu b and a lunch menu l.

There are two main principles governing the immediate selection of items from a menu. In the case of a choice $[a_1 + a_2 + \cdots]$, each of $a_1, a_2, \ldots$ is an immediate selection; and in the case of a combination $[a_1.a_2. \cdots]$, $a_1, a_2, \ldots$ (together) is an immediate selection. A simple account of selection (later to be modified) can then be obtained through the repeated chaining of immediate selection: $[a + b], c$, for example, will be an immediate selection from $[[a+b].c]$ and a an immediate selection from $[a + b]$; and so a, c will be a selection from $[[a + b].c]$. So, in the example above eggs, bacon will be a selection from the meta-menu $[b + l]$.

It is important to bear in mind that we have done nothing to rule out non-trivial identities between combinations or choices. Some of these identities may be structural in origin. Thus we might think of a menu not as a list but as a set of items. We would then want $[a.b]$, for example, to be identical to $[b.a]$ and for $[a + b]$ to be identical to $[b + a]$. But other identities may have a more substantive basis. When one orders eggs and bacon at a restaurant, one is served particular eggs and particular rashers of bacon (and, indeed, might be disappointed to be served the types rather than the tokens). Consider now the combination $[e_1.e_2.r_1.r_2]$ of some particular eggs e_1, e_2 and some particular rashers of bacon r_1, r_2 and consider some other combination $[e_3.e_4.r_3.r_4]$ of particular eggs and rashers. The particular items from which the combinations are formed are different. But one might still want to treat the combinations themselves as, in effect, identical. After all, it is presumably a matter of indifference, if one opts for the combination $[e_1.e_2.r_1.r_2]$, whether one is served e_1, e_2, r_1, r_2 rather than e_3, e_4, r_3, r_4. This means that even though a, b, for example, is an immediate selection from $[a.b]$ and each of a and b is an immediate selection from $[a + b]$, $[a.b]$ and $[a + b]$ may, through their identity with other forms of combination and choice, enjoy other immediate selections as well.

The application of the theory of menus to the current semantics will rest upon taking truth- and falsity-conditions to be menus and taking ground to be selection. Roughly speaking, disjunction will tell us to make a choice of truth-conditions, while conjunction will require us to make a combination of truth-conditions. However, the viability of this application will depend upon making two significant modifications to the simple account of selection presented above.

We must, in the first place, allow two-sided menus, which we might represent as ordered pairs (a, b) of items a and b; and we might, in a more general context, allow vector menus $(a, b, \dots)$ of arbitrary length.[4]

We might, intuitively, think of a two-sided menu as a 'positive' menu of items to be included, on the one side, and a 'negative' menu of items to be excluded, on the other side (as in a kosher chicken platter, which includes the combination of items making up the chicken platter while excluding dairy products). Within the intended semantical application, conditions will correspond to one-sided menus and contents to two-sided menus, with truth-conditions on the one side and falsity-conditions on the other side.

However, allowing two-sided menus calls for a slight complication in our account of selection. For immediate selection is most naturally defined as a relation between two-sided menus (which correspond to contents, or pairs of conditions) and a one sided menu (which corresponds to a condition). So, for example, in making a selection from a kosher chicken platter, all that counts is what may be selected from the chicken platter. But we would like selection to be a relation between two-sided menus so that it can be repeatedly chained. We do this by appeal to the following principle (corresponding to 'Basis' in Definition 2.1 below):

Positive Bias Some two-sided menus (or contents) will be an immediate selection from a given two-sided menu (or content) just in case they are an immediate selection from its positive side (or truth-condition).

We can still say that a two-sided menu (a, b) (or content) is a selection from a one-sided menu c (or condition), but this must now be taken to mean that (a, b) is a selection from (c, d) for some item d. Suppose, for example, that a, b, c, d and e are conditions. Then the pairs (a, b) and (c, d) are contents, while the choice $[(a, b) + (c, d)]$ is another condition. The content (a, b) will then be an immediate selection from the content $([(a, b) + (c, d)], e)$ and, for this reason, an immediate selection from the condition $[(a, b) + (c, d)]$.

The other modification is more radical. For we want to introduce a notion of *weak* selection, corresponding to weak ground, in addition to the previous notion of strict selection, which corresponded to strict ground. Weak selection, however exactly it is understood, is plausibly taken to be subject to the following principle (corresponding to ASCENT in Definition 2.1):

Subsumption Any case of strict selection is a case of weak selection.

Weak selection is also plausibly taken to be subject to a principle of Cut (corresponding to Lower and Upper Cut in Definition 2.1). Say that the set of (two-sided) menus G is a strict (or weak) selection from the set of menus $H = \{v_1, v_2, \dots\}$ if G can

[4]One possible application of vector menus is to many valued logics where, for each truth-value v, there should be a v-maker. Another possible application is to voting. Suppose n people vote on the options $a_1, a_2, \dots, a_m$. Then the menu in this case is the n-dimensional vector $([a_1 + a_2, \dots, a_m], [a_1 + a_2, \dots, a_m], \dots, [a_1 + a_2, \dots, a_m])$ and an immediate selection is of the form $(a_{k_1}, a_{k_2}, \dots, a_{k_n})$. Of course, the options $a_1, a_2, \dots, a_m$ may themselves take the form of further menus, as when $a_1, a_2, \dots, a_m$ are representatives who must themselves choose among different options.

be split up into subsets $G_1, G_2, \ldots$ such that G_1 is a strict (weak) selection from v_1, G_2 is a strict (weak) selection from $v_2, \ldots$. Thus the menus G must, collectively, be a distributive selection from H. The principle then states:

Cut if G is a weak selection from H and H a strict selection from v then G is a strict selection from v, and if G is a strict selection from H and H a weak selection from v then G is a strict selection from v.

Thus items that are strictly selected from a given item can be replaced by weak selections and items from which a strict selection is made can be replaced by an item from which they are weakly selected – in each case preserving strict selection.

These principles do not, of course, provide us with a definition, or even an implicit definition, of weak selection in terms of strict selection. Indeed, they are compatible with weak and strict selection being the same thing. However, there is a further plausible assumption we may make, which does allow us to define the one in terms of the other. This is the following maximality principle:

> Any items that constitute a strict selection from $[v]$ will constitute a weak selection from v (where the corresponding ground-theoretic principle is that if Δ strictly grounds $\neg\neg A$ then Δ weakly grounds A)

Now we know that v is a strict selection from $[v]$; and so this assumption tells us that v is the maximal such item in the sense that any other items that constitute a strict selection from $[v]$ will constitute a weak selection from it. One cannot do better than v, so to speak, in making a strict selection from $[v]$. The converse of this assumption:

> any menus that constitute a weak selection from v will constitute a strict selection from $[v]$

follows from the other assumptions. For v is a strict selection from $[v]$ and so, given that G is a weak selection from v, it is, by Cut, a strict selection from $[v]$.

On the basis of these assumptions, we are therefore justified in adopting the following definition of weak selection in terms of strict selection:

(W/S) G is a weak selection from v iff G is a strict selection from $[v]$ (or, to put it ground-theoretically, Δ weakly grounds A iff Δ strictly grounds $\neg\neg A$).[5]

There is another assumption that may plausibly be taken to relate weak and strict selection. Say that u is a *weak partial* selection from v if it is one of the items in a weak selection from v and that u is a *strict partial* selection from v if u is a weak partial selection from v but v is not a weak partial selection from v; and say that the weak selection G from v is *irreversible* if v is not a weak partial selection from any item of G. The assumption then states:

---

[5]We should note that this definition of weak ground will imply the purely ground-theoretic definition of weak ground proposed in [7, 52], viz. that Δ weakly grounds A iff Δ, Γ strictly grounds B whenever A, Γ strictly grounds B. For the left-to-right direction of the definition follows from Cut. Suppose now that the right-hand side of the definition holds. Since A strictly grounds $\neg\neg A$, Δ strictly grounds $\neg\neg A$ and so, by (W/S) Δ weakly grounds A. As [4, 16],[5, 727-8] observe, the purely ground-theoretic definition is not compatible with the "flat" semantics that [6] provides for the pure logic of ground.

Irreversibility Any irreversible weak selection is a strict selection (where the corresponding ground-theoretic principle is that any irreversible weak ground is a strict ground).

We might take the converse:

Any strict selection from an item is an irreversible weak selection

as an additional assumption (as in definition 2.1). Alternatively, it might be derived from some further assumptions. For suppose the menus G are a strict selection from v. By the above principle of Subsumption, G is a weak selection from v. Now suppose, for *reductio*, that v is a weak partial selection from some item w in G. By Cut, v is a strict selection, on its own or with other items, from v. But this, given:

Non-Circularity No item is part of a strict selection of itself

is a contradiction.

We are therefore justified in adopting the following definition of strict selection in terms of weak selection:

(S/W) The strict selections are the irreversible weak selections (or, put ground-theoretically, Δ strictly grounds A iff Δ irreversibly weakly grounds A).

Thus, given these various assumptions, weak and strict selection – and also weak and strict ground – will be inter-definable.

There are two other assumptions we will need to make, connecting weak and strict selection to combination and choice:

Maximality

Any items which constitute a strict selection from $[v^0.v^1. \cdots]$ will constitute a weak selection from $v^0, v^1, \ldots$;
Any items which constitute a strict selection from $[v^0 + v^1 + \cdots]$ will constitute a weak selection from some subset of $v^0, v^1, \ldots$.

These assumptions generalize the previous maximality principle for $[v]$ and state, in the case of the combination $[v^0.v^1. \cdots]$, that $v^0, v^1, \ldots$ constitute a maximal strict ground, so that any selection must be "at" or "below" $v^0, v^1, \ldots$, and, in the case of the choice $[v^0 + v^1 + \cdots]$, that the subsets of $v^0, v^1, \ldots$ constitute a maximal strict "cover", so that any selection must be "at" or "below" some of $v^0, v^1, \ldots$.

In the above account of the semantics, we have listed various assumptions which we would like to hold. These, in addition to Maximality, are:

Positive Bias The immediate selections from a given two sided menu are the immediate selections from its first component;
Subsumption Any strict selection is a weak selection;
Cut any weak selection from a strict selection and any strict selection from a weak selection of a given item is a strict selection from that item;
Irreversibility The strict and the irreversible weak selections coincide.

However, we have provided no assurance that these assumptions do, or even can, hold.

It is actually rather easy to provide a model in which they hold. For we might take combinations to be formulas of the form $\bigwedge(A_1, A_2, \ldots, A_n)$ and choices to be formulas of the form $\bigvee(A_1, A_2, \ldots, A_n)$ (for $n \geq 0$ and with $\bigwedge(A_1) = \bigvee(A_1)$ and with G a weak selection from A when it is a strict selection from $\bigwedge(A)$). It is then relatively straightforward to show that the various conditions on selection that we have laid down will be satisfied.

Unfortunately, such a model is not enough for the purposes of establishing completeness, for we need to show that, for any consistent set of ground-theoretic claims, there should be a model in which they are true. It is consistent, for example, to suppose that $(A \wedge A)$, $(A \vee A)$ and $\neg\neg A$ are ground-theoretically equivalent or that there is an infinite descending chain of grounds, with A_2 a strict ground of A_1, A_3 a strict ground of A_2, and so on *ad infinitum*. But neither set of claims can be satisfied in the "canonical" model above. We therefore need to allow for a more flexible conception of propositional identity; and, indeed, a large part of the difficulty in the completeness proof results from our having to show how underlying identities in the combinations and choices are capable of accounting for the required ground-theoretic truths.

Some related semantical approaches are to be found in Krämer [12, 13] and Correia [3]. A detailed comparison of our own semantics with these other approaches is beyond the scope of the present paper. But we should note that there are some significant differences relating to (i) the underlying conception of propositional content, (ii) the semantical treatment of the truth-functional connectives, (iii) the account of strict ground and its relation to weak ground, and (iv) the resulting logic of ground. We should note,in particular, that although these other approaches validate our "minimal" system GG, they validate much more and therefore lack the flexibility of our own semantics.

2 Semantics

We set out the proposed semantics in terms of selection systems, define the notion of a model, the content of a truth-functional formula in a model, and the truth of a grounding claim in a model.

A *selection system* is a triple $\mathfrak{F} = \langle \Sigma, \Pi, F \rangle$, where Σ and Π are each operations on finite sequences (including the empty sequence) of ordered pairs of members of F, taking each such sequence into a member of F, with $\Sigma(\langle v \rangle) = \Pi(\langle v \rangle)$. We use lower case letters 'a'-'g' (sometimes with numerical superscripts) for members of F, lower case letters 'u'-'z' (sometimes with numerical superscripts) for pairs of members of F, and upper case letters 'G'-'K' (sometimes with numerical subscripts or superscripts) for sets of pairs of members of F. Thus, if $G = F \times F$, then $\Sigma, \Pi : G^{<\omega} \rightarrow F$. For a pair v, we write $v_{\oplus}$ for v's first element, and $v_{\ominus}$ for its second element. Intuitively, F is a set of *conditions*, and pairs of such conditions are *contents*. Abusing notation, we indicate unions of sets of contents by comma-separated lists, and we often omit brackets for singletons of contents in these lists. So, for instance, G, H, v is used for $G \cup H \cup \{v\}$.

Write $[v^0 + v^1 + \cdots]$ for $\Sigma(\langle v^0, v^1, \ldots \rangle)$ and $[v^0.v^1. \cdots]$ for $\Pi(\langle v^0, v^1, \ldots \rangle)$. $[v^0 + v^1 + \cdots]$ is the *choice* of $v^0, v^1, \ldots$, and $[v^0.v^1. \cdots]$ the *combination* of

$v^0, v^1, \ldots$. Distinct sequences of contents can be taken by either the combination or choice operations to the same condition; a single sequence can be taken by the two operations, respectively, to either the same condition or to different conditions; and the choice of one sequence can be the very same as the combination of a different one. So, choices and combinations need not be uniquely decomposable into (sequences of) contents. We use '$\ll_{\mathfrak{F}}$' to indicate the relation of *immediate selection* between sets (not sequences) of contents and choices and combinations, where $v^i \ll_{\mathfrak{F}} [v^0 + v^1 + \cdots]$ for each i, and $v, w, \cdots \ll_{\mathfrak{F}} [v.w. \cdots]$ (and that is all). We drop the suffix '$\mathfrak{F}$' on '$\ll_{\mathfrak{F}}$' when it is evident from context (and will likewise drop suffixes on the other notions of selection defined below when no confusion will result). Since the *choice* of a single content v is just the same as the *combination* of v, we denote it by $[v]$, which is neutral between the '$+$' notation for choice and the '.' notation for combination.

Given a selection system $\mathfrak{F} = \langle \Sigma, \Pi, F \rangle$, the relation of *strict selection* $<_{\mathfrak{F}}$ between a set of contents G and a content v is defined inductively in terms of immediate selection. In this definition, the *weak selection* relation $G \leq_{\mathfrak{F}} v$ abbreviates $(\exists d)G <_{\mathfrak{F}} ([v], d)$:

Definition 2.1

1. **Basis:** if $G \ll_{\mathfrak{F}} v_{\oplus}$, then $G <_{\mathfrak{F}} v$;
2. **Ascent:** if $G <_{\mathfrak{F}} w$ and $[w] = v_{\oplus}$, then $G <_{\mathfrak{F}} v$;
3. **Lower Cut:** if $G^0 \leq_{\mathfrak{F}} v^0, G^1 \leq_{\mathfrak{F}} v^1, \ldots, G^n \leq_{\mathfrak{F}} v^n$, and $v^0, v^1, \ldots, v^n <_{\mathfrak{F}} v$, then $G^0, G^1, \ldots, G^n <_{\mathfrak{F}} v$; and
4. **Upper Cut:** if $G^0 <_{\mathfrak{F}} v^0, G^1 <_{\mathfrak{F}} v^1, \ldots, G^n <_{\mathfrak{F}} v^n$ and $v^0, v^1, \ldots, v^n \leq_{\mathfrak{F}} v$, then $G^0, G^1, \ldots, G^n <_{\mathfrak{F}} v$.

Relations of partial selection are defined in terms of $<_{\mathfrak{F}}$:

- $w \preceq_{\mathfrak{F}} v$ iff there is an H such that $w, H \leq_{\mathfrak{F}} v$; and
- $w \prec_{\mathfrak{F}} v$ iff $w \preceq_{\mathfrak{F}} v$ but $v \not\preceq_{\mathfrak{F}} w$.

Let a *covering* of G be a family of sets $G_0, G_1, \ldots$ such that $G = G_0 \cup G_1 \cup \ldots$.

Definition 2.2 A *frame* is a selection system $\mathfrak{F}$ meeting two constraints:

1. **Irreversibility**: $G <_{\mathfrak{F}} v$ iff $G \leq_{\mathfrak{F}} v$ and $(\forall w \in G) v \not\preceq_{\mathfrak{F}} w$; and
2. **Maximality**:
   (a) $G <_{\mathfrak{F}} ([v^0.v^1. \cdots], d)$ only if there is a covering $G_0, G_1, \ldots$ of G such that $G_i \leq_{\mathfrak{F}} v^i$, for each i; and
   (b) $G <_{\mathfrak{F}} ([v^0 + v^1 + \cdots], d)$ only if there is a non-empty subset $w^0, w^1, \ldots$ of $v^0, v^1, \ldots$ and a covering $G_0, G_1, \ldots$ of G such that $G_i \leq_{\mathfrak{F}} w^i$ for each i.

Suppose we are given a propositional language $\mathscr{L}$, whose connectives are conjunction, disjunction, and negation. We will identify $\mathscr{L}$ with the set of its sentences. Let $<$, $\leq$, $\prec$, and $\preceq$ be *fresh* symbols. (That is, they are pairwise distinct from one another and from every sentence of $\mathscr{L}$.) The *grounding claims of* $\mathscr{L}$ then consist of the following:

$$\Delta < \phi \qquad \Delta \leq \phi \qquad \phi \prec \psi \qquad \phi \preceq \psi$$

for any $\Delta \subseteq \mathscr{L}$ and any sentences ϕ, ψ of $\mathscr{L}$. We will continue to use the lower-case Greek letters ϕ, ψ, δ, and θ (sometimes with superscripts) for sentences of $\mathscr{L}$ and upper-case Greek letters Δ, Γ, Σ, and Θ (sometimes with superscripts) for sets of such sentences. The Greek letters σ and τ (sometimes with subscripts) are used for grounding claims of $\mathscr{L}$, and upper-case letters S, T, and U (sometimes with subscripts or superscripts) for sets of grounding claims of $\mathscr{L}$. An *interpretation* for a language $\mathscr{L}$ into a frame $\mathfrak{F} = \langle \Sigma, \Pi, F \rangle$ is a function $\bar{\bar{\cdot}}$ mapping each atomic sentence ϕ in $\mathscr{L}$ to a content $\bar{\bar{\phi}}$. We extend interpretations to molecular sentences by means of the following recursive clauses:

1. $\overline{\overline{\neg\phi}} = (\bar{\bar{\phi}}_{\ominus}, [\bar{\bar{\phi}}])$;
2. $\overline{\overline{(\phi\wedge\psi)}} = ([\bar{\bar{\phi}} . \bar{\bar{\psi}}], [\overline{\overline{\neg\phi}} + \overline{\overline{\neg\psi}}])$; and
3. $\overline{\overline{(\phi\vee\psi)}} = ([\bar{\bar{\phi}} + \bar{\bar{\psi}}], [\overline{\overline{\neg\phi}} . \overline{\overline{\neg\psi}}])$.

We extend the notion of an interpretation to sets of sentences of $\mathscr{L}$ in the standard way: $\bar{\bar{\Delta}} = \{\bar{\bar{\delta}} | \delta \in \Delta\}$.

Definition 2.3 A *model* $\mathfrak{M}$ for a language $\mathscr{L}$ is a tuple $\langle \Sigma, \Pi, F, \bar{\bar{\cdot}} \rangle$, where $\mathfrak{F} = \langle \Sigma, \Pi, F \rangle$ is a frame, and $\bar{\bar{\cdot}}$ is an interpretation for $\mathscr{L}$ into $\mathfrak{F}$.

If $\mathfrak{M} = \langle \Sigma, \Pi, F, \bar{\bar{\cdot}} \rangle$ is a model and $\mathfrak{F}$ is the frame $\langle \Sigma, \Pi, F \rangle$, we write $\leq_{\mathfrak{M}}$ for $\leq_{\mathfrak{F}}$, and, similarly, for the other relations of ground.

Definition 2.4 Let $\mathfrak{M}$ be a model $\langle \Sigma, \Pi, F, \bar{\bar{\cdot}} \rangle$. *Truth in a model* for grounding claims is defined by the following clauses:

1. $\mathfrak{M} \vDash \Delta \leq \phi$ iff $\bar{\bar{\Delta}} \leq_{\mathfrak{M}} \bar{\bar{\phi}}$;
2. $\mathfrak{M} \vDash \Delta < \phi$ iff $\bar{\bar{\Delta}} <_{\mathfrak{M}} \bar{\bar{\phi}}$;
3. $\mathfrak{M} \vDash \phi \preceq \psi$ iff $\bar{\bar{\phi}} \preceq_{\mathfrak{M}} \bar{\bar{\psi}}$; and
4. $\mathfrak{M} \vDash \phi \prec \psi$ iff $\bar{\bar{\phi}} \prec_{\mathfrak{M}} \bar{\bar{\psi}}$.

$S \vDash T$ iff, for every model $\mathfrak{M}$, if $\mathfrak{M} \vDash \sigma$ for each $\sigma \in S$, then $\mathfrak{M} \vDash \tau$, for some $\tau \in T$. So, sets of grounding claims are treated conjunctively on the left-hand side and disjunctively on the right-hand side of $\vDash$. $\mathfrak{M} \vDash S$ iff $\mathfrak{M} \vDash \sigma$, for some $\sigma \in S$.

3 The System GG

We specify the system GG and establish soundness and consistency. The system comprises the following rules and axioms, which inductively define a derivabiliy relation $\Vdash$ among *finite sets* of grounding claims:

Structural rules:

THINNING	If $T \Vdash S$, then $T, T' \Vdash S, S'$
SNIP	If $\sigma, S \Vdash T$ and $S' \Vdash T', \sigma$, then $S, S' \Vdash T, T'$

(In the statement of the structural rules, T' and S' are finite sets of grounding claims. Since $\Vdash$ relates sets, contraction and permutation rules are not needed.)

The Pure Logic of Ground:

IDENTITY	$\sigma \Vdash \sigma$	
SUBSUMPTION	$(\leq / \preceq)$: $\Delta, \phi \leq \psi \Vdash \phi \preceq \psi$	$(< / \leq)$: $\Delta < \phi \Vdash \Delta \leq \phi$
	$(< / \prec)$: $\Delta, \phi < \psi \Vdash \phi \prec \psi$	$(\prec / \preceq)$: $\phi \prec \psi \Vdash \phi \preceq \psi$
TRANSITIVITY	$(\preceq / \preceq)$: $\phi \preceq \psi;\ \psi \preceq \theta \Vdash \phi \preceq \theta$	$(\preceq / \prec)$: $\phi \preceq \psi;\ \psi \prec \theta \Vdash \phi \prec \theta$
IRREVERSIBILITY	$\phi \preceq \psi \Vdash \phi \prec \psi;\ \psi \preceq \phi$	
REFLEXIVITY	$\Vdash \phi \leq \phi$	
NON-CIRCULARITY	$\phi \prec \phi \Vdash \emptyset$	
CUT	$\Delta \leq \phi;\quad \phi, \psi_0, \psi_1, \ldots, \psi_n \leq \psi \quad \Vdash \quad \Delta, \psi_0, \psi_1, \ldots, \psi_n \leq \psi$	
REVERSE SUBSUMPTION	$\phi_0, \phi_1, \ldots, \phi_n \leq \psi;\ \phi_0 \prec \psi;\ \phi_1 \prec \psi;\ \cdots;\ \phi_n \prec \psi \quad \Vdash \quad \phi_0, \phi_1, \ldots, \phi_n < \psi$	

The pure logic differs from Fine's [6] system by the replacement of TRANSITIVITY($\prec$ / $\preceq$) with IRREVERSIBILITY. The latter rule could not be formulated in the system of derivation used in [6], which did not allow derivation of multiple conclusions. TRANSITIVITY($\prec$ / $\preceq$) can be derived from the pure logic above using IRREVERSIBILITY, SUBSUMPTION($\prec$ / $\preceq$), the other TRANSITIVITY rules, and SNIP.

Let $S_0, S_1, \ldots$ be finite sets of grounding claims. Then $S \Vdash (S_0|S_1|\ldots)$ is defined to hold iff $S \Vdash \sigma_0, \sigma_1, \ldots$ for each set $\sigma_o, \sigma_1, \ldots$ such that $\sigma_i \in S_i$. It is easily shown that a model $\mathfrak{M}$ verifies every such set $\sigma_0, \sigma_1, \ldots$ iff, for some S_i, $\mathfrak{M}$ verifies every grounding claim in S_i.

Introduction Rules:

$$\Vdash \phi < \neg\neg\phi$$

$$\Vdash \phi < (\phi \vee \psi) \qquad \Vdash \psi < (\phi \vee \psi)$$

$$\Vdash \phi, \psi < (\phi \wedge \psi)$$

$$\Vdash \neg\phi < \neg(\phi \wedge \psi) \qquad \Vdash \neg\psi < \neg(\phi \wedge \psi)$$

$$\Vdash \neg\phi, \neg\psi < \neg(\phi \vee \psi)$$

Elimination Rules:

$$\Delta < \neg\neg\phi \Vdash \Delta \leq \phi$$

$$\Delta < (\phi{\wedge}\psi) \Vdash (\quad \Delta^0_\phi \leq \phi;\, \Delta^0_\psi \leq \psi \quad | \quad \Delta^1_\phi \leq \phi;\, \Delta^1_\psi \leq \psi \quad | \quad \ldots \quad)$$

$$\Delta < (\phi{\vee}\psi) \Vdash \Delta \leq \phi; \quad \Delta \leq \psi; \quad \Delta < (\phi{\wedge}\psi)$$

$$\Delta < \neg(\phi{\vee}\psi) \Vdash (\quad \Delta^0_\phi \leq \neg\phi;\, \Delta^0_\psi \leq \neg\psi \quad | \quad \Delta^1_\phi \leq \neg\phi;\, \Delta^1_\psi \leq \neg\psi \quad | \quad \ldots \quad)$$

$$\Delta < \neg(\phi{\wedge}\psi) \Vdash \Delta \leq \neg\phi; \quad \Delta \leq \neg\psi; \quad \Delta < (\neg\phi{\wedge}\neg\psi)$$

In the statement of the elimination rules for $\wedge$ and $\neg\vee$, $\langle \Delta^0_\phi, \Delta^0_\psi \rangle, \langle \Delta^1_\phi, \Delta^1_\psi \rangle, \ldots$ are taken to be all of the ordered pairs $\langle \Delta^n_\phi, \Delta^n_\psi \rangle$ for which $\Delta = \Delta^n_\phi \cup \Delta^n_\psi$. For any sets S and T of grounding claims, let $S \vdash T$ iff there are $S' \subseteq S$ and $T' \subseteq T$ such that $S' \Vdash T'$.

Theorem 3.1 *(Soundness) If $S \vdash T$, then $S \vDash T$.*

Proof Suppose $S \vdash T$, and let $\mathfrak{M} = \langle \Sigma, \Pi, F, \bar{\bar{\cdot}} \rangle$ be a model such that $\mathfrak{M} \vDash \sigma$, for each $\sigma \in S$. There are finite subsets S' and T' of S and T, respectively, such that $S' \Vdash T'$. We show that $\mathfrak{M} \vDash T'$ (and hence $\mathfrak{M} \vDash T$) by induction on the derivation of $S' \Vdash T'$. The results in each of the basis cases are easy consequences of D2.1, D2.2, D2.3, and D2.4. We consider the cases of TRANSITIVITY($\preceq$ / $\prec$) and $\wedge$-ELIMINATION by way of illustration.

(Transitivity)($\preceq$ / $\prec$): Suppose $\mathfrak{M} \vDash \phi \preceq \psi$ and $\mathfrak{M} \vDash \psi \prec \theta$. By the definition of $\prec_{\mathfrak{M}}$, $\mathfrak{M} \vDash \psi \preceq \theta$, and so $\bar{\bar{\phi}} \preceq_{\mathfrak{M}} \bar{\bar{\theta}}$. Suppose (for *reductio*) that $\bar{\bar{\theta}} \preceq_{\mathfrak{M}} \bar{\bar{\phi}}$. Then $\bar{\bar{\theta}} \preceq_{\mathfrak{M}} \bar{\bar{\phi}} \preceq_{\mathfrak{M}} \bar{\bar{\psi}}$. But, since $\mathfrak{M} \vDash \psi \prec \theta$, $\bar{\bar{\theta}} \not\preceq_{\mathfrak{M}} \bar{\bar{\psi}}$. $\bot$.

($\wedge$−**Elimination**): Suppose $\mathfrak{M} \vDash \Delta < (\phi{\wedge}\psi)$, so that $\bar{\bar{\Delta}} <_{\mathfrak{M}} \overline{\overline{(\phi{\wedge}\psi)}}$. $\overline{\overline{(\phi{\wedge}\psi)}}_{\oplus} = [\bar{\bar{\phi}}.\bar{\bar{\psi}}]$. By D2.2(MAXIMALITY), there is a covering $\bar{\bar{\Delta}}_\phi, \bar{\bar{\Delta}}_\psi$ of $\bar{\bar{\Delta}}$ such that $\bar{\bar{\Delta}}_\phi \leq_{\mathfrak{M}} \bar{\bar{\phi}}$ and $\bar{\bar{\Delta}}_\psi \leq_{\mathfrak{M}} \bar{\bar{\psi}}$. So, $\mathfrak{M} \vDash \Delta_\phi \leq \phi$ and $\mathfrak{M} \vDash \Delta_\psi \leq \psi$. Let $(\Gamma^0_\phi, \Gamma^0_\psi), \ldots, (\Gamma^n_\phi, \Gamma^n_\psi)$ be exactly the pairs of binary coverings of Δ. Then $(\Delta_\phi, \Delta_\psi) = (\Gamma^i_\phi, \Gamma^i_\psi)$ for some i $(0 \leq i \leq n)$. As observed when introducing the $|$ notation, it is then easily verified that $\mathfrak{M} \vDash \sigma^0, \ldots, \sigma^n$, for every set $\sigma^0, \ldots, \sigma^n$ of grounding claims such that $\sigma^i \in \{\Gamma^i_\phi \leq \phi \,;\, \Gamma^i_\psi \leq \psi\}$ for each i.

The result in each of the cases of the structural rules is a trivial consequence of IH, using D2.3 and D2.4. □

It turns out not to be altogether straightforward to show that GG is consistent. This could be shown by constructing a 'free' model along the lines of D4.2. But we can also make use of a simpler, less indirect, construction, which will have the additional benefit of presenting the rules in a way that highlights the affinities between GG and more familiar natural deduction systems.[6]

We adopt the following introduction rules for the connectives (where '( )' indicates that either premise may be used):

$$\frac{\phi}{\neg\neg\phi} \qquad \frac{\phi, \psi}{(\phi\wedge\psi)} \qquad \frac{\neg\phi\ (\neg\psi)}{\neg(\phi\wedge\psi)} \qquad \frac{\phi\ (\psi)}{(\phi\vee\psi)} \qquad \frac{\neg\phi, \neg\psi}{\neg(\psi\vee\psi)}$$

These rules correspond, of course, to the Introduction Rules of GG, though they have now been stated as direct rules of inference without the use of $<$.

A *derivation* of the formula ϕ from the set of formulas Δ is a sequence of formulas $\phi_1, \phi_2, \ldots, \phi_n$, where $\phi_n = \phi$ and ϕ_k, for each $k = 1, 2, \ldots, n$, is either a member of Δ or follows from preceding formulas in the sequence by one of the above rules. We should note that each of the formulas ϕ_k in a derivation $\phi_1, \phi_2, \ldots, \phi_n$ will have a *justification* (not necessarily unique), which consists of a status as assumed or derived and a specification, in case it is derived, of the rule by which it is derived. Given a derivation $\phi_1, \phi_2, \ldots, \phi_n$, say that ϕ_k *figures as a premise* if, for some $m > k$, ϕ_k/ϕ_m is an instance of one of the above rules or if, for some $m > k$ and $l < m$, $\phi_k, \phi_l/\phi_m$ is an instance of one of the above rules. A derivation $\phi_1, \phi_2, ..., \phi_n = \phi$ is said to be *relevant* when each non-terminal formula ϕ_k for $k < n$ figures as a premise in the derivation. The derivation $\phi_1, \phi_2, \ldots, \phi_n = \phi$ of ϕ from Δ is said to be *strict* when it is relevant and when each formula of Δ has a non-terminal occurrence in the derivation and it said to be *weak* when it is relevant and when each formula of Δ has a terminal or non-terminal occurrence in the derivation. So, for instance, p, q is a non-relevant derivation of q from p, q, while $p, q, (p\wedge q)$ is a strict derivation of $(p\wedge q)$ from p, q and also a weak derivation of $(p\wedge q)$ from $p, q, (p\wedge q)$.

Note that a strict derivation may be annotated with justifications for each step in such a way that members of Δ are derived and so do not figure as assumptions. For consider the following derivation $p, q, (p\wedge q), r, (p\wedge q)\wedge r$ of $(p\wedge q)\wedge r$ from $p, q, (p\wedge q), r$. We may here take the third formula $(p\wedge q)$ to be derived from the previous formulas p and q. However, we still have a strict derivation of $(p\wedge q)\wedge r$ from $p, q, (p\wedge q), r$ since $(p\wedge q)$ is used as a premise in deriving $(p\wedge q)\wedge r$. Note also that $p, q, (p\vee q)$ is a strict derivation of $(p\vee q)$ from p, q. Indeed, if $\phi_1, \phi_2, \ldots, \phi_m = \phi$ is a strict (weak) derivation of ϕ from Δ and $\psi_1, \psi_2, \ldots, \psi_n = \phi$ a strict (weak) derivation of ϕ from Γ then $\phi_1, \phi_2, ..., \phi_{m-1}, \psi_1, \psi_2, \ldots, \psi_n$ is a strict (weak) derivation of ϕ from $\Delta \cup \Gamma$; and so Amalgamation can be seen to be built into the definition of derivation.

We say that the formula ϕ is *(strictly, weakly) derivable from* the set of formulas Δ if there exists a (strict, weak) derivation of ϕ from Δ; and we say that ϕ is *strictly*

[6] See [15, 16], and [14] for treatments that draw strong connections between natural deduction systems and the impure logic of ground.

(weakly) partially derivable from ψ if ϕ is strictly (weakly) derivable from a set of formulas Δ that includes ψ.

It will be convenient to use a somewhat stronger notion of partial derivability. Suppose $D = \phi_1, \phi_2, \ldots, \phi_n = \phi$ is an arbitrary sequence of formulas. We say ϕ_k is of direct use in deriving ϕ_m (in the sequence D) if $k < m \leq n$ and either ϕ_k/ϕ_m is an instance of a one-premise rule or, for some $l < m$, $\phi_k, \phi_l/\phi_m$ is an instance of a two-premise rule; and we say ϕ_k is *of (indirect) use in deriving* ϕ_m (in D) if $k < m \leq n$ and there is a sub-sequence $\phi_k = \phi_{k_1}, \ldots, \phi_{k_p} = \phi_m$ in which each non-terminal term ϕ_{k_j} is of direct use in deriving its successor $\phi_{k_{j+1}}$. We may also say that ϕ is *of use in deriving* ψ if $\phi = \phi_k$ is of use in deriving $\psi = \phi_m$ in some sequence in $D = \phi_1, \phi_2, \ldots, \phi_n$.

For later purposes, we note some basic facts about derivations.

Lemma 3.2

1. *If* ϕ *is of use in deriving* ψ *then* ψ *is strictly partially derivable from* ϕ*;*
2. *If* ψ *is strictly partially derivable from* ϕ *then* ϕ *is less complex (contains fewer occurrence of connectives) than* ψ.

Proof An easy induction in each case. □

The converse of (i) also holds though not when made relative to a given sequence. In the sequence $(p \wedge q), r, q, (p \wedge q) \wedge r$, for example, q is not of use in deriving $(p \wedge q) \wedge r$ even though $(p \wedge q) \wedge r$ is strictly partially derivable from q.

Lemma 3.3 *In any relevant derivation* $D = \phi_1, \phi_2, \ldots, \phi_n$*, each* ϕ_k *for* $k < n$ *is of use in deriving* ϕ_n.

Proof Take a ϕ_k for $k < n$ and set $k_1 = k$. Suppose ϕ_k is not of use in deriving ϕ_n in D. Since D is relevant, ϕ_{k_1} figures as a premise and so is of direct use in deriving ϕ_{k_2} for some $k_2 > k_1$. But ϕ_{k_2} cannot be identical to ϕ_n or of use in deriving ϕ_n in D; and so, for some $k_3 > k_2$, ϕ_{k_2} is of use in deriving ϕ_{k_3}. We produce in this way an infinite sequence $\phi_{k_1}, \phi_{k_2}, \ldots$ of members of D which, by the previous lemma, are of increasing complexity. But there are only finitely many formulas in D. $\perp$. □

Lemma 3.4 *Suppose* $D = \phi_1, \phi_2, \ldots, \phi_n = \phi$ *is a derivation of* ϕ *from* Δ*. Let* $D' = \phi_{k_1}, \phi_{k_2}, \ldots, \phi_{k_m}$ *be the subsequence of formulas that are of use in deriving* $\phi_n = \phi$ *in D. Then* D', ϕ *is a weak (relevant) derivation of* ϕ *from* $\Delta \cap \phi_{k_1}, \phi_{k_2}, \ldots, \phi_{k_m}, \phi$.

Proof D' is a derivation of ϕ from Δ and hence from $\Delta \cap \{\phi_{k_1}, \phi_{k_2}, \ldots, \phi_{k_m}\}$, since the justification of any formula of $\phi_1, \phi_2, \ldots, \phi_n$ that is identical to ϕ_n or of use in deriving ϕ_n in D will either be in terms of the formula being a member of Δ or by reference to previous formulas that are of use in deriving ϕ_n and hence the justification will carry over to D', ϕ. Moreover, D', ϕ is relevant. For any formula ϕ_{k_j} is of use in deriving ϕ_n and hence figures as a premise. □

Lemma 3.5 *Suppose that $\phi_1, \phi_2, \ldots, \phi_n$ is a relevant derivation. Then each ϕ_k for $k < n$ is distinct from ϕ_n.*

Proof By L3.3, each ϕ_k for $k < n$ is of use in deriving ϕ_n and so, by L3.2, is distinct from ϕ_n. □

We now introduce a notion of *L-truth* for grounding claims:

$\phi_1, \phi_2, \ldots, \phi_n < \phi$ is L-true if ϕ is strictly derivable from $\phi_1, \phi_2, \ldots, \phi_n$;
$\phi_1, \phi_2, \ldots, \phi_n \leq \phi$ is L-true if ϕ is weakly derivable from $\phi_1, \phi_2, \ldots, \phi_n$;
$\phi \prec \psi$ is L-true if ψ is strictly partially derivable from ϕ;
$\phi \preceq \psi$ is L-true if ψ is weakly partially derivable from ϕ.

In effect, we interpret 'ground' to mean ground in virtue of logical form.

Finally, given two sets of grounding claims T and S, we say that the sequent $T \vdash S$ is *valid* if either a member of T is not L-true or a member of S is L-true. This is a very weak "material" interpretation of validity. The sequent $T, p < q \vdash S$, for example, will always be valid since $p < q$ is not L-true.

We can establish by induction:

Theorem 3.6 *Each derivable sequent of GG is valid.*

Proof We show that the axioms of GG are valid and that the rules of inference of GG preserve validity. THINNING,SNIP, and IDENTITY follow by truth-functional considerations alone. SUBSUMPTION, REFLEXIVITY, and IRREVERSIBILITY follow straightforwardly from the definitions of (full) derivability and partial derivability. The introduction rules fall out from our having adopted the corresponding introduction rules.

(Reverse Subsumption): Suppose $\psi_1, \psi_2, \ldots, \psi_m = \psi$ is a weak derivation of ψ from $\Delta = \{\phi_1, \phi_2, \ldots, \phi_n\}$ and that ψ is strictly partially derivable from each ϕ_k for $k = 1, 2, \ldots, n$. By lemma 1(ii), ψ is distinct from each ϕ_k. But then $\psi_1, \psi_2, \ldots, \psi_m$ is a strict derivation of ψ from Δ.

(Transitivity)($\preceq$ / $\preceq$): Suppose that $\phi_1, \phi_2, \ldots, \phi_m = \psi$ is a weak derivation of ψ from Δ with $\phi \in \Delta$ and that $\psi_1, \psi_2, \ldots, \psi_n = \theta$ is a weak derivation of θ from Γ with $\psi \in \Gamma$. If $\phi = \psi$ or if $\psi = \theta$ then it trivially follows that $\phi \preceq \theta$ is L-true. So suppose $\phi \neq \psi$ and $\psi \neq \theta$. Then $\phi_1, \phi_2, \ldots, \phi_m$ is a strict derivation of ψ from $\Delta \setminus \{\psi\}$ with $\phi \in \Delta \setminus \{\psi\}$ and $\psi_1, \psi_2, \ldots, \psi_n$ is a strict derivation of θ from $\Gamma \setminus \{\theta\}$ with $\psi \in \Gamma \setminus \{\theta\}$. But then $\phi_1, \phi_2, \ldots, \phi_m, \psi_1, \psi_2, \ldots, \psi_n$ is a strict derivation of θ from $(\Delta \setminus \{\psi\}) \cup (\Gamma \setminus \{\theta\})$. So, θ is strictly, hence weakly, partially derivable from ϕ. The case ($\preceq$ / $\prec$) is proved similarly.

(Non-Circularity): Suppose $\phi_1, \phi_2, \ldots, \phi_n = \phi$ is a strict derivation of ϕ from Δ. It then follows from lemma 3.5 that $\phi \notin \Delta$.

(Cut): Suppose $\phi_1, \phi_2, \ldots, \phi_m = \phi$ is a weak derivation of ϕ from Δ and $\psi_1, \psi_2, \ldots, \psi_n = \psi$ a weak derivation of ψ from ϕ, Γ. We show $D = \phi_1, \phi_2, \ldots, \phi_{m-1}, \psi_1, \psi_2, \ldots, \psi_n = \psi$ is a weak derivation of ψ from $\Delta \cup \Gamma$. Clearly, D is a derivation of ψ from Δ, Γ, since the occurrences of ϕ among $\psi_1, \psi_2, \ldots, \psi_n$ can be justified by appeal to the previous derivation

$\phi_1, \phi_2, \ldots, \phi_{m-1}$. D is also a relevant derivation. The only problem case for relevance is one in which ϕ_k figures as a premise in $\phi_1, \phi_2, \ldots, \phi_m$ to an inference whose conclusion is $\phi_m = \phi$. But we know that $\phi = \psi_l$ for some l and so we can establish relevance by appeal to ψ_l instead of ϕ_m. Finally, each member of $\Delta \cup \Gamma$ occurs in D since each member of Δ other than ϕ occurs in $\phi_1, \phi_2, \ldots, \phi_{m-1}$, while ϕ and each member of Γ occurs in $\psi_1, \psi_2, \ldots, \psi_n$.

Elimination Rules: We deal with $\wedge$-ELIMINATION by way of illustration. Suppose that $D = \phi_1, \phi_2, \ldots, \phi_n, \phi_{n+1}$ is a strict derivation of $(\phi \wedge \psi)$ from Δ. Then, for some $k, l \leq n$, $\phi_k = \phi$ and $\phi_l = \psi$. Choose k and l to be maximal. This means that one of k or l is n, since otherwise ϕ_n would not figure as a premise. By lemma 3.3, each ϕ_j, $j = 1, 2, \ldots, n$, is of use in deriving $\phi_{n+1} = (\phi \wedge \psi)$ in D. We may then show by an easy induction that each ϕ_j is identical to ϕ or to ψ or of use in deriving $\phi_k = \phi$ or $\phi_l = \psi$. Look now at the sub-sequence $\phi_{p_1}, \phi_{p_2}, \ldots, \phi_{p_k}$ of formulas which can be used in deriving $\phi_k = \phi$ and at the subsequence $\phi_{q_1}, \phi_{q_2}, \ldots, \phi_{q_1}$ of formulas which can be used in deriving $\phi_l = \psi$. Let Δ_1 be the subset of members of Δ that are identical to ϕ_k or are of use in deriving ϕ_k and Δ_2 the subset of members of Δ that are identical to ϕ_l or are of use in deriving ϕ_l. Then $\Delta = \Delta_1 \cup \Delta_2$ and it follows from lemma 3.4 that $\phi_{p_1}, \phi_{p_2}, \ldots, \phi_{p_k}, \phi$ is a weak derivation of ϕ from Δ_1, and $\phi_{q_1}, \phi_{q_2}, \ldots, \phi_{q_1}, \psi$ a weak derivation of ψ from Δ_2. □

Corollary 3.7

1. $\emptyset \vdash \Delta < \phi$ *iff* ϕ *is strictly derivable from* Δ*;*
2. $\emptyset \vdash \Delta \leq \phi$ *iff* ϕ *is weakly derivable from* Δ*;*
3. $\emptyset \vdash \psi \prec \phi$ *iff* ϕ *is strictly partially derivable from* ψ*;*
4. $\emptyset \vdash \psi \preceq \phi$ *iff* ϕ *is weakly partially derivable from* ψ*.*

Proof The right to left directions may be established by induction on the length of the relevant derivations. Suppose now that $\emptyset \vdash \Delta < \phi$. By T3.6, $\Delta < \phi$ is L-true and so ϕ is strictly derivable from Δ. The left to right directions for the other cases are established similarly. □

We also get:

Corollary 3.8 *GG is consistent*

Proof $p < q$ is not L-true and so the sequent $\emptyset \vdash p < q$ is not valid. □

Indeed, we may use the theorem to establish a stronger consistency result. Say that a grounding claim $\phi_1, \phi_2, \ldots, \phi_n < \psi$ is *simple* if $n > 0$ and each of $\phi_1, \phi_2, \ldots, \phi_n, \psi$ is an atom; and say that a set S of grounding claims is *simple* if each of its members is simple. The set S of strict full grounding claims is said to be *closed* if it is closed under CUT (for strict full ground) and AMALGAMATION; and a closed set S of strict full grounding claims is said to be *acyclic* if it does not contain a grounding claim of the form $\Delta < \phi$ with $\phi \in \Delta$. Finally, given a set A of atoms, let S_A be the set of simple grounding claims that can be formed from the members of A.

We may now show that, for closed, acyclic S, we can consistently suppose, not only that every member of S holds, but that the members of S are *exactly* the grounding claims that hold:

Corollary 3.9 *Suppose that S is a closed acyclic set of simple grounding claims formed from the atoms in A. Then the sequent $S \vdash (S_A \setminus S)$ is not derivable in GG.*

Proof It suffices to establish the result for finite S.[7]

Given S, say $p \prec_S q$ if, for some Δ, $p \in \Delta$ and $\Delta < q \in S$. Since S is acyclic, we can assign a depth $d(p)$ to each atom p of A, where $d(p) = 0$ if for no q is $q \prec_S p$ and otherwise $d(p) = max\{d(q) : q \prec_S p\} + 1$. With each atom p of A, we associate a fresh atom p' not in A. We now define a function f from the atoms of A to formulas:

1. when $d(q) = 0$, $f(q) = q$;
2. when $d(q) > 0$, $f(q) =$

   $$(f(p_{1_1}) \wedge f(p_{1_2}) \wedge \ldots \wedge f(p_{1_{k_1}})) \vee \ldots \vee (f(p_{n_1}) \wedge f(p_{n_2}) \wedge \ldots \wedge f(p_{n_{k_n}})) \vee q'$$

   where $\{p_{1_1}, p_{1_2}, \ldots, p_{1_{k_1}}\}, \ldots, \{p_{n_1}, p_{n_2}, \ldots, p_{n_{k_n}}\}$ constitute the Δ for which $\Delta < q \in S$.

To guarantee the uniqueness of $f(q)$ in (2), we suppose that the atoms of A occur in a fixed order and that conjunctions and disjunctions are associated from left to right.

Let $f(\Delta) = \{f(q) : q \in \Delta\}$, and extend f to sets of grounding claims in the obvious way. The function f maps grounding claims in S to L-truths and non-members to non-L-truths, allowing us to bring T3.6 to bear. That is, for any grounding claim $\Delta < q \in S_A$:

$$\Delta < q \in S \text{ iff } f(\Delta) < f(q) \text{ is L-true.}$$

Thus each grounding claim in $f(S)$ is L-true and each grounding claim in $f(S_A \setminus S)$ is not L-true. So the sequent $f(S) \vdash f(S_A \setminus S)$ is not valid and hence, by the theorem, is not derivable in GG. Since GG is closed under uniform substitutions, $S \not\vdash (S_A \setminus S)$. □

It follows, in particular, that the closure of the set $\{p_2 < p_1, p_3 < p_2, \ldots\}$ is consistent. Indeed, we can consistently suppose that these are the only simple grounding claims to hold. It is turtles all the way down!

We might define a sequent $S \vdash T$ to be *super-valid* if every uniform substitution instance of it is valid. So, for example, $p < q \vdash \emptyset$ is valid though not super-valid, since the substitution instance $p < \neg\neg p \vdash \emptyset$ is not valid. By the theorem, the logic of super-validity is at least GG, since GG is closed under uniform substitution. In fact, it properly extends GG since $p \wedge (p \wedge p) \preceq (p \wedge p) \wedge p \vdash \emptyset$ is super-valid and yet not derivable in GG. It would be interesting to determine the logic of super-validity. Indeed, there is a whole range of questions here, since we might add further

[7] See L8.2 below.

principles, such as $\emptyset \vdash (\phi\wedge\psi) \leq (\psi\wedge\phi)$, to GG and then attempt to determine the logic for the resulting notion of super-validity. There is also a connection here with the previously mentioned semantics of Correia [3]; for we might take him to be adopting a substitutional conception of validity under something akin to a free interpretation of the truth-functional formulas.

4 The Canonical Model: Definition and Elucidation

We define and motivate the canonical model that will be used to establish completeness. We first extend the language by adding certain sentences used for the construction; we then define the notion of a "free" condition or content over the resulting set of sentences; and we finally specify the representative conditions in terms of which the canonical model is defined. We close the section by discussing some features of the construction.

In what follows, we will refer to an indexed set using standard notation, writing $(x_i)_{i<\alpha}$ for $\{x_i | i < \alpha\}$. We will almost always omit the limit ordinal α, and we will often write (x_i), omitting the subscripted restriction '$i < \alpha$' entirely. We indicate co-indexed sets by using the same subscripts. Where there are two subscripts, the first may sometimes depend on the second, and these abbreviations may be embedded. Some examples:

Abbreviation	**Expansion**
(x_i)	$x_0, x_1, \ldots$
$(\Delta_i \leq \phi_i)$	$\Delta_0 \leq \phi_0; \Delta_1 \leq \phi_1, \ldots$
(x_{ij})	$x_{00}, x_{10}, \ldots \quad x_{01}, x_{11}, \ldots, \quad x_{0j}, x_{1j}, \ldots, x_{ij}, \ldots, \quad , \ldots$
$((\delta_{ij})_i, \gamma_j)_j$	$\delta_{00}, \delta_{10}, \ldots, \gamma_0, \quad \delta_{01}, \delta_{11}, \ldots, \gamma_1, \quad \ldots$

Suppose S is a set of grounding claims of $\mathscr{L}$ that is prime ($S \vdash T \Rightarrow (\exists\tau \in T)\tau \in S$). Intuitively, if S is prime, then whenever S "takes" the disjunction of a set T of grounding claims to be true, there is also some specific member σ of T that it already "takes" to be true. The primeness of S implies that it is consistent ($S' \nvdash \emptyset$) and that it is closed under derivability (if $S \vdash \sigma$, then $\sigma \in S$). S will remain fixed for the discussion in this section and throughout Sections 5–7. In what follows, we will sometimes justify claims about S's members by appeal to *the closure of* S to indicate the fact that S is closed under derivability. So, for instance, we may say that, if $S \vdash \phi < \psi$, then $S \vdash \phi \prec \psi$ by "the closure of S."

Before describing the construction in detail, it may be useful to give a rough and intuitive outline that makes our proof strategy clearer. We are given a prime, and so consistent, set of grounding claims S. Since our proof of completeness will be Henkin-style, we will ultimately be constructing a model that verifies exactly those grounding claims of $\mathscr{L}$ that are members of S. For this purpose, S may need to be supplemented in a number of ways. First, note that a partial weak grounding claim of the form $\phi \preceq \psi$, intuitively, says that there is some set of contents, Δ which, together with the content of ϕ, fully weakly grounds the content of ψ. S may contain partial weak grounding claims $\phi \preceq \psi$ without containing any full weak grounding claim of the form $\phi, \Delta \leq \psi$ that witnesses this existential generalization. So, whenever $\phi \preceq \psi \in S$, we will add a fresh sentence w^{ψ} to our language for the purpose of adding a full weak grounding claim $\phi, w^{\psi} \leq \psi$ to S. We will call these new sentences *witnessing constants*.

Second, it would have been convenient if, whenever S had contained a full grounding claim $\Delta < \phi$, S had also contained a corresponding grounding claim $\bigwedge \Delta < \neg\neg\phi$. For then we could have constructed a model $\mathfrak{M}$ in which $[\overline{\overline{\phi}}]$ (i.e., the truth condition for $\neg\neg\phi$) is identified with $[\overline{\overline{\phi}} + \overline{\overline{\bigwedge \Delta}}]$. Then, using the definition of selection in a model, we would have had $\overline{\overline{\bigwedge \Delta}} \leq_{\mathfrak{M}} \overline{\overline{\phi}}$, and so also $\overline{\overline{\Delta}} <_{\mathfrak{M}} \overline{\overline{\phi}}$. Unfortunately, the rules of GG do not guarantee that there is any such conjunction $\bigwedge \Delta$. Our original language does not generally contain multi-grade conjunctions, and, even where the language does contain the relevant conjunction, simply adding the relevant grounding claim will not generally yield a conservative extension of S. Instead, we will expand the language $\mathscr{L}$ to enable us to add the next best thing: a conjunction $v^{\Delta,\phi}$ whose conjuncts include both the members of Δ and some zero-grounded elements z_1, z_2, described below. For this purpose, we must of course expand the language to allow multigrade conjunctions. Then we add $v^{\Delta,\phi} \leq \phi$ to S, and use a construction similar to the one described above to get a model with the selections: $\overline{\overline{\delta}}_0, \overline{\overline{\delta}}_1, \ldots, z_1, z_2 <_{\mathfrak{M}} \overline{\overline{\phi}}$ (where $\Delta = (\delta_i)$); $\emptyset <_{\mathfrak{M}} z_1$; and $\emptyset <_{\mathfrak{M}} z_2$. We can then use ASCENT and UNDER CUT to get the desired selection $\overline{\overline{\delta}}_0, \overline{\overline{\delta}}_1, \cdots <_{\mathfrak{M}} \overline{\overline{\phi}}$.

We cannot add *only* the grounding claim $v^{\Delta,\phi} \leq \phi$ to S. IRREVERSIBILITY requires us also to add either some other weak grounding claim witnessing $\phi \preceq v^{\Delta,\phi}$ or $v^{\Delta,\phi} < \phi$. Adding $v^{\Delta,\phi} < \phi$ would be foolish. Since ϕ might itself have logical structure, MAXIMALITY may then require that we add some weak grounding claims linking, say, $v^{\Delta,\phi}$ and some conjunct of ϕ, requiring us, in turn, to add strict partial grounding claims linking each δ in Δ to that conjunct. Thus we may, again, fail to conservatively extend S. Instead, we satisfy IRREVERSIBILITY by adding a grounding claim that ensures that $\overline{\overline{\phi}} \preceq_{\mathfrak{M}} \overline{\overline{v^{\Delta,\phi}}}$ in the model we construct. That way, IRREVERSIBILITY gets satisfied without having to add any strict partial grounding claim of $\mathscr{L}$. For this purpose, we add to our language a "shadow" $/v^{\Delta,\phi}/$ of $v^{\Delta,\phi}$, and we throw the grounding claim $\phi, /v^{\Delta,\phi}/ \leq v^{\Delta,\phi}$ into S.

We now have $\phi \preceq v^{\Delta,\phi} \preceq \phi$. So, ensuring that we have a conservative extension will require us to distinguish $v^{\Delta,\phi}$ and $v^{\Delta,\psi}$ whenever $\phi \neq \psi$. Otherwise, our additions may yield $\phi \preceq v^{\Delta,\phi} = v^{\Delta,\psi} \preceq \psi$, and so the resulting set of grounding claims will fail to be consistent, much less conservative, when $\psi \prec \phi \in S$. So, the zero-grounded elements we use as conjuncts of $v^{\Delta,\phi}$ must be both zero-grounded and unique for a given sentence ϕ of our original language $\mathscr{L}$. The zero-grounding part is easy: we add to our original language a new sentence $\top^{\wedge}$, whose truth condition is the combination of the empty sequence. Intuitively, $\top^{\wedge}$ is the conjunction of zero sentences. To uniquely mark $v^{\Delta,\phi}$ for each $\phi \in \mathscr{L}$, we also add to our language a "shadow" $/\phi/$. Now, one of the conjuncts z_1 of $v^{\Delta,\phi}$ can be $(\top^{\wedge} \vee /\phi/)$, which will be both zero-grounded and unique to ϕ.

But this addition requires another new conjunct z_2. To illustrate, we may have $\phi < \neg\neg\chi \in S$. After adding $v^{\phi,\neg\neg\chi} \leq \neg\neg\chi$, we will have $\phi, (\top^{\wedge} \vee /\phi/) < \neg\neg\chi$.

Thus, MAXIMALITY requires $(\top^{\wedge}\vee/\phi/) \preceq \chi$. So, we need a way to ensure that we have our marker $(\top^{\wedge}\vee/\phi/)$ as a partial weak ground of χ. For this purpose, we add to our language another new sentence $\top^{\vee}$, which behaves like the disjunction of the markers $(\top^{\wedge}\vee/\psi/)$ for all ψ in our original language $\mathscr{L}$. $\top^{\vee}$ is zero-grounded, and so is suitable as our additional conjunct z_2. Since $\chi < \neg\neg\chi \in S$, the construction we are describing requires that we add $v^{\Delta,\neg\neg\chi} \leq \neg\neg\chi$. Thus, we have

$$\chi, (\top^{\wedge}\vee/\chi/), \top^{\vee} < \neg\neg\chi \quad \vdash \quad \chi, (\top^{\wedge}\vee/\phi/) < \neg\neg\chi \quad \vdash \quad (\top^{\wedge}\vee/\phi/) \preceq \chi$$

as desired.

We have solved the original problem occasioned by the absence from S of a grounding claim $\bigwedge \Delta < \neg\neg\phi$ that would allow us to construct a selection corresponding to a given grounding claim $\Delta < \phi \in S$. But, in doing so, we have created a problem of exactly the same sort. For we have added $\phi, /v^{\Delta,\phi}/ \leq v^{\Delta,\phi}$, which in turn requires that we also add $\phi, /v^{\Delta,\phi}/ < \neg\neg v^{\Delta,\phi}$, and, of course, there is not already a grounding claim $(\phi\wedge/v^{\Delta,\phi}/) \leq \neg\neg v^{\Delta,\phi}$ ($\dashv\vdash$ $(\phi\wedge/v^{\Delta,\phi}/) < \neg\neg\neg\neg v^{\Delta,\phi}$) in our augmented set of grounding claims. But we already know how to solve this problem: we iterate the procedure. This, again, creates a further problem of exactly the same sort. If we iterate out to the limit, then all such problems are solved.

As it turns out, the result of our efforts is a set S^*, which conservatively extends S. We call S^* the *canonical model basis* for S. Like S, S^* will be prime and so consistent. We will then use S to define a selection space (as described above) and an interpretation function whose selections correspond (under the interpretation) to the grounding claims in S^*. We will show that that selection space is a frame, meeting the Irreversibility and Maximality constraints. That frame, together with the interpretation of S^*, is the *canonical model* for S, which verifies exactly the grounding claims of $\mathscr{L}$ that are members of S. As the informal reflections above illustrate, the construction is far from trivial.

We start by extending the language $\mathscr{L}$ to $\mathscr{L}^+$ to include all of the new sentences we need:

Definition 4.1 The *proto-language* $\mathscr{P}^{\mathscr{L}}$ is the smallest set of sentences such that:

1. If ϕ is an atomic sentence of $\mathscr{L}$, then ϕ is an atomic sentence of $\mathscr{P}^{\mathscr{L}}$;
2. If $\phi \in \mathscr{L}$ (whether atomic or molecular), then w^{ϕ} is an atomic sentence of $\mathscr{P}^{\mathscr{L}}$;
3. $\top^{\wedge}$ and $\top^{\vee}$ are each fresh atomic sentences of $\mathscr{P}^{\mathscr{L}}$;
4. if $\phi \in \mathscr{P}^{\mathscr{L}}$, then $/\phi/$ is a sentence of $\mathscr{P}^{\mathscr{L}}$;
5. if $\phi \in \mathscr{P}^{\mathscr{L}}$, then $\neg\phi \in \mathscr{P}^{\mathscr{L}}$; and
6. if $1 \leq n \in \omega$ and $\phi^0, (\phi^i)_{1\leq i\leq n}$ are each sentences of $\mathscr{P}^{\mathscr{L}}$, then $(\phi^0\wedge\phi^1\wedge\ldots\wedge\phi^n)$ and $(\phi^0\vee\phi^1\vee\ldots\vee\phi^n)$ are each sentences of $\mathscr{P}^{\mathscr{L}}$.

The atomic sentences of the language $\mathscr{L}^+$ are the atomic sentences and "shadows" $/\phi/$ of $\mathscr{P}^{\mathscr{L}}$, and $\mathscr{L}^+$ itself is the closure of the atomic sentences of $\mathscr{L}^+$ under negation, multigrade conjunction, and multigrade disjunction, as specified in clauses (5) and (6) above.

Remark We add the witnessing atomic sentences w^{ϕ} for each sentence ϕ of $\mathscr{L}$ (not $\mathscr{L}^{+}$). By contrast, we add atomic sentences $/\phi/$, which in each case is a "shadow" of ϕ, for each sentence (atomic or molecular) of $\mathscr{L}^{+}$.

Remark Notice that clause (6) applies to finite sequences of sentences of length ≥ 2 to yield conjunctions and disjunctions of any finite -arity ≥ 2. The symbols '$\wedge$', '$\vee$', ')', and '(' mentioned in clause D4.1(6) to specify n-ary conjunctions and disjunctions are the very same symbols used in the specification of the original language $\mathscr{L}$. So, for sentences ϕ^0 and ϕ^1 of $\mathscr{L}$, the conjunction $(\phi^0 \wedge \phi^1)$ of our original language $\mathscr{L}$ is the very same string as the binary conjunction $\mathscr{L}^{+}$ specified in D4.1(6) when ϕ^0, (ϕ^i) is just the pair ϕ^0, ϕ^1. Similarly, disjunctions of $\mathscr{L}$ are identical with corresponding binary disjunctions of $\mathscr{L}^{+}$. We do not allow conjunctions with fewer than 2 conjuncts, and, similarly, for disjunctions.

We use the language to specify the elements of our selection space. We start with a space that is "free" of interesting identifications among conditions or contents.

Definition 4.2 The Free Selection Space: Assume that $+$, $.$, $\uparrow$, and the atomic sentences of $\mathscr{L}$ are pair-wise distinct ur-elements. We define the notions of a *free condition* and a *free content* inductively:

1. If ϕ is an atomic sentence of $\mathscr{L}^{+}$, then ϕ and $\neg\phi$ are free conditions.
2. If a and b are free conditions, then the ordered pair (a, b) is a free content.
3. If $X = \langle v, w, \dots \rangle$ is a sequence of free contents of length l ($l \neq 1, l \in \omega$), then $(+, X)$ and $(., X)$ are each free conditions (written $[v + w + \cdots]$ and $[v.w.\cdots]$, respectively, where convenient).
4. If v is a free content, then $(\uparrow, \langle v \rangle)$ is a free condition (written $[v]$).

For any free content $v = (a, b)$, $v_{\oplus} = a$ and $v_{\ominus} = b$.

Remark We may think of $.$, $+$, and $\uparrow$ as operations which take finite sequences X and Y of free contents of appropriate length into the free conditions $(., X)$, $(+, X)$, and $(\uparrow, Y)$, respectively. The operations . and $+$ can each be applied to the null sequence, so that $(., \emptyset)$ and $(+, \emptyset)$ are each free conditions.

Now we link the sentences of $\mathscr{L}^{+}$ to elements of the free selection space.

Definition 4.3 We define a function $\bar{\bar{\cdot}}$ from $\mathscr{L}^{+}$ into the set of free contents recursively as follows:

1. For ϕ atomic, $\bar{\bar{\phi}} = (\phi, \neg\phi)$;
2. $\overline{\overline{\neg\phi}} = (\bar{\bar{\phi}}_{\ominus}.[\bar{\bar{\phi}}])$;
3. $\overline{\overline{(\phi \wedge \psi \wedge \dots)}} = ([\bar{\bar{\phi}}.\bar{\bar{\psi}}.\cdots], [\overline{\overline{\neg\phi}} + \overline{\overline{\neg\psi}} + \cdots])$; and
4. $\overline{\overline{(\phi \vee \psi \vee \dots)}} = ([\bar{\bar{\phi}} + \bar{\bar{\psi}} + \cdots], [\overline{\overline{\neg\phi}}.\overline{\overline{\neg\psi}}.\cdots])$.

We now define a relation $\Rrightarrow$ among sentences of $\mathscr{L}^+$. This relation indicates the new full weak grounding claims linking conjunctions like $v^{\Delta,\phi}$ and the corresponding sentence ϕ that we will be adding to S to yield S^*. As indicated informally above, the process is iterative, so the relation is defined inductively.

Definition 4.4 Fix an enumeration (ϕ^i) of the sentences of $\mathscr{L}$. We take the *natural order* on the sentences of $\mathscr{L}$ to be the corresponding ordering. If $\Delta \subseteq \mathscr{L}$, then we take (δ^i) to be the *natural enumeration* of Δ, *i.e.*, the restriction of the natural order on $\mathscr{L}$ to Δ. If (δ^i) is the natural enumeration of Δ and $\Delta < \phi \in S$, set

$$v^{\Delta,\phi} = (\delta^0 \wedge \delta^1 \wedge \ldots \wedge (\top^\wedge \vee /\phi/) \wedge \top^\vee).$$

We inductively define the relation $\Rrightarrow$ on sentences of $\mathscr{L}^+$ by:

(S): $v^{\Delta,\phi} \Rrightarrow \phi$, if $\Delta < \phi \in S$;
(W): $(\psi \wedge w^\phi) \Rrightarrow \neg\neg\phi$, if $\psi \preceq \phi \in S$;
(Max): $(w^\phi \wedge \phi) \Rrightarrow \neg\neg w^\phi$ for $\phi \in \mathscr{L}$;
(∅) : $(\top^\wedge \wedge (\top^\wedge \vee /\phi/)) \Rrightarrow \top^\vee$, if $\phi \in \mathscr{L}$; and
(Induction): if $\phi \Rrightarrow \psi$, then $(\psi \wedge /\phi/) \Rrightarrow \neg\neg\phi$ and $(\phi \wedge /\phi/) \Rrightarrow \neg\neg/\phi/$.

Remark The definition of $\Rrightarrow$ says nothing in general about arbitrary atomic sentences, negations, conjunctions, or disjunctions. So, many sentences of $\mathscr{L}^+$ appear on neither the RHS nor the LHS of any instance of $\Rrightarrow$.

Remark Recall that $v^{\Delta,\phi}$ is the conjunction that we will use to enable the transparent derivation of $\Delta < \phi$ from $\Delta < v^{\Delta,\phi} \leq \phi \in S^*$. Note that $v^{\Delta,\phi} = v^{\Delta',\phi'}$ iff $\Delta = \Delta'$ and $\phi = \phi'$.

As indicated informally above, whenever $\phi \Rrightarrow \psi$, we will identify the truth-condition for $\neg\neg\psi$ (which is the result $[\bar{\bar{\psi}}]$ of "raising" the truth-condition for ψ) with the truth-condition $[\bar{\bar{\psi}} + \bar{\bar{\phi}}]$ for $(\psi \vee \phi)$. We now define the equivalence relation among free conditions and contents corresponding to this identification.

Definition 4.5 We inductively define a relation $\sim$ as the smallest equivalence relation meeting the following conditions:

($\top^\wedge$): $\langle ., \emptyset \rangle \sim \top^\wedge$;
($\Rrightarrow$): If $\phi \Rrightarrow \psi$, then $[\bar{\bar{\psi}}] \sim [\bar{\bar{\psi}} + \bar{\bar{\phi}}]$;
(Pairing): if $a \sim c$ and $b \sim d$, then $(a, b) \sim (c, d)$;
(Comp): 1. if $(v^i \sim w^i)$, then $[v^0 + v^1 + \cdots] \sim [w^0 + w^1 + \cdots]$;
2. if $(v^i \sim w^i)$, then $[v^0.v^1. \cdots] \sim [w^0.w^1. \cdots]$;
3. if $v \sim w$, then $[v] \sim [w]$.

Remark An immediate import of D4.5($\Rrightarrow$) is that $\overline{\overline{\neg\neg\psi}}_\oplus = [\bar{\bar{\psi}}] \sim [\bar{\bar{\psi}} + \bar{\bar{\phi}}] = \overline{\overline{(\psi \vee \phi)}}_\oplus$, whenever $\phi \Rrightarrow \psi$.

Definition 4.6 The Canonical Model $\mathfrak{M}_S$ is the ordered tuple $\langle F_s, \Sigma_S, \Pi_S, \bar{\cdot} \rangle$ whose elements are defined as follows. Pick a "representative" function g on free conditions a, such that $g(a) \in \{b|a \sim b\}$ and $g(a) = g(b)$ if $a \sim b$. Then:

1. F_S is the range of g.
2. The choice $\Sigma_S(X)$ of any length l sequence of $X = \langle v, w, \dots \rangle$ of members of $F_S \times F_S$ ($l \neq 1, l \in \omega$)is $g([v + w + \cdots])$ (written as $[v + w + \cdots]_g$ when convenient).
3. The combination $\Pi_S(X)$ of any length l sequence of $X = \langle v, w, \dots \rangle$ of members of $F_S \times F_S$ ($l \neq 1, l \in \omega$) is $g([v.w. \cdots])$ (written as $[v.w. \cdots]_g$ when convenient).
4. $\Sigma(\langle v \rangle) = \Pi(\langle v \rangle) = g([v])$ (written as $[v]_g$), for any member v of $F_S \times F_S$.

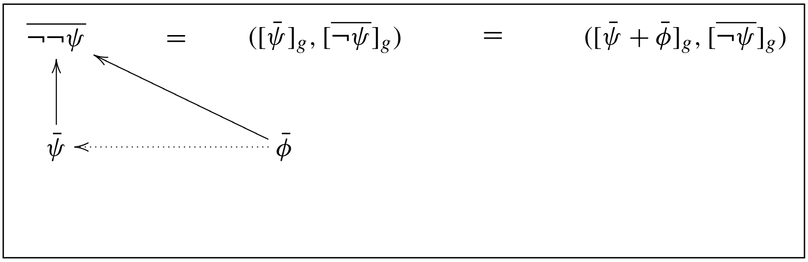
Let $g((a, b)) = (g(a), g(b))$ for all free contents (a, b). Then $\bar{\cdot}$ is the function from $\mathscr{L}^+$ into $F_S \times F_S$ such that $\bar{\phi} = g(\bar{\bar{\phi}})$.

Remark Clearly, since Σ_S and Π_S are defined on all finite sequences of members of $F_S \times F_S$ and $\Sigma\langle v \rangle = \Pi\langle v \rangle = [v]_g$ for all $v \in F_S$, $\mathfrak{M}_S$ is a selection system. The burden of the following three sections is to show that $\mathfrak{M}_S$ is, in fact, a model, thereby meriting the label "canonical model", and that a grounding claim σ of the original language $\mathscr{L}$ is true in $\mathfrak{M}_S$ iff $\sigma \in S$.

Remark $\sim$ is stipulated to be an equivalence relation on free conditions. It then easily follows that it will also be an equivalence relation on free contents. The clauses (PAIRING) and (COMP) in D4.5 will ensure that $\sim$ is a congruence under pairing, choice, and combination.

($\top^\wedge$): This clause will guarantee that $\overline{\top^\wedge_\oplus}$ is equivalent to the combination of nothing (the "zero-combination"). So, $\emptyset <_{\mathfrak{M}_S} \overline{\top^\wedge} <_{\mathfrak{M}_S} \overline{(\top^\wedge \vee/\phi/)}$.

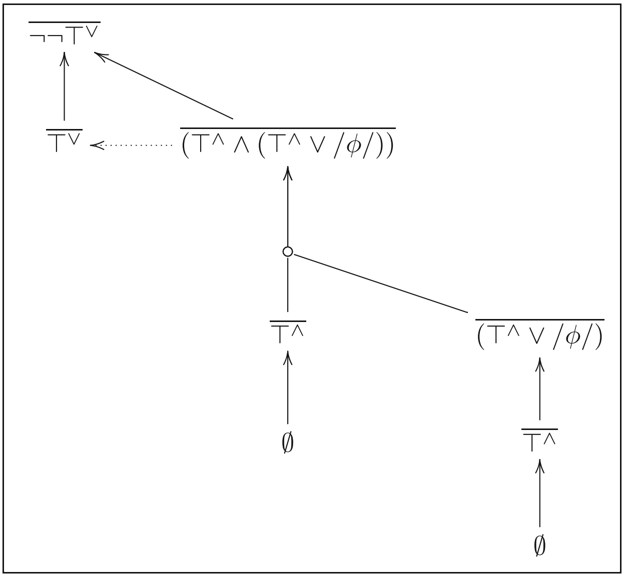
($\Rrightarrow$): This clause is the key to the construction. First, it guarantees that $\phi \Rrightarrow \psi$ implies $\bar{\phi} \leq_{\mathfrak{M}_S} \bar{\psi}$. A picture illustrates the structure:



The solid arrows indicate relations of strict selection. The dotted arrow indicates a relation of weak selection between $\bar{\phi}$ and $\bar{\psi}$, and is warranted by the definition of $\bar{\phi} \leq_{\mathfrak{M}_S} \bar{\psi}$ as $(\exists\, d)\bar{\phi} <_{\mathfrak{M}_S} ([\bar{\psi}]_g, d)$.

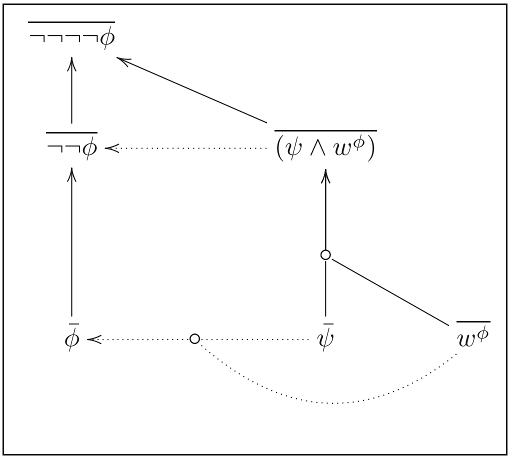
Specific comment is merited on the consequences of the individual clauses in the definition D4.4 of ($\Rrightarrow$). The top two levels of each of the pictures below have the general form indicated in the picture above.

$(\Rrightarrow)(\boldsymbol{\emptyset})$: Since $(\top^\wedge \wedge (\top^\wedge \vee /\phi/)) \Rrightarrow \top^\vee$ for each $\phi \in \mathscr{L}$ we have:



Here, the fact that solid arrows from $\overline{\top^\wedge}$ and $\overline{(\top^\wedge\vee\phi)}$ meet at $\circ$ indicates that they are *jointly* a strict selection from $\overline{(\top^\wedge\wedge(\top^\wedge\vee/\phi/))}$. As the picture indicates, $\emptyset <_{\mathfrak{M}_S} \overline{(\top^\wedge\wedge(\top^\wedge\vee/\phi/))} \leq_{\mathfrak{M}_S} \overline{\top^\vee}$. In effect, as we have said, $\top^\vee$ behaves like the disjunction of all $(\top^\wedge\vee/\phi/)$, for $\phi \in \mathscr{L}$.

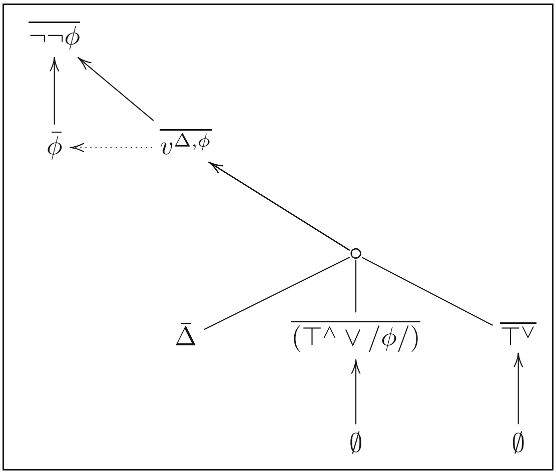
$(\Rrightarrow)(\mathbf{W})$: This clause guarantees that $\bar{\psi}, \overline{w^\phi} \leq_{\mathfrak{M}_S} \bar{\phi}$ whenever $\psi \preceq \phi \in S$:



Here, the dotted arrows from $\bar{\psi}$ and $\overline{w^\phi}$ meet at $\circ$ and continue to $\bar{\phi}$, indicating that $\bar{\psi}$ and $\overline{w^\phi}$ are *jointly* a weak selection from $\bar{\phi}$. This weak selection is guaranteed by the fact that $\bar{\psi}, \overline{w^\phi} <_{\mathfrak{M}_S} \overline{(\psi\wedge w^\phi)} \leq_{\mathfrak{M}_S} \overline{\neg\neg\phi}$. together with the definition of $\bar{\phi} \leq_{\mathfrak{M}_S} \bar{\psi}$ as $(\exists d)\bar{\phi} <_{\mathfrak{M}_S} ([\bar{\psi}]_g, d)$. This ensures that any partial grounding claim $\psi \preceq \phi \in S$ has a corresponding partial weak selection in $\mathfrak{M}_S$.

($\Rrightarrow$)(**Max**): As in the previous case, this clause guarantees that $\overline{w^{\phi}}, \bar{\phi} \leq_{\mathfrak{M}_S} \overline{w^{\phi}}$. The picture above shows that $\bar{\psi}, \overline{w^{\phi}} \leq_{\mathfrak{M}_S} \bar{\phi}$ whenever $\psi \preceq \phi \in S$. IRREVERSIBILITY demands that that either $\bar{\psi}, \overline{w^{\phi}}$ is also a strict selection from $\bar{\phi}$, or that $\bar{\phi}$ is a partial weak selection from one of $\bar{\psi}, \overline{w^{\phi}}$. This clause satisfies IRREVERSIBILITY in this case by guaranteeing the latter alternative. The former alternative needs to be avoided. In particular, we need to avoid the strict selection $\bar{\psi}, \overline{w^{\phi}} <_{\mathfrak{M}_S} \bar{\phi}$, since attempting to meet IRREVERSIBILITY by adding this strict selection might require further additions corresponding to grounding claims that are not in S. Suppose, to illustrate, that $\chi \preceq (\phi \wedge \psi) \in S$, but neither $\chi \preceq \phi$ nor $\chi \preceq \psi$ are in S. If we had (foolishly) attempted to satisfy IRREVERSIBILITY by adding the strict selection $\bar{\chi}, \overline{w^{(\phi\wedge\psi)}} <_{\mathfrak{M}_S} \overline{(\phi\wedge\psi)}$, then MAXIMALITY would require us to add either $\bar{\chi} \preceq_{\mathfrak{M}_S} \bar{\phi}$ or $\bar{\chi} \preceq_{\mathfrak{M}_S} \bar{\psi}$ as well.

($\Rrightarrow$)(**S**): This clause guarantees that $\overline{v^{\Delta,\phi}} \leq_{\mathfrak{M}_S} \bar{\phi}$ whenever $\Delta < \phi \in S$. We have thereby obtained the selection $\bar{\Delta} <_{\mathfrak{M}_S} \overline{v^{\Delta,\phi}} \leq_{\mathfrak{M}_S} \bar{\phi}$ whenever $\Delta < \phi \in S$.



As desired, $v^{\Delta,\phi}$ behaves like the conjunction of the sentences in Δ, except that it has two "zero-grounded" conjuncts $(\top^{\wedge} \vee /\phi/)$ and $\top^{\vee}$. Recall that the inclusion of the "shadow" $/\phi/$ of ϕ as a disjunct in $(\top^{\wedge} \vee /\phi/)$ guarantees that $v^{\Delta,\phi} \neq v^{\Delta,\psi}$ when $\phi \neq \psi$.

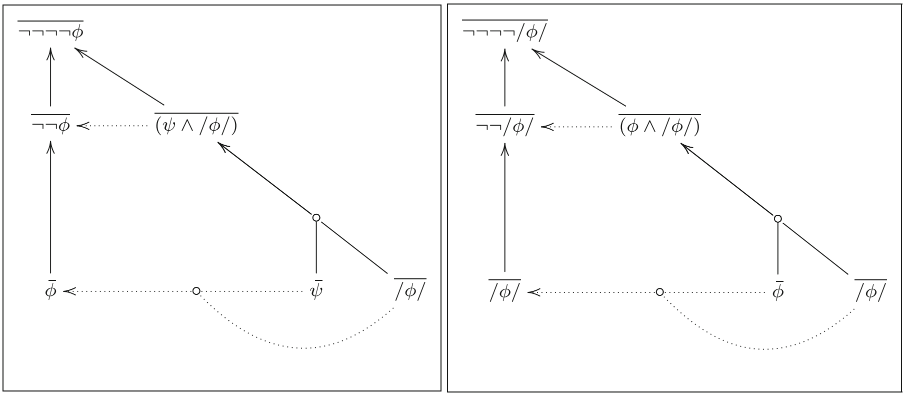
($\Rrightarrow$)(**Induction**): Another function of ($\Rrightarrow$) is to guarantee that

$$\bar{\phi} \preceq_{\mathfrak{M}_S} \bar{\psi} \preceq_{\mathfrak{M}_S} \overline{/\phi/} \preceq_{\mathfrak{M}_S} \bar{\phi}$$

whenever $\phi \Rrightarrow \psi$. The first partial weak selection $\bar{\phi} \preceq_{\mathfrak{M}_S} \bar{\psi}$ is secured immediately, as illustrated by the first picture above.

The other weak selections require us to go up a level. By ($\Rrightarrow$)**(Induction)**, whenever $\phi \Rrightarrow \psi$, we also have $(\psi \wedge /\phi/) \Rrightarrow \neg\neg\phi$. So, $[\overline{\neg\neg\phi}]_g = [\overline{\neg\neg\phi} + \overline{(\psi \wedge /\phi/)}]_g$. Similarly, by ($\Rrightarrow$)**(Induction)**, $(\phi \wedge /\phi/) \Rrightarrow \neg\neg/\phi/$ and so $[\overline{\neg\neg/\phi/}]_g = [\overline{\neg\neg/\phi/} +$

$\overline{(\phi\wedge/\phi/)}]_g$. These two facts secure the partial weak selection relations indicated. Again, pictures summarize the construction:



The partial weak selections

$$\bar{\psi} \preceq_{\mathfrak{M}_S} \bar{\phi} \preceq_{\mathfrak{M}_S} \overline{/\phi/} \preceq_{\mathfrak{M}_S} \bar{\phi}$$

are represented in the bottom rows of the two pictures.

A special case of this circle of partial weak selections is that

$$\bar{\phi} \preceq_{\mathfrak{M}_S} \overline{v^{\Delta,\phi}} \preceq_{\mathfrak{M}_S} \overline{/v^{\Delta,\phi}/} \preceq_{\mathfrak{M}_S} \bar{\phi}.$$

Thus, the weak selection $\overline{v^{\Delta,\phi}} \le_{\mathfrak{M}_S} \bar{\phi}$ is reversible: $\bar{\phi}, \overline{/v^{\Delta,\phi}/} \le_{\mathfrak{M}_S} \overline{v^{\Delta,\phi}}$. This enables $\mathfrak{M}_S$ to simultaneously satisfy (IRREVERSIBILITY) and (MAXIMALITY), as described informally above.

5 Witnessing, $\top^\wedge$, $\top^\vee$, and Theorems in $\mathscr{L}^+$

Recall that S is a prime (and hence consistent) set of grounding claims. We use syntactic methods to extend S. The ultimate goal is to get a well-behaved extension S^* which is prime (in $\mathscr{L}^+$) and whose grounding claims exactly correspond to the selections of $\mathfrak{M}_S$. This extension is described and its adequacy proved in the next two sections. In the present section, we expand S to include:

- full weak grounding claims $\psi, w^\phi \le \phi$ corresponding to the partial grounding claims $\psi \preceq \phi \in S$;
- the grounding claim $\emptyset < \top^\wedge$;
- claims of the form $(\top^\wedge\wedge(\top^\wedge\vee/\phi/)) \le \top^\vee$, for $\phi \in \mathscr{L}$; and
- *theorems* of $\mathscr{L}^+$, *i.e.*, grounding claims of $\mathscr{L}^+$ derivable from the null set: $\phi^i < (\phi^1\vee\phi^2\vee\phi^3\vee\ldots\vee\phi^n)$, $(\neg\phi^i) < \neg(\phi^1\wedge\phi^2\wedge\phi^3\wedge\ldots\wedge\phi^n)$, $/\phi/ \le /\phi/$, and the like.

Adding grounding claims corresponding to instances of ($\Rrightarrow$) will be deferred until the next section.

It is convenient for demonstrating primeness and conservativity to define syntactic objects that, in effect, represent normal forms for derivations in GG of grounding claims from S together with grounding claims for witnessing constants, $\top^{\wedge}$, and $\top^{\vee}$; see D5.3 and D5.6 below. These syntactic objects are called S-derivations.

Definition 5.1 The class of *S-derivations* is given by the following axioms and CUT rule:

(S): If $\Delta \leq \phi \in S$, then $\Delta, \top^{\vee} \leq \phi$ is an axiom;
(W): if $\delta \preceq \phi \in S$, then $\delta, w^{\phi} \leq \phi$ is an axiom;
(Max): $\phi, w^{\phi} \leq w^{\phi}$ is an axiom, for $\phi \in \mathscr{L}$;
($\top^{\wedge}$): $\emptyset \leq \top^{\wedge}$ is an axiom;
($\top^{\vee}$): If $\phi \in \mathscr{L}$, then $(\top^{\wedge}\vee/\phi/) \leq \top^{\vee}$ is an axiom;
(ID): $\phi \leq \phi$ is an axiom;
(Determination): The following are axioms:

$$\phi, \psi, \ldots \leq (\phi\wedge\psi\wedge\ldots) \qquad \phi^{i} \leq (\phi^{0}\vee\phi^{1}\vee\ldots) \qquad \phi \leq \neg\neg\phi$$

$$\neg\phi, \neg\psi, \ldots \leq \neg(\phi\vee\psi\vee\ldots) \qquad \neg\phi^{i} \leq \neg(\phi^{0}\wedge\phi^{1}\wedge\ldots)$$

(Cut): $$\frac{(\Delta^{i} \leq \psi^{i})_{i<n\in\omega} \quad (\psi^{i}), \Gamma \leq \phi}{(\Delta^{i}), \Gamma \leq \phi}$$

If $\Delta \leq \phi$ is the conclusion of an S-derivation, then it is said to be *derivable* or an *S-connection*. We will often simply write $\Delta \leq \psi$ to indicate that $\Delta \leq \psi$ is an S-connection. $(\Delta^{i} \leq \psi^{i})$ are the *minor premises* of the application of (CUT), $(\psi^{i}), \Gamma \leq \phi$ is its *major premise*, (ψ^{i}) are its *cut formulae*, and Γ contains its *side formulae*. The *major premise* of an S-derivation $\mathcal{D}$ that terminates in an application of (CUT) is the major premise of that terminal application, and, similarly, for $\mathcal{D}$'s *minor premises*, *cut formulae*, and *side formulae*. An S-derivation is an *axiom* iff it consists of a single application of an axiom rule.

We will use calligraphic capital letters $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ (sometimes with subscripts or accents) for S-derivations. We will often represent the form of an S-derivation of $\Delta \leq \phi$ that is a subderivation of another S-derivation in tabular form, using

$$\boxed{\frac{\mathcal{D}}{\Delta \leq \phi}}$$

So, for instance, if $\mathcal{D}$ is an axiom $\phi, \psi \leq (\phi\wedge\psi)$, then we may represent $\mathcal{D}$ in tabular form by

$$\boxed{\frac{\mathcal{D}}{\phi, \psi \leq (\phi\wedge\psi)}}$$

D5.2–D5.9 define some notions and establish some facts concerning the application of (CUT) in S-derivations. With the exception of L5.7, these definitions and results do not depend on the particular choice of axioms for S-derivations. Although (CUT) cannot be eliminated from S-derivations, the results show that its application can be severely restricted.

Definition 5.2 The *depth Depth($\mathcal{D}$)* of an S-derivation $\mathcal{D}$ is defined inductively:

1. If $\mathcal{D}$ is an axiom, Depth($\mathcal{D}$) = 1;
2. if $\mathcal{D}$ has the form

$$\dfrac{\left(\dfrac{\mathcal{E}_i}{\Delta_i \leq \phi_i}\right) \quad \dfrac{\mathcal{E}}{(\phi_i), \Gamma \leq \phi}}{(\Delta_i), \Gamma \leq \phi}$$

then Depth($\mathcal{D}$) $= sup((Depth(\mathcal{E}_i)), Depth(\mathcal{E})) + 1$.

Definition 5.3 An S-derivation $\mathcal{D}$ is in *semi-normal form* iff every major premise of every application of (CUT) in $\mathcal{D}$ is an axiom.

Lemma 5.4 ***(Semi-Normal Form Lemma)*** *If $\mathcal{D}$ is an S-derivation of $\Delta \leq \phi$, then there is an S-derivation of $\Delta \leq \phi$ in semi-normal form.*

Proof We prove the result by induction on the depth of S-derivations. It is obvious that every application of (CUT) with more than one minor premise can be split into a series of applications of (CUT) with exactly one minor premise. So, we may assume (wlog) that the terminal instance of (CUT) in $\mathcal{D}$ has exactly one minor premise.

Axioms: Trivially, if $\mathcal{D}$ is an axiom, then it is in semi-normal form.

(Cut): Suppose $\mathcal{D}$ terminates in an application of (CUT). We prove the result by a subsidiary induction on the depth of the S-derivation $\mathcal{F}$ of the major premise of $\mathcal{D}$.

Axioms: Suppose $\mathcal{F}$ is an axiom. By the outermost IH, the S-derivation $\mathcal{E}$ of $\mathcal{D}$'s minor premise is in semi-normal form. So, $\mathcal{D}$ is already in semi-normal form.

(Cut): Let $\mathcal{F}$ be the S-derivation of the major premise in $\mathcal{D}$. Suppose $\Delta \leq \phi$ is the minor premise of $\mathcal{D}$, so that ϕ is the cut formula of $\mathcal{D}$. By the outermost IH, we may assume that $\mathcal{F}$ is semi-normal. (Note that semi-normal derivations will not generally have only one minor premise.) There are three cases: (A) ϕ occurs only as a side formula in $\mathcal{F}$; (B) ϕ occurs only on the left-hand side of some minor premises of $\mathcal{F}$; or (C) ϕ occurs both as a side formula in $\mathcal{F}$ and on the left-hand-side of some minor premises of $\mathcal{F}$.

(A): $\mathcal{D}$ has the form

$$\dfrac{\dfrac{\mathcal{E}}{\Delta \leq \phi} \qquad \dfrac{\left(\dfrac{\mathcal{F}_i^*}{\Gamma_i \leq \gamma_i}\right) \quad (\gamma_i), \phi, \Sigma \leq \psi}{(\Gamma_i), \phi, \Sigma \leq \psi}}{(\Gamma_i), \Delta, \Sigma \leq \psi}$$

where $\mathcal{E}$ and $\mathcal{F}^*$ are each semi-normal. Then

$$\dfrac{\dfrac{\mathcal{E}}{\Delta \leq \phi} \quad \left(\dfrac{\mathcal{F}_i^*}{\Gamma_i \leq \gamma_i}\right) \quad (\gamma_i), \phi, \Sigma \leq \psi}{(\Gamma_i), \Delta, \Sigma \leq \psi}$$

is semi-normal.

(B): $\mathcal{D}$ has the form

$$\dfrac{\dfrac{\mathcal{E}}{\Delta \le \phi} \qquad \dfrac{\left(\dfrac{\mathcal{F}'_j}{\phi, \Gamma_j \le \gamma_j}\right) \quad \left(\dfrac{\mathcal{G}_i}{\Sigma_i \le \chi_i}\right) \quad (\gamma_j), (\chi_i), \Gamma' \le \psi}{\phi, (\Gamma_j), (\Sigma_i), \Gamma' \le \psi}}{\Delta, (\Gamma_j), (\Sigma_i), \Gamma' \le \psi}$$

where $(\phi, \Gamma_j \le \gamma_j)$ are exactly the minor premises of $\mathcal{F}$ with ϕ on the left-hand side, and $(\Sigma_i \le \chi_i)$ are exactly the other minor premises of $\mathcal{F}$. By the outer IH, we may assume (wlog) that $(\mathcal{F}'_j)$, $(\mathcal{G}_i)$ are each semi-normal, and $\mathcal{E}$ is semi-normal. For each j, consider the S-derivation

$$\mathcal{E}^* = \dfrac{\dfrac{\mathcal{E}}{\Delta \le \phi} \qquad \dfrac{\mathcal{F}'_j}{\phi, \Gamma_j \le \gamma_j}}{\Delta, \Gamma_j \le \gamma_j}$$

Notice that Depth($\mathcal{E}^*$) $\le$ Depth($\mathcal{D}$), and Depth($\mathcal{F}'_j$) $<$ Depth($\mathcal{F}$). So, by the inner IH, there is a semi-normal S-derivation $\mathcal{E}'_j$ of $\Delta, \Gamma_j \le \gamma_j$. So,

$$\dfrac{\left(\dfrac{\mathcal{E}'_j}{\Delta, \Gamma_j \le \gamma_j}\right) \quad \left(\dfrac{\mathcal{G}_i}{\Sigma_i \le \chi_i}\right) \quad (\gamma_j), (\chi_i), \Gamma' \le \psi}{\Delta, (\Gamma_j), (\Sigma_i), \Gamma' \le \psi}$$

is semi-normal.

(C): $\mathcal{D}$ has the form

$$\dfrac{\dfrac{\mathcal{E}}{\Delta \le \phi} \qquad \dfrac{\left(\dfrac{\mathcal{F}'_j}{\phi, \Gamma_j \le \gamma_j}\right) \quad \left(\dfrac{\mathcal{G}_i}{\Sigma_i \le \chi_i}\right) \quad (\gamma_j), (\chi_i), \phi, \Gamma' \le \psi}{\phi, (\Gamma_j), (\Sigma_i), \Gamma' \le \psi}}{\Delta, (\Gamma_j), (\Sigma_i), \Gamma' \le \psi}$$

where $(\phi, \Gamma_j \le \gamma_j)$ are exactly the minor premises of $\mathcal{F}$ with ϕ on the left-hand side, and $(\Sigma_i \le \chi_i)$ are exactly the other minor premises of $\mathcal{F}$. As in case (B), we may assume (wlog) that $(\mathcal{G}_J)$ and $\mathcal{E}$ are each semi-normal, and, for each j, there is a semi-normal S-derivation $\mathcal{E}'_j$ of $\Delta, \Gamma_j \le \gamma^j$. So,

$$\dfrac{\dfrac{\mathcal{E}}{\Delta \le \phi} \quad \left(\dfrac{\mathcal{E}'_j}{\Delta, \Gamma_j \le \gamma_j}\right) \quad \left(\dfrac{\mathcal{G}_i}{\Sigma_i \le \chi_i}\right) \quad (\gamma_j), (\chi_i), \phi, \Gamma' \le \psi}{\Delta, (\Gamma_j), (\Sigma_i), \Gamma' \le \psi}$$

is semi-normal. □

Definition 5.5 The *principal connection* of an S-derivation consisting of a single axiom $\Delta \le \phi$ is $\Delta \le \phi$. The *principal connection* of an application of (CUT) is its major premise. The *principal connection* of an S-derivation is the principal connection of its terminal application of an inference rule. An S-connection $\Delta \leq \phi$ is *based on S* iff it is an instance of (S) (i.e. the connection corresponds directly to some member of S), or an instance $\Delta \leq \phi$ of (ID) or (DETERMINATION), where $\Delta, \phi \in \mathscr{L}$.

Definition 5.6 An S-derivation $\mathcal{D}$ is in *normal form* iff it is in semi-normal form and every application of (CUT) in $\mathcal{D}$ with a major premise based on S has no immediate S-subderivation whose principal connection is also based on S.

Intuitively, S-derivations in normal form are semi-normal S-derivations which never use axioms based on S consecutively.

Lemma 5.7 *(Normal Form Lemma) If there is an S-derivation of $\Delta \leq \phi$, then there is an S-derivation of $\Delta \leq \phi$ in normal form.*

Proof We prove the result by induction on the depth of S-derivations $\mathcal{D}$. By L5.4, we may assume (wlog) that $\mathcal{D}$ is semi-normal.

Axioms: Suppose $\mathcal{D}$ is an axiom. Then, trivially, $\mathcal{D}$ is normal.

(Cut): Suppose $\mathcal{D}$ terminates in (CUT). Since $\mathcal{D}$ is semi-normal, its major premise is an axiom. By IH, all proper S-subderivations of $\mathcal{D}$ are normal. So, if the major premise of $\mathcal{D}$ is not based on S, then $\mathcal{D}$ is normal. Suppose, instead, that the major premise of $\mathcal{D}$ is based on S. Then it has the form

$$\frac{(\Delta_i \leq \phi_i) \quad \left(\frac{\mathcal{E}_j}{\Gamma_j \leq \psi_j} \right) \quad \left(\frac{\mathcal{F}_k}{\Sigma_k \leq \delta_k} \right) \quad (\phi_i), (\psi_j), (\delta_k), \Theta \leq \phi}{(\Delta_i), (\Gamma_j), (\Sigma_k), \Theta \leq \phi}$$

where:

- $(\phi_i), (\psi_j), (\delta_k), \Theta \leq \phi$ is based on S;
- $(\Delta_i \leq \phi_i)$ are exactly the minor premises of $\mathcal{D}$ which are axioms based on S;
- $\Gamma_j \leq \psi_j$ are exactly the minor premises of $\mathcal{D}$ derived by a terminal application of (CUT) whose principal connection is based on S; and
- $(\Sigma_k \leq \delta_k)$ are the remaining minor premises of $\mathcal{D}$.

Also, by the closure of S and REFLEXIVITY, $\gamma \leq \gamma \in S$, for each $\gamma \in (\psi_j), (\delta_k), \Theta$. Since $(\Delta_i \leq \phi_i)$ are each based on S, $\Delta_i \setminus \{\top^\vee\} \leq \phi^i \in S$, for each i. So, by the closure of S and CUT (for $\Vdash$, not $\leq$),

$$(\Delta_i), (\psi_j), (\delta_k), \Theta \setminus \{\top^\vee\} \leq \phi \in S.$$

For each j, $\mathcal{E}_j$ has the form

$$\frac{\left(\frac{\mathcal{E}_{lj}}{\Gamma_{lj} \leq \gamma_{lj}} \right)_l \quad (\gamma_{lj})_l, \Gamma'_j \leq \psi_j}{(\Gamma_{lj})_l, \Gamma'_j \leq \psi_j}$$

where $(\gamma_{lj})_l, \Gamma'_j \setminus \{\top^\vee\} \le \psi_j \in S$ and $(\Gamma_{lj})_l, \Gamma'_j \setminus \{\top^\vee\} = \Gamma_j \setminus \{\top^\vee\}$. By IH, we may assume (wlog) that $\mathcal{E}_j$ is in normal form, for each j. So, for each l, j, the principal connection of $\mathcal{E}_{lj}$ is not based on S. So, we have the following members of S: $((\gamma_{lj})_l, \Gamma'_j \setminus \{\top^\vee\} \le \psi_j)_j$, (Δ_i), (ψ_j), (δ_k), $\Gamma \setminus \{\top^\vee\} \le \phi$, and (by the closure of S and REFLEXIVITY) $\gamma \le \gamma$ for each $\gamma \in (\Delta_i), (\delta_k), \Gamma \setminus \{\top^\vee\})$. So, by the closure of S and CUT (for $\Vdash$),

$$(\Delta_i), (\gamma_{lj}), (\Gamma'_j), (\delta_k), \Theta \setminus \{\top^\vee\} \le \phi \in S.$$

So, the S-derivation

$$\dfrac{\left(\dfrac{\mathcal{E}_{lj}}{\Gamma_{lj} \le \gamma_{lj}}\right) \quad \left(\dfrac{\mathcal{F}_k}{\Sigma_k \le \delta_k}\right) \quad (\Delta_i), (\gamma_{lj}), (\Gamma'_j), (\delta_k), \Theta, \top^\vee \le \phi}{(\Delta_i), (\Gamma_{lj}), (\Gamma'_j), (\Sigma_k), \Theta, \top^\vee \le \phi}$$

is in normal form. Since $(\Gamma_{lj})_l, \Gamma'_j \setminus \{\top^\vee\} = \Gamma_j \setminus \{\top^\vee\}$, for each j, this is an S-derivation in normal form of $(\Delta_i), (\Gamma_j), (\Sigma_k), \Theta, \top^\vee \le \phi$. If $\top^\vee \in (\Delta_i), (\Gamma_j), (\Sigma_k), \Theta$, this yields the result. Otherwise, let $\mathcal{D}^\emptyset$ be the normal S-derivation

$$\dfrac{\dfrac{\emptyset \le \top^\wedge \quad \top^\wedge \le (\top^\wedge\vee/\phi/)}{\emptyset \le (\top^\wedge\vee/\phi/)} \qquad (\top^\wedge\vee/\phi/) \le \top^\vee}{\emptyset \le \top^\vee}$$

Then an S-derivation similar to the one above, except with $\mathcal{D}^\emptyset$ used to derive the additional minor premise $\emptyset \le \top^\vee$ of the terminal application of (CUT), is normal and yields the result. □

We now define a way of "telescoping" an S-derivation $\mathcal{D}$, so that, intuitively its applications of (CUT) with more than one minor premise are split up into a series of applications of (CUT), with each having only a single minor premise. Working with "telescoped" S-derivations simplifies some of the proofs in the remainder of this section.

Definition 5.8 If $\mathcal{D}$ is an S-derivation, the result $\mathcal{D}^T$ of *telescoping* $\mathcal{D}$ is defined inductively:

1. If $\mathcal{D}$ is an axiom or $\mathcal{D}$ has the form

$$\dfrac{\dfrac{\mathcal{E}}{\Gamma \le \gamma} \qquad \dfrac{\mathcal{F}}{\gamma, \Sigma \le \phi}}{\Gamma, \Sigma \le \phi}$$

then $\mathcal{D}^T = \mathcal{D}$;

2. If $\mathcal{D}$ has the form

$$\dfrac{\dfrac{\mathcal{E}}{\Gamma \le \gamma} \quad \left(\dfrac{\mathcal{F}^i}{\Delta^i \le \delta^i}\right) \quad \dfrac{\mathcal{G}}{\gamma, (\delta^i), \Sigma \le \phi}}{\Gamma, (\Delta^i), \Sigma \le \phi}$$

and $\mathcal{G}^*$ is the result of telescoping

$$\frac{\left(\dfrac{\mathcal{F}^i}{\Delta^i \leq \delta^i}\right) \qquad \dfrac{\mathcal{G}}{\gamma, (\delta^i), \Sigma \leq \phi}}{\gamma, (\Delta^i), \Sigma \leq \phi}$$

$$\text{then } \mathcal{D}^T = \boxed{\frac{\dfrac{\mathcal{E}^T}{\Gamma \leq \gamma} \qquad \dfrac{\mathcal{G}^*}{\gamma, (\Delta^i), \Sigma \leq \phi}}{\Gamma, (\Delta^i), \Sigma \leq \phi}}$$

Definition 5.9 The *head connection* of an S-derivation $\mathcal{D}$ (Head($\mathcal{D}$)) is defined inductively:

1. If $\mathcal{D}$ is an axiom of the form $\Delta \leq \phi$, then Head($\mathcal{D}$) $= \Delta \leq \phi$; and
2. If $\mathcal{D}$ terminates in an application of (CUT) and $\mathcal{E}$ is the subderivation of $\mathcal{D}$'s major premise, then Head($\mathcal{D}$) =Head($\mathcal{E}$).

Remark Some obvious facts:

1. $\mathcal{D}^T$ and $\mathcal{D}$ have the same conclusion;
2. if $\mathcal{D}^T$ is an S-derivation of $\Delta \leq \phi$, then Head($\mathcal{D}^T$) has the form $\Sigma \leq \phi$.
3. if $\mathcal{D}$ is in semi-normal form, then Head($\mathcal{D}^T$) = the principal connection of $\mathcal{D}$;
4. If $\mathcal{D}$ is semi-normal, and $\mathcal{D}^T$ terminates in an application of (CUT) whose minor premise is $\Gamma \leq \gamma$, then Head($\mathcal{D}^T$) has the form $\gamma, \Delta \leq \phi$.
5. If $\mathcal{D}$ is normal, $\mathcal{D}^T$ terminates in an application of (CUT), and Head($\mathcal{D}^T$) is based on S, then the immediate sub-derivation $\mathcal{E}$ of $\mathcal{D}$'s minor premise is not such that Head($\mathcal{E}$) is also based on S.

For convenience, we will often use the result of "telescoping" normal and semi-normal S-derivations in our proofs. In particular, we will do inductive proofs on the results $\mathcal{D}^T$ of "telescoping" normal S-derivations $\mathcal{D}$, so that, in the induction step, we need consider only applications of (CUT) with a single minor premise.

In the remainder of this section, we show that the set of grounding claims corresponding to S-connections is prime, witnessed, and conservative over S. We read off grounding claims from S-connections in the obvious way:

Definition 5.10 A grounding claim σ of $\mathscr{L}^+$ is $\leq$-*constructible* ($\leq$-con) iff:

1. $\sigma = \Delta \leq \phi$ and $\Delta \leq \phi$ is an S-connection;
2. $\sigma = \delta \preceq \phi$ and $\sigma, \Gamma \leq \delta$ is $\leq$-constructible, for some Γ;
3. $\sigma = \delta \prec \phi$, $\delta \preceq \phi$ is $\leq$-constructible, and $\phi \preceq \delta$ is not $\leq$-constructible; or
4. $\sigma = \Delta < \phi$, $\Delta \leq \phi$ is $\leq$-constructible, and $(\forall \delta \in \Delta)\delta \prec \phi$ is $\leq$-constructible.

Different sentences of $\mathscr{L}^+$ will need to be given different treatment. We have already distinguished witnessing constants and sentences of the original language $\mathscr{L}$. We now define another subclass, the class of *nullities*. Intuitively, nullities are either the zero-grounded sentences we introduced into $\mathscr{L}^+$ or sentences that can be relaced in $\leq$-con grounding claims by those zero-grounded elements. In effect, we will show (see L5.16) that nullities can simply be deleted from the LHS's of S-connections.

Definition 5.11 A sentence ϕ of $\mathscr{L}^+$ is a *nullity* iff $\phi, \Delta \leq \top^\vee$, for some Δ. The set $\mathscr{L}^w$ is the union of the set of sentences of $\mathscr{L}$ with $\{w^\psi | \psi \in \mathscr{L}\}$. The set $\mathscr{L}^0$ is the union of $\mathscr{L}^w$ and the set of nullities.

Notice that $\mathscr{L}^w$ is not a language, since, for instance, $w^\psi \in \mathscr{L}^w$ but $\neg w^\psi \notin \mathscr{L}^w$. Similarly, $\mathscr{L}^0$ is not a language.

It is clear by inspection of the definitions D5.11 of nullities and D5.1 of S-derivations that:

Lemma 5.12

1. *If ϕ is a nullity, and $\delta, \Sigma \leq \phi$, then δ is nullity.*
2. *If $\phi \in \mathscr{L}^0$ and $\delta, \Sigma \leq \phi$, then $\delta \in \mathscr{L}^0$.*

For the purposes of showing that nullities may be removed from the LHS's of S-connections, it is useful to demonstrate strict constraints on the conditions under which they may occur on the RHS's.

Lemma 5.13

1. *If $\delta, \Delta \leq \top^\wedge$, then $\delta = \top^\wedge$;*
2. *If $\delta, \Delta \leq /\phi/$, then $\delta = /\phi/$;*
3. *If $\delta, \Delta \leq (\top^\wedge \vee /\phi/)$, then $\delta = \top^\wedge$ or $\delta = /\phi/$ or $\delta = (\top^\wedge \vee /\phi/)$.*
4. *If $\delta, \Delta \leq \top^\vee$, then $\delta \in \{\top^\vee, \top^\wedge, /\phi/, (\top^\wedge \vee /\phi/)\}$, for some $\phi \in \mathscr{L}$.*
5. *δ is a nullity iff $\delta \in \{\top^\vee, \top^\wedge, /\phi/, (\top^\wedge \vee /\phi/)\}$, for some $\psi \in \mathscr{L}$.*

Proof (1.)-(3.) are easily established by a routine induction on S-derivations. (4.) follows from (1.)-(3.) by a simple induction on S-derivations. (5.) follows from (4.) and D5.1. □

The following two lemmas constrain the form of S-connections containing witnessing constants. In effect, they show that, once witnessing constants occur in an S-connection, that they cannot be cut out. Thus, a witnessing constant is an ineliminable "trace" of an application of (W) in a derivation of an S-connection to a sentence ϕ of $\mathscr{L}$. The first is verified by a straightforward induction on S-derivations of $\Delta \leq w^\psi$:

Lemma 5.14 ***(Persistence Lemma I)*** *If $\Delta \leq w^\psi$, then $w^\psi \in \Delta$.*

Lemma 5.15 ***(Persistence Lemma II)*** *If $\mathcal{D}$ is an S-derivation of $\Gamma \leq \phi$, and the head connection of $\mathcal{D}$ has the form $w^\psi, \Delta \leq \phi$, then $w^\psi \in \Gamma$.*

Proof We prove the result by induction on $\mathcal{D}^T$. Suppose $\mathcal{D}^T$ is an S-derivation of $\Gamma \leq \phi$ with head connection $w^\psi, \Delta \leq \phi$. If $\mathcal{D}^T$ is an axiom, then $w^\psi, \Delta = \Gamma$, so

$w^{\psi} \in \Gamma$. Suppose, then, that $\mathcal{D}^T$ terminates in an application of (CUT), with a major premise of the form $\Delta' \leq \phi$. By IH, $w^{\psi} \in \Delta'$. So, w^{ψ} is either the cut formula or a side formula in $\mathcal{D}^T$. If w^{ψ} is a side formula of $\mathcal{D}$, then $w^{\psi} \in \Gamma$. Suppose, then, that w^{ψ} is the cut formula of $\mathcal{D}^T$. Then the minor premise of $\mathcal{D}^T$ has the form $\Gamma' \leq w^{\psi}$. By L5.14, $w^{\psi} \in \Gamma'$. □

Now we can show that nullities can simply be deleted from the LHS of any S-connection to a sentence $\phi \in \mathscr{L}$. This will help us to show conservativity.

Lemma 5.16 *Suppose* $\Gamma, \Delta \leq \phi$; $\Delta, \phi \subseteq \mathscr{L}$ *(not* $\mathscr{L}^+$*); and* $(\forall \gamma \in \Gamma)\gamma$ *is a nullity. Then* $\Delta \leq \phi \in S$.

Proof We prove the result by induction on S-derivations. By L5.7 we may assume (wlog) that the S-derivation $\mathcal{D}$ of $\Gamma, \Delta \leq \phi$ is in normal form. We prove the result by induction on $\mathcal{D}^T$.

(S): D5.1.
(W): $w^{\psi} \notin \mathscr{L}$ and w^{ψ} is not a nullity. $\perp$.
(Max): $w^{\psi} \notin \mathscr{L}$. $\perp$.
(ID): S is prime + REFLEXIVITY.
($\top^{\wedge}$): $\top^{\wedge} \notin \mathscr{L}$. $\perp$.
($\top^{\vee}$): $\top^{\vee} \notin \mathscr{L}$. $\perp$.
(Determination): S is prime + Introduction Rules + SUBS($<$ / $\leq$).
(Cut): Suppose $\mathcal{D}^T$ terminates in an application of (CUT) of the form

$$\dfrac{\dfrac{\mathcal{E}}{\Theta' \leq \theta} \qquad \dfrac{\mathcal{F}}{\theta, \Theta, \leq \phi}}{\Theta', \Theta \leq \phi}$$

where $\Theta, \Theta' = \Delta, \Gamma$. Since $\phi \in \mathscr{L}$, Head($\mathcal{D}$) =Head($\mathcal{F}$) is either based on S or is an instance of (W). By L5.15(Persistence Lemma II), if Head($\mathcal{D}$) is an instance of (W), then $w^{\psi} \in \Delta, \Gamma$. Since w^{ψ} is neither a nullity nor a sentence of $\mathscr{L}$, $w^{\psi} \notin \Delta, \Gamma$, for any w^{ψ}. So, Head($\mathcal{D}$) must be based on S. So, since $\mathcal{D}$ is normal, Head($\mathcal{E}$) is not based on S. But, also, Head($\mathcal{E}$) has the form $\Sigma \leq \theta$, where $\theta \in \mathscr{L}$ or $\theta = \top^{\vee}$. Suppose (for *reductio*) that $\theta \in \mathscr{L}$. Since Head($\mathcal{E}$) is not based on S and $\theta \in \mathscr{L}$, Head($\mathcal{E}$) cannot be an instance of (S), (ID), or (DETERMINATION). Since $\theta \in \mathscr{L}$, Head($\mathcal{E}$) cannot be an instance of (MAX), (W), ($\top^{\wedge}$), or ($\top^{\vee}$). So, Head($\mathcal{E}$) must be an instance of (W). By L5.15(Persistence Lemma II), if Head($\mathcal{E}$) is an instance of (W), then $w^{\theta} \in \Theta' \subseteq \Delta, \Gamma$. For the same reasons as above, then, Head($\mathcal{E}$) is not an instance of (W). $\perp$. So, $\Theta' \leq \theta$ has the form $\Sigma \leq \top^{\vee}$. By L5.12(1.), $\Theta' \subseteq \Gamma$. So, $\Theta = \Delta, \Gamma'$, where $\Gamma' \subseteq \Gamma$. By IH, $\Delta \leq \phi \in S$. □

If we are to establish the conservativity over S of the $\leq$-con grounding claims, we also need to show that the rules involving witnessing constants don't introduce new grounding claims for the original language. We do this by mapping $\leq$-con grounding claims involving witnessing constants into grounding claims of our original language $\mathscr{L}$, and showing that those grounding claims already belong to S.

Definition 5.17 The *$\mathscr{L}$-reduction* $\phi^{\mathscr{L}}$ of a sentence ϕ of $\mathscr{L}^+$ is the result of replacing each occurence in ϕ of any atom w^χ with χ.

We can now show that grounding claims involving witnessing constants get mapped to members of S.

Lemma 5.18 *If $\delta, \Delta \leq \phi$, and $\delta, \phi \in \mathscr{L}^w$, then $\delta^{\mathscr{L}} \preceq \phi^{\mathscr{L}} \in S$.*

Proof Suppose $\mathcal{D}$ is an S-derivation of $\delta, \Delta \leq \phi$ and $\delta, \phi \in \mathscr{L}^w$. By L5.4(Semi-Normal Form Lemma), we may assume that $\mathcal{D}$ is semi-normal. We prove the result by induction on $\mathcal{D}^T$.

(S): S is prime + SUBSUMPTION($\leq / \preceq$).
(W): The result is trivial if $\psi \preceq \phi \in S$ and $\delta = \psi$. Otherwise $\delta = w^\phi$ and $\phi \preceq \phi \in S$.
(Max): $\delta^{\mathscr{L}} = \phi$ and $\phi \preceq \phi \in S$.
(ID): $\delta^{\mathscr{L}} = \phi$ and $\phi \preceq \phi \in S$.
$\top^\wedge$: $\top^\wedge \notin \mathscr{L}^w$. $\bot$.
($\top^\vee$): $\top^\vee \notin \mathscr{L}^w$. $\bot$.
(Determination): S is prime + SUBSUMPTION($< / \leq$)($\leq / \preceq$).
(Cut): If δ is a side formula of $\mathcal{D}$, then IH implies the result. Suppose, then, that δ is not a side formula of $\mathcal{D}$, and so the minor premise of $\mathcal{D}$ has the form $\delta, \Delta \leq \psi$. Since $\top^\vee, \top^\wedge \notin \mathscr{L}^w$, Head($\mathcal{D}^T$) is an instance of neither ($\top^\vee$) nor ($\top^\wedge$). So, there are two cases: (A) Head($\mathcal{D}^T$) is an instance of (S); or (B) Head($\mathcal{D}^T$) has the form $\psi, \Gamma \leq \phi$, where $\psi, \Gamma \subseteq \mathscr{L}^w$.

- **(A):** Either $\psi \in \mathscr{L}^w$ or $\psi = \top^\vee$. In the former case, IH implies $\delta^{\mathscr{L}} \preceq \psi^{\mathscr{L}} \preceq \phi^{\mathscr{L}} \in S$, and the result follows by the closure of S. If $\psi = \top^\vee$, then, by L5.12(1.) δ is a nullity. So, $\delta \notin \mathscr{L}^w$. $\bot$.
- **(B):** By IH, $\delta^{\mathscr{L}} \preceq \psi^{\mathscr{L}} \preceq \phi^{\mathscr{L}} \in S$, and the result follows by the closure of S.

□

We can now establish conservativity. We do this separately for weak and strict grounding claims.

Lemma 5.19 *For $\Delta, \delta, \phi \subseteq \mathscr{L}$:*

1. *$\Delta \leq \phi$ is $\leq$-constructible iff $\Delta \leq \phi \in S$;*
2. *$\delta \preceq \phi$ is $\leq$-constructible iff $\delta \preceq \phi \in S$;*

Proof

1. $\Rightarrow$: L5.16.
   $\Leftarrow$: $\Delta \leq \phi \in S \xRightarrow{D5.1} \Delta, \top^\vee \leq \phi \xRightarrow{D5.1(\emptyset \leq \top^\vee)} \Delta \leq \phi \xRightarrow{D5.10} \Delta \leq \phi$ is $\leq$-con.

2. $\Rightarrow$: By L5.18, $\delta^{\mathscr{L}} \preceq \phi^{\mathscr{L}} \in S$. Since $\delta, \phi \subseteq \mathscr{L}$, $\phi^{\mathscr{L}} = \phi$ and $\delta^{\mathscr{L}} = \delta$.
   $\Leftarrow$: $\delta \preceq \phi \in S \xRightarrow{D5.1} \delta, w^{\phi} \leq \phi \xRightarrow{D5.10} \delta \preceq \phi$ is $\leq$-con.

□

Lemma 5.20 *Suppose $\delta, \phi \subseteq \mathscr{L}$.*

1. *$\delta \prec \phi$ is $\leq$-constructible iff $\delta \prec \phi \in S$.*
2. *$\Delta < \phi$ is $\leq$-constructible iff $\Delta < \phi \in S$.*

Proof

(1.)$\Leftarrow$: Suppose $\delta \prec \phi \in S$.
$\delta \prec \phi \in S \xRightarrow{S \text{ is prime}} \delta \preceq \phi \in S \xRightarrow{L5.19} \delta \preceq \phi$ is $\leq$-con.
Suppose (for *reductio*) that $\phi \preceq \delta$ is $\leq$-con.
$\phi \preceq \delta$ is $\leq$-con$\xRightarrow{L5.19} \phi \preceq \delta \in S$
$\xRightarrow{\text{S is prime}} \delta \prec \delta \in S \xRightarrow{\text{S is consistent}} \bot$.
(1.)$\Rightarrow$: Suppose $\delta, \phi \subseteq \mathscr{L}$ and $\delta \prec \phi$ is $\leq$-con.

$\delta \prec \phi$ is $\leq$-con$\xRightarrow{D5.10} \delta \preceq \phi$ is $\leq$-con$\xRightarrow{L5.19} \delta \preceq \phi \in S$
$\xRightarrow{S \text{ is prime}} (\delta \prec \phi \in S \quad \vee \quad \phi \preceq \delta \in S)$.
Also, $\phi \preceq \delta \in S \xRightarrow{L5.19} \phi \preceq \delta$ is $\leq$ -con $\xRightarrow{D5.10} \delta \prec \phi$ is not $\leq$-con$\Rightarrow \bot$.
(2.): D5.10, L5.19, (1.), and the clsoure of S.

□

Lemma 5.21 ***(Conservativity)*** *For any grounding claim σ of $\mathscr{L}$, σ is $\leq$-constructible iff $\sigma \in S$.*

Proof L5.19 and L5.20. □

Let Complexity(ϕ) be the standard syntactic complexity function for $\mathscr{L}^+$, so that, *e.g.*, Complexity(ϕ) is less than Complexity($\neg\phi$), Complexity($\phi \vee \psi$), and Complexity($\phi \wedge \psi$). The next lemma says that, for sentences ϕ that are neither nullities nor in $\mathscr{L}$, S-connections $\Delta \leq \phi$ correspond to increasing syntactic complexity. This result is useful for establishing that such connections are irreversible.

Lemma 5.22 *If $\phi \notin \mathscr{L}^0$, $\delta, \Delta \leq \phi$, and $\delta \neq \phi$, then either $(\exists \psi \in \mathscr{L}^0)\delta, \Gamma^1 \leq \psi$ and $\psi, \Gamma^2 \leq \phi$, for some Γ^1, Γ^2, or Complexity(δ) <Complexity(ϕ).*

Proof We prove the result by induction on S-derivations. Suppose $\mathcal{D}$ is an S-derivation of $\delta, \Delta \leq \phi$, $\delta, \phi \notin \mathscr{L}^0$, and $\delta \neq \phi$. If $\mathcal{D}$ is an axiom, it is an instance of (DETERMINATION). It is easy to check in that case that Complexity(δ) < Complexity(ϕ). Suppose that $\mathcal{D}$ terminates in an instance of (CUT). By L5.4 (Semi-Normal Form Lemma), we may assume (wlog) that $\mathcal{D}$ is in semi-normal form. There are only two cases: (A) Head($\mathcal{D}^T$) is an instance of (ID), or (B) Head($\mathcal{D}^T$) is an instance of (DETERMINATION). If δ is a side formula in $\mathcal{D}^T$, then the result follows by the application of IH to the major premise. Suppose, instead, that δ occurs on the LHS of the minor premise of D^T.

(A): The minor premise of $\mathcal{D}^T$ has the form $\delta, \Gamma \leq \phi$. So, the result follows by the application of IH to the minor premise.

(B): All of the cases are proved similarly. We do the case in which ϕ is a conjunction for illustration. Suppose $\phi = (\phi^1 \wedge \phi^2 \wedge \ldots)$. The minor premise of $\mathcal{D}^T$ has the form $\delta, \Gamma \leq \phi^i$. If $\phi^i \in \mathscr{L}^0$, then $\delta, \Gamma \leq \phi^i$ and $\phi^1, \phi^2, \ldots, \phi^i, \ldots \leq \phi$. Otherwise, IH applies to $\delta, \Gamma \leq \phi^i$. Together with D5.1(DETERMINATION) and (CUT), this implies the result. □

We establish irreversibility, and hence the claim that $\leq$-constructibility conforms to the introduction rules of GG.

Lemma 5.23

1. *$\neg\neg\phi, \Gamma \not\leq \phi$;*
2. *$(\phi \wedge \psi \wedge \ldots), \Gamma \not\leq \phi$ and $(\phi \wedge \psi \wedge \ldots), \Gamma \not\leq \psi$ and $\ldots$;*
3. *$(\phi \vee \psi \vee \ldots), \Gamma \not\leq \phi$ and $(\phi \vee \psi \vee \ldots), \Gamma \not\leq \psi$ and $\ldots$;*
4. *$\neg(\phi \wedge \psi \wedge \ldots), \Gamma \not\leq \neg\phi$ and $\neg(\phi \wedge \psi \wedge \ldots), \Gamma \not\leq \neg\psi$ and $\ldots$;*
5. *$\neg(\phi \vee \psi \vee \ldots), \Gamma \not\leq \neg\phi$ and $\neg(\phi \vee \psi \vee \ldots), \Gamma \not\leq \neg\psi$ and $\ldots$.*

Proof The cases are all proved very similarly. We will show the first conjunct of (2), that $(\phi \wedge \psi \wedge \ldots), \Gamma \not\leq \phi$, for illustration. There are two cases: (A) $\phi \in \mathscr{L}^w$ or (B) $\phi \notin \mathscr{L}^w$.

(A): Suppose $(\phi \wedge \psi \wedge \ldots) \notin \mathscr{L}$. Then $(\phi \wedge \psi \wedge \ldots) \notin \mathscr{L}^w$. Nor, by L5.13(5.), is $(\phi \wedge \psi \wedge \ldots)$ a nullity. So, $(\phi \wedge \psi \wedge \ldots) \notin \mathscr{L}^0$. Since $\phi \in \mathscr{L}^0$, the result follows by L5.12(2). Suppose, instead, that $(\phi \wedge \psi \wedge \ldots) \in \mathscr{L}$. Then $(\phi \wedge \psi \wedge \ldots), \Gamma \leq \phi \xRightarrow{L5.18} (\phi \wedge \psi \wedge \ldots) \preceq \phi \in S \xRightarrow{S \text{ is prime}} \bot$.

(B): Suppose (for *reductio*) that $(\phi \wedge \psi \wedge \ldots), \Gamma \leq \phi$. By L5.22, since Complexity$(\phi \wedge \psi \wedge \ldots) \not<$Complexity$(\phi)$, either ϕ is a nullity, or there is a $\chi \in \mathscr{L}^0$ such that $(\phi \wedge \psi \wedge \ldots), \Gamma^1 \leq \chi$ and $\chi, \Gamma^2 \leq \phi$, for some Γ^1, Γ^2. By L5.13, no conjunction is a nullity, and so, by L5.12(1.) and D5.1(DETERMINATION), ϕ is not a nullity. So, $(\phi \wedge \psi \wedge \ldots) \notin \mathscr{L}^0$. By L5.12(2), $(\phi \wedge \psi \wedge \ldots), \Gamma \not\leq \chi$. $\bot$. □

Say that $\Delta \leq \{(\phi^i)\}$ is an S-connection when there is a covering (Δ^i) of Δ such that $(\Delta^i \leq \phi^i)$.

Lemma 5.24 ***(Amalgamation)***

1. *if $\Delta^1 \leq \phi, \ldots, \Delta^n \leq \phi$, then $\Delta^1, \ldots, \Delta^n \leq \phi$.*
2. *If $\Delta \leq \{\phi, \psi, \ldots\}$ and $\Gamma \leq \{\phi, \psi, \ldots\}$, then $\Delta, \Gamma \leq \{\phi, \psi, \ldots\}$.*

Proof By D5.1 the following is an S-derivation if $\mathcal{E}$ and $\mathcal{F}$ are:

$$\dfrac{\dfrac{\mathcal{F}}{\Delta^2 \leq \phi} \qquad \dfrac{\dfrac{\mathcal{E}}{\Delta^1 \leq \phi} \qquad \phi \leq \phi}{\Delta^1, \phi \leq \phi}}{\Delta^1, \Delta^2 \leq \phi}$$

This establishes (1.) by an obvious induction. (2.) follows from (1.) and the definition of a covering. □

L5.25–L5.27 show that the $\leq$-con grounding claims conform to the elimination rules of GG, and hence meet the demands imposed by MAXIMALITY. We start with the case of the newly introduced complex sentences, and deal with the more difficult case of complex sentences in $\mathscr{L}$ later.

Lemma 5.25 *Suppose $\psi \notin \mathscr{L}^w$, $\Delta \leq \psi$ is an S-connection, and $\psi \notin \Delta$.*

1. $\psi = (\phi^1 \wedge \phi^2 \wedge \ldots) \Rightarrow \Delta \leq \{\phi^1, \phi^2, \ldots\}$;
2. $\psi = (\phi^1 \vee \phi^2 \vee \ldots) \Rightarrow (\exists \Sigma \subseteq (\phi^i))\Delta \leq \Sigma$;
3. $\psi = \neg\neg\phi \Rightarrow \Delta \leq \phi$;
4. $\psi = \neg(\phi^1 \vee \phi^2 \vee) \Rightarrow \Delta \leq \{\neg\phi^1, \neg\phi^2, \ldots\}$;
5. $\psi = \neg(\phi^1 \wedge \phi^2 \wedge \ldots) \Rightarrow \Delta \leq \{(\neg\psi^i)\}$, *for some* $(\psi^i) \subseteq (\phi^i)$.

Proof Suppose $\psi \notin \mathscr{L}^w$ and $\Delta \leq \psi$. All of the cases are proved similarly. We do (1.) for illustration. (1.) follows straighforwardly from L5.23 and

$$(\star) \quad \psi = (\phi^1 \wedge \phi^2 \wedge \ldots) \Rightarrow (\Delta = \psi \quad \vee \quad \Delta \setminus \{\psi\} \leq \{\phi^1, \phi^2. \cdots\}$$

We prove ($\star$) by induction on S-derivations of $\Delta \leq \psi$. The cases of the axioms are straightforward. Suppose the S-derivation $\mathcal{D}$ of $\Delta \leq (\phi^1 \wedge \phi^2 \wedge \ldots)$ terminates in an application of (CUT). By L5.4 (Semi-Normal Form Lemma), we may assume (wlog) that $\mathcal{D}$ is in semi-normal form. $\mathcal{D}^T$'s major premise has the form $\theta, \Gamma \leq (\phi^1 \wedge \phi^2 \wedge \ldots)$ and $\mathcal{D}^T$'s minor premise has the form $\Sigma \leq \theta$, where $\Delta = \Sigma, \Gamma$. By IH, either $\theta, \Gamma = (\phi^1 \wedge \phi^2 \wedge \ldots)$ or $\theta, \Gamma \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\} \leq \{\phi^1, \phi^2, \ldots\}$. If $\theta, \Gamma = (\phi^1 \wedge \phi^2 \wedge \ldots)$, then the result follows immediately by IH applied to the minor premise. Suppose $\theta, \Gamma \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\} \leq \{\phi^1, \phi^2, \ldots\}$. $\mathcal{D}^T$'s head connection is an axiom $\Delta' \leq (\phi^1 \wedge \phi^2 \wedge \ldots)$. By D5.1, there are only two cases: (A) $\Delta' = (\phi^1 \wedge \phi^2 \wedge \ldots)$, or (B) $\Delta' = (\phi^i)$.

(A): By D5.3 and D5.8, $\theta = (\phi^1 \wedge \phi^2 \wedge \ldots)$. So, IH applies to the minor premise: either (I) $\Sigma = (\phi^1 \wedge \phi^2 \wedge \ldots)$ or (II) $\Sigma \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\} \leq \{\phi^1, \phi^2, \ldots\}$.

(I): The result follows trivially.

(II) $\theta, \Gamma \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\} = \Gamma \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\}$. By L5.24(AMALGAMATON), since $\Sigma \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\} \leq \{\phi^1, \phi^2, \ldots\}$, $\Sigma, \Gamma \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\} \leq \{\phi^1, \phi^2, \ldots\}$.

(B): Recall that $\theta, \Gamma \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\} \leq \{\phi^1, \phi^2, \ldots\}$. Since $\mathcal{D}$ is semi-normal, $\theta \in (\phi^i)$. By L5.23, $(\phi^1 \wedge \phi^1 \wedge \ldots) \notin \Sigma$. So,

$$(\maltese)\quad \Sigma \cup (\Gamma \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\}) = (\Sigma \cup \Gamma) \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\}.$$

So, for some ϕ^i, we have $\Sigma \leq \phi^i$, and $\phi^i, \Gamma \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\} \leq (\phi^i)$. By ($\maltese$) and an application of CUT, $\Sigma, \Gamma \setminus \{(\phi^1 \wedge \phi^2 \wedge \ldots)\} \leq \{\phi^1, \phi^2\}$. □

Recall that we showed above (L5.14(Persistence Lemma I) and L5.15(Persistence Lemma II)) that w^ϕ was an ineliminable "trace" of an application of (W) in a derivation of an S-connection to a sentence ϕ of $\mathscr{L}$. We now use this to show that whenever we get an S-connection $\Delta \leq \phi$ by successive applications of CUT starting with an S-connection $\Gamma \leq \phi$ based on S, the LHS Δ contains a "trace" of every sentence of $\mathscr{L}$ in Γ. Thus, as we show in the next lemma, Δ can be partitioned into nullities, sentences of $\mathscr{L}$ cut in by (reversible) applications of (W), witnessing constants cut in by (reversible) applications of (W), and members of Γ. This helps us establish (in conjunction with the normal form lemma) that the S-derivation of any irreversible S-connection with a sentence of $\mathscr{L}$ on the RHS has to have "gone through" some irreversible S-connection based on S. So, we can use the fact that S itself conforms to the elimination rules of GG (and so the demands of MAXIMALITY) to show that the set of $\leq$-con grounding claims does, too.

Lemma 5.26 ***(Persistence Lemma III)*** *If $\mathcal{D}$ is an S-derivation of $\Delta \leq \phi$ in normal form, and the head connection $\Gamma \leq \phi$ of $\mathcal{D}^T$ is based on S, then*

$$(\forall \gamma \in \Gamma)(\gamma = \top^\vee \quad \vee \quad \gamma \in \Delta \quad \vee \quad w^\gamma \in \Delta)$$

.

Proof Suppose $\mathcal{D}$ is an S-derivation of $\Delta \leq \phi$ in normal form, and the head connection $\Gamma \leq \phi$ of $\mathcal{D}^T$ is based on S. Suppose $\mathcal{D}$ is an axiom. Then $\Gamma = \Delta$. Suppose, instead, that $\mathcal{D}$ terminates in an application of (CUT). Then $\mathcal{D}^T$ has the form

$$\dfrac{\dfrac{\mathcal{E}}{\Delta' \leq \delta} \qquad \dfrac{\mathcal{F}}{\delta, \Sigma \leq \phi}}{\Delta', \Sigma \leq \phi}$$

where $\Delta = \Delta', \Sigma$ and $\delta \in \Gamma$. Suppose $\gamma \in \Gamma$ and $\gamma \neq \top^\vee$. By IH, $(\exists \gamma' \in \{\gamma, w^\gamma\})\gamma' \in \delta, \Sigma$. Suppose $\gamma' \neq \delta$. Then $\gamma' \in \Sigma \subseteq \Delta', \Sigma$. Suppose, instead, that $\gamma' = \delta$. If $\gamma' = w^\gamma$, then by L5.14(Persistence Lemma I), $w^\gamma \in \Delta' \subseteq \Delta', \Sigma$. So, we may assume that $\gamma' = \gamma \in \mathscr{L}$. Since $\mathcal{D}$ is in normal form, the Head($\mathcal{E}$) is not based on S, So, Head($\mathcal{E}$) is an axiom, not based on S, whose RHS is $\gamma \in \Gamma \subseteq \mathscr{L}$. By D5.1, Head($\mathcal{E}$) has the form $\chi, w^\gamma \leq \delta$. By L5.15 (Persistence Lemma II), $w^\gamma \in \Delta' \subseteq \Delta', \Sigma$. □

Now we can show that the $\leq$-con grounding claims conform to the elimination rules of GG, and thus the demands of MAXIMALITY.

Lemma 5.27 ***(Constructibility Lemma)***

1. *if $\Delta < (\phi^1 \wedge \phi^2 \wedge \ldots)$ is $\leq$-constructible, then there is a covering (Δ^i) of Δ such that $(\Delta^i \leq \phi^i)$ are each $\leq$-constructible.*

2. *if* $\Delta < (\phi^1 \vee \phi^2 \vee \ldots)$ *is* $\leq$*-constructible, then there are* $(\psi^j) \subseteq (\phi^i)$ *and a covering* (Δ^j) *of* Δ *such that* $(\Delta^j \leq \psi^j)$ *are each* $\leq$*-constructible.*
3. *if* $\Delta < \neg\neg\phi$ *is* $\leq$*-constructible, then* $\Delta \leq \phi$ *is* $\leq$*-constructible.*
4. *if* $\Delta < \neg(\phi^1 \vee \phi^2 \vee \ldots)$ *is* $\leq$*-constructible, then then there is a covering* (Δ^i) *of* Δ *such that* $(\Delta^i \leq \neg\phi^i)$ *are each* $\leq$*-constructible.*
5. *if* $\Delta < \neg(\phi^1 \wedge \phi^2 \wedge \ldots)$ *is* $\leq$*-constructible, then there are* $(\psi^j) \subseteq (\phi^i)$ *and a covering* (Δ^j) *of* Δ *such that* $(\Delta^j \leq \neg\psi^j)$ *are each* $\leq$*-constructible.*

Proof All of the cases are proved similarly. We do (1.) for illustration. Suppose $\phi = (\phi^1 \wedge \phi^2 \wedge \ldots)$ and $\Delta < \phi$ is $\leq$-constructible. Then there is an S-derivation $\mathcal{D}$ of $\Delta \leq \phi$. By L5.7 (Normal Form Lemma), we may assume (wlog) that $\mathcal{D}$ is in normal form. There are two cases: (A) $\phi \in \mathscr{L}^w$ or (B) $\phi \notin \mathscr{L}^w$.

(A): Since ϕ is not atomic, $\phi \in \mathscr{L}$. So ϕ is a binary conjunction $(\phi^1 \wedge \phi^2)$. There are two sub-cases: (I) The head connection $\Gamma \leq (\phi^1 \wedge \phi^2)$ of $\mathcal{D}^T$ is based on S, or (II) The head connection $\Gamma \leq (\phi^1 \wedge \phi^2)$ of $\mathcal{D}^T$ has the form $w^{(\phi^1 \wedge \phi^2 \wedge \ldots)}, \chi \leq (\phi^1 \wedge \phi^2 \wedge \ldots)$, for some χ such that $\chi \preceq \phi \in S$.

(I): Take any $\gamma \in \Gamma$. By L5.26(Persistence Lemma III), either $\gamma = \top^\vee$, $\gamma \in \Delta$, or $w^\gamma \in \Delta$. So, by D5.10, since $(\phi^1 \wedge \phi^2), \Theta \not\leq \delta$ for any Θ and any $\delta \in \Delta$, either $\gamma = \top^\vee$ or ($\gamma \in \mathscr{L}$ and $\gamma \prec (\phi^1 \wedge \phi^2)$ is $\leq$-con). Let $\Gamma' = \Gamma \setminus \{\top^\vee\}$, so that $\Gamma = \Gamma', \top^\vee$ and $\Gamma' \subseteq \mathscr{L}$. By D5.1, since $\emptyset \leq \top^\vee$, $\Gamma' \leq \phi$. So, by D5.10, $\Gamma' < \phi$ is $\leq$-con. So, by L5.21(Conservativity), $\Gamma' < \phi \in S$. By the closure of S, there is a covering Γ'^1, Γ'^2 of Γ' such that $(\Gamma'^1 \leq \phi^1) \in S$ and $(\Gamma'^2 \leq \phi^2) \in S$. If $\top^\vee \in \Gamma$, let $\Gamma^1 = \Gamma'^1, \top^\vee$ and $\Gamma^2 = \Gamma'^2, \top^\vee$. Otherwise, let $\Gamma^1 = \Gamma'^1$ and $\Gamma^2 = \Gamma'^2$. Γ^1, Γ^2 is a covering of Γ. We prove by induction on the depth of $\mathcal{D}$ that there is a covering Δ^1, Δ^2 of Δ such that $\Delta^1 \leq \phi^1$ and $\Delta^2 \leq \phi^2$ are each $\leq$-constructible. Suppose $\mathcal{D}$ is an axiom. Then $\Delta = \Gamma$, and by D5.1(S), we have S-derivations $\overline{\Gamma^1, \top^\vee \leq \phi^1}$ and $\overline{\Gamma^2, \top^\vee \leq \phi^2}$. If $\top^\vee \notin \Gamma^i$, by D5.1(Cut), since $\emptyset \leq \top^\vee$, $\Gamma^i \leq \phi^i$ for $i \in \{1, 2\}$. Suppose instead that $\mathcal{D}^T$ terminates in an application of Cut. Then $\mathcal{D}^T$ has the form

$$\dfrac{\dfrac{\mathcal{E}}{\Sigma \leq \gamma} \qquad \dfrac{\mathcal{F}}{\gamma, \Theta \leq (\phi^1 \wedge \phi^2)}}{\Theta, \Sigma \leq (\phi^1 \wedge \phi^2)}$$

By IH, γ, Θ has a covering Θ^1, Θ^2 such that $\Theta^1 \leq \phi^1$ and $\Theta^2 \leq \phi^2$ are each $\leq$-con, and so there are S-derivations $\mathcal{G}^1, \mathcal{G}^2$ of $\Theta^1 \leq \phi^1$ and $\Theta^2 \leq \phi^2$, respectively. There are three cases: (a) $\gamma \in \Theta^1$ and $\gamma \notin \Theta^2$, (b) $\gamma \notin \Theta^1$ and $\gamma \in \Theta^2$, or (c) $\gamma \in \Theta^1$ and $\gamma \in \Theta^2$. The arguments in each case are very similar, so we do (a) for illustration. In this case, $\Theta^1 = \gamma, (\Theta \cap \Theta^1)$, and $\Theta = (\Theta \cap \Theta^1), \Theta^2$. $\Theta^2 \leq \phi^2$ is an S-connection, and the following is an S-derivation:

$$\dfrac{\dfrac{\mathcal{E}}{\Sigma \leq \gamma} \qquad \dfrac{\mathcal{G}^1}{\gamma, (\Theta^1 \cap \Theta) \leq \phi^1}}{\Sigma, (\Theta^1 \cap \Theta) \leq \phi^1}$$

(II): By L5.15 (Persistence Lemma II), $w^{(\phi^1\wedge\phi^2)} \in \Delta$. So, by D5.10, $(\phi^1\wedge\phi^2) \preceq w^{(\phi^1\wedge\phi^2)}$ is not $\leq$-con. By D5.1(MAX), $(\phi^1\wedge\phi^2), w^{(\phi^1\wedge\phi^2)} \leq w^{(\phi^1\wedge\phi^2)}$. $\bot$.

(B): By L5.25, either $\phi \in \Delta$ or $\Delta \leq \{\phi^1, \phi^2, \ldots\}$. Suppose (for *reductio*) that $\phi \in \Delta$. Then, by D5.10, $\phi \prec \phi$ is $\leq$-con. But, by D5.1(ID), $\phi \leq \phi$. $\bot$. □

Definition 5.28 Let the relation $\Vdash^+$ between sets of grounding claims of $\mathscr{L}^+$ be defined by the axioms and rules for GG specified in Section 3, with the following changes:

1. Add axioms

   (⊤^∧): $\Vdash^+ \emptyset \leq \top^\wedge$
   (⊤^∨): If $\phi \in \mathscr{L}$, then $\Vdash^+ (\top^\wedge \vee /\phi/) < \top^\vee$

2. Replace the axiom for ∧-INTRODUCTION with a generalization suitable for finite multigrade conjunctions of $\mathscr{L}^+$:

$$\Vdash^+ (\phi^i) < (\phi^0\wedge\phi^1\wedge\ldots)$$

   and, similarly, replace the introduction rules for ∨, ¬∧, and ¬∨ with generalizations suitable for finite multi-grade conjunctions and disjunctions;
3. Replace the axiom for ∧-ELIMINATION with a generalization suitable for finite multigrade conjunctions of $\mathscr{L}^+$:

$$\Delta < (\phi^0\wedge\phi^1\wedge\ldots) \Vdash^+ (\quad \Delta^{0,0} \leq \phi^0; \Delta^{0,1} \leq \phi^1; \ldots \quad | \quad \Delta^{1,0} \leq \phi^0; \Delta^{1,1} \leq \phi^1; \ldots \quad | \quad \ldots \quad)$$

   and, similarly, replace the elimination rules for ∨, ¬∧, and ¬∨ with generalizations suitable for finite multi-grade conjunctions and disjunctions

Let $S \vdash^+ T$ iff there are $S' \subseteq S$ and $T' \subseteq T$ such that $S' \Vdash^+ T'$. A set S of grounding claims is *prime in* $\mathscr{L}^+$ iff $S \vdash^+ T \Rightarrow (\exists\tau \in T)(\tau \in S)$.

Now we can show that the set of S-constructible grounding claims is prime.

Lemma 5.29 ***(Primeness)*** *If S is a prime set of grounding claims of $\mathscr{L}$, $T^1 \vdash^+ T^2$ and $(\forall\sigma \in T^1)\sigma$ is $\leq$-constructible, then $(\exists\tau \in T^2)\tau$ is $\leq$-constructible.*

Proof Suppose $S^1 \vdash^+ S^2$. Then there are $T^1 \subseteq S^1$ and $T^2 \subseteq T^2$ such that $T^1 \Vdash^+ T^2$. We prove the result by induction on the definition of $T^1 \Vdash^+ T^2$. The basis cases are all easy consequences of D5.10, D5.1, L5.13, L5.23, and L5.27(Constructibility Lemma). We do the cases of TRANSITIVITY(⪯/⪯), (⊤^∨), NON-CIRCULARITY, ∧-INTRODUCTION and ∧-ELIMINATION for illustration.

(Transitivity)(⪯ / ⪯): Suppose $\phi \preceq \psi$ and $\psi \preceq \theta$ are both $\leq$-con. By D5.10, there are $\leq$-connections of the form $\phi, \Sigma \leq \psi$ and $\psi, \Gamma \leq \theta$. By D5.1(CUT) $\phi, \Sigma, \Gamma \leq \theta$ is an $\leq$-connection. So, by D5.10 $\phi \preceq \theta$ is $\leq$-con.
(⊤^∨): $(\top^\wedge \vee /\phi/) \leq \top^\vee$ is $\leq$-con by D5.1 and D5.10. By L5.13(3.), $\top^\vee, \Delta \not\leq (\top^\wedge \vee /\phi/)$, for any Δ. So, $(\top^\wedge \vee /\phi/) < \top^\vee$ is $\leq$-con by D5.10.
(Non-Circularity): $\phi \prec \phi$ is not $\leq$-con by D5.1(ID) and D5.10, since $\phi \leq \phi$.
(∧-Introduction): L5.23 + D5.1(DETERMINATION) + D5.10.

($\wedge$-Elimination): Suppose $\Delta < (\phi^1 \wedge \phi^2 \wedge \ldots)$ is $\leq$-con. By L5.27 (Constructibility Lemma), there is a covering $\{\Delta^1, \Delta^2, \ldots\}$ of Δ such that each member of $\{\Delta^1 \leq \phi^1, \quad \Delta^2 \leq \phi^2, \quad \ldots\}$ is $\leq$-con. Suppose $\Delta < (\phi^1 \wedge \phi^2 \wedge \ldots) \Vdash T$ is an instance of $\wedge$-ELIMINATION. Then T has the form $\sigma^0, \sigma^1, \ldots,$ where $(\langle \Gamma_1^i, \Gamma_2^i, \ldots \rangle)$, are exactly the ordered tuples such that $\Delta = \Gamma_1^i \cup \Delta_1^i \cup \ldots$ and, for each i, $\sigma^i \in \{\Gamma_1^i \leq \phi^1, \quad \Gamma_2^i \leq \phi^2, \quad \ldots\}$. Since $\Delta^1, \Delta^2, \ldots$ is a covering of Δ, $\langle \Delta^1, \Delta^2, \ldots \rangle = \langle \Gamma_1^i, \Gamma_2^i, \ldots \rangle$, for some i. So, $\Gamma_1^i \leq \phi^1, \Gamma_2^i \leq \phi^2, \ldots$ are each $\leq$-con, for some i. So, σ^i is $\leq$-con, for some i.

The induction step involves two cases: THINNING and SNIP. Both cases are very easy. We do SNIP for illustration.

(Snip): Suppose every grounding claim $\sigma' \in S', S''$ is $\leq$-con, $\sigma, S' \Vdash T'$ and $S'' \Vdash T'', \sigma$. By IH, there is a grounding claim $\tau \in T'', \sigma$ such that τ is $\leq$-con. Either $\tau = \sigma$ or $\tau \in T''$ If $\tau = \sigma$, then every member of σ, S' is $\leq$-con, and IH applies to $\sigma, S' \Vdash T'$ to entail that there is a $\tau' \in T'$ such that τ' is $\leq$-con. Otherwise, $\tau \in T''$ and τ is $\leq$-con. So, in each case, there is a $\leq$-con grounding claim that is a member of T', T''. □

Theorem 5.30 ***(Extension Theorem)*** *If S is a prime set of grounding claims of $\mathscr{L}$, then the set S^+ of $\leq$-constructible grounding claims is witnessed and prime in $\mathscr{L}^+$, and for grounding claims σ of $\mathscr{L}$, $\sigma \in S^+ \Leftrightarrow \sigma \in S$.*

Proof S^+ is prime in $\mathscr{L}^+$ by L5.29. S^+ is witnessed by D5.10. By L5.21, for grounding claims σ of $\mathscr{L}$, $\sigma \in S^+ \Leftrightarrow \sigma \in S$. □

6 The Canonical Model Basis

Suppose S is a prime (and so consistent) set of grounding claims in $\mathscr{L}$. Let the language $\mathscr{L}^+$ be the language defined in D4.1. By T5.30, the set S^+ of $\leq$-constructible claims is witnessed, prime in $\mathscr{L}^+$, and conservative over S, *i.e.*, for any grounding claim σ of $\mathscr{L}$, $\sigma \in S^+ \Leftrightarrow \sigma \in S$. However, S^+ leaves us with our initial difficulty for constructing our canonical model: it may contain strict grounding claims $\Delta < \phi$ but no corresponding weak grounding claim $\bigwedge \Delta \leq \phi$. Recall that the relation $\Rrightarrow$ indicates exactly the new full weak grounding claims that we need to add. We are now going to throw those into S and show that the result has the desired properties. That is, we now extend S^+ to include grounding claims arising from our definition of $\Rrightarrow$. The result is the canonical model basis for S. In this section, we show that the canonical model basis is prime, witnessed, and conservative over S. In the next section, we show that the canonical model basis contains exactly those grounding claims which are true in the canonical model.

First, we extend the definition of S-connections to include connections required by $\Rrightarrow$. We define a broader set of S-derivations and the corresponding relation $\leq'$ for a relation containing $\leq$, by adding to the definition D5.1 additional axioms for instances of $\Rrightarrow$:

Definition 6.1

($\Rrightarrow$): If $\phi \Rrightarrow \psi$, then:

$$\phi \leq' \psi \qquad \psi, /\phi/ \leq' \phi \qquad \phi, /\phi/ \leq' /\phi/$$

are each axioms.

Remark Intuitively, $\leq'$ extends $\leq$ by simply throwing in connections corresponding to instances of $\Rrightarrow$ and closing under CUT. The new connections $\psi, /\phi/ \leq' \phi$ and $\phi, /\phi/ \leq' /\phi/$ are added to ensure that all of the full weak grounding claims are reversible, so that IRREVERSIBILITY can be satisfied without adding further strict grounding claims. Trivially, if $\Delta \leq \phi$, then $\Delta \leq' \phi$.

Remark In proofs, we will indicate justifications for particular claims about $\Rrightarrow$ that appeal to clause (S) of D4.4 using the notation $(\Rrightarrow)(S)$, and, similarly, for the other clauses. Likewise, we will indicate justifications for particular claims about $\leq'$ (in the sense of D6.1), using $(\leq')(S)$, and the like. In cases which appeal to D6.1($\Rrightarrow$), we will indicate more specific justification using $(\leq')(\Rrightarrow)(S)$, and, similarly, for other clauses of the definition D4.4 of $\Rrightarrow$. So, for instance, we will say that if $\Delta < \phi \in S$, then $v^{\Delta,\phi} \leq' \phi$ by $(\leq')(\Rrightarrow)(S)$. Finally, we will indicate justification by stacking when convenient, as in

$$v^{\Delta,\phi} \underset{(\Rrightarrow)(\mathrm{S})}{\leq'} \phi.$$

The next lemma establishes some useful properties of $\Rrightarrow$.

Lemma 6.2

1. *If $\phi \Rrightarrow \psi$, then ϕ has the form $(\phi^1 \wedge \phi^2 \wedge \ldots)$.*
2. *If $\phi \Rrightarrow \psi$, then $\phi \notin \mathscr{L}$.*
3. *If $\phi \Rrightarrow \psi$, then $\phi \neq \psi$.*
4. *If $\phi \Rrightarrow \psi$ and $\phi \Rrightarrow \psi$ is an instance of neither* (S) *nor* (∅)*, then ψ has the form $\neg\neg\psi'$.*
5. *If $\phi \Rrightarrow \psi$ and $\psi \in \mathscr{L}$, then $\phi \Rrightarrow \psi$ is an instance of either* (S) *or* (W)
6. *If $\chi \Rrightarrow \theta$ and $\chi \Rrightarrow \theta'$, then $\theta = \theta'$.*

Proof (1)–(5) are proved by routine inductions on $\Rrightarrow$. We also prove (6) by induction on $\Rrightarrow$. All of the basis cases are proved similarly. We prove the basis case (W) for illustration.

(W): Suppose $\psi \preceq \phi \in S$, $\chi = (\psi \wedge w^{\phi})$, and $\theta = \neg\neg\phi$. Suppose also $\chi \Rrightarrow \theta'$. Now, $\chi (= (\psi \wedge w^{\phi}))$ does not have any of the following forms:

$$v^{\Delta,\phi'}, \quad (w^{\phi'} \wedge \phi'), \quad (\top^{\wedge} \wedge (\top^{\wedge} \vee /\phi'/)), \quad (\psi' \wedge /\phi'/), \quad (\psi' \wedge /\phi'/).$$

So, for some ϕ', ψ', $\psi' \preceq \phi' \in S$ and $\chi = (\psi' \wedge w^{\phi'})$ and $\theta' = \neg\neg\phi'$. But then $/\phi/ = /\phi'/$, and so by D4.1 $\phi = \phi'$. So, $\theta = \neg\neg\phi = \neg\neg\phi' = \theta'$.

(Induction)(1): Suppose $\phi \Rrightarrow \psi$, $\chi = (\psi \wedge /\phi/)$, and $\theta = \neg\neg\phi$. Suppose $(\psi \wedge /\phi/) \Rrightarrow \theta'$. As in the case (W) above, χ does not have any of the forms required for $\chi \Rrightarrow \theta'$ to come by any of the basis cases for $\Rrightarrow$. Suppose (for *reductio*) that, for some ϕ', ψ', $\phi' \Rrightarrow \psi'$, $\psi = \phi'$, and $/\phi/ = /\phi'/$, so that $\chi = (\phi' \wedge /\phi'/)$ and $\theta' = \neg\neg/\phi'/$. (Intuitively, we are supposing that $\chi \Rrightarrow \theta'$ comes by the other induction step.) By D4.1, since $/\phi/ = /\phi'/$, $\phi = \phi'$. Since $\phi' = \phi$, by IH, $\psi' = \psi$. *So*, $\phi = \phi' \Rrightarrow \psi' = \psi = \phi$. But, by (3), $\phi \not\Rrightarrow \phi$. $\perp$. So, for some ϕ', ψ', $\phi' \Rrightarrow \psi'$, $\chi = (\psi' \wedge /\phi'/)$, and $\theta' = \neg\neg\phi'$. Then $/\phi'/ = /\phi/$. So, by D4.1, $\phi' = \phi$. So, $\theta = \neg\neg\phi = \neg\neg\phi' = \theta'$.

(Induction)(2): Suppose $\phi \Rrightarrow \psi$, $\chi = (\phi \wedge /\phi/)$, and $\theta = \neg\neg/\phi/$. Suppose $(\phi \wedge /\phi/) \Rrightarrow \theta'$. As in the case (W) above, χ does not have any of the forms required for $\chi \Rrightarrow \theta'$ to come by any of the basis cases for $\Rrightarrow$. As in the previous induction case, $\chi \Rrightarrow \theta'$ cannot come by there being a ϕ' and ψ' such that $\phi' \Rrightarrow \psi'$, where $\psi' = \phi$, $/\phi'/ = /\phi/$, and $\theta' = \neg\neg\phi$. So, for some ϕ', ψ', $\phi' \Rrightarrow \psi'$, $\chi = (\phi' \wedge /\phi'/)$, and $\theta' = \neg\neg/\phi'/$. Then $\phi = \phi'$, and so, by D4.1, $/\phi/ = /\phi'/$. So, $\theta = \neg\neg/\phi/ = \neg\neg/\phi'/ = \theta'$. □

We now define the set of grounding claims corresponding to $\leq'$ in the obvious way. We also define the notion of a super-normal S-derivation, for the purposes of managing notational complexity. Applications of CUT in this sort of S-derivation contain no side-formulae that we need to track as parameters in proofs.

Definition 6.3 Define σ *is* $\leq'$*-constructible* ($\leq'$-con) in a manner similar to D5.10, except using the relation $\leq'$ defined in D6.1, instead of $\leq$. So, for instance, $\Delta \leq \phi$ is $\leq'$-con iff $\Delta \leq' \phi$. Define the notions of *major premise*, *minor premise*, *cut formulae*, *side formulae*, *principal connection*, *semi-normal form*, *normal form*, and $\mathcal{D}^T$ in the obvious ways. Say that $\mathcal{D}$ is in *super-normal form* (or is *super-normal*) iff it is the result of adding minor premises of the form $\phi \leq' \phi$ to an S-derivation in normal form to yield an S-derivation in which no application of CUT has any side formulae.

Remark Super-normal S-derivations just fill out normal S-derivations with identity axioms. To illustrate, if

$$\boxed{\dfrac{\Delta \leq' \phi \qquad \phi, \gamma^1, \gamma^2 \leq' \psi}{\Delta, \gamma^1, \gamma^2 \leq' \psi}}$$ is in normal form, then

$$\boxed{\dfrac{\Delta \leq' \phi \qquad \gamma^1 \leq' \gamma^1 \qquad \gamma^2 \leq' \gamma^2 \qquad \phi, \gamma^1, \gamma^2 \leq' \psi}{\Delta, \gamma^1, \gamma^2 \leq' \psi}}$$ is super-normal.

We can prove a normal form theorem in a way similar to L5.7:

Lemma 6.4 *if* $\Delta \leq' \phi$*, then there is an* S*-derivation of* $\Delta \leq' \phi$ *in normal form.*

We will establish conservativity of the canonical model basis over S^+ by defining a function f that maps the LHS and RHS of an instance of $\Rrightarrow$ (and the "shadow" of the LHS) to the same formula. This function assimilates instances of $\leq'$ required by D6.1($\Rrightarrow$) to instances of $\leq$(ID). Thus, f "undoes" the new connections of ground required by D6.1($\Rrightarrow$). The lemma immediately after the definition of f shows, intuitively, that nothing is thereby lost. Define the function $f : \mathscr{L}^+ \to \mathscr{L}^+$ as follows:

Definition 6.5

1. If $\phi \in \mathscr{L}$, then $f(\phi) = \phi$;
2. If ϕ is atomic and not of the form $/\phi'/$, where $\phi' \Rrightarrow \psi$ for some ψ, then $f(\phi) = \phi$;
3. $f(\neg\phi) = \neg f(\phi)$;
4. if $\phi \Rrightarrow \psi$, then $f(\phi) = f(\psi)$ and $f(/\phi/) = f(\psi)$;
5. if $f(\phi\wedge\psi\wedge\ldots) \notin \mathscr{L}$ and $(\phi\wedge\psi\wedge\ldots) \not\Rrightarrow \psi$, for any ψ, then $f(\phi\wedge\psi\wedge\ldots) = (f(\phi)\wedge f(\psi)\wedge\ldots)$; and
6. If $(\phi\vee\psi\vee\ldots) \notin \mathscr{L}$, then $f(\phi\vee\psi\vee\ldots) = (f(\phi)\vee f(\psi)\vee\ldots)$.

Let $f(\Delta) = \{f(\delta)|\delta \in \Delta\}$.

Remark f is well-defined by L6.2. First, $\Rrightarrow$ is a functional relation, by L6.2(6.). Second, in the basis cases of the definition D4.4 of $\Rrightarrow$, the formulae on the RHS of $\Rrightarrow$ are all either members of $\mathscr{L}$, the atomic sentence $\top^\vee$, or double-negations of witnessing constants w^ϕ. So, the result of applying f in each of these cases is defined by clauses (1)-(3) above. Thus, the result of applying f to the LHS of basis cases for $\Rrightarrow$ is well-defined. Third, in the inductive clause of the definition D4.4 of $\Rrightarrow$, the RHS is always a double-negation of either some lower-level LHS ϕ of an instance of $\Rrightarrow$, or a "shadow" $/\phi/$ of some such LHS. In this case, the application of f to $\neg\neg\phi$ (or $\neg\neg/\phi/$) is handled by a "previous" application of clause (4) above to ϕ (or $/\phi/$), together with clause (3).

Lemma 6.6 *If* $\Gamma \leq' \chi$, *then* $f(\Gamma) \leq f(\chi)$

Proof We prove the result by induction on the definition D6.1 of $\leq'$. The case of (ID) is trivial. The cases of (S), (W), (MAX), ($\top^\wedge$), and ($\top^\vee$), are proved similarly, using L6.2 and D6.1. We will prove the result in the case of ($\top^\wedge$) for illustration.

($\top^\wedge$): $f(\emptyset) = \emptyset$. $\top^\wedge$ is atomic, so $f(\top^\wedge) = \top^\wedge$. D5.1 implies the result.

(Determination): We prove the case in which $\chi = (\chi^1\wedge\chi^2\wedge\ldots)$. The other cases are proved similarly. Suppose $\chi = (\chi^1\wedge\chi^2\wedge\ldots)$ and $\Gamma = \chi^1, \chi^2, \ldots$. There are three cases: (A) $\chi \in \mathscr{L}$, (B) $\chi \Rrightarrow \psi$, for some ψ, or (C) neither.

- **(A):** Trivial, by D5.1, since $f(\Gamma) = \Gamma$ and $f(\chi) = \chi$.
- **(C):** By D6.5 $f((\chi^1\wedge\chi^2\wedge\ldots)) = (f(\chi^1)\wedge f(\chi^2)\wedge\ldots)$. So, the result is trivial, by D5.1.
- **(B):** We prove the result by a subsidiary induction on $\Rrightarrow$:
  - **(S):** Suppose $\chi = v^{\Delta,\phi}$, where $\Delta < \phi \in S$. Then $\Gamma = \Delta, (\top^\wedge\vee/\phi/), \top^\vee$. So, $f(\chi) = \phi$, and it's easy to see by L6.2(1.) that $f(\Gamma) = \Gamma$, since neither

$(\top^{\wedge}\vee/\phi/)$ nor $\top^{\vee}$ is a conjunction. Moreover, by the closure of S, $\Delta \leq \phi \in S$. So, $\Delta, \top^{\vee} \leq \phi$. Since $(\top^{\wedge}\vee/\chi/) \leq \top^{\vee}$, the result follows by D5.1(CUT).

(W): Suppose $\chi = (\psi \wedge w^{\phi})$, where $\psi \preceq \phi \in S$. Then $\Gamma = \psi, w^{\phi} = f(\Gamma)$, and $f(\chi) = \neg\neg\phi$.

$$\psi, w^{\phi} \underset{(W)}{\leq} \phi \underset{(\text{DETER.})}{\leq} \neg\neg\phi.$$

(Max): Suppose $\chi = (w^{\phi} \wedge \phi)$, where $\phi \in \mathscr{L}$. Then $\Gamma = \phi, w^{\phi} = f(\Gamma)$, and $f(\chi) = \neg\neg w^{\phi}$.

$$\phi, w^{\phi} \underset{(\text{MAX})}{\leq} w^{\phi} \underset{(\text{DETER.})}{\leq} \neg\neg w^{\phi}.$$

(∅): Suppose $\chi = (\top^{\wedge} \wedge (\top^{\wedge}\vee/\phi/))$. Then $\Gamma = \top^{\wedge}, (\top^{\wedge}\vee/\phi/)$. $f(\chi) = \top^{\vee}$ and $f(\Gamma) = \Gamma$. $(\top^{\wedge}\vee/\phi/) \leq \top^{\vee}$ and $\top^{\wedge} \leq (\top^{\wedge}\vee/\phi/) \leq \top^{\vee}$. So, by (Amalgamation), $\top^{\wedge}, (\top^{\wedge}\vee/\phi/) \leq \top^{\vee}$.

Induction Step: Suppose that $\phi \Rrightarrow \psi$ and $\chi = (\psi \wedge /\phi/)$. Then $f(\chi) = f(\neg\neg\phi) = \neg\neg f(\phi)$. $\Gamma = \psi, /\phi/$, and $f(\phi) = f(/\phi/) = f(\psi)$. Thus, $f(\Gamma) = f(\psi)$ and $f(\chi) = \neg\neg f(\psi)$. D5.1(DETERMINATION) implies the result. A similar argument yields the result if $\chi = (\phi \wedge /\phi/)$.

$\Rrightarrow$: Suppose $\Gamma \leq' \chi$ is an instance of ($\Rrightarrow$). There are three cases: for some ϕ, ψ, $\phi \Rrightarrow \psi$ and either (A) $\chi = \psi$ and $\Gamma = \phi$, (B) $\chi = \phi$ and $\Gamma = \psi, /\phi/$, or (C) $\chi = /\phi/$ and $\Gamma = \phi, /\phi/$, In each of these cases, since $f(\phi) = f(/\phi/) = f(\psi)$, $f(\Gamma) = f(\chi) = f(\psi)$. The result follows by D5.1,(ID).

(Cut): IH and D5.1,(CUT). □

Recall from (D5.11) that $\mathscr{L}^{w}$ is the union of the set of sentences of $\mathscr{L}$ with the set of witnessing constants $\{w^{\psi} | \psi \in \mathscr{L}\}$.

Lemma 6.7 *Suppose* $\delta, \Delta, \phi \subseteq \mathscr{L}^{w}$. *Then*

1. *if* $\Delta \leq' \phi$, *then* $\Delta \leq \phi$; *and*
2. *if* $(\exists\Gamma)\delta, \Gamma \leq' \phi$, *then* $(\exists\Delta)\delta, \Delta \leq \phi$.

Proof D6.5 and L6.6. □

Lemma 6.8 ***(Conservativity)*** *If* σ *is a grounding claim of* $\mathscr{L}^{w}$ *and* σ *is* $\leq'$*-con, then* $\sigma \in S^{+}$.

Proof Suppose σ is a grounding claim of $\mathscr{L}^{w}$ and is $\leq'$-con.

1. Suppose $\sigma = \Delta \leq \phi$. Then $\Delta \leq \phi$. L6.7 and D5.10 imply the result.
2. Suppose $\sigma = \delta \preceq \phi$. L6.7 and D5.10 imply the result.
3. Suppose $\sigma = \delta \prec \phi$. By (2.) above, $\delta \preceq \phi \in S^{+}$. By the closure of S^{+}(IRREVERSIBILITY), either $\phi \preceq \delta \in S^{+}$ or $\delta \prec \phi \in S^{+}$. Suppose (for *reductio*) that $\phi \preceq \delta \in S^{+}$. Since S^{+} is witnessed, $(\exists\Gamma)\phi, \Gamma \leq \delta \in S^{+}$. So, $\phi, \Gamma \leq \delta$, and so $\phi, \Gamma, \leq \delta$. By D6.3, $\phi \preceq \delta$ is $\leq$-con. $\perp$.
4. Suppose $\sigma = \Delta < \phi$. (1.) above, (3.) above, and the closure of S^{+}(REVERSE SUBSUMPTION) imply the result. □

The following two lemmas are immediate by the definition of $\leq'$-con.

Lemma 6.9 ***(Consistency)*** *$\phi \prec \phi$ is not $\leq'$-con*

Lemma 6.10 ***(Witnessing)*** *if $\phi \preceq \psi$ is $\leq'$-con, then $(\exists\Gamma)\phi, \Gamma \leq \psi$ is $\leq'$-con.*

It is straightforward to show that grounding claims corresponding to introduction rules in GG are $\leq'$-con.

Lemma 6.11 *The following are $\leq'$-con:*

1. $\phi, \psi, \dots < (\phi\wedge\psi\wedge\dots)$*;*
2. $\phi^i < (\phi^0\vee\phi^1\vee\dots)$*;*
3. $\phi < \neg\neg\phi$*;*
4. $\neg\phi, \neg\psi, \dots < \neg(\phi\vee\psi\vee\dots)$*; and*
5. $\neg\phi^i < \neg(\phi^0\wedge\phi^1\wedge\dots)$.

Proof All of the cases are proved similarly. We do (1.) for illustration. By $(\leq')$(DETERMINATION), $\phi, \psi, \dots \leq' (\phi\wedge\psi\wedge\dots)$. Suppose (for *reductio*) that $(\phi\wedge\psi\wedge\dots), \Gamma \leq' \phi$, for some Γ. (Cases of other conjuncts are proved similarly.) By L6.6, $f(\phi\wedge\psi\wedge\dots), f(\Gamma) \leq f(\phi)$. By D6.5, there are seven cases bearing on the value of $f(\phi\wedge\psi\wedge\dots)$:

(A): $f(\phi\wedge\psi\wedge\dots) = (f(\phi)\wedge f(\psi)\wedge\dots)$. Then, by L5.23(2.), it is not the case that $(f(\phi)\wedge f(\psi)\wedge\dots), f(\Gamma) \leq f(\phi)$. $\bot$.

(B): $(\phi\wedge\psi\wedge\dots) = v^{\Delta,\delta}$ and $\Delta < \delta \in S$. Then $f(v^{\Delta,\delta}) = \delta$. Either $\phi \in \Delta$ or ϕ is a nullity. If $\phi \in \Delta$, then $f(\phi) = \phi$. So, $\delta, f(\Gamma) \leq \phi$. By the consistency of S^+, it is not the case that $\delta, f(\Gamma) \leq \phi$. $\bot$. Suppose, then, that ϕ is a nullity, so that by L5.13(5.) $f(\phi) = \phi$. By L5.12, since $\delta, f(\Gamma) \leq \phi$, δ is a nullity. But δ is not nullity. $\bot$.

(C): $(\phi\wedge\psi\wedge\dots) = (\chi\wedge w^\theta)$ and $\chi \preceq \theta \in S$. Then $f(\chi\wedge w^\theta) = \neg\neg\theta$. Also, $\phi = \chi$, and $f(\phi) = \phi$. Since $\chi \preceq \theta \in S$ $w^\theta, \chi \underset{(W)}{\leq} \theta$, we have $\neg\neg\theta, f(\Gamma) \leq \chi$ and so, by D5.1(CUT)$\neg\neg\theta, f(\Gamma), \theta \leq \theta$. But S^+ is prime, and so consistent. $\bot$.

(D): $(\phi\wedge\psi\wedge\dots) = (w^\chi\wedge\chi)$ and $\chi \in \mathscr{L}$. Then $f(w^\chi\wedge\chi) = \neg\neg w^\chi$. So, $\phi = w^\chi$, and $f(\phi) = \phi$. It is easy to see that D5.1 and the closure and consistency of S^+ imply $\bot$.

(E): $(\phi\wedge\psi\wedge\dots) = (\top^\wedge\wedge(\top^\wedge\vee/\chi/))$ and $\chi \in \mathscr{L}$. Then $f((\top^\wedge\wedge(\top^\wedge\vee/\chi/))) = \top^\vee$ and $f(\top^\wedge) = \top^\wedge$. By L5.13(1.), it is not the case that $\top^\vee, f(\Gamma) \leq \top^\wedge$. $\bot$.

(F): $(\exists\chi, \theta)$ such that $\chi \Rrightarrow \theta$ and $(\phi\wedge\psi\wedge\dots) = (\phi\wedge\psi) \in \{(\theta\wedge/\chi/), (\chi\wedge/\chi/)\}$, so that $\phi \in \{\chi, \theta\}$ and $\psi = /\chi/$. Then, by D6.5, $f(\phi) = f(\theta)$, and $f(\phi\wedge\psi\wedge\dots) = \neg\neg f(\theta)$. By L5.23, it is not the case that $\neg\neg f(\theta), f(\Gamma) \leq f(\theta)$. $\bot$.

(G): $(\phi\wedge\psi\wedge\dots) \in \mathscr{L}$. Then $f(\phi\wedge\psi\wedge\dots) = (\phi\wedge\psi\wedge\dots) = (f(\phi)\wedge f(\psi)\wedge\dots)$, so this case reduces to case (A). □

Showing that $\leq'$-con grounding claims conform to the elimination rules of GG, and so the demands of MAXIMALITY, is much less straightforward. Demonstrating this fact is the burden of L6.12–L6.17.

It is clear by inspection of D4.4 of $\Rrightarrow$ and then D6.1 that there is no instance of $(\leq')(\Rrightarrow)$ with w^{ϕ} on the RHS. Moreover, whenever $/\phi/$ occurs on the RHS of such an instance, it also occurs on the LHS. Thus, the following results can be proved by an easy induction on S-derivations. They are useful because they show that S-connections to w^{ϕ} and $/\phi/$ are always reversible and so never correspond to full, strict grounding claims.

Lemma 6.12 ***(Persistence)***

1. *If $\Delta \leq' /\phi/$, then $/\phi/ \in \Delta$.*
2. *If $\Delta \leq' w^{\phi}$, then $w^{\phi} \in \Delta$.*
3. *If $\mathcal{D}$ is a semi-normal S-derivation of $\Gamma \leq' \phi$, and the principal connection of $\mathcal{D}$ has the form $w^{\psi}, \Delta \leq' \phi$, then $w^{\psi} \in \Gamma$.*

Remark **(Amalgamation)** By D6.1 the following is an instance of CUT.

$$\frac{(\Delta_i \leq' \phi)_{i \leq n \in \omega} \quad \phi \leq' \phi}{(\Delta_i) \leq' \phi}$$

So, if $(\Delta^i \leq' \phi)_{i \leq n < \omega}$, then $(\Delta^i) \leq' \phi$.

It is convenient to define a distributive notion of S-derivability:

Definition 6.13 $\Gamma \leq' (\delta^i)$ iff there is a covering (Γ^i) of Γ such that $(\Gamma^i \leq' \delta^i)$.

It is easy to see that this distributive extension of $\leq'$ is transitive and closed under unions (*i.e.*, $(\Delta^i \leq \Gamma^i) \Rightarrow (\Delta^i) \leq (\Gamma^i)$).

Say that the S-connection $\Delta \leq \phi$ is *reversible* iff $(\exists \delta \in \Delta)(\exists \Sigma)\phi, \Sigma \leq \delta$.

Remark Clearly, if $\Delta \leq' \psi \leq' \phi$, $\Delta \leq \psi$ is reversible, and $\psi \leq' \phi$ is reversible, then $\Delta \leq' \phi$ is reversible. Equivalently, if $\Delta \leq' \phi$ is irreversible, $\Delta \leq' \psi \leq' \phi$, and $\psi \leq \phi$ is reversible, then $\Delta \leq' \psi$ is irreversible.

Remark Every instance of $(\leq')(\Rrightarrow)$ is reversible.

We can now show that S-connections to our new conjunctions, including $v^{\Delta,\phi}$ meet the demands of MAXIMALITY.

Lemma 6.14

1. *If $\Gamma \leq' v^{\Delta,\phi}$, then either $\Gamma \leq' v^{\Delta,\phi}$ is reversible, or $\Gamma \leq' \Delta, (\top^{\wedge}\vee/\phi/), \top^{\vee}$.*
2. *If $\Gamma \leq' (\psi \wedge w^{\phi})$, then either $\Gamma \leq' (\psi \wedge w^{\phi})$ is reversible, or $\Gamma \leq' \phi$.*
3. *If $\Gamma \leq' (w^{\phi} \wedge \phi)$, then either $\Gamma \leq' (w^{\phi} \wedge \phi)$ is reversible, or $\Gamma \leq' w^{\phi}$.*

Proof Each of (1.)–(3.) is proved similarly. We do (1.) for illustration. Let $v = v^{\Delta\phi}$, and assume $\Gamma \leq v$. We prove the result by induction on S-derivations. By L6.4, we may assume that the derivation $\mathcal{D}$ of $\Gamma \leq v$ is in super-normal form. By D6.1, if $\mathcal{D}$ is an axiom, it is an instance of $(\leq)$(DETERMINATION), $(\leq)(\Rrightarrow)$, or $(\leq)$(ID). So, (S), (W), (MAX), $(\top^{\wedge})$, and $(\top^{\vee})$ are not relevant.

(ID): Trivial.
$(\Rrightarrow)$: Every instance of $(\Rrightarrow)$ is reversible.
(Determination): Suppose $\mathcal{D}$ is an axiom of the form $\Delta, (\top^{\wedge}\vee/\phi/), \top^{\vee} \leq' v$. The result is immediate by $(\leq')$(ID).
(Cut): Suppose $\mathcal{D}$ terminates in an application of CUT. The principal connection of $\mathcal{D}$ is an instance of either (A) (ID), (B) $(\Rrightarrow)$, or (C) (DETERMINATION).

- **(A):** The minor premises of $\mathcal{D}$ have the form $(\Gamma_i \leq' v)$. By IH, either $(\exists\gamma \in \Gamma_i))v \preceq \gamma$ is $\leq'$-con, or $\Gamma_i \leq' \Delta, (\top^{\wedge}\vee/\phi/), \top^{\vee}$ for each i. The result follows by (Amalgamation).
- **(B):** L6.12(Persistence) implies that $/v/ \in \Gamma$. Since $v, /v/ \leq' /v/, \Gamma \leq' v$ is reversible.
- **(C):** Trivial. □

The following lemma says that the only way to get an irreversible S-connection to some ϕ in our original language $\mathscr{L}$ is to "go through" some strict grounding claim in our original set S. This allows us to show that the canonical model basis meets the demand imposed by MAXIMALITY for sentences of $\mathscr{L}$ by appealing to the fact that our original set S is prime, and so already conforms to the elimination rules of GG.

Lemma 6.15 ***(Interpolation)*** *If $\Delta \leq' \phi$ and $\phi \in \mathscr{L}$, then either $\Delta \leq' \phi$ is reversible, or*

$$(\exists\Gamma)(\Gamma < \phi \in S \textit{ and } \Delta \leq' v^{\Gamma,\phi} \leq' \phi)$$

Proof We prove the result by induction on $\Delta \leq \phi$. By L6.4, we may assume (wlog) that the S-derivation $\mathcal{D}$ of $\Delta \leq \phi$ is in super-normal form.

(S): Suppose $\Delta = \Delta', \top^{\vee}$ and $\Delta' \leq \phi \in S$. Either (A) $(\exists\delta \in \Delta')\phi \preceq \delta \in S$ or (B) not.

- **(A):** $\phi, w^{\delta} \underset{(W)}{\leq} \delta$. So, $\Delta \leq' \phi$ is reversible.
- **(B):** By the closure of S, $\Delta' < \phi \in S$.

$$\emptyset \underset{(\top^{\wedge})}{\leq} \top^{\wedge} \underset{(\text{DETER.})}{\leq} (\top^{\wedge}\vee/\phi/).$$

So,

$$(\Delta =) \quad \Delta', \emptyset, \top^{\vee} \quad \leq' \quad \Delta', (\top^{\wedge}\vee/\phi/), \top^{\vee} \quad \underset{(\text{DETER.})}{\leq'} \quad v^{\Delta',\phi} \quad \underset{(\Rrightarrow)}{\leq'} \quad \phi$$

.

(W): Trivially, by $(\leq')$(MAX), $(\Delta =)\ \delta, w^{\phi} \leq' \phi$ is reversible.
(Max): $w^{\psi} \notin \mathscr{L}$. $\perp$.

(ID): Trivially, $\phi \leq \phi$ is reversible.

($\Rightarrow$): By L6.2(5.), there are only two relevant instances of $\Rightarrow$: (S) $v^{\Gamma,\phi} \leq' \phi$, where $\Gamma < \phi \in S$; or (W) $(\psi \wedge w^{\chi}) \leq' \neg\neg\chi$, where $\psi \preceq \chi \in S$ and $\phi = \neg\neg\chi$.

- **(S):** $v^{\Gamma,\phi} \leq' v^{\Gamma,\phi} \leq' \phi$.
- **(W):** By $(\leq)(\Rightarrow)$, $\neg\neg\chi, /(\psi \wedge w^{\chi})/ \quad \leq' \quad (\psi \wedge w^{\chi})$, so the S-connection $(\psi \wedge w^{\chi}) \leq' \neg\neg\chi$ is reversible.

(Determination): Suppose $\phi = (\phi^1 \wedge \phi^2)$ and $\Delta = \phi^1, \phi^2$. By the closure of S($\wedge$-INTRODUCTION), $\Delta < \phi \in S$. As in the case (S)(B) above, this implies that $\Delta \leq' v^{\Delta,\phi} \leq' \phi$. The more general cases for $\wedge$, $\vee$, and $\neg$ are proved similarly.

(Cut): Suppose $\mathcal{D}$ terminates in an application of CUT. The principal connection of $\mathcal{D}$ is an instance $\Delta' \leq' \phi$ of either (S), (ID), (W), ($\Rightarrow$)(S), ($\Rightarrow$)(W), or (DETERMINATION). Since $\mathcal{D}$ is super-normal, $\Delta \leq' \Delta'$. So, the arguments in the basis cases for ($\Rightarrow$)(S) and (DETERMINATION) imply that $\Delta \leq' \Delta' \leq' v^{\Delta',\phi} \leq' \phi$. That leaves the cases (W), ($\Rightarrow$)(W), (ID) and (S):

- **(W):** Suppose the principal connection of $\mathcal{D}$ is $\psi, w^{\phi} \leq' \phi$. By L6.12(2.) (Persistence), $w^{\phi} \in \Delta$. By $(\leq')$(MAX), $\phi, w^{\phi} \leq' w^{\phi}$, so $\Delta \leq' \phi$ is reversible.
- **(($\Rightarrow$))(W):** The principal connection of $\mathcal{D}$ has the form $(\psi \wedge w^{\chi}) \leq' \neg\neg\chi$. Since $\mathcal{D}$ is super-normal, the minor premises have the form $(\Gamma^i \leq' (\psi \wedge w^{\chi}))$. By L6.14(2.), there are two cases: (A) $\Gamma^i \leq' (\psi \wedge w^{\chi})$ is reversible, for some i, or (B) $\Gamma^i \leq' \chi$ for all i.

  - **(A):** For some $\delta \in \Gamma^i \subseteq \Delta$ and some Θ, $(\psi \wedge w^{\chi}), \Theta \leq' \delta$. Since, by $(\leq)(\Rightarrow)$, $\neg\neg\chi, /(\psi \wedge w^{\chi})/ \leq' (\psi \wedge w^{\chi})$,
    $\neg\neg\chi, /(\psi \wedge w^{\chi})/, \Theta \leq' \delta$. So, $\Delta \leq' \phi$ is reversible.
  - **(B):** By the closure of S($\neg\neg$-INTRODUCTION), $\chi < \neg\neg\chi \in S$. So, by $\Rightarrow$(S), $v^{\chi,\neg\neg\chi} \Rightarrow \neg\neg\chi$, and thus

    ($\star$) $\quad v^{\chi,\neg\neg\chi} \leq' \neg\neg\chi$.

    Also, by (AMALGAMATION)

    ($\maltese$) $\quad (\Delta =) \quad (\Gamma^i) \leq' \chi$.

    Since

    $$\emptyset \underset{(\top^{\wedge})}{\leq'} \top^{\wedge} \underset{(\text{DETER.})}{\leq'} (\top^{\wedge}\vee/\neg\neg\chi/) \underset{(\top^{\vee})}{\leq'} \top^{\vee}$$

    ($\maltese$) and (AMALGAMATION) imply:

    ($\star\star$) $\quad (\Delta =) \quad \Delta, \emptyset, \emptyset \quad \leq' \quad \chi, (\top^{\wedge}\vee/\neg\neg\chi/), \top^{\vee}$

    Putting this all together, we have:

    $$\Delta \underset{(\star\star)}{\leq'} \chi, (\top^{\wedge}\vee/\neg\neg\chi/), \top^{\vee} \underset{(\text{DETER.})}{\leq'} v^{\chi,\neg\neg\chi} \underset{(\star)}{\leq'} \neg\neg\chi$$

    .

- **(ID):** Suppose the principal connection of $\mathcal{D}$ is $\phi \leq' \phi$. Then the minor premises of $\mathcal{D}$ have the form $(\Delta^i \leq' \phi)$. Assume that $\Delta \leq \phi$ is irreversible, and so, for each i $\Delta^i \leq' \phi$ is irreversible. By IH, for each i, there is a Γ^i such that $\Gamma^i < \phi \in S$ and $\Delta^i \leq' v^{\Gamma^i,\phi} \leq' \phi$. By ($\leq'$),($\Rrightarrow$)(S),

($\bigstar$) $\phi, /v^{\Gamma^i,\phi}/ \leq' v^{\Gamma^i,\phi}$.

Suppose (for *reductio*) that $\Delta^i \leq' v^{\Gamma^i,\phi}$ is reversible, for some i. By ($\bigstar$) and ($\leq'$)(CUT), $\Delta^i \leq' \phi$ is reversible. $\bot$. So, $\Delta^i \leq' v^{\Gamma^i,\phi}$ is irreversible for all i. By L6.14(1.),

(✢) $\Delta^i \leq' \Gamma^i, (\top^\wedge\vee/\phi/), \top^\vee$.

Since $\Gamma^i < \phi \in S$, for each i, the closure of S,(SUBSUMPTION,CUT,REVERSE SUBSUMPTION) implies that $(\Gamma^i) < \phi \in S$. Let $\Gamma = (\Gamma^i)$. Then,

($\bigstar\bigstar$) $v^{\Gamma,\phi} \leq' \phi$.

by ($\leq$)($\Rrightarrow$). Putting all of this together:

$$(\Delta^i) \underset{(✢)}{\leq'} (\Gamma^i), (\top^\wedge\vee/\phi/), \top^\vee \underset{(\text{DETER.})}{\leq'} v^{\Gamma,\phi} \underset{(\bigstar\bigstar)}{\leq'} \phi.$$

(S): The principal connection of $\mathcal{D}$ has the form $\Delta', \top^\vee \leq' \phi$, where $\Delta' \leq \phi \in S$. We are going to divide the formulae $\delta \in \Delta'$ (and, correlatively, the minor premises of $\mathcal{D}$) according to whether $\delta \prec \phi$ is $\leq'$-con or not. On this division, $\Delta', \top^\vee \leq' \phi$ has the form $(\theta^i), (\gamma^j), \top^\vee \leq' \phi$, and the minor premises have the form $(\Theta^i \leq' \theta^i), (\Gamma^j \leq' \gamma^j), (\Sigma^k \leq' \top^\vee)$, where:

- $(\theta^i), (\gamma^j) \leq \phi \in S$;
- **strict:** for each j, there is no Ξ such that $\phi, \Xi \leq' \gamma^j$; and
- **merely weak:** for each i, $\phi, \Xi \leq \theta^i$, for some Ξ.

By D6.3 and L6.8(Conservativity), for each j, $\gamma^j \prec \phi$ is $\leq'$-con, and so $\gamma^j \prec \phi \in S^+$. By L5.21(Conservativity), $\gamma^j \prec \phi \in S$. If $(\Delta =)(\Theta^i)(\Gamma^j)(\Sigma^k) \leq \phi$ is reversible, then we are done. So, assume that it is irreversible. Suppose (for *reductio*) that, for some i, $\Theta^i \leq \theta$ is reversible. Then $(\Theta^i)(\Gamma^j)(\Sigma^k) \leq' \phi$ is reversible. $\bot$. So, IH applies to the minor premises $(\Theta^i \leq' \theta^i)$: for each i, there is a v^{Ω^i,θ^i} such that $\Theta^i \leq v^{\Omega^i,\theta^i} \leq \theta^i$ and $\Omega^i < \theta^i \in S$. By the closure of S, $(\Omega^i), (\gamma^i) < \phi \in S$. Let $\Gamma = (\Omega^i), (\gamma^j)$, so that $\Gamma < \phi \in S$. Also, by ($\leq'$)($\Rrightarrow$), $v^{\Gamma,\phi} \leq \phi$.

Since, for each i, $\Theta^i \leq' \theta^i$ is irreversible, and by ($\leq'$)($\Rrightarrow$), $v^{\Omega^i,\theta^i} \leq' \theta^i$ is reversible, $\Theta^i \leq' v^{\Omega^i,\theta}$ is irreversible. So, L6.14(1.) applies: we have, for each i,

$$\Theta^i \leq' \Omega^i, (\top^\wedge\vee/\theta^i/), \top^\vee.$$

Since $(\top^\wedge\vee/\theta^i/) \underset{(\top^\vee)}{\leq'} \top^\vee$, this implies, for each i

($\bigstar$) $\Theta^i \leq' \Omega^i, \top^\vee$

So, for each i, j, k, we have the S connections:

$$\Theta^i \underset{(\bigstar)}{\leq'} \Omega^i, \top^\vee$$

$$\Gamma^j \leq' \gamma^j$$
$$\Sigma^k \leq' \top^{\vee}.$$

By (AMALGAMATION):

(✢) $(\Theta^i), (\Gamma^j), (\Sigma^k) \leq' (\Omega^i), (\gamma^j), \top^{\vee} \quad (= \Gamma, \top^{\vee})$

Since $\emptyset \leq' \top^{\wedge} \underset{\text{(DETER.)}}{\leq'} (\top^{\wedge}\vee/\phi/)$,

(★★) $\Gamma, \top^{\vee}, \emptyset \quad \leq' \quad \Gamma, (\top^{\wedge}\vee/\phi/), \top^{\vee}$

Putting all of this together, we have:

$$(\Theta^i), (\Gamma^j), (\Sigma^k) \underset{(✢)}{\leq'} \Gamma, \top^{\vee} \underset{(\star\star)}{\leq'} \Gamma, (\top^{\wedge}\vee/\phi/), \top^{\vee} \underset{\text{(DETER.)}}{\leq'} v^{\Gamma,\phi} \underset{(\Rrightarrow)}{\leq'} \phi$$

□

We now use L6.15(Interpolation) to demonstrate that MAXIMALITY is satisfied for sentences in $\mathscr{L}$.

Lemma 6.16 *Suppose $\phi \in \mathscr{L}$, and $\Delta \leq' \phi$ is irreversible. Then,*

1. *When $\phi = (\phi^1 \wedge \phi^2)$, $\Delta \leq' \{\phi^1, \phi^2\}$;*
2. *When $\phi = \neg\neg\phi$, $\Delta \leq' \phi$;*
3. *When $\phi = (\phi^1 \vee \phi^2)$, either $\Delta \leq' \phi^1$, $\Delta \leq' \phi^2$, or $\Delta \leq' \{\phi^1, \phi^2\}$;*
4. *When $\phi = \neg(\phi^1 \vee \phi^2)$, $\Delta \leq' \{\neg\phi^1, \neg\phi^2\}$; and*
5. *When $\phi = \neg(\phi^1 \wedge \phi^2)$, either $\Delta \leq' \neg\phi^1$, $\Delta \leq' \neg\phi^2$, or $\Delta \leq' \{\neg\phi^1, \neg\phi^2\}$.*

Proof

(1): By L6.15(Interpolation), $\Delta \leq' v^{\Theta,\phi} \leq' \phi$ and $\Theta < \phi \in S$, for some Θ. By the primeness of S ($\wedge$-ELIMINATION), there is a covering Θ^1, Θ^2 of Θ such that $\Theta^1 \leq \phi^1 \in S$ and $\Theta^2 \leq \phi^2 \in S$. By $(\leq)$(S),

(★) $\Theta^1, \Theta^2, \top^{\vee} \leq' \{\phi^1, \phi^2\}$.

Also, $v^{\Theta,\phi} \leq \phi$ is reversible, by $(\leq)(\Rrightarrow)$. Since $\Delta \leq \phi$ is irreversible, $\Delta \leq v^{\Theta,\phi}$ is irreversible. So, L6.14(1.) implies

(★★) $\Delta \leq \Theta, (\top^{\wedge}\vee/\phi/), \top^{\vee}$.

By $(\leq')(\top^{\vee})$, $(\top^{\wedge}\vee/\phi/) \leq' \top^{\vee}$, so

(✢) $\Delta \leq' \Theta, (\top^{\wedge}\vee/\phi/), \top^{\vee} \leq' \Theta, \top^{\vee}$.

Putting all of this together, we have

$$\Delta \underset{(✢)}{\leq'} \Theta, \top^{\vee} \underset{(\star)}{\leq'} \{\phi^1, \phi^2\}.$$

(2)-(5): Arguments similar to that for (1) yield the results, applying different elimination rules to $\Theta < \phi$. The argument for (2) uses ($\neg\neg$-elimination) where the argument for (1) uses ($\wedge$-elimination), and, similarly, for the other cases. □

It is now straightforward to extend L6.16 beyond the special case in which the RHS is in $\mathscr{L}$:

Lemma 6.17 *Suppose $\Delta \leq' \chi$ is irreversible.*

1. *When $\chi = (\phi^1 \wedge \phi^2 \wedge \dots)$, $\Delta \leq' \{\phi^1, \phi^2, \dots\}$;*
2. *When $\chi = (\phi^1 \vee \phi^2 \vee \dots)$, there is a non-empty subset (ψ^i) of (ϕ^j) such that $\Delta \leq' (\psi^i)$;*
3. *When $\chi = \neg\neg\phi$, $\Delta \leq' \phi$;*
4. *When $\chi = \neg(\phi^1 \vee \phi^2 \vee \dots)$, $\Delta \leq' \{\neg\phi^1, \neg\phi^2, \dots\}$; and*
5. *When $\chi = \neg(\phi^1 \wedge \phi^2 \wedge \dots)$, there is a subset (ψ^i) of (ϕ^j) such that $\Delta \leq' (\neg\psi^i)$.*

Proof Easy inductions on S-derivations yield (4.) and (5.).

1. We prove the result by induction on S-derivations. Instances of (MAX), ($\top^\wedge$), and ($\top^\vee$) do not have conjunctions on the RHS.

   (S): L6.16.
   (W): By ($\leq'$)(MAX), $\chi, w^\chi \leq' w^\chi$. $\perp$.
   (ID): $\perp$.
   ($\Rrightarrow$): Suppose $\chi \Rrightarrow \psi$, and $\mathcal{D}$ is an instance of ($\leq'$)($\Rrightarrow$). Every instance of ($\leq'$)($\Rrightarrow$) is reversible. $\perp$.
   (Determination): ($\leq'$)(ID).
   (Cut): Suppose $\mathcal{D}$ terminates in an instance of CUT. By L6.4 we may assume (wlog) that $\mathcal{D}$ is in normal form. The principal connection of $\mathcal{D}$ cannot be an instance of ($\top^\wedge$), ($\top^\vee$), or (MAX). If the principal connection is an instance of (S) or (W), then $\chi \in \mathscr{L}$, and so L6.16 implies the result. If it is an instance of (ID), then IH and (Amalgamation) imply the result. If it is an instance of (DETERMINATION), then ($\leq'$)(ID) and (Amalgamation) imply the result. If it is an instance of ($\Rrightarrow$) then, by L6.2(4.), the major premise of $\mathcal{D}$ has either the form $v^{\Gamma,\chi} \leq' \chi$ (where $\chi \in \mathscr{L}$) or the form

   $$\chi', /\chi/ \leq' \chi$$

   (where $\chi \Rrightarrow \chi^*$, for some χ^*). In the former case, L6.16 implies the result. In the latter case, by L6.12(1.) (Persistence), $/\chi/ \in \Delta$. $\chi, /\chi/ \underset{(\Rrightarrow)}{\leq'} /\chi/$, so $\Delta \leq' \chi$ is reversible. $\perp$.

2. We prove the result by induction on S-derivations. All of the cases are similar to the corresponding cases for (1.) above, except the case in which $\mathcal{D}$ is an instance of ($\Rrightarrow$), and the case in which $\mathcal{D}$ terminates in (CUT) and has as its principal connection an instance of ($\Rrightarrow$).

   ($\Rrightarrow$): The only relevant case is one in which the axiom has the form $v^{\Gamma,\chi} \leq' \chi$ and $\chi \in \mathscr{L}$. As above, L6.16 implies the result.
   (Cut): As in (1.) above, we assume our S-derivation $\mathcal{D}$ is in normal form. We need only check the case in which the principal connection of $\mathcal{D}$ is an instance of ($\Rrightarrow$). Again, there is only one relevant case: the principal connection has the form $v^{\Gamma,\chi} \leq' \chi$, where $\chi \in \mathscr{L}$. As above, L6.16 delivers the result.

3. As in (1.) and (2.) above, the key cases are those involving instances of ($\Rrightarrow$).

($\Rrightarrow$): Suppose $\mathcal{D}$ is an instance of ($\leq'$)($\Rrightarrow$). Every instance of ($\leq'$)($\Rrightarrow$) is reversible. $\perp$.

(Cut): We assume our S-derivation $\mathcal{D}$ is in super-normal form, so the application of (CUT) has no side formulae. We need only check the case in which the principal connection of $\mathcal{D}$ is an instance of ($\Rrightarrow$). There are five cases concerning the form of the principal connection of $\mathcal{D}$, corresponding to the five clauses in the definition D4.4 of $\Rrightarrow$ in which the RHS may be a double-negation: (S), (W), (MAX), and each of the two (Induction) cases.

(S): The principal connection has the form $v^{\Gamma,\chi} \leq' \chi$ and $\chi \in \mathscr{L}$. L6.16.

(W): The principal connection has the form $(\psi \wedge w^{\phi}) \leq' \neg\neg\phi$, where $\psi \preceq \phi \in S$. $\neg\neg\phi \in \mathscr{L}$, so L6.16 yields the result.

(Max): The principal connection has the form $(w^{\gamma} \wedge \gamma) \leq' \neg\neg w^{\gamma}$, where $\gamma \in \mathscr{L}$. Since $\mathcal{D}$ is in super-normal form, the minor premises of $\mathcal{D}$ have the form $(\Delta^i \leq' (\gamma \wedge w^{\gamma}))$ and $\Delta = (\Delta^i)$. Since $\Delta \leq' \neg\neg w^{\gamma}$ is irreversible and the principal connection $(w^{\gamma} \wedge \gamma) \underset{(\Rrightarrow)}{\leq'} \neg\neg w^{\gamma}$ is reversible, for each i $\Delta^i \leq' (w^{\gamma} \wedge \gamma)$ is irreversible. So, the result follows by L6.14(3.) and (Amalgamation).

(Induction)(1): The principal connection has the form $(\psi^2 \wedge /\psi^1/) \leq' \neg\neg\psi^1$, where $\psi^1 \Rrightarrow \psi^2$, and so (by ($\Rrightarrow$)(Induction)) $(\psi^2 \wedge /\psi^1/) \Rrightarrow \neg\neg\psi^1$. Since $\mathcal{D}$ is in super-normal form, the minor premises of $\mathcal{D}$ have the form $(\Delta^i \leq' (\psi^2 \wedge /\psi^1/))$ and $\Delta = (\Delta^i)$. Also, $\neg\neg\psi^1, /(\psi^2 \wedge /\psi^1/)/ \underset{(\Rrightarrow)}{\leq'} (\psi^2 \wedge /\psi^1/)$, so the principal connection is reversible. Since $\Delta \leq' \chi$ is irreversible, none of the minor premises are reversible.. By (1) above and (Amalgamation), $(\Delta^i) \leq' \{\psi^2, /\psi^1/\}$. Also, since $\psi^1 \Rrightarrow \psi^2$, $\psi^2, /\psi^1/ \underset{(\Rrightarrow)}{\leq} \psi_1$. So, $(\Delta^i) \leq' \psi^2, /\psi^1/ \leq' \psi^1$.

(Induction)(2): The principal connection has the form $(\psi^1 \wedge /\psi^1/) \leq' \neg\neg/\psi^1/$, where $\psi^1 \Rrightarrow \psi^2$, for some ψ^2. An argument similar to that in case (D) shows that $(\Delta^i) \leq' \{\psi^2, /\psi^1/\} \leq' /\psi^1/$. □

The following lemma shows, as we have repeatedly claimed, that $\top^{\vee}$ behaves as if it is the disjunction of $(\top^{\wedge} \vee /\phi/)$ for all $\phi \in \mathscr{L}$. It is useful for proving the adequacy of the construction of the canonical model in the next section.

Lemma 6.18 *If* $\Delta \leq' \top^{\vee}$, *then either* $\Delta \leq' \top^{\vee}$ *is reversible, or* $\Delta \leq' \{(\top^{\wedge} \vee /\phi^0/), (\top^{\wedge} \vee /\phi^1/), \ldots\}$, *for some* $(\phi^i)_{i \leq n \in \omega} \subset \mathscr{L}$.

Proof We prove the result by induction on S-derivations $\mathcal{D}$ of $\Delta \leq' \top^{\vee}$. Suppose $\mathcal{D}$ is an S-derivation of $\Delta \leq' \top^{\vee}$ and $\Delta \leq' \top^{\vee}$ is irreversible. By L6.4 we may assume (wlog) that $\mathcal{D}$ is in super-normal form. If $\mathcal{D}$ is an axiom, it is an instance of ($\top^{\vee}$), ($\Rrightarrow$), or (ID).

($\top^{\vee}$): Trivial.

($\Rrightarrow$): Every instance of ($\leq'$)($\Rrightarrow$) is reversible. $\perp$.

(ID): $\top^{\vee} \leq \top^{\vee}. \perp$.
(Cut): The principal connection of $\mathcal{D}$ is an instance of $(\top^{\vee})$, $(\Rrightarrow)$, or (ID).

- $(\top^{\vee})$: The principal connection has the form $(\top^{\wedge}\vee/\phi/) \leq' \top^{\vee}$, for some $\phi \in \mathscr{L}$. Since $\mathcal{D}$ is super-normal, all of the minor premises have the form $(\Delta^i \leq' (\top^{\wedge}\vee/\phi/))$, and $\Delta = (\Delta^i)$. (Amalagamation) implies the result.
- $(\Rrightarrow)$: The only relevant instance has the form $(\top^{\wedge}\wedge(\top^{\wedge}\vee/\phi/)) \leq' \top^{\vee}$, for some $\phi \in \mathscr{L}$. Since $\mathcal{D}$ is super-normal, the minor premises have the form $(\Delta^i \leq' (\top^{\wedge}\wedge(\top^{\wedge}\vee/\phi/)))$ and $\Delta = (\Delta^i)$. For each i, $\top^{\vee}, /(\top^{\wedge}\wedge(\top^{\wedge}\vee/\phi/))/ \underset{(\Rrightarrow)}{\leq'} (\top^{\wedge}\wedge(\top^{\wedge}\vee/\phi/))$. So, the principal connection of $\mathcal{D}$ is reversible, and thus the minor premises are each irreversible.. So, L6.17 applies:

  $(\bigstar) \quad (\Delta^i) \quad \leq' \quad \{\top^{\wedge}, (\top^{\wedge}\vee/\phi/)\}$.

  So,

  $$(\Delta^i) \underset{(\bigstar)}{\leq'} \top^{\wedge}, (\top^{\wedge}\vee/\phi/) \underset{(\text{DETER.})}{\leq'} (\top^{\wedge}\vee/\phi/), (\top^{\wedge}\vee/\phi/).$$

- **(ID):** The minor premises have the form $(\Delta^i \leq' \top^{\vee})$. IH yields the result.

□

Let the *canonical model basis* S^* for S be $\{\sigma|\sigma$ is $\leq'$-con$\}$.

Theorem 6.19 *S^* is prime in $\mathscr{L}^+$ and has the following features:*

Conservativity *For grounding claims σ of $\mathscr{L}$, $\sigma \in S^*$ iff $\sigma \in S$.*
Witnessing *If $\delta \preceq \phi \in S^*$, then $(\exists\Gamma)\delta, \Gamma \leq \phi \in S^*$.*
Irreversibility

1. *$\Delta < \phi \in S^*$ iff $\Delta \leq \phi \in S^*$ and $(\forall\delta \in \Delta)\delta \prec \phi \in S^*$; and*
2. *if $\delta \preceq \phi \in S^*$, then either $\delta \prec \phi \in S^*$ or $\phi \preceq \delta \in S^*$.*

Maximality

1. *$\Delta < \neg\neg\phi \in S^*$ iff $\Delta \leq \phi \in S^*$;*
2. *$\Delta < (\phi^0\wedge\phi^1\wedge\ldots) \in S^*$ iff there is a covering (Δ^i) of Δ such that $\Delta^i \leq \phi^i \in S^*$ for each i;*
3. *$\Delta < (\phi^0\vee\phi^1\vee\ldots) \in S^*$ iff there is a covering (Δ^i) of Δ and a subset (ψ^i) of (ϕ^j) such that $\Delta^i \leq \psi^i \in S^*$ for each i;*
4. *$\Delta < \neg(\phi^0\vee\phi^1\vee\ldots) \in S^*$ iff there is a covering (Δ^i) of Δ such that $\Delta^i \leq \neg\phi^i \in S^*$ for each i; and*
5. *$\Delta < \neg(\phi^0\wedge\phi^1\wedge\ldots) \in S^*$ iff there is a covering (Δ^i) of Δ and a subset $(\neg\psi^i)$ of $(\neg\phi^j)$ such that $\Delta^i \leq \neg\psi^i \in S^*$ for each i.*

Proof

(Conservativity): L6.8 and L5.21 imply $\Rightarrow$. $\Leftarrow$ follows from $\Rightarrow$ by $(\leq')$(S), D6.3, and the fact that S is prime, since $\emptyset \leq' \top^{\vee}$.
(Witnessing): L6.10.

(Irreversibility): Immediate by D6.3.
(Maximality): L6.17 and D6.3 imply $\Rightarrow$. L6.11, and D6.3 imply $\Leftarrow$.
(Primeness): The primeness of S^* in $\mathscr{L}^+$ is proved straightforwardly in a manner similar to the proof of L5.29, using D6.1, D6.3, Irreversibility of S^*, and Maximality of S^*. □

7 The Canonical Model Justified

We are given a prime set S of grounding claims of the language $\mathscr{L}$. In this section, we show that $\mathfrak{M}_S$ satisfies the definition D2.3 of a model and that the grounding claims of $\mathscr{L}$ verified by $\mathfrak{M}_S$ are exactly the members of S (justifying the label "canonical model for S").

We extend S to its canonical model basis S^* as defined in the previous section. S^* is a set of grounding claims of the language $\mathscr{L}^+$, which extends $\mathscr{L}$. S^* is witnessed and prime (in $\mathscr{L}^+$), by T6.19.

Remark Recall that $\sim$ is an equivalence relation on conditions and contents. Intuitively, we identify conditions and contents when they are $\sim$-related. Lemmas 7.1-7.10 concern the structure of $\sim$ and the relationship between $\sim$ and $\bar{\bar{\cdot}}$.

Recall that the function g selects, for any given condition a or content, some representative of the equivalence class of a given by $\sim$; see D4.6. The following facts are immediate consequences of D4.5, D4.6, and D4.3:

Lemma 7.1

1. $a \sim g(a)$ *and* $v \sim g(v)$*;*
2. $g(g(a)) = g(a)$ *and* $g(g(v)) = g(v)$*;*
3. $a \sim b$ *iff* $g(a) = g(b)$*, and* $v \sim w$ *iff* $g(v) = g(w)$*;*
4. *For* $\circ \in \{+, .\}$, $[v^0 \circ v^1 \circ \ldots] \sim [g(v^0) \circ g(v^1) \circ \ldots] \sim [g(v^0) \circ g(v^1) \circ \ldots]_g$*;*
5. $[v] \sim [g(v)] \sim [g(v)]_g$*;*
6. *For* $\circ, \otimes \in \{+, .\}$, $[v^0 \circ v^1 \circ \ldots] \sim [w^0 \otimes w^1 \otimes \ldots]$ *iff* $[g(v^0) \circ g(v^1) \circ \ldots]_g = [g(w^0) \otimes g(w^1) \otimes \ldots]_g$*;*
7. $[v] \sim [w]$ *iff* $[g(v)]_g = [g(w)]_g$*;*
8. $\overline{\neg\phi} = (\bar{\phi}_\ominus, [\bar{\phi}]_g)$*;*
9. $\overline{(\phi \wedge \psi \wedge \ldots)} = ([\bar{\phi}.\bar{\psi}. \cdots]_g, [\overline{\neg\phi} + \overline{\neg\psi} + \cdots]_g)$*; and*
10. $\overline{(\phi \vee \psi \vee \ldots)} = ([\bar{\phi} + \bar{\psi} + \cdots]_g, [\overline{\neg\phi}.\overline{\neg\psi}. \cdots]_g)$.

The next few lemmas constrain decomposition of combinations and choices. L7.2 says that combinations are uniquely decomposable (up to $\sim$); and L7.4 says that choices of three or more contents are uniquely decomposable (up to $\sim$). Neither choices $[v + w]$ of two contents nor singletons $[v]$ are uniquely decomposable, since, by D4.5($\Rrightarrow$), whenever $\phi \Rrightarrow \psi$, $[\bar{\bar{\psi}}] \sim [\bar{\bar{\psi}} + \bar{\bar{\phi}}]$. L7.5 constrains decomposition in this crucial case.

Lemma 7.2 *If* $[v^1.v^2.\cdots] \sim c$*, then* $c = [w^1.w^2.\cdots]$*, and*$(v^i \sim w^i)$*, for some* (w^i).

Proof We show by induction on $\sim$ that, if $[v^1.v^2.\cdots] \sim c$ or $c \sim [v^1.v^2.\cdots]$, then $(\exists w^1, w^2, \ldots)c = [w^1.w^2.\cdots]$, and $(v^i \sim w^i)$. The effect of this proof procedure is to make the case of symmetry (which is implicit in our requirement that $\sim$ be an equivalence relation) a trivial consequence of IH. (We often employ this simple technique implicitly in proving results concerning $\sim$ below.)

(Pairing): Not relevant.
(Comp): Trivial.
($\top^\wedge$): $[v.w.\cdots] \neq \top^\wedge$ and $[v.w.\cdots] \neq (., \emptyset)$.
($\Rrightarrow$): $[v.w.\cdots]$ has neither the form $[\bar{\bar{\phi}}]$ nor the form $[\bar{\bar{\phi}} + \bar{\bar{\psi}}]$.
(Transitivity): Suppose $[v.w.\cdots] \sim b \sim c$. IH implies the result. Similarly, IH implies the result if $c \sim b \sim [v.w.\cdots]$ □

A simple induction on $\sim$ establishes the following lemma. Note that $\psi \neq \bar{\bar{\psi}}$: no sentence ψ is a free content, only literals are free conditions, and no literal is a free choice or combination.

Lemma 7.3

1. *If* $\psi \sim a$*, then either* $a = \psi$ *or* $(\psi = \top^\wedge$ *and* $a = (., \emptyset))$.
2. *If* $(., \emptyset) \sim a$*, then either* $a = (., \emptyset)$ *or* $a = \top^\wedge$.
3. *If* $\neg\psi \sim a$*, then* $a = \neg\psi$.

A simple induction on $\sim$ also establishes the following lemma. Note that $[v^1 + v^2 + v^3 + \cdots]$ has at least three constituents, and so does not have the form $[v + w]$.

Lemma 7.4

1. *If* $[v^1 + v^2 + v^3 \ldots] \sim c$*, then* $(c = [w^1 + w^2 + \cdots]$*, and* $(v^i \sim w^i))$*, for some* (w^i).
2. *If* $[v] \sim c$ *or* $[v + w] \sim c$*, then either* $c = [v']$ *or* $c = [v' + w']$*, for some* v', w'.

Lemma 7.5

1. *if* $[v + w] \sim [v' + w']$*, then either*
   (a) $v \sim v'$ *and* $w \sim w'$*; or*
   (b) $v \sim v' \sim \bar{\bar{\psi}}$*,* $w' \sim \bar{\bar{\phi}}$ *and* $\phi \Rrightarrow \psi$*, for some* ϕ, ψ.
2. *If* $[v + w] \sim [v']$ *or* $[v'] \sim [v + w]$*, then* $v' \sim v \sim \bar{\bar{\psi}}$*,* $w \sim \bar{\bar{\phi}}$ *and* $\phi \Rrightarrow \psi$*, for some* ϕ, ψ.
3. *if* $[v] \sim [v']$*, then* $v \sim v'$.

Proof We prove all three results simultaneously by induction on $\sim$. Each of the cases in D4.5 is either irrelevant or trivial, except for transitivity. For the case of transitivity, by L7.4(2.), there are 8 cases: for some v', w', v'', w''

1. $[v + w] \sim [v' + w'] \sim [v'' + w'']$;
2. $[v + w] \sim [v' + w'] \sim [v'']$;
3. $[v + w] \sim [v'] \sim [v'' + w'']$;
4. $[v + w] \sim [v'] \sim [v'']$;
5. $[v] \sim [v' + w'] \sim [v'' + w'']$;
6. $[v] \sim [v' + w'] \sim [v'']$;
7. $[v] \sim [v'] \sim [v'' + w'']$; or
8. $[v] \sim [v'] \sim [v'']$;

By considerations of symmetry, there are essentially only four cases, typified by (1), (2), (4), and (8).

(1.): By IH(1.), $v' \sim v$ and either (A) $w' \sim w$ or (B) $v' \sim \bar{\bar{\psi}}'$, $w' \sim \bar{\bar{\phi}}'$, and $\phi' \Rrightarrow \psi'$.

(A): By IH (1.) applied to $[v' + w'] \sim [v'' + w'']$, $v \sim v' \sim v''$, and either $w'' \sim w'$ or ($v'' \sim \bar{\bar{\psi}}''$, $w'' \sim \bar{\bar{\phi}}''$, and $\phi'' \Rrightarrow \psi''$), for some ϕ'', ψ''. In the former case, the result (1.)(a.) is satisfied. In the latter case, the result (1.)(b.) is satisfied.

(B): By IH (1.) again, $v \sim v' \sim v''$, and either $w'' \sim w'$ or ($v'' \sim \bar{\bar{\psi}}''$, $w'' \sim \bar{\bar{\phi}}''$, and $\phi'' \Rrightarrow \psi''$) for some ϕ'', ψ''. In the former case, $v \sim v' \sim v'' \sim \bar{\bar{\psi}}'$, $w'' \sim w' \sim \bar{\bar{\psi}}'$, and $\phi' \Rrightarrow \psi'$. So, the result (1.)(b.) is satisfied. In the latter case, the result (1.)(b.) is also satisfied.

(2.): By IH(1.), $v' \sim v$ and either (A) $w' \sim w$ or (B) $v' \sim \bar{\bar{\psi}}'$, $w' \sim \bar{\bar{\phi}}'$, and $\phi' \Rrightarrow \psi'$, for some ϕ', ψ'.

(A): By IH(2.), $v'' \sim v' \sim v$; also by IH(2), $v \sim v' \sim \bar{\bar{\psi}}$, $w \sim w' \sim \bar{\bar{\phi}}$ and $\phi \Rrightarrow \psi$, for some ϕ, ψ, so the result (2.) is satisfied;

(B): By IH(2.), $v'' \sim v' \sim v$, so the result (2.) is satisfed.

(4.): By IH, $\bar{\bar{\phi}} \sim v \sim v' \sim v''$, $\bar{\bar{\phi}} \sim w$, and $\phi \Rrightarrow \psi$, for some ϕ, ψ. So, the result (2.) is satisfied.

(8.): By IH(3.), $v \sim v' \sim v''$. So, the result (3.) is satisfied. □

The next series of lemmas culminate in L7.10, which says that our interpretation function $\bar{\cdot}$ (and also $\bar{\bar{\cdot}}$) is one-one.

Lemma 7.6 *If ϕ is atomic and $\bar{\bar{\phi}}_\oplus \sim \bar{\bar{\psi}}_\oplus$, then $\phi = \psi$.*

Proof Suppose ϕ is atomic and $\bar{\bar{\phi}}_\oplus = \bar{\bar{\psi}}_\oplus$. By D4.3, $\phi \sim \bar{\bar{\psi}}_\oplus$. By L7.3, either $\bar{\bar{\psi}}_\oplus = \phi$ or $\bar{\bar{\psi}}_\oplus = (., \emptyset)$. By D4.3, $\bar{\bar{\psi}}_\oplus = \phi$. Since, for all $\chi, \theta, \ldots, \phi \notin \{\neg\chi, [\bar{\bar{\chi}}], [\bar{\bar{\chi}}.\bar{\bar{\theta}}. \cdots], [\bar{\bar{\chi}} + \bar{\bar{\theta}} + \cdots]\}$, ψ is atomic and $\bar{\bar{\psi}}_\oplus = \psi$. □

Lemma 7.7 *If* $\overline{\overline{\neg\phi}} \sim \bar{\bar{\psi}}$*, then* ψ *has the form* $\neg\chi$*, where* $\bar{\bar{\phi}} \sim \bar{\bar{\chi}}$.

Proof Suppose $\overline{\overline{\neg\phi}} \sim \bar{\bar{\psi}}$. We first show that ψ has the form $\neg\chi$. It is useful to prove

(★) If $\overline{\overline{\neg\phi}} \sim \overline{\overline{\psi}}$, then ψ has either the form $\neg\chi$ or the form $(\psi^1 \wedge \psi^2 \wedge \ldots)$

by induction on the complexity of ψ:

$\boldsymbol{\psi}$ **atomic:** By L7.6, $\psi = \neg\phi$. $\bot$.
$\boldsymbol{\psi = (\psi^1 \vee \psi^2 \vee \ldots)}$: Then $[\overline{\overline{\neg\psi^1}}.\overline{\overline{\neg\psi^2}}. \cdots] = \bar{\bar{\psi}}_\ominus \sim \overline{\overline{\neg\phi}}_\ominus = [\bar{\bar{\phi}}]$. By L7.2, $[\overline{\overline{\neg\psi^1}}.\overline{\overline{\neg\psi^2}}. \cdots] \not\sim [\bar{\bar{\phi}}]$. $\bot$.

We can now use (★) to prove the result by induction on the complexity of ϕ. Suppose (for *reductio*) that ψ has the form $(\psi^1 \wedge \psi^2 \wedge \ldots)$.

$\boldsymbol{\phi}$ **atomic:** $\neg\phi = \overline{\overline{\neg\phi}}_\oplus \sim \bar{\bar{\psi}}_\oplus = [\bar{\bar{\psi}}^1.\bar{\bar{\psi}}^2. \cdots]$. By L7.3, $[\bar{\bar{\psi}}^1.\bar{\bar{\psi}}^2. \cdots] = \neg\phi$. $\bot$.
$\boldsymbol{\phi = \neg\phi'}$: $[\bar{\bar{\phi}}'] = \overline{\overline{\neg\phi}}_\oplus \sim \bar{\bar{\psi}}_\oplus = [\bar{\bar{\psi}}^1.\bar{\bar{\psi}}^2. \cdots]$. By L7.2, $[\bar{\bar{\phi}}'] \not\sim [\bar{\bar{\psi}}^1.\bar{\bar{\psi}}^2. \cdots]$. $\bot$.
$\boldsymbol{\phi = (\phi^1 \wedge \phi^2 \wedge \ldots)}$: $[\overline{\overline{\neg\phi^1}} + \overline{\overline{\neg\phi^2}} + \cdots] = \overline{\overline{\neg\phi}}_\oplus \sim \bar{\bar{\psi}}_\oplus = [\bar{\bar{\psi}}^1.\bar{\bar{\psi}}^2. \cdots]$. By L7.2, $[\overline{\overline{\neg\phi^1}} + \overline{\overline{\neg\phi^2}} + \cdots] \not\sim [\bar{\bar{\psi}}^1.\bar{\bar{\psi}}^2. \cdots]$. $\bot$.
$\boldsymbol{\phi = (\phi^1 \vee \phi^2 \vee \ldots)}$: $[\bar{\bar{\phi}}] = \overline{\overline{\neg\phi}}_\ominus \sim \bar{\bar{\psi}}_\ominus = [\overline{\overline{\neg\psi^1}} + \overline{\overline{\neg\psi^2}} + \cdots]$. By L7.5, $\bar{\bar{\phi}} \sim \overline{\overline{\neg\psi^1}}$. By (★), ϕ is either a negation or a conjunction. $\bot$.

So $\overline{\overline{\neg\phi}} \sim \overline{\overline{\neg\chi}}$, for some χ. Then $[\bar{\bar{\phi}}] = \overline{\overline{\neg\phi}}_\ominus \sim \overline{\overline{\neg\chi}}_\ominus = [\bar{\bar{\chi}}]$. By L7.5(3.), $\bar{\bar{\phi}} \sim \bar{\bar{\chi}}$. □

Lemma 7.8 *If* $\overline{\overline{(\phi^1 \wedge \phi^2 \wedge \ldots)}} \sim \bar{\bar{\psi}}$*, then* ψ *has the form* $(\psi^1 \wedge \psi^2 \wedge \ldots)$*, for* $(\bar{\bar{\phi}}^i \sim \bar{\bar{\psi}}^i)$.

Proof Suppose $\overline{\overline{(\phi^1 \wedge \phi^2 \wedge \ldots)}} \sim \bar{\bar{\psi}}$. We first prove by induction on the complexity of ψ that ψ has the form $(\psi^1 \wedge \psi^2 \wedge \ldots)$.

$\boldsymbol{\psi}$ **atomic:** By L7.6, $\psi = (\phi^1 \wedge \phi^2 \wedge \ldots)$. $\bot$.
$\boldsymbol{\psi = \neg\chi}$: By L7.7, $(\phi^1 \wedge \phi^2 \wedge \ldots)$ has the form $\neg\phi$. $\bot$.
$\boldsymbol{\psi = (\phi^1 \vee \phi^2 \vee \ldots)}$: $[\bar{\bar{\psi}}^1 + \bar{\bar{\psi}}^2 + \cdots] = \bar{\bar{\psi}}_\oplus \sim \bar{\bar{\phi}}_\oplus = [\bar{\bar{\phi}}^1.\bar{\bar{\phi}}^2. \cdots]$. By L7.2, $[\bar{\bar{\psi}}^1 + \bar{\bar{\psi}}^2 + \cdots] \not\sim [\bar{\bar{\phi}}^1.\bar{\bar{\phi}}^2. \cdots]$. $\bot$.

By D4.3 and L7.2, $(\bar{\bar{\phi}}^i \sim \bar{\bar{\psi}}^i)$. □

Lemma 7.9 *If* $\overline{\overline{(\phi^1 \vee \phi^2 \vee \ldots)}} \sim \bar{\bar{\psi}}$*, then* ψ *has the form* $(\psi^1 \vee \psi^2 \vee \ldots)$*, where* $(\bar{\bar{\phi}}^i \sim \bar{\bar{\psi}}^i)$.

Proof Suppose $\overline{\overline{\phi^1 \vee \phi^2 \vee \ldots}} \sim \bar{\bar{\psi}}$. We first prove by induction on the complexity of ψ that ψ has the form $(\psi^1 \vee \psi^2 \vee \ldots)$.

$\boldsymbol{\psi}$ **atomic:** By L7.6, $\psi = (\phi^1 \vee \phi^2 \vee \ldots)$. $\bot$.

$\boldsymbol{\psi = \neg\chi}$: By L7.7, $(\phi^1\vee\phi^2\vee\dots)$ has the form $\neg\phi$. $\bot$.
$\boldsymbol{\psi = (\psi^1\wedge\psi^2\wedge\dots)}$: By L7.8, $(\phi^1\vee\phi^2\vee\dots)$ has the form $(\chi^1\wedge\chi^2\wedge\dots)$. $\bot$.

So, $\overline{\overline{(\phi^1\vee\phi^2\vee\dots)}} \sim \overline{\overline{(\psi^1\vee\psi^2\vee\dots)}} = \bar{\bar{\psi}}$. Then $[\overline{\overline{\neg\phi^1}}.\overline{\overline{\neg\phi^2}}.\cdots] = \overline{\overline{(\phi^1\vee\phi^2\vee\dots)}}_{\ominus} \sim \bar{\bar{\psi}}_{\ominus} = [\overline{\overline{\neg\psi^1}}.\overline{\overline{\neg\psi^2}}.\cdots]$. By L7.2, $(\overline{\overline{\neg\phi^i}} \sim \overline{\overline{\neg\psi^i}})$. By L7.7, $(\bar{\bar{\phi}}^i \sim \bar{\bar{\psi}}^i)$. □

Lemma 7.10

1. *If* $\bar{\bar{\phi}} \sim \bar{\bar{\psi}}$, *then* $\phi = \psi$.
2. *If* $\bar{\phi} = \bar{\psi}$, *then* $\phi = \psi$.

Proof Suppose $\bar{\bar{\phi}} = \bar{\bar{\psi}}$. We prove (1) by induction on the complexity of ϕ.

$\boldsymbol{\phi}$ **atomic:** L7.6.
$\boldsymbol{\phi = \neg\chi}$: By L7.7, ψ has the form $\neg\gamma$ and $\bar{\bar{\chi}} \sim \bar{\bar{\gamma}}$. IH implies that $\chi = \gamma$.
$\boldsymbol{\phi = (\phi^1\wedge\phi^2\wedge\dots)}$: By L7.8, ψ has the form $(\psi^1\wedge\psi^2\wedge\dots)$ and $(\bar{\bar{\phi}}^i \sim \bar{\bar{\psi}}^i)$. By IH, $(\phi^i = \psi^i)$.
$\boldsymbol{\phi = (\phi^1\vee\phi^2\vee\dots)}$: As in the previous case, L7.9 and IH imply $\phi = \psi$.

(2) is an immediate consequence of (1) and D4.3 since, if $\bar{\phi} \sim \bar{\psi}$ then $\bar{\bar{\phi}} \sim \bar{\phi} \sim \bar{\psi} \sim \bar{\bar{\psi}}$. □

Definition 7.11 (Immediate Selection) We define a relation $\ll_S$ between sets of contents of F_S and conditions of F_S in the obvious way:

1. $v^i \ll_S [v^0 + \cdots]_g$ for each i,
2. $v, \cdots \ll_S [v. \cdots]_g$,
3. $v \ll_S [v]_g$ and
4. $G \not\ll_S g((+, \emptyset))$

We allow the special case in which $\langle v, \dots\rangle = \emptyset$. In this case, $\emptyset \ll_S [\,]_g$.

Because choices $[v^0 + v^1 + \cdots]_g$ and singleton choices/combinations $[v]_g$ may not be uniquely decomposable, there can be surprising immediate selections. For instance, there can be cases in which $v \ll_S [w]_g$, but $v \not\sim w$. But there are sharp limits on such surprises. Lemmas 7.12–7.21 below specify those limits. Lemmas 7.12–7.16 specify the limitations on immediate selection when the right-hand relatum is the truth-condition $\bar{\bar{\psi}}_{\oplus}$ of some sentence ψ of $\mathscr{L}^+$. Lemmas 7.18–7.21 specify those limitations in the other cases.

Lemma 7.12 *If* $G \ll_S [\bar{\phi}^0.\bar{\phi}^1.\cdots]_g$, *then* $G = (\bar{\phi}^i)$.

Proof Suppose that $G \ll_S [\bar{\phi}^0.\bar{\phi}^1.\cdots]_g$. There are two cases: (I) $[\bar{\phi}^0.\bar{\phi}^1.\cdots]_g = [v^0.v^1.\cdots]_g$ and $G = (v^i)$, for some $(v^i) \subseteq (F_S \times F_S)$; or (II) $[\bar{\phi}^0.\bar{\phi}^1.\cdots]_g \in \{[v^0 + v^1 + \cdots]_g, [v]_g\}$ and $G \in \{v, (v^i)\}$, for some $v, (v^i) \subseteq (F_S \times F_S)$. By L7.2, case (II) does not occur. In case (I), $[\bar{\phi}^0.\bar{\phi}^1.\cdots]_g = g([v^0.v^1.\cdots]) \sim [v^0.v^1.\cdots]$.

By L7.2, $(\bar{\phi}^i \sim v^i)$, so (by L7.1), $(g(v^i) = g(\bar{\phi}^i) = \bar{\phi}^i)$. Since $(v^i) \subseteq (F_S \times F_S)$, $(v^i = g(v^i))$. So, $G = (v^i) = (\bar{\phi}^i)$. □

The following lemma is proved in a way similar to L7.12, using L7.4 in place of L7.2:

Lemma 7.13 *If* $G \ll_S [\bar{\phi}^0 + \bar{\phi}^1 + \bar{\phi}^2 + \cdots]$, *then* $G = \bar{\phi}^i$, *for some* i.

Lemma 7.14 *If* $G \ll_S [\bar{\phi}^1 + \bar{\phi}^2]_g$, *then* $(\exists\delta, i)(G = \bar{\delta}$ *and* $\delta \leq \phi^i \in S^*)$.

Proof If $G = \bar{\phi}^i$, for some $i = 1, 2$, then we may set $\delta = \phi^i$ and $\phi^i \leq \phi^i \in S^*$ by D6.3. So, it is enough to show that, if $G \ll_S [\bar{\phi}^1 + \bar{\phi}^2]_g$, then either $G \in \{\bar{\phi}^1, \bar{\phi}^2\}$ or $(\exists\delta)(G = \bar{\delta}$ and $\delta \leq \phi^1 \in S^*$. By D7.11 there are three cases: (A) $[\bar{\phi}^1 + \bar{\phi}^2]_g = [v^0.v^1. \cdots]_g$ and $G = (v^j)$ for some $(v^j) \subseteq (F_S \times F_S)$; (B) $[\bar{\phi}^1 + \bar{\phi}^2]_g = [v]_g$ and $G = v$, for some $v \in (F_S \times F_S)$; or (C) $[\bar{\phi}^1 + \bar{\phi}^2]_g = [v^1 + v^2 + \cdots]_g$ and $G = v^j$, for some $(v^j) \in (F_S \times F_S)$ and some j. By L7.2, case (A) does not occur.

(B): By L7.5, there is a ψ such that $v \sim \bar{\phi}^1 \sim \bar{\bar{\psi}}$. So, $G = v = g(v) = \bar{\phi}^1$.
(C): By L7.4, $[v^1 + v^2 + \cdots]$ has the form $[v^1 + v^2]$. By L7.5 there are two cases: (I) $v^1 \sim \bar{\phi}^1$ and $v^2 \sim \bar{\phi}^2$ or (II) $v^1 \sim \phi^1 \sim \bar{\bar{\psi}}$, $v^2 \sim \bar{\bar{\delta}}$, and $\delta \Rrightarrow \psi$, for some δ, ψ.

(I): Either $G = v^1 = g(v^1) = \bar{\phi}^1$ or $G = v^2 = g(v^2) = \bar{\phi}^2$.
(II): Since $\bar{\phi}^1 \sim \bar{\bar{\psi}}$, by L7.10, $\phi^1 = \psi$. If $G = v^1$, then $G = v^1 = g(v^1) = \bar{\phi}^1$. If $G = v^2$, then $G = v^2 = g(v^2) = \bar{\delta}$ and $\delta \Rrightarrow \phi^1$, for some δ. By D6.1($\Rrightarrow$), $\delta \leq \phi^1 \in S^*$. □

The following lemma is proved similarly to L7.14.

Lemma 7.15 *If* $G \ll_S [\bar{\phi}]_g$, *then* $(\exists\delta)(G = \bar{\delta}$ *and* $\delta \leq \phi \in S^*)$.

Lemma 7.16 *If* $G \ll_S \bar{\phi}_\oplus$, *then* $(\exists\Delta)(G = \bar{\Delta}$ *and* $\Delta < \phi \in S^*)$.

Proof Suppose $G \ll_S \bar{\phi}_\oplus$. We prove the result by induction on the complexity of ϕ.

ϕ atomic: By L7.3, $G = \emptyset$ and $\phi = \top^\wedge$. By $(\leq)(\top^\wedge)$, $\emptyset \leq \top^\wedge$. Also, trivially, $\emptyset \leq \top^\wedge$ is irreversible. So, by D6.3, $\emptyset < \top^\wedge \in S^*$.
$\phi = \neg\chi$, χ atomic: By L7.7, $\overline{\neg\chi}_\oplus = \neg\chi \notin \{[v]_g, [v + \cdots]_s, [v. \cdots]_g\}$, for any v, w. So, $G \not\ll_S \bar{\phi}_\oplus$. $\perp$.
$\phi = \neg\neg\chi$: $\overline{\neg\neg\chi}_\oplus = [\bar{\chi}]_g$. By L7.15, $(\exists\delta)(G = \bar{\delta}$ and $\delta \leq \chi \in S^*)$. By T6.19, $\delta < \phi \in S^*$.
$\phi = (\phi^1 \wedge \phi^2 \wedge \ldots)$: L7.12 and T6.19 imply the result.
$\phi = \neg(\phi^1 \vee \phi^2 \vee \ldots)$: L7.12 and T6.19 imply the result.
$\phi = (\phi^1 \vee \phi^2 \vee \ldots)$: L7.14 and T6.19imply the result.
$\phi = \neg(\phi^1 \wedge \phi^2 \wedge \ldots)$: L7.14 and T6.19 imply the result. □

Definition 7.17 a is *formularic* iff $a \sim \bar{\bar{\phi}}_\oplus$, for some $\phi \in \mathscr{L}^+$. (a, b) is *formularic* iff a and b are each formularic.

The following lemma is proved by an easy induction on $\sim$:

Lemma 7.18

1. *If $a \sim b$ and a is formularic, then b is formularic.*
2. *If $v \sim w$ and v is formularic, then w is formularic.*

Lemma 7.19

1. *If $[v]$ is formularic, then $v \sim \bar{\bar{\psi}}$, for some $\psi \in \mathscr{L}^+$.*
2. *If $[w + \cdots + v + \cdots]$ is formularic, then $v \sim \bar{\bar{\psi}}$, for some $\psi \in \mathscr{L}^+$.*
3. *If $[w. \cdots .v. \cdots]$ is formularic, then then $v \sim \bar{\bar{\psi}}$, for some $\psi \in \mathscr{L}^+$.*

Proof Suppose $a \in \{[v], [w. \cdots .v. \cdots], [w + \cdots + v + \cdots]\}$ and a is formularic. Then $a \sim \bar{\bar{\phi}}_\oplus$, for some $\phi \in \mathscr{L}^+$. We prove the result by induction on the complexity of ϕ.

ϕ atomic: Then $a \sim \phi$. By L7.3, $a = \phi$ or $a = (., \emptyset)$. $\perp$.
$\phi = \neg\chi$, χ atomic: By L7.3, $a = \neg\chi$. $\perp$.
$\phi = \neg\neg\chi$: Then $a \sim \bar{\bar{\phi}}_\oplus = [\bar{\bar{\chi}}]$. By L7.5, either $v \sim \bar{\bar{\chi}}$ or $v \sim \bar{\bar{\theta}}'$, for some θ'.
$\phi = (\phi^1 \wedge \phi^2 \wedge \ldots)$: L7.2.
$\phi = \neg(\phi^1 \vee \phi^2 \wedge \ldots)$: L7.2.
$\phi = (\phi^1 \vee \phi^2 \vee \ldots)$: By L7.4 and L7.5, $v \sim \bar{\bar{\theta}}$, for some θ.
$\phi = \neg(\phi^1 \wedge \phi^2 \wedge \ldots)$: By L7.4 and L7.5, $v \sim \bar{\bar{\theta}}$, for some θ. □

Remark An immediate consequence of the defintion D4.2 is that, if a is not formularic, then $a = (+, \emptyset)$ or a has one of the forms: $[v], [v^0 + v^1 + \cdots]$, or $[v^0.v^1. \cdots]$.

Lemma 7.20 ***(Unique Decomposition)***

1. *If $[v]$ is not formularic and $[v] \sim a$, then, for some v', $a = [v']$ and $v' \sim v$;*
2. *If $[v^0 + v^1 + \cdots]$ is not formularic and $[v^0 + v^1 + \cdots] \sim a$, then, for some (w^i), $a = [w^0 + w^1 + \cdots]$ and $(v^i \sim w^i)$;*
3. *If $[v^0.v^1. \cdots]$ is not formularic and $[v^0.v^1. \cdots] \sim a$, then, for some (w^i), $a = [w^0.w^1. \cdots]$ and $(v^i \sim w^i)$; and*
4. *If $(+, \emptyset) \sim a$, then $a = (+, \emptyset)$.*

Proof All of the cases are proved similarly. We do (2.) for illustration. Suppose $[v^0 + v^1 + \cdots]$ is not formularic and $[v^0 + v^1 + \cdots] \sim a$. We prove the result by induction on $\sim$:

(Pairing): $[v^0 + v^1 + \cdots]$ does not have the form (a, b), for free conditions a, b.
(Comp): Trivial.
(⊤^): Neither $\top^\vee$ nor $(., \emptyset)$ have the form $[v^0 + v^1 + \cdots]$.
(⇛): Both $[\bar{\bar{\psi}}](= \overline{\overline{\neg\neg\psi}}_\oplus)$ and $[\bar{\bar{\psi}} + \bar{\bar{\phi}}](= \overline{\overline{(\psi \vee \phi)}}_\oplus)$ are formularic.

(Transitivity): IH immediately implies the result. □

An immediate consequence of L7.20 is that the free conditions a exhaustively partition into: (i): $(+, \emptyset)$, (ii) the formularic conditions $\bar{\bar{\phi}}_{\oplus}$ for sentences ϕ of $\mathscr{L}^+$, and (iii) uniquely decomposable (up to $\sim$), non-formularic choices and combinations. Thus, L7.16 and the following lemma together provide helpful necessary and sufficient conditions for the immediate selection relation $\ll_S$ to obtain.

Lemma 7.21 *For* $v, (v^i) \subseteq (F_S \times F_S)$*:*

1. *If* $G \ll_S [v]_g$ *and* $[v]$ *is not formularic, then* $G = v$*;*
2. *If* $G \ll_S [v^0 + v^1 + \cdots]_g$ *and* $[v^0 + v^1 + \cdots]$ *is not formularic, then* $G = v^i$*, for some* i*; and*
3. *If* $G \ll_S [v^0.v^1. \cdots]_g$ *and* $[v^0.v^1. \cdots]$ *is not formularic, then* $G = (v^i)$

Proof Each of the claims is proved similarly, using L7.20. We do (1.) for illustration. Suppose $G \ll_S [v]_g$ and $[v]$ is not formularic. Then there are three cases: (A) $[v] \sim [w]$ and $G = w = g(w)$, for some w; (B) $[v] \sim [w^0 + w^1 + \cdots]$ and $G = w^i = g(w^i)$, for some (w^i) and some i; or (C) $[v] \sim [w^0.w^1. \cdots]$ and $G = (w^i) = (g(w^i))$, for some (w^i). By L7.20, cases (B) and (C) do not occur, and $G = w = g(w) \sim v$, for some w. So, $G = w = g(w) = g(v) = v$. □

We are now ready to show that there is an exact corespondence between selection in $\mathfrak{M}_S$ and the members of the canonical model basis S^*. D7.22–T7.30 establish this result.

We define the class of $\mathfrak{M}_S$-derivations of selections $G <_S v$ using the following axiom and rules, which correspond to the clauses of the definition D2.1 of selection. As before, a selection is of the form $G \leq_S v$ iff it is of the form $G <_S ([v]_g, d)$, for some d:

Definition 7.22

1. **Basis**: $G <_S v$ is an axiom whenever $G \ll_S v_{\oplus}$;
2. **Ascent**: $\dfrac{G <_S v}{G <_S ([v]_g, d)}$ for any d.
3. **Lower Cut**: $\dfrac{(G^i \leq_S v^i)_{i<n\in\omega} \quad (v^i) <_S v}{(G^i) <_S v}$
4. **Upper Cut**: $\dfrac{(G^i <_S v^i)_{i<n\in\omega} \quad (v^i) \leq_S v}{(G^i) <_S v}$

The notions of the major premise and minor premises of applications of UPPER CUT and LOWER CUT are defined in the obvious way. We will often write $G <_S v$ to indicate that there is an $\mathfrak{M}_S$-derivation of $G <_S v$, and $G \leq_S v$ to indicate that there is an $\mathfrak{M}_S$-derivation of $G <_S ([v]_g, d)$, for some free condition d.

Remark **(Amalgamation)**: Since $v \ll_S [v]_g$, $v <_{\mathfrak{M}_S} ([v]_g, d)$ (*i.e.* $v \leq_S v$) is an axiom for all $v \subset F_S \times F_S$ and $d \in F_S$, it follows that

$$\boxed{\frac{(G^i <_S v)_{i<n\in\omega} \quad v \leq_S v}{(G^i) <_S v}}$$

is an instance of Upper Cut and

$$\boxed{\frac{(G^i \leq_S v)_{i<n\in\omega} \quad v <_S v}{(G^i) \leq_S v}}$$

is an instance of Lower Cut. So, if $(G^i <_S v)$, then $(G^i) <_S v$; and if $(G^i \leq_S v)$, then $(G^i) \leq_S v$.

Definition 7.23 An $\mathfrak{M}_S$-derivation is in *semi-normal form* (or *is semi-normal*) iff every major premise of every application of upper cut or lower cut is an axiom.

An argument broadly similar to the proof of L5.4(Semi-Normal Form Lemma) yields a similar result for $\mathfrak{M}_S$-derivations:

Lemma 7.24 ***(Semi-Normal Form Lemma)*** *If there is an $\mathfrak{M}_S$-derivation of $G <_S v$, then there is a semi-normal $\mathfrak{M}_S$-derivation of $G <_S v$.*

Lemma 7.25 *If $\bar{\bar{\phi}}_\oplus \sim \bar{\bar{\psi}}_\oplus$ and $\Delta < \phi \in S^*$, then $\Delta < \psi \in S^*$.*

Proof Suppose $\bar{\bar{\phi}}_\oplus \sim \bar{\bar{\psi}}_\oplus$ and $\Delta < \phi \in S^*$. There are six syntactic forms ϕ may have.

$\boldsymbol{\phi}$ atomic: By L7.6, $\psi = \phi$.
$\boldsymbol{\phi = \neg\chi, \chi}$ atomic: By L7.3(3.) and D4.3, $\psi = \phi$.
$\boldsymbol{\phi = \neg\neg\phi'}$: $\bar{\bar{\phi}}_\oplus = [\bar{\bar{\phi}}'] \sim \bar{\bar{\psi}}_\oplus$. By L7.4(2.), either (A) $\bar{\bar{\psi}}_\oplus = [v]$, for some v, or (B) $\bar{\bar{\psi}}_\oplus = [v + w]$, for some v and w.

- **(A):** By L7.5(3.), $v \sim \bar{\bar{\phi}}'$. Inspection of D4.3 shows that ψ must have the form $\neg\neg\psi'$, where $v = \overline{\overline{\psi'}}$. To illustrate, suppose (for *reductio*) that ψ is a disjunction $(\psi^1 \vee \psi^2 \vee \dots)$. Then $\bar{\bar{\psi}}_\oplus = [\bar{\bar{\psi}}^1 + \bar{\bar{\psi}}^2 + \cdots] = [v]$. $\perp$. Similar arguments show that ψ cannot be a literal, a conjunction, the negation of a disjunction, nor the negation of a conjunction. So, $\psi = \neg\neg\psi'$, for some ψ', and so $\bar{\bar{\psi}}_\oplus = [\overline{\overline{\psi'}}] = [v]$. So, $\overline{\overline{\psi'}} \sim \overline{\overline{\phi'}}$. By L7.10, $\psi' = \phi'$ and so $\psi = \phi$.
- **(B):** By L7.5(2.), $v \sim \bar{\phi}'$. As in the previous case, the fact that $\bar{\bar{\psi}}_\oplus$ has the form $[v + w]$, implies that there are two cases: (I) $v = \bar{\bar{\psi}}^1$, $w = \bar{\bar{\psi}}^2$, and $\psi = (\psi^1 \vee \psi^2)$, for some ψ^1, ψ^2; or (II): $v = \overline{\overline{\neg\psi^1}}$, $w = \overline{\overline{\neg\psi^2}}$, and $\psi = \neg(\psi^1 \wedge \psi^2)$, for some ψ^1, ψ^2.
  - **(I):** By L7.10, $\psi^1 = \phi'$. By T6.19, since $\Delta < \neg\neg\phi' \in S^*$, $\Delta \leq \phi' \in S^*$, and so $\Delta < (\phi' \vee \psi^2)(= (\psi^1 \vee \psi^2) = \psi) \in S^*$.
  - **(II):** By L7.10, $\neg\psi^1 = \phi'$. By T6.19, since $\Delta < \neg\neg\phi' \in S^*$, $\Delta \leq \neg\psi^1 \in S^*$, and so $\Delta < \neg(\psi^1 \wedge \psi^2) \in S^*$.

$\phi = (\phi^1 \wedge \phi^2 \wedge \ldots)$: As above, L7.2, L7.10, and T6.19 imply the result.
$\phi = \neg(\phi^1 \vee \phi^2 \wedge \ldots)$: As above, L7.2, L7.10, and T6.19 imply the result.
$\phi = (\phi^1 \vee \phi^2 \vee \ldots)$: There are two cases: (A) (ϕ^i) has more than two members, so that ϕ has the form $(\phi^1 \vee \phi^2 \vee \phi^3 \vee \ldots)$ or (B.) ϕ has the form $(\phi^1 \vee \phi^2)$.

(A): As above, L7.4(1.), D4.3, L7.10, and T6.19 imply the result.
(B): $\phi = (\phi^1 \vee \phi^2)$ and $\bar{\bar{\psi}}_\oplus \sim \bar{\bar{\phi}}_\oplus = [\bar{\bar{\phi}}^1 + \bar{\bar{\phi}}^2]$. So, by L7.4(2.) and L7.5(1.),(2.), there are two cases: (I) $\bar{\bar{\psi}}_\oplus$ has the form $[v^1 + v^2]$, where $(v^i \sim \bar{\bar{\phi}}^i)$; or (II) $\bar{\bar{\psi}}_\oplus$ has the form $[v]$, where there are θ^1, θ^2 such that $v \sim \bar{\bar{\phi}}^1 \sim \theta^1$, $\phi^2 \sim \theta^2$, and $\theta^1 \Rrightarrow \theta^2$.

(I): The argument in case (A) above yields the result.
(II): By L7.10, $\phi^1 = \theta^1$, $\phi^2 = \theta^2$, and so $\phi^2 \Rrightarrow \phi^1$. By $(\leq')(\Rrightarrow)$, $\phi^2 \leq \phi^1 \in S^*$. By T6.19, since $\Delta < (\phi^1 \vee \phi^2) \in S^*$, either $\Delta \leq \phi^1$, $\Delta \leq \phi^2$, or both $\Delta^1 \leq \phi^1$ and $\Delta^2 \leq \phi^2$ (where $\Delta = \Delta^1, \Delta^2$) is a member of S^*. In each case, the closure of S^* implies that $\Delta < \neg\neg\phi^1 \in S^*$. As above, since $\bar{\bar{\psi}}_\oplus$ has the form $[v]$, D4.3 constrains the form of ψ: ψ must be of the form $\neg\neg\chi$, where $v = \bar{\bar{\chi}}$. Since $\bar{\bar{\chi}} = v \sim \bar{\bar{\phi}}^1$, L7.10 implies that $\chi = \phi^1$. So, $\Delta < \neg\neg\phi^1 (= \neg\neg\chi = \psi) \in S^*$.

$\phi = \neg(\phi^1 \wedge \phi^2 \wedge \ldots)$: As above, L7.4, L7.5, L7.10, and T6.19 imply the result.

□

Lemma 7.26

1. *If $G <_S (\bar{\phi}_\oplus, d)$, then there is a $\Delta \subseteq \mathscr{L}^+$ such that $G = \bar{\Delta}$ and $\Delta < \phi \in S^*$.*
2. *If $G <_S \bar{\phi}$, then there is a $\Delta \subseteq \mathscr{L}^+$ such that $G = \bar{\Delta}$ and $\Delta < \phi \in S^*$.*
3. *If $G \leq_S \bar{\phi}$, then there is a $\Delta \subseteq \mathscr{L}^+$ such that $G = \bar{\Delta}$ and $\Delta \leq \phi \in S^*$.*

Proof (2.) and (3.) follow from (1.) and T6.19. We prove (1.) by induction on $\mathfrak{M}_S$-derivations $\mathcal{D}$ of $G <_S (\bar{\phi}_\oplus, d)$. By L7.24, we may assume that $\mathcal{D}$ is semi-normal.

(Basis): Suppose $G \ll_S (\bar{\phi}_\oplus, d)$. L7.16.
(Ascent): Suppose $\mathcal{D}$ has the form:

$$\dfrac{\dfrac{\mathcal{E}}{G <_S w}}{G <_S ([w]_g, b)}$$ where $[w]_s = \bar{\phi}_\oplus$.

By L7.19, $g(w) = \bar{\psi}$, for some $\psi \in \mathscr{L}^+$. So, IH applies to $\mathcal{E}$: $G = \bar{\Delta}$ and $\Delta < \psi \in S^*$, for some Δ. Also, $\bar{\psi} = w \ll_S (\bar{\phi}_\oplus, d)$, so L7.16 implies that $\psi < \phi \in S^*$. The result follows by T6.19.
(Lower Cut): Suppose $\mathcal{D}$ has the form

$$\dfrac{\left(\dfrac{\mathcal{F}^i}{G^i \leq_S v^i}\right) \quad \dfrac{}{(v^i) <_S (\bar{\phi}_\oplus, d)}}{(G^i) <_S (\bar{\phi}_\oplus, d)}$$

By IH, for some $(\bar{\delta}^i)$, $((v^i) = (\bar{\delta}^i)$ and $(\delta^i) < \phi \in S^*)$; and (since $[\bar{\delta}^i]_g = \overline{\neg\neg\delta^i_\oplus}$) $(\exists\bar{\Delta}^i)(G^i = \bar{\Delta}^i$ and $\Delta^i < \neg\neg\delta^i \in S^*)$, for each i. The result follows by T6.19.

(Upper Cut): Suppose $\mathcal{D}$ has the form

$$\dfrac{\left(\dfrac{\mathcal{F}^i}{G^i <_S v^i}\right) \qquad \dfrac{}{(v^i) \leq_S (\bar{\phi}_\oplus, d)}}{(G^i) <_S (\bar{\phi}_\oplus, d)}$$

Since $(v^i) \leq_S (\bar{\phi}_\oplus, d)$ is an axiom, $(v^i) \ll_S [(\bar{\phi}_\oplus, d)]$. There are two cases: (A) $(\bar{\phi}_\oplus, d) \sim \bar{\bar{\psi}}$, for some ψ, or (B) not.

(A): $[(\bar{\phi}_\oplus, d)]_g = [\bar{\bar{\psi}}]_g = \overline{\neg\neg\psi}_\oplus$. By L7.16, since $(v^i) \ll_S \overline{\neg\neg\psi}_\oplus$, $((v^i) = (\bar{\delta}^i))$ and $(\delta^i) < \neg\neg\psi \in S^*)$, for some $(\bar{\delta}^i)$. Also, IH, applies to $(G^i <_S v^i)$ to imply that $(\exists\bar{\Delta}^i)(G^i = \bar{\Delta}^i$ and $\Delta^i < \delta^i \in S^*)$, for each i. By T6.19, $(\Delta^i) < \psi \in S^*$.Since $\bar{\psi}_\oplus = \bar{\phi}_\oplus =$, the result follows by L7.25.

(B): Since $(v^i) \ll_S [(\bar{\phi}_\oplus, d)]_g$, by L7.19 and L7.20(Unique Decomposition), $(v^i) = (\bar{\phi}_\oplus, d)$. So, IH applies to the minor premises: for each i $(\exists\Delta^i)(G^i = \bar{\Delta}^i$ and $\Delta^i < \phi \in S^*)$. The result follows by T6.19. □

Lemma 7.27 *If $\Delta < \phi \in S$ (not merely S^*), then (1.) $\overline{v^{\Delta,\phi}} \leq_S \bar{\phi}$, and (2.) $\bar{\Delta}, \overline{\top^\vee} <_S \bar{\phi}$.*

Proof

1. Suppose $\Delta < \phi \in S$. Then, by D4.4, $v^{\Delta,\phi} \Rrightarrow \phi$. By D4.5 and D4.3, $[\bar{\phi}]_g = [\bar{\phi} + \overline{v^{\Delta,\phi}}]_g$. So, $\overline{v^{\Delta,\phi}} \ll_S [\bar{\phi}]_g$. The result follows by D7.22.
2. By D4.3, D4.5, D4.6 and D7.22, $\bar{\Delta}$, $\overline{(\top^\wedge\vee/\phi/)}$, $\overline{\top^\vee} <_S \overline{v^{\Delta,\phi}}$ is an axiom. Also, by D7.22, $\emptyset <_S \overline{\top^\wedge} <_S \overline{(\top^\wedge\vee/\phi/)}$. So, the result follows by (1.) and D7.22.

□

Lemma 7.28 *If $\Delta \leq' \phi$, then there is an $\mathfrak{M}_S$-derivation of $\bar{\Delta} \leq_S \bar{\phi}$.*

Proof By induction on $\Delta \leq' \phi$. It is useful to first prove

(★) If $\phi \Rrightarrow \psi$, then $\bar{\phi} \leq_S \bar{\psi}$; $\bar{\psi}, \overline{/\phi/} \leq_S \bar{\phi}$; and $\bar{\phi}, \overline{/\phi/} \leq_S \overline{/\phi/}$.

Suppose $\phi \Rrightarrow \psi$. Then, by D4.5 $[\bar{\bar{\psi}}] \sim [\bar{\bar{\psi}} + \bar{\bar{\phi}}]$. So, $\bar{\phi} \ll_S [\bar{\psi}]_g$. So, $\bar{\phi} \leq_S \bar{\psi}$ is an axiom. Since $\phi \Rrightarrow \psi$, $(\psi\wedge/\phi/) \Rrightarrow \neg\neg\phi$ by D4.4(Induction). So, $\overline{(\psi\wedge/\phi/)} \leq_S \overline{\neg\neg\phi}$. D4.3 and D7.22 then imply that there is an $\mathfrak{M}_S$-derivation (using (UPPER CUT)) of $\bar{\psi}, \overline{/\phi/} \leq_S \bar{\phi}$. A similar argument establishes that $\bar{\phi}, \overline{/\phi/} \leq_S \overline{/\phi/}$.

(ID): Immediate by D7.22, since $\bar{\phi} \ll_S [\bar{\phi}]_g$.

($\top^\wedge$): By D4.5 and D4.3, $\emptyset \ll_S \overline{\top^\vee}_\oplus$. The result follows by D7.22.

(Determination): All of the cases are proved similarly. We do the case of conjunction for illustration. $\phi, \psi, \cdots \leq' (\phi\wedge\psi\wedge\ldots)$. By D4.3, $\overline{(\phi\wedge\psi\wedge\ldots)}_\oplus = [\bar{\phi}.\bar{\psi}.\cdots]_g$. So, $\bar{\phi}, \bar{\psi}, \cdots \ll_S \overline{(\phi\wedge\psi\wedge\ldots)}_\oplus$. The result is immediate by D7.22.

($\Rrightarrow$): ($\bigstar$) implies the result.

(W): Suppose $\psi \preceq \phi \in S$. By D4.4, $(\psi \wedge w^{\phi}) \Rrightarrow \neg\neg\phi$. By ($\bigstar$), $\overline{(\psi \wedge w^{\phi})} \leq_S \overline{\neg\neg\phi}$. By the argument for the case (DETERMINATION) above, $\bar{\psi}, \overline{w^{\phi}} <_S \overline{(\psi \wedge w^{\phi})}$. So, by D7.22(UPPER CUT), $\bar{\psi}, \overline{w^{\phi}} <_S \overline{\neg\neg\phi}$. $\overline{\neg\neg\phi}$ has the form $([\bar{\phi}], d)$.

(Max): An argument similar to that for the case (W) above yields the result.

($\top^{\vee}$): By D4.4, $(\top^{\wedge} \wedge (\top^{\wedge} \vee /\phi/)) \Rrightarrow \top^{\vee}$. By ($\bigstar$), $\overline{(\top^{\wedge} \wedge (\top^{\wedge} \vee /\phi/))} \leq_S \overline{\top^{\vee}}$. By the argument for the case (DETERMINATION) above, $\overline{\top^{\wedge}}, \overline{(\top^{\wedge} \vee /\phi/)} \leq_S \overline{(\top^{\wedge} \wedge (\top^{\wedge} \vee /\phi/))}$. So, by D7.22(LOWER CUT), $\overline{\top^{\wedge}}, \overline{(\top^{\wedge} \vee /\phi/)} \leq_S \overline{\top^{\vee}}$. By the argument for the case ($\top^{\wedge}$) above, $\emptyset \leq_S \overline{\top^{\wedge}}$. So, by D7.22, $\overline{(\top^{\wedge} \vee /\phi/)} \leq_S \overline{\top^{\vee}}$.

(S): Suppose $\Delta \leq \phi \in S$. By the closure of S, $\Delta < \neg\neg\phi \in S$. By L7.27 $\bar{\Delta}, \overline{\top^{\vee}} <_S \overline{\neg\neg\phi}$. $\overline{\neg\neg\phi}$ has the form $([\bar{\phi}], d)$.

(Cut): IH and D7.22 immediately imply the result. □

Lemma 7.29

1. *If $\Delta < \phi \in S^*$, then there is an $\mathfrak{M}_S$-derivation of $\bar{\Delta} <_S \bar{\phi}$.*
2. *If $\Delta \leq \phi \in S^*$, then there is an $\mathfrak{M}_S$-derivation of $\bar{\Delta} \leq_S \bar{\phi}$.*

Proof (2.) follows from D6.3 and L7.28. Suppose $\Delta < \phi \in S^*$. We prove (1.) by induction on the complexity of ϕ.

ϕ is a literal: A simple induction on D6.1 shows that every S-connection of the form $\Delta \leq' \neg\top^{\wedge}$ is reversible. So, $\Delta < \neg\top^{\wedge} \notin S^*$. Similar arguments show that $\Delta < \neg\top^{\vee} \notin S^*$, $\Delta < \neg/\chi/ \notin S^*$, for any $\chi \in \mathscr{L}^+$, and $\Delta < \neg w^{\chi} \notin S^*$, for any $\chi \in \mathscr{L}$. So, there are five cases: (A) $\phi = /\chi/$, for some $\chi \in \mathscr{L}^+$, (B) $\phi = \top^{\wedge}$, (C) $\phi = \top^{\vee}$; (D) $\phi \in \mathscr{L}$ (and not merely $\mathscr{L}^+$); or (E) $\phi = w^{\chi}$, for some $\chi \in \mathscr{L}$.

(A): By L6.12 $/\chi/ \in \Delta$. By T6.19(Irreversibility), $/\chi/ \notin \Delta$. $\perp$.

(B): A simple induction on D6.1 shows that, if $\Delta \leq' \top^{\wedge}$, then either $\Delta = \top^{\wedge}$ or $\Delta = \emptyset$. By T6.19(Irreversibility), $\Delta \neq \top^{\wedge}$. So, $\Delta = \emptyset$. By D4.5 and D4.3, $\emptyset \ll_S \overline{\top^{\wedge}}_{\oplus}$, so $\emptyset <_S \overline{\top^{\wedge}}$ is an axiom.

(C): By L6.18, $\Delta \leq' ((\top^{\wedge} \vee /\phi^i/))$, for some $(\phi^i) \subset \mathscr{L}$. So, by L7.28, there is a covering (Δ^i) of Δ such that, for each i, $\bar{\Delta}^i \leq_S \overline{(\top^{\wedge} \vee /\phi^i/)}$. Also by L7.28, $\overline{(\top^{\wedge} \wedge (\top^{\wedge} \vee /\phi^i/))} \leq_S \overline{\top^{\vee}}$. By D7.22, $\overline{\top^{\wedge}}, \overline{(\top^{\wedge} \vee /\phi^i/)} <_S \overline{(\top^{\wedge} \wedge (\top^{\wedge} \vee /\phi^i/))}$ and $\emptyset \leq_S \overline{\top^{\wedge}}$; so $\overline{(\top^{\wedge} \vee /\phi^i/)} <_S \overline{(\top^{\wedge} \wedge (\top^{\wedge} \vee /\phi^i/))}$. Putting this all together, we have

$$\bar{\Delta}^i \leq_S \overline{(\top^{\wedge} \vee /\phi^i/)} <_S \overline{(\top^{\wedge} \wedge (\top^{\wedge} \vee /\phi^i/))} \leq_S \top^{\vee}.$$

It's easy to see that D7.22 implies that $\bar{\Delta}^i <_S \top^{\vee}$, for each i. The result follows by (Amalgamation).

(D): By L6.14(1.) and L6.15(Interpolation), there is a (γ^i) and a covering (Γ^i), Δ^1, Δ^2 of Δ such that $(\Gamma^i \leq' \gamma^i)$, $\Delta^1 \leq' (\top^{\wedge} \vee /\phi/)$, $\Delta^2 \leq' \top^{\vee}$, and $v^{(\gamma^i),\phi} \leq' \phi$. By L7.28, $(\bar{\Gamma}^i \leq_S \bar{\gamma}^i)$, $\bar{\Delta}^1 \leq_S \overline{(\top^{\wedge} \vee /\phi/)}$, $\bar{\Delta}_2 \leq_S \overline{\top^{\vee}}$, and $\overline{v^{(\gamma^i),\phi}} \leq' \bar{\phi}$. By D4.3, $(\bar{\gamma}^i)$, $\overline{(\top^{\vee} \wedge /\phi/)}$, $\overline{\top^{\vee}} \ll_S \overline{v^{(\gamma^i),\phi}}_{\oplus}$. So, D7.22 implies the result.

(E): By L6.12(Persistence), $w^{\chi} \in \Delta$. By T6.19(Irreversibility), $w^{\chi} \notin \Delta$. $\perp$.

ϕ is molecular, and not a literal: ϕ has one of the following forms: $\neg\neg\psi$, $\neg(\psi^1\wedge\psi^2\wedge\ldots)$, $\neg(\psi^1\vee\psi^2\vee\ldots)$, $(\psi^1\wedge\psi^2\wedge\ldots)$, or $(\psi^1\vee\psi^2\vee\ldots)$. Each of the cases is proved similarly. We consider the case of $\neg(\psi^1\vee\psi^2\vee\ldots)$ for illustration. By T6.19(Maximality), Δ has a covering (Δ^i) such that $(\Delta^i \leq' \neg\psi^i)$. So, by L7.28, $(\bar{\Delta}^i \leq_S \overline{\neg\psi^i})$. By D4.3, $(\overline{\neg\psi^i}) \ll_S \overline{\neg(\psi^1\vee\psi^2\vee\ldots)}_{\oplus}$. The result follows by D7.22. □

Theorem 7.30 ***(Conservativity)***

1. $\Delta \leq \phi \in S^*$ *iff* $\bar{\Delta} \leq_S \bar{\phi}$;
2. $\Delta < \phi \in S^*$ *iff* $\bar{\Delta} <_S \bar{\phi}$;
3. $\delta \preceq \phi \in S^*$ *iff* $\bar{\delta}, H \leq_S \bar{\phi}$; *and*
4. $\delta \prec \phi \in S^*$ *iff* $\bar{\delta}, H \leq_S \bar{\phi}$, *and there is no* I *such that* $\bar{\phi}, I \leq_S \bar{\delta}$.

Proof By L7.10, $\bar{\Delta} = \bar{\Gamma}$ iff $\Delta = \Gamma$. So, (1.) and (2.) are immediate consequences of L7.29 and L7.26. (4.) follows from (3.) and D6.3. By T6.19(Witnessing), if $\delta \preceq \phi \in S^*$, then $\delta, \Delta \leq \phi \in S^*$, for some Δ. By (1.) above, there is an $\mathfrak{M}_S$-derivation of $\bar{\delta}, \bar{\Delta} \leq_S \bar{\phi}$. For the converse, suppose that $\bar{\delta}, H \leq_S \bar{\phi}$. By L7.26, there is a Δ such that $\bar{\delta}, H = \bar{\Delta}$ and $\Delta \leq \phi \in S^*$. Since $\bar{\delta} = \bar{\gamma}$, for some $\gamma \in \Delta$, L7.10 implies that $\delta = \gamma \in \Delta$. The result follows by T6.19. □

The remainder of this section (L7.31–T7.36) establishes that $\mathfrak{M}_S$ meets the constraints of MAXIMALITY and IRREVERSIBILITY, and so qualifies as a model. When we were dealing on the proof-theoretic side with the canonical model basis, MAXIMALITY was harder and IRREVERSIBILITY easier. Now that we are dealing on the semantic side with the canonical model, the situation is reversed. We start by demonstrating that $\mathfrak{M}_S$ satisfies MAXIMALITY.

Lemma 7.31 ***(Maximality)*** *Suppose* $(v^i) \subseteq (F_s \times F_s)$.

1. *If* $G <_S ([v^0. \cdots]_g, d)$, *then there is a covering* (G^i) *of* G *such that* $(G^i \leq_S v^i)$.
2. *If* $G <_S ([v^0 + \cdots]_g, d)$, *then there is a subset* (u^j)*of* (v^i) *and a covering* (G^j) *of* G *such that* $(G^j \leq_S u^j)$.

Proof The proofs of (1.) and (2.) are similar. We do (1.) for illustration. Suppose that $G <_S ([v^o. \cdots)_g, d)$, for some d. If $\langle v^0, \ldots\rangle = \langle v\rangle$, for some v, then $G <_S ([v]_g, d)$, i.e. $G \leq_S v$. The trivial covering G of G yields the result. Suppose, then, that $\langle v^0, \ldots\rangle \neq \langle v\rangle$. There are two cases: (A) $[v^0. \cdots]$ is formularic, or (B) not.

(A): By L7.19 for each i, $v^i = \bar{\psi}^i$, for some $\psi^i \in \mathscr{L}^+$. So, $[v^0. \cdots] = \overline{(\psi^0\wedge\psi^1\wedge\ldots)}_{\oplus}$. By L7.26, $G = \bar{\Delta}$ and $\Delta < (\psi^0\wedge\psi^1\wedge\ldots) \in S^*$, for some Δ. By T6.19(Maximality), there is a covering (Δ^i) of Δ such that $(\Delta^i \leq \psi^i) \subseteq S^*$. By L7.29, there are $\mathfrak{M}_S$-derivations of $(\bar{\Delta}^i \leq_s \bar{\psi}^i)$. Since $G = \bar{\Delta}$, $(\bar{\Delta}^i)$ is a covering of G.

(B): We prove the result by induction on $\mathfrak{M}_S$-derivations $\mathcal{D}$. By L7.24, we may assume (wlog) that $\mathcal{D}$ is semi-normal.

(Basis): Suppose that $G <_S ([v^0. \cdots]_g, d)$ is an axiom, so that $G \ll_S [v^0. \cdots]_g$. By L7.21, $G = (v^i)$. Each instance of $(v^i \leq_S v^i)$ is an axiom.

(Ascent): Suppose $\mathcal{D}$ has the form

$$\dfrac{\dfrac{\mathcal{E}}{G <_S w}}{G <_S ([w]_g, b)} \quad \text{where } [w]_g = [v^0. \cdots]_g.$$

By L7.20, $\langle v^i \rangle = w$. $\perp$.

(Lower Cut): Suppose $\mathcal{D}$ has the form

$$\dfrac{\left(\dfrac{\mathcal{F}^j}{G^j \leq_S w^j} \right) \qquad \dfrac{}{(w^j) <_S ([v^0. \cdots]_g, d)}}{(G^j) <_S ([v^0. \cdots]_g, d)}$$

Since $(w^j) <_S ([v^0. \cdots]_g, d)$ is an axiom, $(w^j) \ll_S [v^0. \cdots]_g$. By L7.21, $(w^j) = (v^i)$. So, $(G^j \leq_S w^j) = (G^i \leq_S v^i)$ by re-indexing.

(Upper Cut): Suppose $\mathcal{D}$ has the form

$$\dfrac{\left(\dfrac{\mathcal{F}^j}{G^j <_S w^j} \right) \qquad \dfrac{}{(w^j) \leq_S ([v^0. \cdots]_g, d)}}{(G^j) <_S ([v^0. \cdots]_g, d)}$$

Since $(w^j) \leq_S ([v^0. \cdots]_g, d)$ is an axiom, $(w^j) \ll_S [([v^0. \cdots]_g, d)]_g$. By L7.21, $(w^j) = ([v^0. \cdots]_g, d)$. So, IH applies to each of the $\mathfrak{M}_S$-derivations $(\mathcal{F}^j)$: for each j there is a covering $(G^{ji})_i$ of G^j such that there are $\mathfrak{M}_S$-derivations of $(G^{ji} \leq_S v^i)_i$. By re-ordering, for each i, we have $\mathfrak{M}_S$-derivations of $(G^{ji} \leq_S v^i)_j$. By D7.22, there are $\mathfrak{M}_S$-derivations of each of the selections $((G^{ji})_j \leq_S v^i)_i$. □

To establish that $\mathfrak{M}_S$ satisfies IRREVERSIBILITY it is useful to show that weak selections from contents that do not correspond to any element of $\mathscr{L}^+$ must take a particularly strong form.

Lemma 7.32

1. *If $v \not\sim \bar{\bar{\psi}}$, for any $\psi \in \mathscr{L}^+$, and $G \leq_S v$, then either $G = v$, or $G <_S v$, or $(G = v, G'$ and $G' <_S v)$.*
2. *If $v \not\sim \bar{\bar{\psi}}$, for any $\psi \in \mathscr{L}^+$, and $w, G \leq_S v$, then either $w = v$ or $w, H <_S v$, for some H.*

Proof (2.) follows immediately from (1.) (setting G in (1.) to G, w). Suppose $v \not\sim \bar{\bar{\psi}}$, for any $\psi \in \mathscr{L}^+$. We prove (1.) by induction on $\mathfrak{M}_S$-derivations $\mathcal{D}$. By L7.24, we may assume (wlog) that $\mathcal{D}$ is semi-normal.

(Basis): Suppose $G \ll_S [v]_g$. By L7.19, $[v]_g$ is not formularic. So, by L7.21, $G = v$.

(Ascent): Suppose there is an $\mathfrak{M}_S$-derivation of $G <_S w$ and $[w]_s = [v]_s$. As above, $w = v$. So, IH yields the result.

(Lower Cut): Suppose that there are $\mathfrak{M}_S$-derivations of each of $(G^i \leq_S v^i)$ and that $(v^i) <_S ([v]_g, d)$ is an axiom. As above, $(v^i) = v$. So, IH applies to each of the selections $(G^i \leq_S v^i)$. It is easy to see that the result follows by D7.22.

(Upper Cut): Suppose that there are $\mathfrak{M}_S$-derivations of each of $(G^i <_S v^i)$ and that $(v^i) <_S ([([v]_g, d)], e)$ is an axiom. As above, $(v^i) = ([v]_s, d)$. So, IH applies to each of the selections $(G <_S v^i)$. It is easy to see that the result follows by D7.22. □

Now we show that the application of choice and combination to some contents yields a condition that is "raised" up a level. This is the key to demonstrating that $\mathfrak{M}_S$ satisfies IRREVERSIBILITY.

Lemma 7.33

1. $v_\oplus \not\sim [w + \cdots + v + \cdots]$.
2. $v_\oplus \not\sim [w. \ \cdots \ .v. \ \cdots]$.
3. $v_\oplus \not\sim [v]$

Proof Each of (1.)-(3.) is proved similarly. We do (1.) for illustration. There are two cases: either (A) $[w + \cdots + v + \cdots]$ is formularic, or (B) it is not.

(A): By L7.19, $[w + \cdots + v + \cdots] \sim [\bar{\bar{\psi}}^0 + \cdots + \bar{\bar{\psi}}^i + \cdots]$, where $v \sim \bar{\bar{\psi}}^i$. So, $g(v) = \bar{\psi}^i$. Suppose (for *reductio*) $v_\oplus \sim [w + \cdots + v + \cdots]$. Then $g(v_\oplus) = [\bar{\psi}^0 + \cdots + \bar{\psi}^i + \cdots]_g$; so

$$\bar{\psi}^i \ll_S g(v_\oplus) \Rightarrow \bar{\psi}^i \ll_S g(v)_\oplus \Rightarrow \bar{\psi}^i <_S g(v) \Rightarrow \bar{\psi}^i <_S \bar{\psi}^i.$$

By T7.30, $\psi^i < \psi^i \in S^*$. But, by D6.3, $\psi^i < \psi^i \notin S^*$. $\bot$.

(B): We prove the result by induction on free conditions $a = v_\oplus$, defined in D4.2. Suppose (for *reductio*) that $v_\oplus \sim [w + \cdots + v + \cdots]$. If $v_\oplus$ is formularic, then the argument in (A) yields the result, so we may assume (wlog) that $v_\oplus$ is not formularic. So, L7.20 implies that $v_\oplus = [w' + \cdots + v' + \cdots]$ and $v' \sim v$, for some $w', \ldots, v, \ldots$.

Basis: $a = v_\oplus$ is a literal of $\mathscr{L}^+$. All such literals are formularic. $\bot$.

Inductive Step: Let $v' = (a'.b')$. IH is that, for all $w^*, \ldots, v^*(= (a', b^*)), \ldots, a' = v^*_\oplus \not\sim [w^* + \cdots + v^*(= (a', b^*)) + \cdots]$. Since $v' \sim v$, $v'_\oplus \sim v_\oplus$. So, $v'_\oplus \sim v_\oplus = [w' + \cdots + v' + \cdots]$. By IH, $v'_\oplus \not\sim [w' + \cdots + v' + \cdots]$. $\bot$.

□

Lemma 7.34 ***(Irreversibility*** $\Rightarrow$***)***

1. *$v, H \not<_S v$.*
2. *If $H <_S v$, then there is no $w \in H$ and no G such that $v, G \leq_S w$.*

Proof (2.) follows from (1.) by D7.22. In regard to (1.), either (A) v is formularic, or (B) it is not.

(A): Suppose (for *reductio*) that $v, H <_S v$. Then, by L7.26, there are δ, Δ such that $v = \bar{\delta}$ and $\delta, \Delta < \delta \in S^*$. By D6.3, $\delta, \Delta < \delta \notin S^*$.

(B): By D4.2 either (I) $v_\oplus = (+, \emptyset)$ or (II) $v_\oplus$ is a choice, combination, or singleton.

(I): An easy induction on $\mathfrak{M}_S$-derivations shows that, for all G, $G \not<_S v$.

(II): We prove the result by induction on free contents. The basis cases are handled by the arguments for (A) and (B)(I). All of the remaining cases are proved similarly. We do the case in which $v_\oplus = [w^0 + w^1 + \cdots]_g$. IH is that, for each i, there is no $\mathfrak{M}_S$-derivation of $w^i, G <_S w^i$. Suppose $G <_S v$. By 7.31 (Maximality), there is a subset (u^j) of (w^i) and a covering (G^j) of G such that $(G^j \leq_S u^j)$. Suppose (for *reductio*) $v \in G^j$, for some j, so that $v, G^j \leq_S u^j$. Then, by L7.26, since $v \not\sim \bar{\bar{\psi}}$ for any $\psi \in \mathscr{L}^+$, $u^j \not\sim \bar{\bar{\phi}}$ for any $\phi \in \mathscr{L}^+$. Also, by L7.33, $u^j_\oplus \not\sim v_\oplus$, so $u^j \neq v$. So, L7.32 applies to the $\mathfrak{M}_S$-derivation of $v, G^j \leq_S u^j$: $v, H <_S u^j$, for some H. So, by D7.22, $u^j, H <_S u^j$. But, by IH, there is no $\mathfrak{M}_S$-derivation of $u^j, H <_S u^j$. $\bot$. □

Lemma 7.35 ***(Irreversibility*** $\Leftarrow$*) If $G \leq_S v$ and $v, H \not\leq_S w$ for any H and any $w \in G$, then $G <_S v$.*

Proof Suppose there is an $\mathfrak{M}_S$-derivation of $G \leq_S v$, but no $\mathfrak{M}_S$-derivation of $v, H \leq_S w$ for any H and any $w \in G$. There are two cases: either (A) $v \sim \bar{\bar{\psi}}$, for some $\psi \in \mathscr{L}^+$, or (B) not.

(A): $v = g(v) = \bar{\psi}$. By L7.26, $G = \bar{\Delta}$ and $\Delta \leq \psi \in S^*$, for some Δ. By T7.30, for all $\delta \in \Delta$, $\psi \preceq \delta \notin S^*$. So, by D6.3, $(\forall \delta \in \Delta)\delta \prec \psi \in S^*$, and so $\Delta < \psi \in S^*$. By T7.30 again, there is an $\mathfrak{M}_S$-derivation of $\bar{\Delta} <_S \bar{\psi}$.

(B): By L7.32, either $v \in G$ or there is an $\mathfrak{M}_S$-derivation of $G <_S v$. Since every instance of $v \leq_S v$ is an axiom, $v \notin G$. □

The restriction of $\bar{\cdot}$ to atomic sentences is an interpretation. By L7.1, the extension of that interpretation to molecular sentences ϕ is just $\bar{\phi}$. Clearly, there is an $\mathfrak{M}_S$-derivation of $G <_S v$ iff $G <_{\mathfrak{M}_S} v$. So, the following theorem is immediate by L7.31(Maximality), L7.34(Irreversibility $\Rightarrow$), and L7.35(Irreversibility $\Leftarrow$):

Theorem 7.36 *$\mathfrak{M}_S$ is a model.*

8 Completeness

Definition 8.1 We have assumed that the sentences of the language $\mathscr{L}$ are well-ordered. It follows that the grounding claims for $\mathscr{L}$ are well-ordered, and so can be indexed to an ordinal α, so that they form a set of of the form $\{\tau_0, \tau_1, \ldots, \tau_\beta, \ldots\}$ $(\beta < \alpha)$. Suppose also that S and T are finite sets of grounding claims such that $S \nvdash T$. For each $\beta < \alpha$, define S_β by recursion:

1. $S_0 = S$;

2. $S_{\beta+1} = \begin{cases} S_\beta \cup \{\tau_\beta\}, & \text{if } S_\beta, \tau_\beta \not\vdash T; \\ S_\beta, & \text{otherwise.} \end{cases}$
3. $S_\lambda = \bigcup_{\beta<\lambda}(S_\beta)$ for limit λ.

Let $S_T^+ = \bigcup_{\beta<\alpha}(S_\beta)$.

Recall that $\vdash$ was defined so that $U \vdash V \Leftrightarrow U' \Vdash V'$, for some $U' \subseteq U$ and $V' \subseteq V$. A simple induction on the definition of $\Vdash$ shows that, if $U' \Vdash V'$, then U' and V' are both finite, yielding the following lemma.

Lemma 8.2 ***(Syntactic Compactness)*** *If $U \vdash V$ then there are finite $U' \subseteq U$ and $V' \subseteq V$ such that $U' \vdash V'$, $U' \vdash V$, and $U \vdash V'$.*

L8.2 and an induction on the cardinality of finite sets of grounding claims U straightforwardly yields

Lemma 8.3

1. *If $S \Vdash U$ and $(\forall\tau \in U)S, \tau \Vdash T$, then $S \Vdash T$.*
2. *If $S \vdash U$ and $(\forall\tau \in U)S, \tau \vdash T$, then $S \vdash T$.*

Standard reasoning from L8.2, L8.3, and L5.21 (Main Witnessing Lemma) then shows

Lemma 8.4 *If S and T are sets of grounding claims of $\mathscr{L}$ such that $S \not\vdash T$, then there is a prime, witnessed extension S^* of S such that $S^* \not\vdash T$.*

Lemma 8.5 *If S and T are sets of grounding claims of $\mathscr{L}$ such that $S \not\vdash T$, then there is a model $\mathfrak{M}$ such that $(\forall\sigma \in S)\mathfrak{M} \vDash \sigma$ and $\mathfrak{M} \nvDash T$.*

Proof Suppose $S \not\vdash T$. By L8.4, there is a prime, witnessed extension S^* of S such that $S^* \not\vdash T$. By T6.19, the canonical model basis S' for S^* is such that, for all grounding claims σ of the language $\mathscr{L}^*$ of S^*, $\sigma \in S'$ iff $\sigma \in S^*$. By T7.36, $\mathfrak{M}_{S^*}$ is a model. By T7.30, $(\forall\sigma \in S^*)\mathfrak{M}_{S^*} \vDash \sigma$. Since $S \subseteq S^*$, $(\forall\sigma \in S)\mathfrak{M}_{S^*} \vDash \sigma$. By T7.30, $\mathfrak{M}_{S^*} \vDash T \Leftrightarrow (\exists\sigma)\sigma \in (S^* \cap T)$. Since $S^* \not\vdash T$, $S^* \cap T = \emptyset$. So, $\mathfrak{M}_{S^*} \nvDash T$. □

Theorem 8.6 ***(Completeness)*** *If $S \vDash T$, then $S \vdash T$.*

Proof Suppose $S \vDash T$. By the definition of $\vDash$, there is no model $\mathfrak{M}$ such that $(\forall\sigma \in S)\mathfrak{M} \vDash \sigma$ and $\mathfrak{M} \nvDash T$. By L8.5, $S \vdash T$. □

9 Further Work

We make some brief suggestions as to how further work on the ideas presented in this paper might proceed.

9.1 Going Infinitary

Our system is finitary: in each of the full grounding claims $\Delta \leq A$ and $\Delta < A$, the set of formulas Δ must be finite; and $T \vdash S$ just in case there are finite sets $T' \subseteq T$ and $S' \subseteq S$ such that $T' \Vdash S'$. It will prove desirable for certain purposes to relax the first of these requirements and allow in principle for a statement to have infinitely many grounds; and once this is done, it will be natural to relax the second of these requirements and to allow the grounding claims to the left and right of $\Vdash$ to be infinite.

This means that the rules of THINNING, SNIP, CUT and REVERSE SUBSUMPTION will need to be revised. It also means that, in the semantics, we must allow for the infinitary application of combination and choice. Proofs of soundness and completeness can then, with suitable modifications, go through much as before.

9.2 Quantification

Our system is sentential; the formulas flanking a grounding claim are those of sentential logic - formed from sentential atoms by means of the usual truth-functional connectives. The question therefore arises as to how to extend the system with quantifiers so that the formulas flanking a grounding claim can be those of an arbitrary first order language.

In order to be able to account for the grounds for a universal statement, we presuppose given a domain D of individuals(as in [7]). Suppose then that $\mathrm{a}_1, \mathrm{a}_2, \ldots$ are the distinct individuals of D; and let $D = \{a_1, a_2, \ldots\}$ be the set of corresponding names for those individuals. An interpretation over D should then assign to every n-place predicate F a function F taking each n-tuple of individuals from D into a content; and the content of the atomic sentence $Fa_{k_1}a_{k_2}\ldots a_{k_n}$ should then be taken to be $\mathrm{F}(\mathrm{a}_{\mathrm{k}_1}, \mathrm{a}_{\mathrm{k}_2}, \ldots, \mathrm{a}_{\mathrm{k}_\mathrm{n}})$.

When it comes to the quantifiers, we might think of a universal statement $\forall x\phi(x)$ as the conjunction $\phi(a_1)\wedge\phi(a_2)\wedge\ldots$ of its instances and of an existential statement $\exists x\phi(x)$ as the disjunction $\phi(a_1)\vee\phi(a_2)\vee\ldots$ of its instances. Since there is an obvious extension of the introduction and elimination rules for binary conjunction and disjunction to conjunctions and disjunctions of arbitrary length, we may read off the introduction and elimination rules for universal and existential quantification from the extended rules for conjunction and disjunction. We are thereby lead to adopt the following pair of positive introduction and elimination rules for the universal quantifier:

∀I $\Vdash \phi(a_1), \phi(a_2), \cdots < \forall x\phi(x)$
∀E $\Delta < \forall x\phi(x) \Vdash \Delta \leq \phi(a_1), \phi(a_2), \ldots.$

In the statement of ∀E, $S \Vdash \Delta \leq \chi_1, \chi_2, \ldots$ abbreviates

$$S \Vdash (\Delta_1^1 \leq \chi_1; \Delta_2^1 \leq \chi_2, \ldots \quad | \quad \Delta_1^2 \leq \chi_1; \Delta_2^2 \leq \chi_2, \ldots \quad | \quad \ldots)$$

where $(\langle \Delta_1^i, \Delta_2^i, \ldots \rangle)$ are exactly the sequences (of appropriate length) for which $\Delta = \Delta_1^i \cup \Delta_2^i \cup \ldots$ [7, 64]. There would be corresponding rules for the existential quantifier. There is a corresponding semantic treatment. For, as we have seen, the semantics for binary conjunction and disjunction may be extended to conjunctions and disjunctions of arbitrary length; and we may then let the semantics for these conjunctions and disjunctions of arbitrary length be our guide in providing a semantics for the quantifiers. However, there is a hitch. For the semantics for $\phi(a_1) \wedge \phi(a_2) \wedge \ldots$ or for $\phi(a) \vee \phi(b) \vee \ldots$ takes account of the order of the conjuncts or of the disjuncts. Thus the truth-condition for $\phi(a_1) \wedge \phi(a_2) \wedge \ldots$, for example, will be the combination of the contents of $\phi(a_1), \phi(a_2), \ldots$ in that order. Since the combination may vary with the order, this makes it unclear what the content of the universal statement should be taken to be.

This is, in fact, a general difficulty for any semantics which is based on the semantic equivalence of $\forall x \phi(x)$ to $\phi(a_1) \wedge \phi(a_2) \wedge \ldots$ and which is sensitive to the order of the conjuncts in a conjunction. There are a number of ways within our own framework of dealing with this difficulty. Perhaps the most conservative option is to suppose given a well-ordering $\mathrm{a}_1, \mathrm{a}_2, \ldots$ of the individuals of D and a corresponding well-ordering $a_1, a_2, \ldots$ of the individual names. We can then stipulate that, for semantical purposes, $\forall x \phi(x)$ is to be taken to be equivalent to $\phi(a_1) \wedge \phi(a_2) \wedge \ldots$ in that very order, so that its truth-condition is to be the combination of the contents of $\phi(a_1), \phi(a_2), \ldots$ in that very order; and similarly for $\exists x \phi(x)$. This is, of course, to introduce an arbitrary element into the semantics, since any other well-ordering of the individuals would have done just as well. But we may think of the combination (or choice) of the specific sequence of contents of $\phi(a_1), \phi(a_2), \ldots$ as representing the combination (or choice) of the corresponding *set* of contents, without our thereby having to extend the existing apparatus of combination and choice to include their application to sets rather than sequences.

Quantifiers with variable domains raise additional complications, since there is then the need for totality facts (as in [7, 59 et seq.]). We believe that the development of the semantics in this direction requires the introduction of *dependent* combinations and choices, but this is not something that we shall pursue here.

9.3 Propositional identities

In [7, 67], it was suggested that one might want to add certain ground-theoretic equivalences to the logic of ground. In the case of conjunction, one might want to insist upon commutativity in the form:

$$(\phi \wedge \psi) \leq (\psi \wedge \phi).$$

and similarly in the case of disjunction. However, the ground-theoretic equivalence of $\phi \wedge \psi$ and $\psi \wedge \phi$ would not guarantee the ground-theoretic equivalence, for example, of $\neg(\phi \wedge \psi)$ and $\neg(\psi \wedge \phi)$; and so, just as we previously suggested in the case of the quantifiers that one might wish to insist upon the ground-theoretic equivalence of any two alphabetic variants, so we might, in the present case, wish to insist upon the ground-theoretic equivalence of θ and θ' whenever θ' could be obtained from θ by replacing a subformula $(\phi \wedge \psi)$ with $(\psi \wedge \phi)$ (and similarly in the case of disjunction).

A corresponding semantic treatment could be obtained by subjecting combination and choice to the corresponding conditions. However, certain propositional equivalences are incompatible with the existing rules. The equivalence of $\neg\neg\phi$ with ϕ, for example, is incompatible with ϕ being a strict ground for $\neg\neg\phi$ and, likewise, the equivalence of $\phi\vee\phi$ with ϕ is incompatible with ϕ being a strict ground for $\phi\vee\phi$; and associativity for either disjunction or conjunction also runs into difficulties. For:

$$\begin{aligned} &((\phi\vee\psi)\vee\psi) \leq (\phi\vee(\psi\vee\psi)) \\ &\qquad \Vdash (\phi\vee\psi) < (\phi\vee(\psi\vee\psi)) \\ &\qquad \Vdash (\phi\vee\psi) \leq (\psi\vee\psi) \qquad \text{(using } \vee\text{-ELIMINATION)} \\ &\qquad \Vdash \phi < (\psi\vee\psi) \\ &\qquad \Vdash \phi \leq \psi. \end{aligned}$$

A similar argument shows that associativity for conjunction implies that $\phi \preceq \psi$. Letting $\phi = \neg\neg A$ and $\psi = A$, we get an inconsistency.

Under a "flat" approach to the semantics, by contrast, these various equivalences will hold. It turns out that our approach can be modified and extended in such a way as to accommodate one such "flat" approach, the theory of content and an associated logic of ground given by Angell's theory of analytic containment.[8]

Angell's theory includes all of the equivalences noted above, as well as DeMorgan equivalences. So, a suitable modification of the approach here, with frames given by choice and combination operations and interpretations assigning contents to formulae, yields the logic of GG if choice and combination are constrained as in the semantics of Section 2, and the logic of the Angellic system if choice and combination are constrained differently. Thus, each logic can be characterized as a special case of a single, general approach. It remains unclear whether other interesting views of propositional identity can be characterized in a similar way.

9.4 Lambda Abstraction

The system of [7] contains some obvious rules for lambda abstraction. In extending the semantics to the closed lambda abstract $\lambda x\phi(x)$, the obvious strategy is to take its semantic value to be a function which assigns, to each individual a of the domain, the content $([\overline{\overline{\phi(a)}}], [\overline{\overline{\neg\phi(a)}}])$. The contents of $\phi(a)$ and $\neg\phi(a)$ are "raised;" and we thereby guarantee that $\phi(a)$ is the immediate strict ground for $\lambda x\phi(x)a$ and $\neg\phi(a)$ the immediate strict ground for $\neg\lambda x\phi(x)a$. However, this has the undesirable consequence that $\lambda x\phi(x)a$ and $\neg\neg\phi(a)$ will always have the same content and hence be intersubstitutable in any ground-theoretic context.

One way round this difficulty is to suppose that there are different ways in which a content can be raised. Thus the semantics for negation involves one form of raising, under which the content of a statement is converted into a falsity condition for its negation, while the semantics for lambda abstraction will involve another form of

[8] Angell [1]. See [2, 7], and [8] for semantic characterizations of Angell's system and the corresponding logic of ground. The specification of the modifications of the present approach to capture GG (under one set of constraints on choice and combination) and Angell's system (under another set of constraints) and the associated proofs are omitted here for reasons of length.

raising, under which the content of a statement or of its negation, is converted into a truth or falsity condition for the corresponding complex predication. From this point of view, our previous identification of $[v]$ with a singleton combination or choice was a harmless simplification which should be dropped once different forms of raising are in play.[9]

References

1. Angell, R. B. (1989). Deducibility, entailment and analytic consequence. In *J. Norman and R. Sylvan, editors, Directions in Relevant Logic* (pp. 119–44). Kluwer.
2. Correia, F. (2010). Grounding and truth-functions. *Logique et Analyse*, *53*(211), 251–79.
3. Correia, F. (2017). An impure logic of representational grounding. *Journal of Philosophical Logic*, *46*, 507–38.
4. deRosset, L. (2013). What is weak ground? *Essays in Philosophy*, *14*(1), 7–18.
5. deRosset, L. (2014). On weak ground. *Review of Symbolic Logic*, *7*(4), 713–44.
6. Fine, K. (2012a). The pure logic of ground. *Review of Symbolic Logic*, *5*(1), 1–25.
7. Fine, K. (2012b). Guide to ground. In *Benjamin Schnieder and Fabrice Correia, editors, Metaphysical Grounding: Understanding the Structure of Reality, pp. 37–80. Cambridge University Press*. reprinted online in 'Philosophers Annual' for 2012 (eds. P. Grim, C. Armstrong, P. Shirreff, N-H Stear).
8. Fine, K. (2016). Angellic content. *Journal of Philosophical Logic*, *45*(2), 199–226. https://doi.org/10.1007/s10992-015-9371-9.
9. Fine, K. (2017a). Survey on truthmaker semantics. In *R. Hale, A. Miller, and C. Wright, editors, Companion to the Philosophy of Language, pp. 556–77. Blackwell*.
10. Fine, K. (2017b). A Theory of truthmaker content I: conjunction, disjunction, and negation. *Journal of Philosophical Logic*, *46*(6), 625–74. https://doi.org/10.1007/s10992-016-9413-y.
11. Fine, K. (2020). Semantics. In *M. Raven, editor, Routledge Handbook of Metaphysical Ground, pp. 501–9. Routledge*.
12. Krämer, S. (2018a). Towards a theory of ground-theoretic content. *Synthese*, *195*, 785–814.
13. Krämer, S. (2018b). Ground-theoretic equivalence. *Synthese*. https://doi.org/10.1007/s11229-019-02154-4.
14. Litland, J. E. (2015). Grounding, explanation, and the limit of internality. *The Philosophical Review*, *124*(4), 481–533.
15. Poggiolesi, F. (2016). On defining the notion of complete and immediate formal grounding. *Synthese*, *193*(3), 3147–67.
16. Poggiolesi, F. (2018). On constructing a logic for the notion of complete and immediate formal grounding. *Synthese*, *195*(3), 1231–54.

[9]Material from this paper was presented at a workshop on truthmaker semantics at NYU, at a conference on truthmaker semantics at the University of Hamburg, and in a Spring 2020 seminar on recent work in philosophical logic at NYU. We would like to thank participants in those events, as well as Fabrice Correia, Stephan Krämer, and two anonymous referees for comments and advice.