Journal of Philosophical Logic (2023) 52:221–266
https://doi.org/10.1007/s10992-022-09671-7

Continuous Accessibility Modal Logics

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Received: 13 January 2022 / Accepted: 20 May 2022 / Published online: 24 September 2022

Abstract

In classical modal semantics, a binary accessibility relation connects worlds. In this paper, we present a uniform and systematic treatment of modal semantics with a continuous accessibility relation alongside the continuous accessibility modal logics that they model. We develop several such logics for a variety of philosophical applications. Our main conclusions are as follows. Modal logics with a continuous accessibility relation are sound and complete in their natural classes of models. The class of Kripke frames where a continuous accessibility relation has a magnitude characterizing its degree of accessibility is not modally definable, and this has unappreciated significance to completeness proofs for such logics, revealing a methodological advantage of using classical multimodal semantics over fuzzy modal semantics. There is a pseudometric space modal logic that is complete in the class of pseudometric spaces, a natural semantic setting for quantitative modal reasoning about similarity. There is a metric space modal logic that is complete in the class of metric spaces, a natural semantic setting for quantitative modal reasoning about neighborhoods and counterfactual stability. There is a real line continuous temporal logic that is canonical for real lines, a natural semantic setting for quantitative modal reasoning about time.

Keywords Continuous modal logic · Classical multimodal semantics · Fuzzy modal semantics · Pseudometric space modal logic · Metric space modal logic · Continuous temporal logic

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1 Introduction

In classical modal semantics, a binary accessibility relation connects worlds. In this paper, we present a uniform and systematic treatment of modal semantics with a continuous accessibility relation alongside the continuous accessibility modal logics that they model. A continuous accessibility relation can be understood as accessibility as a matter of degree. In this way, necessity and possibility (and other modalities) can in turn be interpreted as a matter of degree.

We develop several such logics for a variety of philosophical applications. Work relevant to this subject up to this point has focused primarily on computational applications related to time (see [1–3], and [4]) and distance (see [5, 6], and [7]), though there have been attempts to apply metric-based approaches to reasoning about similarity (see [8] and [9]). Others have considered continuous modality by interpreting the *truth* of modal sentences via a continuum of truth values, modeling, for example, *how true* it is that something is possible (see the probabilistic semantics of [10] or a modal fragment of continuous logic given in [11]). Below, we interpret truth classically, but assume that the modalities themselves have a degree structure, modeling, for example, *how possible* something is. Our logics include a basic continuous accessibility modal logic along with simple extensions, interactive extensions such as a pseudometric space modal logic and a metric space modal logic, and continuous temporal logics.

The main conclusions of this paper are as follows.

- Modal logics with a continuous accessibility relation are sound and complete in their natural classes of models. Philosophers should therefore feel free to use modal semantics that include a continuous accessibility relation, or any modal notion that can be modeled thereby.
- The class of Kripke frames where a continuous accessibility relation is well-founded (has a magnitude characterizing its degree of accessibility) is not modally definable. This result, along with its unappreciated significance to completeness proofs, is revealed by using classical semantics instead of fuzzy semantics to model continuous accessibility modal logics, illustrating a methodological advantage of the former over the latter.
- Pseudometric spaces provide a natural semantic setting for quantitative modal reasoning about similarity. There is a pseudometric space modal logic that is complete in the class of pseudometric spaces, interpreted as multimodal Kripke frames. Moreover, such frames satisfy the modal logic $S5$ in the ∞-indexed modal operator.
- Metric spaces provide a natural semantic setting for quantitative modal reasoning about neighborhoods and counterfactual stability. There is a metric space modal logic that is complete in the class of metric spaces, interpreted as multimodal Kripke frames. Moreover, any dense-in-itself metric space, interpreted as a multimodal Kripke frame, simultaneously satisfies the modal logic $S4$ in the interior operator and the modal logic $S5$ in the ∞-indexed modal operator.
- The real line provides a natural semantic setting for quantitative modal reasoning about time. There is a real line continuous temporal logic that is canonical for

real lines. Constructing this logic requires stipulating frame properties beyond well-foundedness in order to have syntactic access to the magnitude of the accessibility relations.

1.1 Motivation

The binary accessibility relation in classical modal semantics leads to discrete modalities. However, when reasoning about similarity, space, time, or probabilities, thinking about modalities as a matter of degree can be more fruitful. Here are three examples where a continuous accessibility relation can model modal reasoning better than a binary accessibility relation.

First, counterfactual conditionals are normally analyzed in terms of material conditionals in nearby possible worlds. Nearness is a measure of the relevant similarity of a possible world with respect to the actual world, and is usually implemented as a brute (partial or total) ordering of the worlds. Whether a world is nearby can then be interpreted as a cut-off in nearness. However, intuitively, the relevant similarity of two worlds is not all-or-nothing, but rather a matter of degree. This can be captured by a continuous accessibility relation: the more relevantly similar the world, the more accessible the world is. How true a counterfactual conditional is can then be read off how necessary the material conditional is. Analyses that depend on real-numbered measures as opposed to a mere partial or total ordering of worlds can thus proceed in a straightforward way.

Second, probability functions can be modeled by a continuous accessibility relation as a continuum-indexed possibility operator. For a subjective probability example, credence can be taken as a judgement of probability that some proposition is true. This can also be captured by a continuous accessibility relation: the higher the subjective probability of the proposition, the more accessible the world where that proposition is true. For an objective probability example, a non-deterministic system in some initial state evolves into one of many final states with some probability. This again can be captured by a continuous accessibility relation: the higher the objective probability of evolving into a given state, the more accessible the world where that state obtains. Structural features of the probabilities can then be applied as constraints on the accessibility relation.

Finally, temporal logic can also benefit from a continuous accessibility relation. Normally, temporal logic has two binary accessibility relations, one each for the future and past operators. But in contexts informed by a scientific treatment of time, it is natural to understand time as a real numbered dimension. In that case, the distance into the future or past can be captured by a continuous accessibility relation for each direction.

1.2 Strategy

A direct and *prima facie* natural approach to developing a continuous accessibility modal logic is to take classical modal logic semantics and replace the binary accessibility relation with a continuous accessibility relation, that is, a continuum-valued binary partial function (as in [12] and [13]). Call this the *fuzzy semantics* approach.

We do not, at least initially, take this approach. Instead, we replace the binary accessibility relation with continuum-many binary accessibility relations, which we index with the non-negative real numbers and ∞. We then impose constraints on these relations to capture the intended accessibility structure. (A similar technique was used in [3] and [7].) Call this the *classical multimodal semantics* approach.

We take the classical multimodal semantics approach for three related reasons. First, it allows us to model the logics in a transparent way using Kripke frames. We begin with the class of all Kripke frames, and the constraints provided by each of our logics correspond to different subclasses of frames. Second, this approach allows us to apply results from classical modal logic to our proofs, significantly simplifying them. Finally, it reveals details about the underlying structure of the logic that are hidden by the fuzzy semantics approach. Nevertheless, we prove in Section 3.4 that, in the cases of interest (*i.e.*, when accessibility is a magnitude represented by a real number), these two approaches yield equivalent results.

We make free use of choice functions. Moreover, when we discuss maps between proper classes, we use the terms "function", "injection", "surjection", and "bijection", leaving out the prefix "class-" for convenience.

2 $[0, \infty]$-Indexed Multimodal Logic

The first step in our treatment of modal logic with a continuous accessibility relation is to develop a multimodal logic with continuum-many modal operators. In classical modal semantics, the number of independent modal operators corresponds to the number of accessibility relations in which a given ordered pair of worlds can stand. For example, an epistemic modal logic might have one knowledge operator for each epistemic agent. A typical multimodal logic has a finite or countably infinite number of such operators, and thus finitely or countably infinitely many accessibility relations. Below, we present a modal logic with continuum-many independent modal operators, and thus continuum-many accessibility relations. To facilitate their interpretation as distance-like, we index these operators with the non-negative real numbers, along with ∞ defined as an upper bound.

2.1 The Basic $[0, \infty]$-Indexed Multimodal Logic

Consider $([0, \infty) \subseteq \mathbb{R}, +, \leq)$ as a linearly ordered monoid. Extend it to $([0, \infty], +, \leq)$ such that for every $x \in [0, \infty)$, $\infty > x$, and for any $x \in [0, \infty]$, $x + \infty = \infty$ and $\infty + x = \infty$. The *language of basic* $[0, \infty]$*-indexed multimodal logic*, denoted $L_{[0,\infty]}$, is the smallest normal multimodal language containing $\{\Box_x, \Diamond_x : x \in [0, \infty]\}$ as its set of modal operator symbols.

We follow the conventions of the basic modal logic given in [14]. Our language includes a set of basic proposition symbols, P. The *set of well-formed formulas*, denoted $W_{[0,\infty]}$, is generated recursively as the smallest set such that the following hold.

- $\mathrm{P} \subseteq W_{[0,\infty]}$.

- If $\varphi \in W_{[0,\infty]}$, then $\neg\varphi \in W_{[0,\infty]}$.
- If $\varphi, \psi \in W_{[0,\infty]}$, then $\varphi \wedge \psi, \varphi \vee \psi, \varphi \to \psi, \varphi \leftrightarrow \psi \in W_{[0,\infty]}$.
- If $\varphi \in W_{[0,\infty]}$ and $x \in [0,\infty]$, then $\Box_x\varphi, \Diamond_x\varphi \in W_{[0,\infty]}$.

The *basic* $[0,\infty]$*-indexed multimodal logic*, denoted $K_{[0,\infty]}$, is the smallest normal multimodal logic in modal operators $\{\Box_x, \Diamond_x : x \in [0,\infty]\}$. That is, $K_{[0,\infty]}$ is the smallest proof system with rules of inference:

(a) *Modus ponens*: If φ and $\varphi\to\psi$ are both provable, then ψ is provable.
(b) *Necessitation*: If φ is provable, then for every $x \in [0,\infty]$, $\Box_x\varphi$ is provable.

and axiom schemata:

(c) All tautologies from propositional logic, closed under universal substitution. (We will use $\top$ to denote some choice of propositional tautology.)
(d) *Modal distribution*: For every $\varphi, \psi \in W_{[0,\infty]}$ and $x \in [0,\infty]$,

$$\Box_x(\varphi\to\psi)\to(\Box_x\varphi\to\Box_x\psi).$$

(e) *Duality*: For every $\varphi \in W_{[0,\infty]}$ and $x \in [0,\infty]$,

$$\Diamond_x\varphi \leftrightarrow \neg\Box_x\neg\varphi.$$

The *normal form* of $\varphi \in W_{[0,\infty]}$, written nf$(\varphi)$, is some choice of formula ψ such that none of $\vee$, $\to$, $\leftrightarrow$, or $\Box_x$ (for any $x \in [0,\infty]$) appear in ψ, φ and ψ are provably equivalent in $K_{[0,\infty]}$, and ψ is of minimal length. By classical results such a function exists. *Consistency* and *maximal consistency* are defined in the standard way on sets of formulas.

2.2 The $[0,\infty]$-Indexed Multimodal Semantics

A *frame* $\mathfrak{F}$ is a pair $(M, \{R_x : x \in [0,\infty]\})$ where M is a nonempty set, and for every $x \in [0,\infty]$, $R_x \subseteq M^2$. A *model* is then a pair $(\mathfrak{F}, V)$ where $\mathfrak{F}$ is a frame and $V : M \times \mathrm{P} \to \{0,1\}$. In other words, these models are just the standard multimodal possible worlds models over modal operators $\{\Box_x, \Diamond_x : x \in [0,\infty]\}$.

Every element $w \in M$ is called a *world*, each R_x is called the *x-accessibility relation*, and V is called the *valuation map*. When $(w,u) \in R_x$, we say that $R_x(w,u)$ *holds*.

Satisfaction is defined in the natural way. That is, a model $\mathfrak{M}$ satisfies $\varphi \in W_{[0,\infty]}$ at $w \in M$, written $\mathfrak{M}, w \models \varphi$, when the following hold.

- If nf$(\varphi) \in \mathrm{P}$, then $V(w, \mathrm{nf}(\varphi)) = 1$.
- If nf$(\varphi) = \neg\psi$ for some $\psi \in W_{[0,\infty]}$, then $\mathfrak{M}, w \not\models \psi$.
- If nf$(\varphi) = \psi \wedge \theta$ for some $\psi, \theta \in W_{[0,\infty]}$, then $\mathfrak{M}, w \models \psi$ and $\mathfrak{M}, w \models \theta$.
- If nf$(\varphi) = \Diamond_x\psi$ for some $\psi \in W_{[0,\infty]}$ and $x \in [0,\infty]$, then there is some $u \in M$ such that $R_x(w,u)$ holds and $\mathfrak{M}, u \models \psi$.

If $\varphi \in W_{[0,\infty]}$ is satisfied at every world in every model, we say that φ is *valid* and write $\models \varphi$.

Frame satisfaction is defined as follows. Frame $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$ satisfies formula φ if and only if every valuation V on that frame provides a model $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{M}, w \vDash \varphi$ for all $w \in M$.

For a set of formulas $T \subseteq W_{[0,\infty]}$, when $\mathfrak{M}, w \vDash \varphi$ for every $\varphi \in T$, we say $\mathfrak{M}, w \vDash T$. The maximal such set for a given model $\mathfrak{M}$ and world w is the *theory of* $\mathfrak{M}$ *at* w, that is, $Th(\mathfrak{M}, w) := \left\{\varphi \in W_{[0,\infty]} : \mathfrak{M}, w \vDash \varphi\right\}$.

We say that a set of formulas $T \subseteq W_{[0,\infty]}$ is *satisfiable* if there is some model $\mathfrak{M} = (M, \{R_x : x \in [0, \infty]\}, V)$ and world w such that $\mathfrak{M}, w \vDash T$. T is *finitely satisfiable* if it is satisfiable on a model such that $|M|$ is finite.

Since $K_{[0,\infty]}$ is a normal multimodal logic and each modal formula of $W_{[0,\infty]}$ is finite in length, it is easy to see that the finite model property holds.

Proposition 1 (Finite model property) *For any* $\varphi \in W_{[0,\infty]}$, φ *is satisfiable if and only if it is finitely satisfiable.*

Proof (*sketch*) The reverse direction follows by definition. For the forward direction, extend the standard finite model property argument, as given in [14], to the multimodal setting. The key reason this argument extends is that any given formula contains only finitely many modal operators. □

Note that this finite model property is not an *effective* finite model property, since the set of all possible relations, and hence all models, on a given finite domain is uncountable. Nevertheless, a naming system on the set of formulas may be sufficient for an effective construction.

2.3 Completeness and Canonicity

Recall the following well-known result.

Theorem 1 (Canonical Model Theorem, Theorem 4.22, [15]) *Any normal modal logic is strongly complete with respect to its canonical model.*

This theorem is extended to the multimodal setting via Exercise 4.2.4 in [15]. From this we derive completeness as a simple corollary.

Corollary 1 (Completeness of $K_{[0,\infty]}$ in possible world models) *The logic* $K_{[0,\infty]}$ *is sound and strongly complete in the class of models given above.*

It follows from completeness that a set of formulas is consistent if and only if it is satisfiable.

The continuous accessibility modal logics of this paper are all extensions of $K_{[0,\infty]}$. To prove their completeness, we recall another well-known result.

Theorem 2 (Sahlqvist Completeness Theorem, Theorem 4.42, [15]) *Every Sahlqvist formula is canonical for the first-order property it defines. Hence, given a set of*

Sahlqvist axioms Σ, the logic $K\Sigma$ is strongly complete with respect to the first order class of frames defined by Σ.

Every axiom we introduce is a Sahlqvist formula. The bulk of the axioms are, moreover, modal in just one of the indices, so the Sahlqvist Completeness Theorem applies transparently. Others, including some of the more interesting ones, are properly multimodal or *interactive*, involving more than one index. So to prove the completeness of the continuous accessibility modal logics, we will prove the canonicity of these axioms.

To this end, we consider a lemma that is a special case of the multimodal version of the Sahlqvist Completeness Theorem. We present the lemma in terms of $K_{[0,\infty]}$ (though it applies to any normal multimodal logic), and we include its simple proof for understanding.

Lemma 1 (Canonicity for generalized transitivity) *Fix $x_1, ..., x_n \in [0, \infty]$. Let ψ be the formula schema: for all $\varphi \in W_{[0,\infty]}$, $\Diamond_{x_1}\Diamond_{x_2}\ldots\Diamond_{x_{n-1}}\varphi \to \Diamond_{x_n}\varphi$. Let a frame $(M, \{R_x : x \in [0, \infty]\})$ have property Ψ if and only if: for all worlds $w_1, w_2, \ldots, w_n \in M$, if each of $R_{x_1}(w_1, w_2)$, $R_{x_2}(w_2, w_3)$, ..., and $R_{x_{n-1}}(w_{n-1}, w_n)$ hold, then $R_{x_n}(w_1, w_n)$ holds.*

Then ψ is canonical for Ψ.

Proof Since $K_{[0,\infty]}$ is a normal multimodal logic, it suffices to show that a frame satisfies ψ if and only if it has property Ψ. The reverse direction is straightforward. For the forward direction, fix a frame $(M, \{R_x : x \in [0, \infty]\})$. Suppose, by contraposition, that the frame does not have Ψ. Then there are worlds $w_1, w_2, \ldots, w_n \in M$ such that all of $R_{x_1}(w_1, w_2)$, $R_{x_2}(w_2, w_3)$, $\ldots$, $R_{x_{n-1}}(w_{n-1}, w_n)$ hold, but $R_{x_n}(w_1, w_n)$ does not. Fix some such $w_1, w_2, \ldots, w_n$, and fix an arbitrary proposition $p \in \mathrm{P}$. Define V such that $V(w_n, p) = 1$ and $V \equiv 0$ everywhere else. Then V provides model $\mathfrak{M}$ such that $\mathfrak{M}, w_n \vDash p$ and $\mathfrak{M}, w \nvDash p$ for all $w \neq w_n \in M$. By construction, $\mathfrak{M}, w_1 \vDash \Diamond_{x_1}\Diamond_{x_2}\ldots\Diamond_{x_{n-1}}p$ but $\mathfrak{M}, w_1 \nvDash \Diamond_{x_n}p$. Therefore, the frame does not satisfy ψ. □

3 Continuous Accessibility Modal Logic

In this section, we present the basic continuous accessibility modal logic.

Consider $K_{[0,\infty]}$. Since each $[0, \infty]$-indexed accessibility relation is independent of all the others, for any nonempty set of worlds and arbitrary combination of $[0, \infty]$-indexed accessibility relations on those worlds, there is some frame satisfying $K_{[0,\infty]}$ that includes precisely those worlds and relations. For example, one world might be both 1-accessible and 3-accessible to another, but not 2-accessible to it. To capture *continuous accessibility*, that is, accessibility as a magnitude with the structure of the real numbers, we need to impose specific constraints on acceptable accessibility combinations that correspond to conditions on the class of all frames. Once the appropriate constraints are applied, we can interpret each accessibility relation as a distinct degree of accessibility. (In our convention, the lower the index on the accessibility relation, the higher the degree of accessibility, so the index is better understood as a degree of *remoteness*.)

Two such constraints are appropriate to impose. The first is upward closure. The idea here is that if something is very possible (true in a world accessible to a high degree), then it is *a fortiori* somewhat possible (true in a world accessible to a low degree). Upward closure guarantees that a world accessible to another at some degree of remoteness is accessible at every higher degree of remoteness. After imposing this condition, the $[0, \infty]$-indexed accessibility relations of $K_{[0,\infty]}$ can be interpreted in the remaining frames as a single continuous accessibility relation indexed by remoteness, with the structure of the real numbers.

The second constraint is well-foundedness. If accessibility is a magnitude, then it makes sense to quantify *how* accessible one world is to another, that is, to assign some real number as the degree of accessibility of that world. Upward closure alone doesn't guarantee this, since the world might be accessible up to, but not including, some magnitude. For example, a world might be x-accessible to another for every $x > 1$ without being 1-accessible, in which case no real number characterizes how accessible the world is. Well-foundedness guarantees that a world accessible to another at some degree of remoteness is accessible to it at some *minimum* degree of remoteness. After imposing this further condition, the remaining frames reflect continuous accessibility as a magnitude, represented by a unique, real-numbered value.

As we prove below, upward closure is straightforward to apply (there is a modal axiom schema that is canonical for it) and gives much of the relevant structure. On the other hand, well-foundedness is not modally definable and requires a first-order constraint on the class of frames.

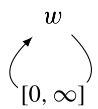
3.1 The Basic Continuous Accessibility Modal Logic

Define the *basic continuous accessibility modal logic*, denoted C, as the smallest extension of $K_{[0,\infty]}$ that contains the axiom schema:

(f) *Upward closure*: For every $\varphi \in W_{[0,\infty]}$ and $x < y \in [0, \infty]$,

$$\Diamond_x \varphi \rightarrow \Diamond_y \varphi.$$

Any frame that satisfies C is called a *continuous accessibility frame*. It's easy to see such frames exist. For example, define the *trivial frame* as the frame with a single world maximally accessible to itself:



The trivial frame is a continuous accessibility frame.

We want to make sure that continuous accessibility frames have the intended structure, specifically that for every $y > x \in [0, \infty]$, each world x-accessible to another is also y-accessible to it. Fortunately, by Lemma 1, upward closure is canonical for the following property.

Definition 1 A frame $(M, \{R_x : x \in [0, \infty]\})$ has **upwardly closed accessibility** if and only if: for every $w, u \in M$ and every $x \in [0, \infty]$, if $R_x(w, u)$ holds, then for every $y > x \in [0, \infty]$, $R_y(w, u)$ holds.

We thereby gain a completeness result.

Corollary 2 (Completeness of C in continuous accessibility frames) *The logic C is sound and strongly complete in the class of continuous accessibility frames.*

Proof Follows directly from Theorem 1 and the canonicity of upward closure. □

3.2 Well-Foundedness and Adequately-Founded Accessibility

Recall that, in addition to upward closure, being well-founded is what allows accessibility to be interpreted as a magnitude.

Definition 2 A frame $(M, \{R_x : x \in [0, \infty]\})$ is **well-founded** if and only if: for every $w, u \in M$, if $R_x(w, u)$ holds for some $x \in [0, \infty]$, then there is a minimum such x.

There is a family of properties had by well-founded continuous accessibility frames and lacked by continuous accessibility frames that are not well-founded. The most inclusive such property is adequately-founded accessibility.

Definition 3 A frame $(M, \{R_x : x \in [0, \infty]\})$ has **adequately-founded accessibility** if and only if: for every $w, u \in M$ and every $x \in [0, \infty]$, if $R_y(w, u)$ does not hold for any $y < x$ and $R_y(w, u)$ holds for every $y > x$, then $R_x(w, u)$ holds.

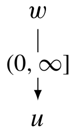
While the class of frames with adequately-founded accessibility is first-order definable (by the above definition), it is not modally definable in our language, nor indeed is the class of frames with any property that distinguishes well-founded continuous accessibility frames from continuous accessibility frames that are not well-founded.

Proposition 2 *For any property had by well-founded continuous accessibility frames and lacked by continuous accessibility frames that are not well-founded, the class of frames with that property is not modally definable.*

Proof If any such property is modally definable, then there is some modal formula in $W_{[0,\infty]}$ that is satisfied by all well-founded continuous accessibility frames and by no continuous accessibility frames that are not well-founded. We show that the consequent is false by providing a counterexample. It is sufficient to give an example of two frames such that: (a) one frame is a continuous accessibility frame that is not well-founded, (b) the other frame is a well-founded continuous accessibility frame, and (c) every valuation on the first frame provides a model that is bisimilar to some model over the second frame.

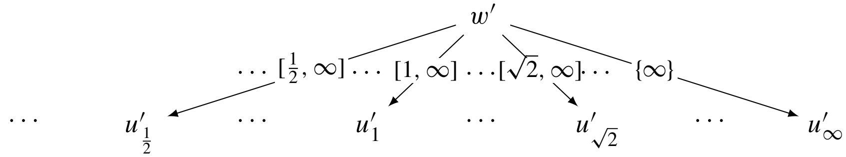
Consider the following two frames. The first, the *committed frame*, is composed of a world (w) that has accessibility to another (u) for each degree of remoteness above 0:

$$\big(\{w, u\}, \big\{\{(w, u)\}_x : x \in (0, \infty]\big\}\big)$$



The second, the *gregarious frame*, is composed of a world (w') that has accessibility to continuum-many worlds, each with a distinct minimum degree of remoteness above 0:

$$(\{w'\} \cup \{u'_x : x \in (0, \infty]\}, \{\{(w', u'_x)\}_y : y \geq x \in (0, \infty]\})$$



The committed frame has upwardly closed accessibility, so is a continuous accessibility frame. It is also not well-founded, since u is not 0-accessible to w despite being accessible at every higher degree of remoteness. So (a) is satisfied. The gregarious frame also has upwardly closed accessibility, so it is a continuous accessibility frame. But it is well-founded, since every u'_x is accessible to w' at the minimum x. So (b) is satisfied. Finally, for every valuation on the committed frame,

$$E := \big\{(w, w')\big\} \cup \big\{(u, u'_x) : x \in (0, \infty]\big\}$$

is a bisimulation from the provided model of the committed frame to some model of the gregarious frame. To see this, fix an arbitrary valuation V on the committed frame, providing model $\mathfrak{M}$. We can select a valuation V' on the gregarious frame such that, for all $p \in \mathrm{P}$, $V'(w', p) = V(w, p)$ and $V'(u'_x, p) = V(u, p)$ for each $x \in (0, \infty]$, providing model $\mathfrak{M}'$. By construction, for each $p \in \mathrm{P}$ and ordered pair of worlds $(v, v') \in E, \mathfrak{M}, v \vDash p$ if and only if $\mathfrak{M}', v' \vDash p$. Moreover, for each $x \in (0, \infty]$, only $R_x(w, u)$ holds in $\mathfrak{M}$, while $R'_x(w', u'_x)$ holds $\mathfrak{M}'$. Finally, for each $x, y \in (0, \infty]$ such that $y \leq x$, only $R'_x(w', u'_y)$ holds in $\mathfrak{M}'$, while $R_x(w, u)$ holds in $\mathfrak{M}$. Thus, $\mathfrak{M}$ is bisimilar to $\mathfrak{M}'$. So (c) is satisfied. □

In well-founded continuous accessibility frames, for each pair of worlds where one is accessible to the other, the minimum degree of remoteness is a unique real number that can be treated as a *magnitude* describing how accessible it is. It is important to note, however, that it does not follow that modalities like necessity and possibility likewise have magnitudes. Consider, for example, the gregarious frame in the proof above. Extend the frame to any model where some proposition p is true at all worlds

u'_x such that $x < 1$, but false at u'_1. In that case, there is no maximum x such that $\Box_x p$ is true at w', and therefore no real-numbered magnitude describing how necessary p is. Indeed, in general, for any formula φ, if there is a maximum x such that $\Box_x \varphi$, then there is no minimum x such that $\Diamond_x \neg\varphi$. So the modalities themselves, while being a matter of degree with the structure of the real numbers, might not have magnitudes.

3.3 Well-Foundedness, Completeness, and Canonicity

Since no axiom schemata induce adequately-founded accessibility, we will stipulate that the frames in question are well-founded. The consequences of this stipulation are surprisingly slight. This is because we can generalize the proof of Proposition 2 to apply to any continuous accessibility frame that is not well-founded, and indeed to those that also satisfy certain extensions of C.

We begin by defining planted models, which are tree-unraveled models except that self-0-accessibility is not unraveled.

Definition 4 For every model $\mathfrak{M} = (M, \{R_x : x \in [0, \infty]\}, V)$ and world $a \in M$, the **planted model** of pointed model $(\mathfrak{M}, a)$ is a tree-like pointed model $\left(\mathfrak{M}^\dagger, \langle a \rangle\right)$ such that $\mathfrak{M}^\dagger = \left(M^\dagger, \{R_x^\dagger : x \in [0, \infty]\}, V^\dagger\right)$, constructed as follows.

The worlds in $M^\dagger$ are all finite paths of worlds in M, where each world in the path after the first is indexed to some $x \in [0, \infty]$. For every such path (as a world of $M^\dagger$), we say that it **corresponds** to the final world (of M) in itself. $M^\dagger$ is defined recursively as follows:

- $\langle a \rangle$ is in $M^\dagger$.
- If path $w^\dagger$ (corresponding to w) is in $M^\dagger$, then for each $x \neq 0$ such that $R_x(w, w)$ holds, the path that results from concatenating $w^\dagger$ and w_x is in $M^\dagger$.
- If path $w^\dagger$ (corresponding to w) is in $M^\dagger$, then for each $u \neq w \in M$: for each x such that $R_x(w, u)$ holds, the path that results from concatenating $w^\dagger$ and u_x is in $M^\dagger$.

The accessibility relations $\{R_x^\dagger : x \in [0, \infty]\}$ are defined as follows:

- For each path $w^\dagger \in M^\dagger$ (corresponding to w), if $R_0(w, w)$ holds, then $R_0^\dagger(w^\dagger, w^\dagger)$ holds.
- For each pair of paths $w^\dagger, u^\dagger \in M^\dagger$ such that $u^\dagger$ is one longer than $w^\dagger$, $R_x^\dagger(w^\dagger, u^\dagger)$ holds, where x is the index of the final world in $u^\dagger$.

Valuation map $V^\dagger$ is defined as follows: For all $w^\dagger \in M^\dagger$ corresponding to $w \in M$ and for all propositions $p \in \mathrm{P}$, $V^\dagger(w^\dagger, p) = V(w, p)$.

Notably, planted models are well-founded, and a simple exercise reveals that a pointed model is bisimilar to its planted model.

Next, we characterize a set of properties had by pointed models that can be successfully applied to their planted models without undoing the latter's well-foundedness or its bisimilarity.

Definition 5 Let Π be a frame property. Let $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$ be an arbitrary frame that has Π. Let $(\mathfrak{M}, a)$ be an arbitrary pointed model extending $\mathfrak{F}$. Let $\left(\mathfrak{M}^\dagger, \langle a \rangle\right)$ be its planted model, with $\mathfrak{M}^\dagger = \left(M^\dagger, \{R_x^\dagger : x \in [0, \infty]\}, V^\dagger\right)$.

Property Π is **plantable** if and only if there exists a set of accessibility relations $\{R'_x : x \in [0, \infty]\}$ such that:

1. for all $x \in [0, \infty]$, $R'_x \supseteq R_x^\dagger$;
2. $\left(M^\dagger, \{R'_x : x \in [0, \infty]\}\right)$ has Π;
3. $\left(M^\dagger, \{R'_x : x \in [0, \infty]\}\right)$ is well-founded; and
4. for all $w^\dagger, u^\dagger \in M^\dagger$ corresponding to w and u, respectively, and for all $x \in [0, \infty]$, $R'_x(w^\dagger, u^\dagger)$ holds only if $R_x(w, u)$ holds.

The idea here is that, for plantable properties, some of the accessibility relations of the pointed model can be added back to the planted model to recover the property while preserving well-foundedness.

Proposition 3 *The class of well-founded frames with a plantable property is modally indistinguishable from the class of frames with that property.*

Proof It suffices to show that every pointed model extending a frame with a plantable property is bisimilar to a pointed model extending a well-founded frame with that property.

By construction, a pointed model $(\mathfrak{M}, a)$ extending a frame with a plantable property is bisimilar to its planted model $\left(\mathfrak{M}^\dagger, \langle a \rangle\right)$. By the definition of a plantable property, there is a set of accessibility relations that extends $\left(\mathfrak{M}^\dagger, \langle a \rangle\right)$ to a well-founded pointed model with that property, $\left(\mathfrak{M}', \langle a \rangle\right)$. Since each of the additional accessibility relations in $\mathfrak{M}'$ relate paths in $\mathfrak{M}^\dagger$ that correspond to worlds in $\mathfrak{M}$ that are likewise related, $(\mathfrak{M}, a)$ is bisimilar to $\left(\mathfrak{M}', \langle a \rangle\right)$. $\square$

Lemma 2 *Upwardly closed accessibility is plantable.*

Proof See Appendix A.1. $\square$

Theorem 3 *The logic C is sound and strongly complete in the class of well-founded continuous accessibility frames.*

Proof Follows directly from Corollary 2, Proposition 3, and Lemma 2. $\square$

3.4 Comparison to the Fuzzy Semantics Approach

Recall the fuzzy semantics approach to continuous accessibility, where the accessibility relation itself is a partial function from pairs of worlds to a real number (or ∞). On this approach, we define a non-standard *fuzzy frame* $\mathfrak{F}^*$ as a pair (M, R^*) where $R^* : M^2 \rightharpoonup [0, \infty]$ is called a *fuzzy accessibility relation*, and a *fuzzy model* $\mathfrak{M}^*$ as $(\mathfrak{F}^*, V)$.

Fuzzy satisfaction, denoted $\vDash^*$, is defined in the standard way for non-modal sentences. For modal sentences, we have:

- $\mathfrak{M}^*, w \vDash^* \Diamond_x \varphi$ if and only if there is some $u \in M$ such that $R^*(w, u) \leq x$ and $\mathfrak{M}^*, u \vDash^* \varphi$.

Given a frame $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$, we can define

$$R^*(w, u) := \inf\{x \in [0, \infty] : R_x(w, u)\}$$

to construct a fuzzy frame $\mathfrak{F}^* = (M, R^*)$. Call this map the *classical–fuzzy map*. Notice also that, given a fuzzy frame $\mathfrak{F}^* = (M, R^*)$, we may construct a frame $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$ where for any worlds $w, u \in M$ and $x \in [0, \infty]$, $R_x(w, u)$ holds if and only if $x \geq R^*(w, u)$. Notably, $\mathfrak{F}$ is a well-founded continuous accessibility frame. Call this the *fuzzy–classical map*. This leads us to the following result.

Theorem 4 *There is a satisfaction-preserving bijection between the class of well-founded continuous accessibility frames and the class of fuzzy frames.*

Proof We claim that the classical–fuzzy map, when restricted to well-founded continuous accessibility frames, witnesses this theorem.

First, we show that it is satisfaction preserving. Fix an arbitrary well-founded continuous accessibility frame $(M, \{R_x : x \in [0, \infty]\})$, which maps to fuzzy frame (M, R^*). Select arbitrary valuation V, and extend the frames to models $\mathfrak{M}$ and $\mathfrak{M}^*$. For non-modal formulas, the definitions of satisfaction and fuzzy satisfaction are the same, so $\mathfrak{M}$ and $\mathfrak{M}^*$ satisfy the same non-modal formulas. For modal formulas, suppose $\mathfrak{M}, w \vDash \Diamond_x \varphi$. Then there must be some $u \in M$ such that $R_x(w, u)$ and $\mathfrak{M}, u \vDash \varphi$. By the definition of the classical–fuzzy map, $R^*(w, u) \leq x$. It follows that $\mathfrak{M}^*, w \vDash^* \Diamond_x \varphi$. So $\mathfrak{M}$ and $\mathfrak{M}^*$ satisfy all the same formulas. Therefore, the classical–fuzzy map is satisfaction-preserving.

Second, we show that it is injective. Fix well-founded continuous accessibility frames $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$ and $\mathfrak{F}' = (M', \{R'_x : x \in [0, \infty]\})$. $\mathfrak{F}$ maps to $\mathfrak{F}^* = (M, R^*)$ and $\mathfrak{F}'$ maps to $\mathfrak{F}'^* = (M', R'^*)$. Suppose that $\mathfrak{F}^* = \mathfrak{F}'^*$. By the definition of fuzzy frames, it must be that $M = M'$ and $R^* = R'^*$. Now fix $w, u \in M$ and $x \in [0, \infty]$ such that $R_x(w, u)$ holds. By the definition of the classical–fuzzy map, it must be that $R^*(w, u) \leq x$, which is just to say $R'^*(w, u) \leq x$ By the definition of the classical–fuzzy map, and since $\mathfrak{F}'$ is well-founded, it must be that $R'_x(w, u)$ holds. Conversely, fix $w, u \in M$ and $x \in [0, \infty]$ such that $R'_x(w, u)$ holds. By a symmetric argument, $R_x(w, u)$ holds. Therefore, for every $x \in [0, \infty]$, $R_x = R'_x$, so $\mathfrak{F} = \mathfrak{F}'$.

Lastly, we show that it is surjective. Fix a fuzzy frame $\mathfrak{F}^* = (M, R^*)$. Then construct $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$ using the fuzzy–classical map. It is clear that $\mathfrak{F}$ has upwardly closed accessibility and adequately-founded accessibility, and is therefore a well-founded continuous accessibility frame. Moreover, by simple calculation, applying the classical–fuzzy map to $\mathfrak{F}$ yields $\mathfrak{F}^*$. □

This theorem proves that, in the cases of interest, classical multimodal semantics and fuzzy semantics are equivalent. Nevertheless, there are benefits to developing continuous modal logic using the classical multimodal semantics approach. First, the classical multimodal semantics approach uses frames and their properties in the standard way. This simplifies the interpretation of continuous accessibility modal logics. Second, and relatedly, using classical multimodal semantics allows us to appeal to the Canonical Model and Sahlqvist Completeness theorems, and transparently situates our logics in the family of multimodal logics. Third, the classical multimodal semantics approach is more revealing than the fuzzy semantics approach about the complexity of these structures. The fuzzy semantics approach obscures the modal undefinability of adequately-founded accessibility in the definition of fuzzy accessibility and the fuzzy satisfaction of modal formulas. By revealing this feature, the advantages of the classical multimodal semantics approach are demonstrated.

For example, completeness proofs in the literature that take the fuzzy semantics approach (or a sufficiently similar approach) rely on non-standard frames and non-standard notions of satisfaction. How such frames are situated in the family of Kripke frames has, up until this point, been unclear. The classical multimodal semantics approach shows that such proofs are effectively taking for granted the well-foundedness of standard Kripke frames. Thus, what is sometimes presented as completeness is, implicitly, a form of relative completeness. Moreover, by taking the classical multimodal semantics approach, we are able to prove (as we did above) that relative completeness is, in many cases of interest, equivalent to completeness.

For another example, those aspects of continuous accessibility modal logic that do not rely on the models being well-founded are purely modally definable, and those that do rely on the models being well-founded are transparently not so. More generally, the nature and extent of imposing first-order conditions on frames can be tailored to suit the particular application.

To stipulate a class of frames via a first-order property is to make an assumption about which frames are worth considering. Methodologically, weaker assumptions should be prioritized over stronger assumptions. Developing a modal logic with a continuous accessibility relation in fuzzy semantics makes a stronger assumption than doing so in classical multimodal semantics. To see this, suppose we combine fuzzy satisfaction with the $[0, \infty]$-indexed relations of the classical multimodal semantics. In that case, we need to represent the fuzzy accessibility relation with a frame property that constrains the $[0, \infty]$-indexed accessibility relations in the appropriate way. The obvious candidate property, since it constrains the set of relations to be a function, is the following.

Definition 6 A frame $(M, \{R_x : x \in [0, \infty]\})$ has **unique accessibility** if and only if: for every $w, u \in M$ and $y \neq x \in [0, \infty]$, if $R_x(w, u)$ holds then $R_y(w, u)$ does not hold.

Unique accessibility, however, like well-foundedness, is not modally definable.[1] Notably, frames with unique accessibility trivially also have adequately-founded

[1] The tree unraveling of the trivial frame provides the proof.

accessibility, but the converse does not hold. It follows, therefore, that unique accessibility is a stronger assumption than adequately-founded accessibility.

Because of this, even though assumptions are required for many of our desired applications, the classical multimodal semantics approach allows us to make weaker assumptions than the fuzzy semantics approach. Most of our applications require that the continuous accessibility frames have adequately-founded accessibility. In these cases, as we stated earlier, we can stipulate that the frames are well-founded. But for one application, real line continuous temporal logic, we require syntactic access to the magnitudes of the accessibility relations. For that, we must stipulate further that the frames have (a version of) unique accessibility. In each case, however, the extent of the assumptions is transparent.

3.5 Continuous Accessibility Modal Logic with Unique Accessibility

The strength of unique accessibility is not needed for most applications developed in this paper. Still, when it is needed, it can be incorporated into a "close cousin" of continuous accessibility modal logic. Since frames with unique accessibility and those with upwardly closed accessibility overlap only when no pairs of worlds are x-accessible for any $x \in [0, \infty)$, we will extend our language to include a second set of modal operators to represent magnitudes. Define the *language of basic μ-extended* $[0, \infty]$*-indexed modal logic*, $L^{\mu}_{[0,\infty]}$, to be the smallest extension of $L_{[0,\infty]}$ that includes the additional modal operator symbols $\left\{\Box^{\mu}_x, \Diamond^{\mu}_x : x \in [0, \infty]\right\}$. The well-formed formulas, $W^{\mu}_{[0,\infty]}$, are defined in the standard way.

The *basic μ-extended* $[0, \infty]$*-indexed multimodal logic*, $K^{\mu}_{[0,\infty]}$, is the smallest normal multimodal logic in modal operators $\left\{\Box_x, \Diamond_x, \Box^{\mu}_x, \Diamond^{\mu}_x : x \in [0, \infty]\right\}$. That is, in addition to the rules of inference and axiom schemata of $K_{[0,\infty]}$, $K^{\mu}_{[0,\infty]}$ has the standard versions of necessitation, modal distribution, and duality for $\Box^{\mu}_x$. Normal form is also defined in the standard way.

Models and frames for $K^{\mu}_{[0,\infty]}$ are defined as for $K_{[0,\infty]}$, except that, in addition to the x-accessibility relations R_x, there is another set of $[0, \infty]$-indexed accessibility relations, the μ-x-accessibility relations R^{μ}_x, such that for every $x \in [0, \infty]$, $R^{\mu}_x \subseteq M^2$. Satisfaction is then defined in the standard way for modal formulas with $\Diamond^{\mu}_x$.

Just as we extended $K_{[0,\infty]}$ to C, we can extend $K^{\mu}_{[0,\infty]}$ to the *basic μ-extended continuous accessibility logic*, C^{μ}, by requiring that it contain upward closure in the $\Diamond_x$ operator (as in C), along with the following axiom schema:

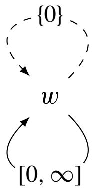
(f^{μ}) *μ-corespondence*: For every $\varphi \in W^{\mu}_{[0,\infty]}$ and $x \in [0, \infty]$,

$$\Diamond^{\mu}_x \varphi \rightarrow \Diamond_x \varphi.$$

By Lemma 1, μ-corespondence is canonical for the following.

Definition 7 A frame $(M, \{R_x : x \in [0, \infty]\})$ has **μ-corresponding accessibility** if and only if: for all $w, u \in M$ and $x \in [0, \infty]$, if $R^{\mu}_x(w, u)$ holds then $R_x(w, u)$ holds.

Any frame that satisfies C^μ is a *μ-extended continuous accessibility frame*. An example of such a frame is the *μ-extended trivial frame*, which is the extension of the trivial frame where w is also μ-0-accessible to itself.



We will stipulate that the frames have two further properties. The first is *unique μ-accessibility*, that is, unique accessibility in the μ-x-accessibility relations. Unique μ-accessibility is not modally definable, even restricting the domain to well-founded μ-extended continuous accessibility frames.[2]

The second is a correlate of well-foundedness:

Definition 8 A frame $\left(M, \{R_x, R_x^\mu : x \in [0, \infty]\}\right)$ has **μ-induced well-foundedness** if and only if: for every $w, u \in M$, if $R_x(w, u)$ holds for some $x \in [0, \infty]$, then there is a minimum $y \leq x$ such that $R_y^\mu(w, u)$ holds.

As with well-foundedness, there is a family of properties had by μ-extended continuous accessibility frames with μ-induced well-foundedness and lacked by μ-extended continuous accessibility frames without μ-induced well-foundedness. The most inclusive such property is μ-induced adequately-founded accessibility.

Definition 9 A frame $\left(M, \{R_x, R_x^\mu : x \in [0, \infty]\}\right)$ has **μ-induced adequately-founded accessibility** if and only if: for every $w, u \in M$ and every $x \in [0, \infty]$, if $R_y(w, u)$ does not hold for any $y < x \in [0, \infty]$ and $R_y(w, u)$ holds for every $y > x \in [0, \infty]$, then $R_x^\mu(w, u)$ holds.

Any μ-extended continuous accessibility frame with μ-induced adequately-founded accessibility is well-founded, and thus has adequately-founded accessibility (in the x-accessibility relations). It turns out that μ-induced adequately-founded

[2]The proof is similar to that of Proposition 2, but with the frame

$$\left(\{w, u\}, \left\{\{(w, u)\}_x, \{(w, u)\}_z^\mu : x \in [0, \infty], z \in \{0, 1\}\right\}\right),$$

which is well-founded but lacks unique μ-accessibility, and the frame

$$\left(\{w', u'_0, u'_1\}, \left\{\left\{(w', u'_0)\right\}_x, \left\{(w', u'_1)\right\}_y, \left\{(w', u'_z)\right\}_z^\mu : x \in [0, \infty], y \in [1, \infty], z \in \{0, 1\}\right\}\right),$$

which is well-founded and has unique μ-accessibility.

accessibility is not modally definable.[3] We conjecture that any property that distinguishes μ-extended continuous accessibility frames with μ-induced well-foundedness from those that lack μ-induced well-foundedness is not modally definable, even restricting the domain to well-founded frames with unique μ-accessibility.[4]

For convenience, we call any μ-extended continuous accessibility frame with unique μ-accessibility and μ-induced adequately-founded accessibility a *well-structured* μ-extended continuous accessibility frame. Well-structured μ-extended continuous accessibility frames are well-founded, and the corresponding minima are encoded in the μ-x-accessibility relations. The μ-extended trivial frame is one such frame.

The classical–fuzzy map can be extended to μ-extended continuous accessibility frames by applying it to the subframe containing only the x-accessibility relations. By construction, any fuzzy frame $\mathfrak{F}^* = (M, R^*)$ can be mapped to a μ-extended continuous accessibility frame via the fuzzy–classical map together with the condition that for any worlds $w, u \in M$ and $x \in [0, \infty]$, $R^\mu_x(w, u)$ holds if and only if $x = R^*(w, u)$. Notably, this μ-extended continuous accessibility frame is well-structured. Call this the *μ-extended fuzzy–classical map*. Extending fuzzy satisfaction in the expected way:

- $\mathfrak{M}^*, w \vDash \Diamond^\mu_x \varphi$ if and only if there is some $u \in M$ such that $R^*(w, u) = x$ and $\mathfrak{M}^*, u \vDash \varphi$;

leads us to the following result.

Corollary 3 *There is a satisfaction-preserving bijection between the class of well-structured μ-extended continuous accessibility frames and the class of fuzzy frames.*

Proof (*sketch*) The μ-extended fuzzy–classical map witnesses the corollary. By construction, the map is injective and surjective. By Theorem 4, the map is satisfaction-preserving over $W_{[0,\infty]}$. Moreover, by the above extension of fuzzy satisfaction, it is satisfaction-preserving over all of $W^\mu_{[0,\infty]}$. □

[3] Again, the proof is similar to that of Proposition 2, but with the frame

$$\left(\{w, u\}, \left\{\{(w, u)\}_x, \{(w, u)\}^\mu_\infty : x \in [0, \infty]\right\}\right),$$

which lacks μ-induced adequately-founded accessibility, and the frame

$$\left(\left\{w', u'_x : x \in [0, \infty]\right\}, \left\{\left\{w', u'_x\right\}_x, \left\{w', u'_\infty\right\}^\mu_\infty : x \in [0, \infty]\right\}\right),$$

which has μ-induced adequately-founded accessibility. But note that the latter frame is not a μ-extended continuous accessibility frame.

[4] The conjecture arises from the following reasoning. Restrict the domain of frames to well-founded μ-extended continuous accessibility frames with unique μ-accessibility. We consider which sorts of modal formulas frames with μ-induced well-foundedness satisfy that frames without the property fail to satisfy. For all formulas $\varphi \in W^\mu_{[0,\infty]}$ and all $x \in [0, \infty]$, the formula $(\exists y \leq x)(\Diamond_x \varphi \rightarrow \Diamond^\mu_y \varphi)$ is satisfied by frames with μ-induced well-foundedness and not by frames without it. However, in general, there is no single y for which a given frame with μ-induced well-foundedness satisfies the formula. Still, while the property may not be modally definable, it is at least first-order definable without quantifying over worlds or accessibility relations.

4 Extensions and Applications

The basic continuous accessibility modal logic can be extended to give each x-accessibility relation the correct structure for a variety of philosophical applications.

4.1 Simple Extensions of Continuous Accessibility Modal Logic

The basic continuous accessibility logic C includes no conditions on frames beyond those, mentioned above, that establish the continuity of the accessibility relation. Analogues to various modal axioms (like D, T, B, 4, and 5) can then be used to extend C to analogues of extensions of K (like D, T, B, $S4$, and $S5$).

For every $x \in [0, \infty]$, define the following extensions of C:

- The *continuous accessibility system-D_x modal logic*, denoted CD_x, is the smallest extension of C that contains the axiom schema:

  (g) *x-seriality (D_x)*: For every $\varphi \in W_{[0,\infty]}$,

  $$\Box_x \varphi \rightarrow \Diamond_x \varphi.$$

- The *continuous accessibility system-T_x modal logic*, denoted CT_x, is the smallest extension of C that contains the axiom schema:

  (g) *x-reflexivity (T_x)*: For every $\varphi \in W_{[0,\infty]}$,

  $$\Box_x \varphi \rightarrow \varphi.$$

For every $X \subseteq [x, \infty]$, define the following extensions of CT_x:

- The *continuous accessibility system-$B_{x,X}$ modal logic*, denoted $CB_{x,X}$, is the smallest extension of CT_x that contains the axiom schema:

  (h) *X-symmetry (B_X)*: For every $\varphi \in W_{[0,\infty]}$ and $x' \in X$,

  $$\Diamond_{x'} \Box_{x'} \varphi \rightarrow \varphi.$$

- The *continuous accessibility $S4_{x,X}$ modal logic*, denoted $CS4_{x,X}$, is the smallest extension of CT_x that contains the axiom schema:

  (h) *X-transitivity (4_X)*: For every $\varphi \in W_{[0,\infty]}$ and $x' \in X$,

  $$\Box_{x'} \varphi \rightarrow \Box_{x'} \Box_{x'} \varphi.$$

- The *continuous accessibility $S5_{x,X}$ modal logic*, denoted $CS5_{x,X}$, is the smallest extension of CT_x that contains the axiom schema:

  (h) *X-euclidean (5_X)*: For every $\varphi \in W_{[0,\infty]}$ and $x' \in X$,

  $$\Diamond_{x'} \varphi \rightarrow \Box_{x'} \Diamond_{x'} \varphi.$$

Corollary 4 *For every x and $X \subseteq [x, \infty]$, each of the logics CD_x, CT_x, $CB_{x,X}$, $CS4_{x,X}$, and $CS5_{x,X}$ are strongly complete in the respective classes of frames defined by them.*

Proof Follows from the Sahlqvist Completeness Theorem. □

4.2 Application to Counterfactual Conditionals

The basic continuous accessibility modal logic, along with these simple extensions, have very general application. Let's work through one prominent example.

Consider the analysis of counterfactual conditionals. In classical modal logic, the counterfactual conditional $\varphi \;\Box\!\!\rightarrow\; \psi$ (read *if* φ *were true, then* ψ *would be true*) is evaluated by checking the truth of $\varphi \rightarrow \psi$ in all nearby possible worlds. Nearness in this context represents how similar a world is to the actual world in some relevant respect, implemented as a partial or total order on the worlds. So $\varphi \;\Box\!\!\rightarrow\; \psi$ is true if and only if $\varphi \rightarrow \psi$ is true in all nearby worlds, *i.e.*, $\Box(\varphi \rightarrow \psi)$ given that only those worlds that meet some nearness threshold are accessible. The largest set of nearby worlds for a given true counterfactual conditional will be the one whose threshold allows access to all and only worlds nearer than the nearest worlds where the material conditional is false. Comparing the strength of two counterfactual conditionals at the actual world can be done by comparing their respective largest sets of nearby worlds, if nearness is a total order. It is difficult to do the same for two counterfactual conditionals at distinct worlds, since each world can in principle have independent nearness relations with all other worlds (that is, each world can have a distinct and unrelated total ordering of all worlds).

Using a continuous accessibility modal logic, the accessibility relation itself (being a measure of remoteness) can take the place of nearness. Since nearness is intuitively a similarity relation, an appropriate extension to use is the continuous accessibility system-$B_{0,[0,\infty]}$ modal logic. We can then say $\varphi \;\Box\!\!\rightarrow\; \psi$ is true if and only if there is some $x \in [0, \infty]$ such that $\Box_x(\varphi \rightarrow \psi)$. Since there may be no largest such x, we can take their supremum as a real-numbered magnitude measuring how true the counterfactual conditional is, non-comparatively on some dimension that characterizes the similarity of worlds. The comparative strength of two counterfactual conditionals at the same world can be straightforwardly quantified. Similar analyses that rely on the structure of the real numbers (like modeling the fuzziness at the threshold of nearness) can also be done. Moreover, unlike in the classical case, we can compare the strength of two counterfactual conditionals at distinct worlds, and this too can be quantified. Other comparisons between worlds that rely on structured nearness relationships across worlds can follow.

Finally, the fact that similarity is encoded directly into the accessibility relation yields the relevant and expected valid modal formulas. For example, applying $\mathrm{B}_{[0,\infty]}$, if at the actual world it is possible at least to degree 1 that some counterfactual conditional is true to degree 1, then the associated material conditional is actually true.

4.3 Counterfactual Stability

Above, we used the supremum of all $x \in [0, \infty]$ such that $\Box_x(\varphi \rightarrow \psi)$ as a measure of how true a counterfactual conditional is. Alternatively, it is a measure of how *counterfactually stable* the truth of the material conditional $\varphi \rightarrow \psi$ is at a given world. These considerations can be nicely generalized for all formulas.

Definition 10 For every continuous accessibility model $\mathfrak{M} = (M, \{R_x : x \in [0, \infty]\}, V)$, $w \in M$, and $\varphi \in W_{[0,\infty]}$, we define the **counterfactual stability** of φ at w in $\mathfrak{M}$ as

$$\text{cs}(\mathfrak{M}, w, \varphi) := \begin{cases} \sup\{x \in [0, \infty] : \mathfrak{M}, w \vDash \Box_x \varphi\} & \text{if } \mathfrak{M}, w \vDash \Box_0 \varphi \\ \varnothing & \text{otherwise.} \end{cases}$$

If $\text{cs}(\mathfrak{M}, w, \varphi) \in (0, \infty]$, we say that φ is **counterfactually stable** at w in $\mathfrak{M}$. Otherwise, it is **counterfactually unstable** at w in $\mathfrak{M}$.

Among other uses, counterfactual stability gives us further insight into the truth of counterfactual conditionals. For example, a minimum threshold of truth for a counterfactual conditional is often taken to be this: $\varphi \mathrel{\Box\!\!\rightarrow} \psi$ is true if and only if ψ is true in the nearest possible worlds where φ is true. A sufficient condition for this is that $\varphi \rightarrow \psi$ is more counterfactually stable than $\neg\varphi$.[5]

4.4 Interactive Extensions and Additive Transitivity

The simple extensions of C described above all share a common feature. Every individual axiom in each extension includes no more than one $[0, \infty]$-indexed operator. It follows that the conditions on frames apply separately to each degree of remoteness. To generate more interesting conditions on frames, these $[0, \infty]$-indexed modalities must interact with one another in some way other than upward closure.

For example, consider a Markov process where each state has an objective probability of transforming into each of a set of successor states, where the probability is independent of its predecessor states. One way to model this process with a continuous accessibility relation is to interpret (the magnitude of) the accessibility relation as the objective probability of one state transforming into a given state. But another way to model it is to interpret (the magnitude of) the accessibility relation as the probability of one state *eventually* transforming into a given state. In the latter case, we want the accessibility relation to be transitive, but not in the way captured by 4_X. For example, if state A eventually transforms into state B with probability 0.5, and state B eventually transforms into state C with probability 0.5, then $4_{[0,\infty]}$ requires that state A eventually transforms into state C with probability 0.5. But this is obviously undesirable; we know no more about the latter probability than that it is at least 0.25 (0.25 because it is the product of independent probabilities, and "at least" because there may be other paths from A to C).

A better transitivity axiom schema would be interactive:

[5] It is not a necessary condition since the counterfactual stability of $\varphi \rightarrow \psi$ might equal that of $\neg\varphi$, and yet ψ is true in all the nearest possible worlds where φ is true. This is because $\neg\varphi$ might not be true in a world x-accessible at the supremum, while $\varphi \rightarrow \psi$ is true in all such worlds. Handling these sorts of cases requires flagging those formulas that are necessary up to, but not including, the supremum, and defining inequality relations that involve them in the expected way.

(h) *Additive transitivity (A4)*: For every $\varphi \in W_{[0,\infty]}$ and $x, y \in [0, \infty]$,

$$\Box_{x+y}\varphi \rightarrow \Box_x \Box_y \varphi.$$

By Lemma 1, A4 is canonical for the following property.

Definition 11 A frame $(M, \{R_x : x \in [0, \infty]\})$ has **additively transitive accessibility** if and only if: for all $w, u, v \in M$ and $x, y \in [0, \infty]$, if both $R_x(w, u)$ and $R_y(u, v)$ hold, then $R_{x+y}(w, v)$ holds.

Continuous accessibility frames with additively transitive accessibility have $[0, \infty]$-indexed accessibility relations that can model distances. But they can also model probabilities. To understand A4 in this context, consider that we can map x-accessibility (where $x \in [0, \infty]$, 0 is most accessible, and ∞ is least accessible) to probability $p(x)$ (where $p(x) \in [0, 1]$, 0 is least probable, and 1 is most probable) using an inverse exponential function: $p(x) = \exp(-x)$, with $\exp(-\infty) := 0$. Now consider the desired probabilistic transitivity condition: if the probability from A to B is $p(x_{AB})$ and the probability from B to C is $p(x_{BC})$, then probability from A to C is at least $p(x_{AC}) = p(x_{AB}) \cdot p(x_{BC})$. Since $p(x_{AC}) = p(x_{AB}) \cdot p(x_{BC})$ if and only if $x_{AC} = x_{AB} + x_{BC}$, the desired condition corresponds to: if A has x_{AB}-accessibility to B and B has x_{BC}-accessibility to C, then there is some $x_{AC} \leq x_{AB} + x_{BC}$ such that A has x_{AC}-accessibility to C. These are exactly the frames with both upwardly closed accessibility and additively transitive accessibility.

The utility of A4 is hard to overstate. For example, to model reasoning about credences or continuous dependence relations, one can produce an additive pre-order modal logic by extending C to include T_0 and A4. In the sections below, we develop several interactive extensions of C that have a variety of uses. For each such extension, the main interactive axiom schemata are upward closure and A4.

5 Pseudometric Space Modal Logic

A *metric space* is a set of points related in the way we ordinarily think about distance. A *pseudometric space* is a generalization of a metric space, where more than one distinct point can be in the same position. In other words, in a pseudometric space, two distinct points might have all the same distance relations to all points.

The basic pseudometric space modal logic presented below can be considered, in some sense, the "workhorse" of continuous accessibility modal logics. Attempts to characterize a relation between worlds as a distance along one or more dimensions of similarity will likely implicitly appeal to the semantics of a pseudometric space modal logic. For example, consider a Hilbert space whose axes are the parameters of a physical theory for each particle in the universe. Each point in the Hilbert space is a description of the physical state of some universe. The distance between points can then be a measure of how similar two universes are. A pseudometric space modal logic captures this similarity.

5.1 The Basic Pseudometric Space Modal Logic

Define the *basic pseudometric space modal logic*, denoted Pseu, as the smallest extension of C that contains the axiom schemata:

(f) 0-*reflexivity* (T_0): For every $\varphi \in W_{[0,\infty]}$,

$$\Box_0\varphi \to \varphi.$$

(g) $[0,\infty]$-*symmetry* ($B_{[0,\infty]}$): For every $\varphi \in W_{[0,\infty]}$ and $x \in [0,\infty]$,

$$\Diamond_x\Box_x\varphi \to \varphi.$$

(h) *Additive transitivity (A4)*: For every $\varphi \in W_{[0,\infty]}$ and $x, y \in [0,\infty]$,

$$\Box_{x+y}\varphi \to \Box_x\Box_y\varphi.$$

Since Pseu contains T_0 and $B_{[0,\infty]}$, it is an extension of $CB_{0,[0,\infty]}$.

Unsurprisingly, since Pseu contains $B_{[0,\infty]}$ and A4, Pseu proves the schema:

- *Additive Euclidean (A5)*: For every $\varphi \in W_{[0,\infty]}$ and $x, y \in [0,\infty]$,

$$\Diamond_x\varphi \to \Box_y\Diamond_{x+y}\varphi.$$

The proof is substantially similar to using B and 4 to prove 5.[6] Pseu thus also proves ∞-euclidean ($5_{\{\infty\}}$), which follows from the A5 schema when $x = y = \infty$. Since Pseu also proves T_∞ (by T_0 and upward closure), it is an extension of $CS5_{\infty,\{\infty\}}$, which is just $S5$ when restricted to $\Box_\infty$.

5.2 Pseudometric Space Frames

Any frame that satisfies Pseu and is universal in ∞-accessibility is called a *pseudometric space frame*. As before, the trivial frame is a pseudometric space frame. Since Pseu is an extension of $S5$ in $\Box_\infty$, any frame that satisfies Pseu is isomorphic to a union of pseudometric space frames.

Pseudometric space frames have three notable properties. First, they are reflexive, that is, reflexive in x-accessibility for all $x \in [0,\infty]$. This follows, by classical results, from T_0 and upward closure. Second, they are symmetric, that is, symmetric in x-accessibility for all $x \in [0,\infty]$. This follows, by classical results, from $B_{[0,\infty]}$. Third, they have additively transitive accessibility. This follows from the canonicity of A4.

Corollary 5 *The logic* Pseu *is sound and strongly complete in the class of pseudometric space frames.*

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[6]For arbitrary $\varphi \in W_{[0,\infty]}$ and $x, y \in [0,\infty]$,

1. $\Diamond_x\varphi \to \Box_y\Diamond_y\Diamond_x\varphi$ (Instance of contraposition of $B_{\{y\}}$ on $\Diamond_x\varphi$)
2. $\Diamond_y\Diamond_x\varphi \to \Diamond_{x+y}\varphi$ (Instance of contraposition of A4, commutativity of addition)
3. $\Box_y\Diamond_y\Diamond_x\varphi \to \Box_y\Diamond_{x+y}\varphi$ (Nec with $\Box_y$, modal dist, and prop logic on 2)
4. $\Diamond_x\varphi \to \Box_y\Diamond_{x+y}\varphi$ (Propositional logic on 1 and 3)

Proof Follows from Sahlqvist Completeness Theorem, Corollary 2, and the canonicity of A4. □

Lemma 3 *Upwardly closed accessibility, reflexivity, symmetry, and additively transitive accessibility are jointly plantable.*

Proof See Appendix A.2. □

Proposition 4 *The logic* Pseu *is sound and strongly complete in the class of well-founded pseudometric space frames.*

Proof Follows from Corollary 5, Proposition 3, and Lemma 3. □

5.3 Pseudometric Spaces as Pseudometric Space Frames

In this section, we examine the relationship between pseudometric spaces and pseudometric space frames.

For our purposes, a *pseudometric space* is a pair (X, d) such that: X is a nonempty set, and the pseudometric function $d : X^2 \to [0, \infty]$ is such that for every $w, u, v \in X$, the following hold.

- *Indiscernibility of identicals*: $d(w, w) = 0$.
- *Symmetry*: $d(w, u) = d(u, w)$.
- *Triangle inequality*: $d(w, u) + d(u, v) \geq d(w, v)$.

Spaces of this form have historically been called ∞-pseudometric spaces, since the extended pseudometric can take on the value of ∞.

Lemma 4 *(a) If a frame $(M, \{R_x : x \in [0, \infty]\})$ is a pseudometric space frame, then under the classical–fuzzy map, (M, R^*) is a pseudometric space. (b) If a fuzzy frame (M, R^*) is a pseudometric space, then under the fuzzy–classical map, $(M, \{R_x : x \in [0, \infty]\})$ is a well-founded pseudometric space frame.*

Proof of (a) Fix a frame $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$, which maps to $\mathfrak{F}^* = (M, R^*)$ using the classical–fuzzy map.

Suppose that $\mathfrak{F}$ is a pseudometric space frame. Because $\mathfrak{F}$ is universal in ∞-accessibility, for every $w, u \in M$, $R_\infty(w, u)$ holds, and therefore $R^*(w, u)$ exists. $R^* : M^2 \to [0, \infty]$ is thus a total function. It remains to be shown that it is pseudometric.

Because $\mathfrak{F}$ is reflexive, for every $w \in M$,

$$R^*(w, w) = \inf\{x \in [0, \infty] : R_x(w, w)\} = 0.$$

Because $\mathfrak{F}$ is symmetric, for every $w, u \in M$,

$$R^*(w, u) = \inf\{x \in [0, \infty] : R_x(w, u)\} = \inf\{x \in [0, \infty] : R_x(u, w)\} = R^*(u, w).$$

Because $\mathfrak{F}$ has additively transitive accessibility, for every $w, u, v \in M$, if $x' \in \{x : R_x(w, u)\}$ and $y' \in \{y : R_y(u, v)\}$, then $x' + y' \in \{z : R_z(w, v)\}$. It follows that

$$\begin{aligned} R^*(w, u) + R^*(u, v) &= \inf\{x : R_x(w, u)\} + \inf\big\{y : R_y(u, v)\big\} \\ &= \inf\big\{x + y : R_x(w, u) \wedge R_y(u, v)\big\} \\ &\geq \inf\{z : R_z(w, v)\} \\ &= R^*(w, v). \end{aligned}$$

Therefore, $\mathfrak{F}^*$ is a pseudometric space. □ *(a)*

Proof of (b) Fix a fuzzy frame $\mathfrak{F}^* = (M, R^*)$, which maps to $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$ using the fuzzy–classical map. Suppose that $\mathfrak{F}^*$ is a pseudometric space. We need to show that $\mathfrak{F}$ is universal in the ∞-accessibility relation and that $\mathfrak{F}$ satisfies Pseu. Since $\mathfrak{F}^*$ is a pseudometric space, for all $w, u \in M$, $R^*(w, u)$ exists. Therefore, by the definition of the fuzzy–classical map, for all $w, u \in M$, $R_\infty(w, u)$ holds. $\mathfrak{F}$ is thus universal in the ∞-accessibility relation. We now show that $\mathfrak{F}$ satisfies Pseu. The proof of Theorem 4 shows that $\mathfrak{F}$ is a well-founded continuous accessibility frame. As a continuous accessibility frame, $\mathfrak{F}$ satisfies C. We therefore need only show that it satisfies T_0, $\mathrm{B}_{[0,\infty]}$, and A4.

First, since R^* is a pseudometric, for every $w \in M$, $R^*(w, w) = 0$. By the definition of the fuzzy–classical map, $R_0(w, w)$ holds. Thus, $\mathfrak{F}$ satisfies T_0.

Second, since R^* is a pseudometric, for arbitrary $w, u \in M$, $R^*(w, u) = R^*(u, w)$. By the definition of the fuzzy–classical map, both $R_x(w, u)$ and $R_x(u, w)$ hold for all $x \geq R^*(w, u) \in [0, \infty]$, and neither $R_x(w, u)$ nor $R_x(u, w)$ hold for any $x < R^*(w, u) \in [0, \infty]$. $\mathfrak{F}$ is therefore symmetric, from which it follows that $\mathrm{B}_{[0,\infty]}$.

Third, since $\mathfrak{F}^*$ is a pseudometric space, for every $w, u, v \in M$, $R^*(w, u) + R^*(u, v) \geq R^*(w, v)$. By the definition of the fuzzy–classical map, for all $x, y, z \in [0, \infty]$: $R_x(w, u)$ holds if and only if $x \geq R^*(w, u)$; $R_y(u, v)$ holds if and only if $y \geq R^*(u, v)$; and $R_z(w, v)$ holds if and only if $z \geq R^*(w, v)$. Now fix arbitrary $x, y \in [0, \infty]$ such that $R_x(w, u)$ and $R_y(u, v)$ hold. Since $x + y \geq R^*(w, u) + R^*(u, v) \geq R^*(w, v)$, $R_{x+y}(w, v)$ also holds. $\mathfrak{F}$ therefore has additively transitive accessibility, and so satisfies A4.

Therefore, $\mathfrak{F}$ is a pseudometric space frame. □ *(b)*

Based on these results, Pseu can be considered the canonical continuous modal logic for pseudometric spaces.

Theorem 5 (Canonicity of Pseu for pseudometric spaces) *A well-founded frame* $\mathfrak{F}$ *satisfies* Pseu *if and only if* $\mathfrak{F}^*$ *is a pseudometric space (or a union of pseudometric spaces), under the classical–fuzzy map.*

Proof Follows directly from Theorem 4 and Lemma 4. □

Theorem 6 (Completeness of Pseu in pseudometric spaces) Pseu *is sound and strongly complete on the image of the class of pseudometric spaces under the fuzzy–classical map.*

Proof Follows directly from Proposition 4 and Theorem 5. □

5.4 Bounded Pseudometric Space Frames

For every $D \in [0, \infty]$, define the *pseudometric space of diameter D modal logic*, denoted Pseu_D, as the smallest extension of Pseu that contains the axiom schema:

(i) *D-boundedness*: For every $\varphi \in W_{[0,\infty]}$,

$$\Diamond_\infty \varphi \rightarrow \Diamond_D \varphi.$$

Fix $D \in [0, \infty]$. A pseudometric space frame that satisfies Pseu_D is called a *pseudometric space of diameter at most D frame*. As before, the trivial frame is one such frame.

By Lemma 1, D-boundedness is canonical for the following.

Definition 12 A frame $(M, \{R_x : x \in [0, \infty]\})$ has ***D*-bounded accessibility** if and only if: for all $w, u \in M$, if $R_\infty(w, u)$ holds then $R_D(w, u)$ holds.

Since pseudometric space frames are universal in ∞-accessibility, by D-bounded accessibility and upwardly closed accessibility, pseudometric space of diameter at most D frames are universal in x-accessibility for all $x \in [D, \infty]$.

Corollary 6 *For every $D \in [0, \infty]$, the logic* Pseu_D *is sound and strongly complete in the class of pseudometric space of diameter at most D frames.*

Proof Follows from Corollary 5, the Sahlqvist Completeness Theorem, and the canonicity of D-boundedness. □

Lemma 5 *Upwardly closed accessibility, reflexivity, symmetry, additively transitive accessibility, and D-bounded accessibility are jointly plantable.*

Proof See Appendix A.3. □

Proposition 5 *The logic* Pseu_D *is sound and strongly complete in the class of well-founded pseudometric space of diameter at most D frames.*

Proof Follows from Corollary 6, Proposition 3, and Lemma 5. □

Now consider again pseudometric spaces. Define the *diameter* of a pseudometric space (X, d) as $\sup\{d(w, u) : w, u \in X\}$.

Lemma 6 *(a) If a frame $(M, \{R_x : x \in [0, \infty]\})$ is a pseudometric space of diameter at most D frame, then under the classical–fuzzy map, (M, R^*) is a pseudometric space of diameter at most D. (b) If a fuzzy frame (M, R^*) is a pseudometric space of diameter at most D, then under the fuzzy–classical map, $(M, \{R_x : x \in [0, \infty]\})$ is a well-founded pseudometric space of diameter at most D frame.*

Proof of (a) Fix a frame $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$, which maps to $\mathfrak{F}^* = (M, R^*)$ using the classical–fuzzy map. Suppose $\mathfrak{F}$ is a pseudometric space of diameter at most D frame. By Lemma 4, $\mathfrak{F}^*$ is a pseudometric space. Moreover, since $\mathfrak{F}$ is a pseudometric space of diameter at most D frame, R_D is universal. Therefore, by the definition of the classical–fuzzy map, the diameter of $\mathfrak{F}^*$ is at most D. □ *(a)*

Proof of (b) Fix a fuzzy frame $\mathfrak{F}^* = (M, R^*)$, which maps to $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$ using the fuzzy–classical map. Suppose that $\mathfrak{F}^*$ is a pseudometric space of diameter at most D. By Lemma 4, $\mathfrak{F}$ is a well-founded pseudometric space frame. Also, since $\mathfrak{F}^*$ is a pseudometric space of diameter at most D, $\sup\{R^*(w, u) : w, u \in M\} \leq D$. Then by the definition of the fuzzy–classical map and upwardly closed accessibility, it must be that for every $w, u \in M$, $R_D(w, u)$ holds. Hence it is a pseudometric space of diameter at most D frame. □ *(b)*

As with Pseu, each Pseu_D can be considered the canonical continuous modal logic for pseudometric spaces of diameter at most D.

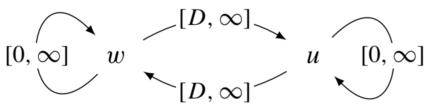
Corollary 7 (Canonicity of Pseu_D for pseudometric spaces of diameter at most D) *A well-founded frame $\mathfrak{F}$ satisfies* Pseu_D *if and only if $\mathfrak{F}^*$ is a pseudometric space of diameter at most D (or a union of such spaces), under the classical–fuzzy map.*

Proof Follows directly from Theorem 4 and Lemma 6. □

Corollary 8 (Completeness of Pseu_D for pseudometric spaces of diameter at most D) Pseu_D *is sound and strongly complete on the image of the class of pseudometric spaces of diameter at most D under the fuzzy–classical map.*

Proof Follows directly from Proposition 5 and Corollary 7. □

We can improve this result by refining the class of frames. A pseudometric space frame that satisfies Pseu_D, but that for each $x \in [0, D)$ does not satisfy Pseu_x, is called a *pseudometric space of diameter D frame*. For each such D, the *D-connected frame* proves that they exist: two worlds, each x-accessible to itself for all $x \in [0, \infty]$, and x-accessible to one another for all $x \in [D, \infty]$.



Each pseudometric space of diameter D frame is a pseudometric space of diameter at most D frame, so the former are also universal in x-accessibility for all $x \in [D, \infty]$. But furthermore, pseudometric space of diameter D frames are not universal in x-accessibility for any $x \in [0, D)$.

Lemma 7 *(a) If a frame $(M, \{R_x : x \in [0, \infty]\})$ is a pseudometric space of diameter D frame, then under the classical–fuzzy map, (M, R^*) is a pseudometric space* *of diameter D. (b) If a fuzzy frame (M, R^*) is a pseudometric space of diameter D, then under the fuzzy–classical map, $(M, \{R_x : x \in [0, \infty]\})$ is a well-founded pseudometric space of diameter D frame.*

Proof of (a) Fix a frame $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$, which maps to $\mathfrak{F}^* = (M, R^*)$ using the classical–fuzzy map. Suppose $\mathfrak{F}$ is a pseudometric space of diameter D frame. By Lemma 6, $\mathfrak{F}^*$ is a pseudometric space of diameter at most D. For each $x \in [0, D)$, since $\mathfrak{F}$ does not satisfy Pseu_x, R_x is not universal. So by the definition of the classical–fuzzy map, $\sup\{d(w, u) : w, u \in M\} \geq x$ for every $x \in [0, D)$. It follows that $\mathfrak{F}^*$ is a pseudometric space of diameter D. □ *(a)*

Proof of (b) Fix a fuzzy frame $\mathfrak{F}^* = (M, R^*)$, which maps to $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$ using the fuzzy–classical map. Suppose that $\mathfrak{F}^*$ is a pseudometric space of diameter D. By Lemma 6, $\mathfrak{F}$ is a well-founded pseudometric space of diameter at most D frame. Moreover, since $\sup\{d(w, u) : w, u \in M\} = D$, by the definition of the fuzzy–classical map, R_x is not universal for any $x \in [0, D)$. Therefore, $\mathfrak{F}$ is a pseudometric space of diameter D frame. □ *(b)*

Unsurprisingly, since pseudometric spaces of diameter at most D can be embedded in pseudometric spaces of diameter D, we get the following lemma:

Lemma 8 *For each* $D \in [0, \infty]$*, the class of pseudometric space of diameter D frames is modally indistinguishable from the class of pseudometric space of at most diameter D frames.*

Proof It suffices to show that every pointed model extending a pseudometric space of diameter at most D frame is bisimilar to a pointed model extending a pseudometric space of diameter D frame.

Let $(\mathfrak{M}, a)$ be an arbitrary pointed model extending a pseudometric space of diameter at most D frame. The proof of Lemma 5 generates $(\mathfrak{M}', \langle a \rangle)$, to which $(\mathfrak{M}, a)$ is bisimilar. By construction, $(\mathfrak{M}', \langle a \rangle)$ has D-bounded accessibility but for each $x \in [0, D)$ does not have x-bounded accessibility, so it is a pseudometric space of diameter D frame. □

Theorem 7 (Completeness of Pseu_D in pseudometric spaces of diameter D) Pseu_D *is sound and strongly complete on the image of the class of pseudometric spaces of diameter D under the fuzzy–classical map.*

Proof Follows directly from Corollary 8, Lemma 7, and Lemma 8. □

6 Metric Space Modal Logics

The difference between a metric space and a pseudometric space is that no two distinct points in a metric space can be in the same position. That is, no two distinct points have all the same distance relations to every point. In the context of modal logic, where the points are worlds, this means that no two distinct worlds are maximally accessible (accessible to a degree of remoteness of 0) to one another.

Attempts to characterize a relation between worlds as a distance along one or more dimensions that together completely characterize a world implicitly appeal to the semantics of a metric space logic. The continuous accessibility relation can then be interpreted as a complete similarity relation: no two distinct worlds are perfectly identical, and therefore no two distinct worlds are perfectly accessible to one another.

6.1 The Basic and Bounded Metric Space Modal Logics

Define the the *basic metric space modal logic*, denoted Met, as the smallest extension of Pseu that contains the axiom schema:

(i) 0-*coreflexivity*: For every $\varphi \in W_{[0,\infty]}$,

$$\varphi \rightarrow \Box_0 \varphi.$$

For every $D \in [0, \infty]$, define the *metric space of diameter D modal logic*, denoted Met_D, as the smallest extension of Met that contains the axiom schema:

(j) *D-boundedness*: For every $\varphi \in W_{[0,\infty]}$,

$$\Diamond_\infty \varphi \rightarrow \Diamond_D \varphi.$$

Any pseudometric space frame that satisfies Met is called a *metric space frame*, and such frames exist (again, the trivial frame is a metric space frame). For every $D \in [0, \infty]$, a pseudometric space frame that satisfies Met_D is called a *metric space of diameter at most D frame*, and such frames exist (again, for each $D \in [0, \infty]$, the trivial frame is a metric space of diameter at most D frame). For every $D \in [0, \infty]$, a pseudometric space frame that satisfies Met_D but fails to satisfy Met_x for every $x \in [0, D)$ is called a *metric space of diameter D frame*, and such frames exist (the trivial frame is a metric space of diameter 0 frame, and for each $D \in (0, \infty]$, the D-connected frame is a metric space of diameter D frame). All such frames are coreflexive in 0-accessibility. This follows, by classical results, from 0-coreflexivity.

Results analogous to those of Section 5 follow without difficulty. The more important ones are presented below.

Corollary 9 *For every $D \in [0, \infty]$, the logic* Met_D *is sound and strongly complete in the class of metric space of diameter at most D frames.*

Proof Follows from Corollary 6 and the Sahlqvist Completeness Theorem. □

Lemma 9 *Upwardly closed accessibility, reflexivity, symmetry, additively transitive accessibility, D-bounded accessibility, and coreflexivity in 0-accessibility are jointly plantable.*

Proof See Appendix A.4. □

Lemma 10 *The class of metric space of diameter D frames is modally indistinguishable from the class of metric space of diameter at most D frames.*

Proof (sketch) Follows from Lemma 8 and the proof of Lemma 9. □

Proposition 6 *For every $D \in [0, \infty]$, the logic* Met_D *is sound and strongly complete in the class of well-founded metric space of diameter D frames.*

Proof Follows from Corollary 9, Proposition 3, Lemma 9, and Lemma 10. □

6.2 Metric Spaces as Metric Space Frames

For our purposes, a *metric space* is a pseudometric space (X, d) such that for all $w, u \in X$,

- *Identity of indiscernables*: If $d(w, u) = 0$ then $w = u$.

Under the classical–fuzzy map, not all metric space frames map to metric spaces. A metric space frame that is not well-founded might have one world accessible to a distinct world at all magnitudes except zero. Since the classical–fuzzy map takes the infimum, the value of the fuzzy accessibility relation between the worlds is zero, violating the identity of indiscernables. So we must stipulate that the metric space frames of interest are those with *adequately-founded accessibility around 0*, that is, for frame $(M, \{R_x : x \in [0, \infty]\})$, for every $w, u \in M$, if $R_x(w, u)$ holds for every $x > 0$, then $R_0(w, u)$ holds.

Lemma 11 *(a) If a frame $(M, \{R_x : x \in [0, \infty]\})$ is a metric space of diameter D frame with adequately-founded accessibility around 0, then under the classical–fuzzy map, (M, R^*) is a metric space of diameter D. (b) If a fuzzy frame (M, R^*) is a metric space of diameter D, then under the fuzzy-classical map, $(M, \{R_x : x \in [0, \infty]\})$ is a well-founded metric space of diameter D frame.*

Proof of (a) Fix a frame $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$, which maps to $\mathfrak{F}^* = (M, R^*)$ using the classical–fuzzy map. Suppose $\mathfrak{F}$ is a metric space of diameter D frame with adequately-founded accessibility around 0. By Lemma 7, $\mathfrak{F}^*$ is a pseudometric space of diameter D. Now fix $w, u \in M$ such that $R^*(w, u) = 0$. Since $\mathfrak{F}$ has adequately-founded accessibility around 0, this implies that $R_0(w, u)$ holds. Since $\mathfrak{F}$ is coreflexive in 0-accessibility, this means that $w = u$. Therefore, $\mathfrak{F}^*$ is a metric space of diameter D frame. □ *(a)*

Proof of (b) Fix a fuzzy frame $\mathfrak{F}^* = (M, R^*)$, which maps to $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$ using the fuzzy–classical map. Suppose $\mathfrak{F}^*$ is a metric space of diameter D. By Lemma 7, $\mathfrak{F}$ is a well-founded pseudometric space of diameter D frame. Now fix $w, u \in M$ such that $R_0(w, u)$ holds. By the definition of the classical–fuzzy map, $R^*(w, u) = 0$. Since $\mathfrak{F}^*$ is a metric space, this implies that $w = u$. Thus, $\mathfrak{F}$ is coreflexive in 0-accessibility. Therefore, $\mathfrak{F}$ is a well-founded metric space of diameter D frame. □ *(b)*

As above, Met can be considered the canonical continuous modal logic for metric spaces, with Met_D the canonical continuous modal logic for metric spaces of diameter at most D.

Theorem 8 (Canonicity of Met_D in metric spaces of diameter at most D) *A well-founded frame $\mathfrak{F}$ satisfies* Met_D *if and only if $\mathfrak{F}^*$ is a metric space of diameter at most D (or a union of such spaces), under the classical–fuzzy map.*

Proof Follows directly from Theorem 4 and Lemma 11. □

Moreover, Met is the complete logic for metric spaces, with Met_D the complete logic for metric spaces of diameter D.

Theorem 9 (Completeness of Met_D in metric spaces of diameter D) Met_D *is sound and strongly complete on the image of the class of metric spaces of diameter D under the fuzzy–classical map.*

Proof Follows from Proposition 6 and Lemma 11. □

6.3 Neighborly Theories

Given a metric space frame $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$, we say that $\mathfrak{F}$ is *discrete* if $\inf\{R^*(w, u) : w \neq u \in M\} > 0$. Notably, every metric space frame with a finite set of worlds is discrete. If a Met-theory is satisfiable in a discrete metric space frame, we say that it is *discretely* satisfiable.

Definition 13 A Met-theory T is **neighborly** if the following conditions are satisfied for all $\varphi \in W_{[0,\infty]}$.

- If $\varphi \in T$, then there is some $x \in (0, \infty]$ such that $\Box_x \varphi \in T$.
- If $\Diamond_x \varphi \in T$ for some $x \in [0, \infty]$, then there is some $y \in (0, \infty]$ such that $\Diamond_x \Box_y \varphi \in T$.

Proposition 7 *Any discretely satisfiable* Met*-theory can be extended to a neighborly, maximally consistent* Met*-theory.*

Proof Fix a discretely satisfiable Met-theory T. Then there is a discrete metric space model $\mathfrak{M} = (M, \{R_x : x \in [0, \infty]\}, V)$ and $w \in M$ such that $T \subseteq Th(\mathfrak{M}, w)$. Let $m := \inf\{R^*(w, u) : w \neq u \in M\}$ and $\overline{T} := Th(\mathfrak{M}, w)$. By definition and completeness, $\overline{T}$ is maximally consistent. We now claim that $\overline{T}$ is also neighborly. Consider that, since $\frac{m}{2} < m$, the only world within $\frac{m}{2}$ of w is w itself. It follows that if $\varphi \in \overline{T}$, then $\Box_{\frac{m}{2}} \varphi \in \overline{T}$.

Moreover, suppose that, for some $\varphi \in W_{[0,\infty]}$ and $x \in [0, \infty]$, $\Diamond_x \varphi \in \overline{T}$. Then there is some $u \in \mathfrak{M}$ such that $R^*(w, u) \leq x$ and $\mathfrak{M}, u \vDash \varphi$. As above, the only world within $\frac{m}{2}$ of u is u itself. It follows that $\mathfrak{M}, u \vDash \Box_{\frac{m}{2}} \varphi$, so $\Diamond_x \Box_{\frac{m}{2}} \varphi \in \overline{T}$. □

By the finite model property, any finite Met-theory is finitely, and hence discretely, satisfiable. Therefore, any finite Met-theory can be extended to a neighborly, maximally consistent Met-theory. However, the same result doesn't hold for even countably infinite theories.

Proposition 8 *There is a countably infinite* Met*-theory that is not neighborly and has no neighborly extensions.*

Proof Consider $T := \{\mathrm{p}\} \cup \left\{\Diamond_{\frac{1}{n}}\neg\mathrm{p} : n \in \mathbb{N}\right\}$. Construct model $\mathfrak{M} = (M, \{R_x : x \in [0,\infty]\}, V)$ such that: $M = [0,1]$; for all $w, u \in M$, $R_x(w,u)$ holds for all $x \geq |w-u|$ and does not hold otherwise; and $V(0,\mathrm{p}) = 1$, with all other valuations being 0. T is clearly consistent since it is satisfied at world 0 in $\mathfrak{M}$. But it is also not neighborly and has no neighborly extensions, since for every $x \in (0,\infty]$, $T \cup \{\Box_x \mathrm{p}\}$ is inconsistent. □

We conjecture that the converse of Proposition 7 does not hold, *i.e.*, we believe there is a neighborly, maximally consistent Met-theory that is not discretely satisfiable.

Proposition 9 *For any* Met*-consistent formula φ, there is a metric space model and world in that model such that φ is counterfactually stable in that model at that world.*

Proof Fix a Met-consistent formula φ. Then $\{\varphi\}$ is a consistent Met-theory. By Proposition 7, extend $\{\varphi\}$ to a maximally consistent and neighborly Met-theory $\overline{\{\varphi\}}$. Since Met is complete, there is a metric space model $\mathfrak{M} = (M, \{R_x : x \in [0,\infty]\}, V)$ and world $w \in M$ such that $\mathfrak{M}, w \vDash \overline{\{\varphi\}}$. Since $\overline{\{\varphi\}}$ is neighborly, for some $x \in (0,\infty]$, $\mathfrak{M}, w \vDash \Box_x\varphi$. Therefore, $\mathrm{cs}(\mathfrak{M}, w, \varphi) \geq x > 0$. □

6.4 Topological Metric Space Modal Logic

Let $L^{\mathrm{top}}_{[0,\infty]}$, the *language of the basic topology and* $[0,\infty]$*-valued multimodal logic*, be $L_{[0,\infty]}$ with the addition of a new modal operator, $\Box^{\mathrm{int}}$, called the *interior operator*. Define the well-formed formulas, $W^{\mathrm{top}}_{[0,\infty]}$, and normal form in the standard way. Models and satisfaction remain the same as before with the following additional condition.

- $\mathfrak{M}, w \vDash \Box^{\mathrm{int}}\varphi$ if and only if for some $x \in (0,\infty]$, $\mathfrak{M}, w \vDash \Box_x\varphi$.

Likewise for fuzzy satisfaction:

- $\mathfrak{M}, w \vDash^* \Box^{\mathrm{int}}\varphi$ if and only if for some $x \in (0,\infty]$, $\mathfrak{M}, w \vDash^* \Box_x\varphi$.

It directly follows from this definition that φ is counterfactually stable at world w in $\mathfrak{M}$ if and only if $\mathfrak{M}, w \vDash \Box^{\mathrm{int}}\varphi$.

For every $D \in [0,\infty]$, define the proof system $\mathrm{Met}^{\mathrm{top}}_D$ to be the smallest extension of Met_D that contains the axiom schema:

(k) *Induced topology*: For every $\varphi \in W^{\mathrm{top}}_{[0,\infty]}$ and $x \in (0,\infty]$,

$$\Box_x\varphi \rightarrow \Box^{\mathrm{int}}\varphi.$$

Proposition 10 *For every $D \in [0, \infty]$, a frame satisfies* Met_D *if and only if it satisfies* $\mathrm{Met}_D^{\mathrm{top}}$.

Proof The reverse direction is trivial. The forward direction follows directly from the definition of the satisfaction relation and the fact that $\mathrm{Met}_D^{\mathrm{top}}$ is an extension of Met_D. □

Corollary 10 *For every $D \in [0, \infty]$, every* Met_D*-theory is also a* $\mathrm{Met}_D^{\mathrm{top}}$*-theory.*

Proof Let T be a Met_D-theory. By Theorem 9, there is a metric space of diameter D model $\mathfrak{M} = (M, \{R_x : x \in [0, \infty]\}, V)$ and world $w \in M$ such that $T \subseteq Th(\mathfrak{M}, w)$. By Proposition 10, $\mathfrak{M}$ extends a frame that satisfies $\mathrm{Met}_D^{\mathrm{top}}$. Therefore, T must be $\mathrm{Met}_D^{\mathrm{top}}$-consistent, and hence a $\mathrm{Met}_D^{\mathrm{top}}$-theory. □

Theorem 10 *For every $D \in [0, \infty]$, any discretely satisfiable* Met_D*-theory T can be extended to a* $\mathrm{Met}_D^{\mathrm{top}}$*-theory $\overline{T}$ such that if $\varphi \in T$, then $\Box^{\mathrm{int}}\varphi \in \overline{T}$.*

Proof Let T be a discretely satisfiable Met_D-theory. By Proposition 7, extend T to a maximally consistent and neighborly Met-theory T'. By Corollary 10, T' is also a $\mathrm{Met}_D^{\mathrm{top}}$-theory. Then define $\overline{T}$ as the set of consequences of T'. $\overline{T}$ is clearly a $\mathrm{Met}_D^{\mathrm{top}}$-theory, since it is $\mathrm{Met}_D^{\mathrm{top}}$-equiconsistent with T'. Then fix $\varphi \in T$. By the construction of T', there is some $x \in (0, \infty]$ such that $\Box_x\varphi \in T'$. So by the definition of $\overline{T}$, $\Box^{\mathrm{int}}\varphi \in \overline{T}$. □

We conclude this subsection with a brief remark that in the topological metric space modal logic, we lose the finite model property.

Proposition 11 *For every $D \in (0, \infty]$, the logic* $\mathrm{Met}_D^{\mathrm{top}}$ *does not have the finite model property.*

Proof Fix arbitrary $D \in (0, \infty]$. For arbitrary $p \in \mathrm{P}$, consider the formula $p \wedge \neg\Box^{\mathrm{int}} p$. The formula is clearly $\mathrm{Met}_D^{\mathrm{top}}$-consistent, since the following metric space of diameter D fuzzy model satisfies it at world 0: (M, R^*, V) such that $M = [0, D]$, R^* is the Euclidean metric, and $V(0, p) = 1$ with all other valuations being 0. We need only show that the formula is not satisfied on any finite model.

Fix a fuzzy model $\mathfrak{M}^* = (M, R^*, V)$ that satisfies $\mathrm{Met}_D^{\mathrm{top}}$ and such that M is finite. Let $m := \min\{R^*(w, u) : w \neq u \in M\}$. Then for every $w \in M$, $\mathfrak{M}^*, w \vDash^* p \leftrightarrow \Box_{\frac{m}{2}} p$. It follows by the induced topology schema that for every $w \in M$, $\mathfrak{M}^*, w \vDash^* p \rightarrow \Box^{\mathrm{int}} p$, from which it follows that $\mathfrak{M}^*, w \nvDash^* p \wedge \neg\Box^{\mathrm{int}} p$. □

6.5 Metrizable Topological Spaces

Let L^{top} be the classical topological modal language found in [14], with modal operator $\Box^{\mathrm{int}}$. The *topological well-formed formulas* are generated naturally, and denoted by W^{top}. By this construction, $W^{\mathrm{top}} \subseteq W_{[0,\infty]}^{\mathrm{top}}$.

A *topological frame*, in the classical sense, is a pair $(M, \mathcal{O})$ where M is a nonempty set and $\mathcal{O}$ is a topology on M. Topological frames can be extended to *topological models* when they are paired with a valuation $V : M \times \mathrm{P} \to \{0, 1\}$. The classical *topological semantics* is defined as usual on the standard propositions and logical connectives, and on modal operator $\Box^{\text{int}}$ as $\mathfrak{M}, w \vDash_{\text{top}} \Box^{\text{int}}\varphi$ if and only if w is in the topological interior of the set of all worlds that satisfy φ.

A *metrizable topological frame* is a topological frame $(M, \mathcal{O})$ such that there is a metric $d : M^2 \to [0, \infty]$ where the open sets of the metric space (M, d) under the induced topology are precisely the open sets $\mathcal{O}$.

Theorem 11 *Over topological well-formed formulas, there is a satisfaction-preserving map from the class of metrizable topological frames* $(M, \mathcal{O})$ *into the class of well-founded metric space frames.*

Proof Fix a metrizable topological frame $(M, \mathcal{O})$. Extend this frame to an arbitrary model $\mathfrak{M} = (M, \mathcal{O}, V)$. Via the axiom of choice, fix a metric $d : M^2 \to [0, \infty]$ such that (M, d) induces the set $\mathcal{O}$ as the open sets of M. Define $R^* := d$. Clearly satisfaction is preserved between $(M, \mathcal{O}, V)$ and (M, R^*, V) for non-modal formulas.

To see that it is preserved for formulas containing $\Box^{\text{int}}$, first fix $\varphi \in W^{\text{top}}$ and $w \in M$ such that $\mathfrak{M}, w \vDash_{\text{top}} \Box^{\text{int}}\varphi$. By the definition of topological interior, there is some $O \in \mathcal{O}$ such that $w \in O$ and for every $u \in O$, $\mathfrak{M}, u \vDash_{\text{top}} \varphi$. Since d induces the same open sets, there must be some ball (neighborhood) of w with radius $\epsilon > 0$ contained in O. Therefore, $\mathfrak{M}, w \vDash^* \Box_{\frac{\epsilon}{2}}\varphi$. Hence, by the induced topology axiom schema, $\mathfrak{M}, w \vDash^* \Box^{\text{int}}\varphi$.

Next fix $\varphi \in W^{\text{top}}$ and $w \in M$ such that $\mathfrak{M}, w \vDash_{\text{top}} \neg\Box^{\text{int}}\varphi$. It must be that for all $O \in \mathcal{O}$ such that $w \in O$ there is some $u \in O$, $\mathfrak{M}, u \vDash_{\text{top}} \neg\varphi$. But again, since d induces the same open sets, then for every ball of w with radius $\epsilon > 0$, there is some u in that ball such that $\mathfrak{M}, u \vDash^* \neg\varphi$. It follows that for every $\epsilon > 0$, $\mathfrak{M}, w \vDash^* \neg\Box_\epsilon\varphi$. Hence, by definition, $\mathfrak{M}, w \vDash^* \neg\Box^{\text{int}}\varphi$.

By Theorem 8 and since $W^{\text{top}} \subseteq W^{\text{top}}_{[0,\infty]}$, there is a unique well-founded metric space model $(M, \{R_x : x \in [0, \infty]\}, V)$ that satisfies the same topological well-formed formulas as (M, R^*, V). $\Box$

Corollary 11 *If fuzzy frame* (M, R^*) *is a dense-in-itself metric space, then under the fuzzy–classical map, frame* $(M, \{R_x : x \in [0, \infty]\})$ *satisfies* $S4$ *in the interior operator.*

Proof It is a well-known result of [16] that $S4$ is sound and complete on the class of dense-in-themselves metric spaces considered as topological frames. The result then follows from the proof of Theorem 11. $\Box$

Remark The rational fragments of the metric space and topological metric space logics become fragments of the logic given in [7] with the identifications of $\Box_\infty \mapsto \forall$, $\Box_q \mapsto \forall^{\leq q}$ for every $q \in \mathbb{Q}$, and $N \mapsto \Box$, along with the elimination of the $\Box_0$ and $\Diamond_0$ operators by 0-reflexivity and 0-coreflexivity. In [7], the decidability problem for that extension is shown to be EXPTIME-complete. It is currently unknown, however, if the same complexity result holds for the rational fragment of the logic given here as well.

7 Continuous Temporal Logics

Even though all the continuous accessibility logics described above are technically multimodal, they are interpreted as having a single continuous accessibility relation. But for some applications, we need more than one continuous accessibility relation. For example, a continuous temporal logic would require two such relations, one for distance into the future, and one for distance into the past. Some interactive multimodal extensions of continuous accessibility modal logic generate difficulties, but these difficulties can be overcome by an appeal to our magnitude operators.

7.1 The Basic Bimodal Continuous Accessibility Logic

Define the *language of basic bimodal* $[0, \infty]$*-indexed multimodal logic*, $L^{\pm}_{[0,\infty]}$, to be the smallest normal multimodal language containing $\{[+]_x, \langle+\rangle_x, [-]_x, \langle-\rangle_x : x \in [0, \infty]\}$ as its set of modal operator symbols. The well-formed formulas, $W^{\pm}_{[0,\infty]}$, are defined in the standard way.

The *basic bimodal* $[0, \infty]$*-indexed multimodal logic*, $K^{\pm}_{[0,\infty]}$, is the smallest normal multimodal logic in modal operators $\{[+]_x, \langle+\rangle_x, [-]_x, \langle-\rangle_x : x \in [0, \infty]\}$. That is, in addition to the modus ponens and the tautologies of modal logic, $K^{\pm}_{[0,\infty]}$ has the standard versions of necessitation, modal distribution, and duality for each of $[+]_x$ and $[-]_x$. Normal form is also defined in the standard way.

Models and frames for $K^{\pm}_{[0,\infty]}$ are defined as expected, with accessibility relations R^+_x and R^-_x for each $x \in [0, \infty]$. Satisfaction is defined in the standard way for modal formulas with $\langle+\rangle_x$ and $\langle-\rangle_x$. $K^{\pm}_{[0,\infty]}$ is sound and complete in this class of frames, as well as in the class of well-founded such frames.

Extend $K^{\pm}_{[0,\infty]}$ to the *basic bimodal continuous accessibility logic*, denoted $C^{\pm}$, to include upward closure in $\langle+\rangle_x$ and in $\langle-\rangle_x$. $C^{\pm}$ satisfies C in each of the $[+]_x$ and $[-]_x$ operators. It is sound and complete in the class of *bimodal continuous accessibility frames*, that is, frames that are continuous accessibility frames in each of the R^+_x and R^-_x relations. It is also sound and complete in the class of well-founded such frames.

The expected relationship holds between bimodal continuous accessibility frames and fuzzy frames. A *bimodal fuzzy frame* is a triple (M, R^{+*}, R^{-*}) such that both (M, R^{+*}) and (M, R^{-*}) are fuzzy frames. For any bimodal continuous accessibility frame $\mathfrak{F} = (M, \{R^+_x, R^-_x : x \in [0, \infty]\})$, by the classical–fuzzy map, both (M, R^{+*}) and (M, R^{-*}) are fuzzy frames, so $\mathfrak{F}^* := (M, R^{+*}, R^{-*})$ is a bimodal fuzzy frame. Call this the *bimodal classical–fuzzy map*. A bimodal fuzzy–classical map can be defined as in Section 3.4, yielding well-founded bimodal continuous accessibility frames.

Corollary 12 *There is a satisfaction-preserving bijection between the class of well-founded bimodal continuous accessibility frames and the class of bimodal fuzzy frames.*

Proof (sketch) The bimodal classical–fuzzy map witnesses the corollary, following as the direct bimodal extension of Theorem 4. □

7.2 Continuous Temporal Logic

Extend $C^{\pm}$ to the *minimal continuous temporal logic*, denoted C_t, to include the following axiom schemata:

(g) $[0, \infty]$-*converse*: For all $\varphi \in W^{\pm}_{[0,\infty]}$ and all $x \in [0, \infty]$,

$$\varphi \rightarrow [+]_x \langle - \rangle_x \varphi \quad \text{and} \quad \varphi \rightarrow [-]_x \langle + \rangle_x \varphi.$$

Note that C_t satisfies the classical minimal temporal logic in the $[+]_x$ and $[-]_x$ operators for each $x \in [0, \infty]$. Given upward closure in $\langle + \rangle_x$ and in $\langle - \rangle_x$, it is natural to interpret $[+]_\infty$ and $[-]_\infty$ as classical G ("it will always be the case that") and H ("it has always been the case that") operators, respectively.

By classical results, frames that satisfy C_t are $\pm$-antisymmetric, that is, for all $w, u \in M$ and $x \in [0, \infty]$, $R^+_x(w, u)$ holds if and only if $R^-_x(u, w)$ holds. Moreover, C_t is sound and complete in the class of $\pm$-antisymmetric bimodal continuous accessibility frames, as well as in the class of well-founded such frames.

Just as with classical minimal temporal logic, C_t can be extended to yield a desired temporal structure. A particularly useful one for our purposes is to include the additive preorder axioms. Extend C_t to the *continuous temporal logic*, denoted C_{TL}, to include 0-reflexivity (T_0) in $[+]_0$ and additive transitivity (A4) in $[+]_x$. C_{TL} proves T_0 in $[-]_0$ and A4 in $[-]_x$.

Note that C_{TL} is equivalent to the pseudometric space modal logic if the following axiom schema is introduced: For every $\varphi \in W^{\pm}_{[0,\infty]}$ and $x \in [0, \infty]$, $[+]_x\varphi \leftrightarrow [-]_x\varphi$. Further introducing the 0-coreflexivity axiom schema makes it equivalent to the metric space modal logic.

Note also that C_{TL} satisfies the classical reflexive and transitive temporal logic in the $[+]_\infty$ and $[-]_\infty$ operators. This is because ∞-reflexivity follows from T_0 and upward closure, while ∞-transitivity follows from additive transitivity when $x = y = \infty$.

Call any frame that satisfies C_{TL} and is universal in the relation $R^+_\infty \cup R^-_\infty$ a *continuous temporal frame*. As with pseudometric space frames, any frame that satisfies C_{TL} is isomorphic to unions of continuous temporal frames.

Continuous temporal frames are x-reflexive in each of the R^+_x and R^-_x relations (by 0-reflexivity and upward closure), and have additively transitive accessibility in the R^+_x relations and in the R^-_x relations. They are therefore reflexive and additively transitive $\pm$-antisymmetric bimodal continuous accessibility frames. C_{TL} is sound and complete in this class of frames, as well as in the class of well-founded such frames.

Moreover, just as with pseudometric spaces, when such frames are well-founded, they are antisymmetric quasi-pseudometric spaces (sometimes called directed pseudometric spaces). The addition of 0-coreflexivity to the axioms yields a logic whose frames (similarly defined) are antisymmetric quasi-metric spaces (sometimes called directed metric spaces).

7.3 The μ-Extended Continuous Temporal Logic

One important application of a continuous temporal logic is time understood as a real-numbered dimension. For this application, not only will the continuous temporal logic need to be well-founded, but we will need syntactic access to the minima. Since $C^{\pm}$ is a continuous accessibility logic in each of the $[+]_x$ and $[-]_x$ operators, we will apply the μ-extension techniques from Section 3.5 separately to each operator.

This procedure yields (for well-formed formulas $W^{\pm,\mu}_{[0,\infty]}$) the *basic μ-extended bimodal continuous accessibility logic*, $C^{\pm,\mu}$, where μ-corespondence applies separately to $\langle+\rangle_x$ and $\langle-\rangle_x$. Frames that satisfy $C^{\pm,\mu}$ are *μ-extended bimodal continuous accessibility frames*. We will be concerning ourselves with those frames that are well-structured, where again unique μ-accessibility and μ-induced adequately-founded accessibility apply separately to the $R^{+,\mu}_x$ and $R^{-,\mu}_x$ relations.[7]

As in Section 3.5, the bimodal classical–fuzzy map can be extended to μ-extended bimodal continuous accessibility frames by applying it to the subframes containing only the R^+_x and R^-_x relations. Again by construction, any bimodal fuzzy frame $\mathfrak{F}^* = \left(M, R^{+*}, R^{-*}\right)$ can be mapped to a μ-extended bimodal continuous accessibility frame via the bimodal fuzzy–classical map together with the conditions that for any worlds $w, u \in M$ and $x \in [0,\infty]$, $R^{+,\mu}_x(w,u)$ holds if and only if $x = R^{+*}(w,u)$ and $R^{-,\mu}_x(w,u)$ holds if and only if $x = R^{-*}(w,u)$. Notably, this μ-extended bimodal continuous accessibility frame is well-structured. Call this the *μ-extended bimodal fuzzy–classical map*.

Corollary 13 *There is a satisfaction-preserving bijection between the class of well-structured μ-extended bimodal continuous accessibility frames and the class of bimodal fuzzy frames.*

Proof (sketch) The bimodal classical–fuzzy map witnesses the corollary, following Corollary 3 and Corollary 12. □

Extend $C^{\pm,\mu}$ to the *μ-extended continuous temporal logic*, C^{μ}_{TL}, by including the following axiom schemata:

(g) μ-$[0,\infty]$-*converse*: For all $\varphi \in W^{\pm,\mu}_{[0,\infty]}$ and all $x \in [0,\infty]$,

$$\varphi \rightarrow [+]^{\mu}_x \langle-\rangle^{\mu}_x \varphi \quad \text{and} \quad \varphi \rightarrow [-]^{\mu}_x \langle+\rangle^{\mu}_x \varphi.$$

(h) μ-0-*reflexivity* (T^{μ}_0): For every $\varphi \in W^{\pm,\mu}_{[0,\infty]}$,

$$[+]^{\mu}_0 \varphi \rightarrow \varphi.$$

(i) μ-*additive transitivity* ($A4^{\mu}$): For every $\varphi \in W^{\pm,\mu}_{[0,\infty]}$ and $x, y \in [0,\infty]$,

$$[+]^{\mu}_{x+y} \varphi \rightarrow [+]^{\mu}_x [+]^{\mu}_y \varphi.$$

C^{μ}_{TL} proves T^{μ}_0 in $[-]^{\mu}_0$ and $\mathrm{A4}^{\mu}$ in $[-]^{\mu}_x$. T_0 follows from T^{μ}_0 and μ-corespondence.

Any frame that satisfies C^{μ}_{TL} and is universal in the $R^+_\infty \cup R^-_\infty$ relation is a *μ-extended continuous temporal frame*. As above, any frame that satisfies C^{μ}_{TL} is isomorphic to a union of μ-extended continuous temporal frames.

A μ-extended continuous temporal frame has three notable properties:

- It is μ-$\pm$-antisymmetric, that is, $\pm$-antisymmetric in the $R^{+,\mu}_x$ and $R^{+,\mu}_x$ relations.

[7] Those familiar with alternative formulations of metric temporal logics (like those in [1] and [2]) may recognize the resultant $\langle+\rangle^{\mu}_x$ and $\langle-\rangle^{\mu}_x$ operators as "punctuality" operators.

- It is reflexive in the $R_0^{+,\mu}$ and $R_0^{-,\mu}$ relations, and in each R_x^+ and R_x^- relation.
- It has additively transitive accessibility in the $R_x^{+,\mu}$ and $R_x^{-,\mu}$ relations.

Since well-structured μ-extended continuous temporal frames have μ-induced adequately-founded accessibility, such frames are also $\pm$-antisymmetric in each R_x^+ and R_x^- relation and additively transitive in the R_x^+ and R_x^- relations. Well-structured μ-extended continuous temporal frames therefore satisfy C_{TL}.

Proposition 12 *If* $(M, \{R_x^+, R_x^-, R_x^{+,\mu}, R_x^{-,\mu} : x \in [0,\infty]\})$ *is a well-structured* μ*-extended continuous temporal frame, then for every* $w, u \in M$*, exactly one of the following is true.*

- *Only* $R_0^{+,\mu}(w,u)$ *and* $R_0^{-,\mu}(w,u)$ *hold;*
- $R_x^{+,\mu}(w,u)$ *holds for exactly one* $x \in [0,\infty]$ *and* $R_x^{-,\mu}(w,u)$ *holds for no* $x \in [0,\infty]$*;*
- $R_x^{-,\mu}(w,u)$ *holds for exactly one* $x \in [0,\infty]$ *and* $R_x^{+,\mu}(w,u)$ *holds for no* $x \in [0,\infty]$.

Proof Fix a well-structured μ-extended continuous temporal frame $\mathfrak{F} = (M, \{R_x^+, R_x^-, R_x^{+,u}, R_x^{-,u} : x \in [0,\infty]\})$, and worlds $w, u \in M$. By definition, $R_\infty^+ \cup R_\infty^-$ is universal, so at least one of $R_\infty^+(w,u)$ or $R_\infty^-(w,u)$ holds. Since $\mathfrak{F}$ is a continuous accessibility frame in R_x^+ and R_x^-, it follows from μ-induced adequately-founded accessibility that either $R_x^{+,\mu}(w,u)$ holds for some $x \in [0,\infty]$ or $R_y^{-,\mu}(w,u)$ holds for some $y \in [0,\infty]$, or both. If $R_x^{+,\mu}(w,u)$ holds, then by unique μ-accessibility, $R_{x'}^{+,\mu}(w,u)$ holds for no $x' \neq x \in [0,\infty]$, and likewise for $R_y^{+,\mu}(w,u)$.

Suppose that both $R_x^{+,\mu}(w,u)$ and $R_y^{-,\mu}(w,u)$ hold. By μ-$\pm$-antisymmetry, $R_y^{+,\mu}(u,w)$ holds. So by additively transitive accessibility, $R_{x+y}^{+,\mu}(w,w)$ holds. It follows from reflexivity in $R_0^{+,\mu}$ and unique μ-accessibility that $x = y = 0$. □

7.4 Real line continuous temporal logic

Extend C_{TL}^{μ} to the *real line continuous temporal logic*, denoted C_{RL}, by including the following axiom schemata:

(j) μ-$[0,\infty]$*-seriality (*$D_{[0,\infty)}^{\mu}$*)*: For every $\varphi \in W_{[0,\infty]}^{\pm,\mu}$ and $x \in [0,\infty)$,

$$[+]_x^{\mu}\varphi \to \langle + \rangle_x^{\mu}\varphi \quad \text{and} \quad [-]_x^{\mu}\varphi \to \langle - \rangle_x^{\mu}\varphi.$$

(k) μ-$[0,\infty]$*-functionality (*$F_{[0,\infty)}^{\mu}$*)*: For every $\varphi \in W_{[0,\infty]}^{\pm,\mu}$ and $x \in [0,\infty]$,

$$\langle + \rangle_x^{\mu}\varphi \to [+]_x^{\mu}\varphi \quad \text{and} \quad \langle - \rangle_x^{\mu}\varphi \to [-]_x^{\mu}\varphi.$$

(l) *No endpoints*:

$$\neg\langle + \rangle_\infty^{\mu}\top \quad \text{and} \quad \neg\langle - \rangle_\infty^{\mu}\top.$$

Notably, C_{RL} proves the schema:

- μ*-subtractive transitivity*: For every $\varphi \in W^{\pm,\mu}_{[0,\infty]}$ and $x, y \in [0, \infty)$,

$$\langle + \rangle^{\mu}_{x+y} \langle - \rangle^{\mu}_{y} \varphi \rightarrow \langle + \rangle^{\mu}_{x} \varphi.$$

The proof relies on μ-$[0, \infty]$-converse, $\mathrm{A4}^{\mu}$, $\mathrm{D}^{\mu}_{[0,\infty)}$, and $\mathrm{F}^{\mu}_{[0,\infty]}$.[8]

Any μ-extended continuous temporal frame that satisfies C_{RL} is called a *real line continuous temporal frame*. By classical results and Lemma 1, here are further notable properties of such a frame:

- It is serial in the real $R^{+,\mu}_{x}$ relations. That is, for each $x \in [0, \infty)$ and $w \in M$, $R^{+,\mu}_{x}(w, u)$ holds for some $u \in M$. It is also serial in the real $R^{-,\mu}_{x}$ relations. (It is also serial in each R^{+}_{x} and R^{-}_{x} relation, as a consequence of x-reflexivity.)
- It is functional in each $R^{+,\mu}_{x}$ relation. That is, for each $x \in [0, \infty]$ and all $w, u \in M$, if $R^{+,\mu}_{x}(w, u)$ holds, then $R^{+,\mu}_{x}(w, v)$ does not hold for any $v \neq u \in M$. It is also functional in each $R^{-,\mu}_{x}$ relation.
- It has *finite* μ*-accessibility* in $R^{+,\mu}_{x}$, that is, for all $w, u \in M$, $R^{+,\mu}_{\infty}(w, u)$ doesn't hold. It also has finite μ-accessibility in $R^{-,\mu}_{x}$.
- It is coreflexive in $R^{+,\mu}_{0}$ and $R^{-,\mu}_{0}$, as a consequence of functionality and μ-0-reflexivity.
- It has *subtractively transitive accessibility* in the $R^{+,\mu}_{x}$ and $R^{-,\mu}_{x}$ relations. That is, if $R^{+,\mu}_{x+y}(w, v)$ holds and $R^{-,\mu}_{y}(v, u)$ holds, then $R^{+,\mu}_{x}(w, u)$ holds.

Well-structured real line continuous temporal frames, moreover, are coreflexive in R^{+}_{0} and R^{-}_{0}, and have subtractively transitive accessibility in the R^{+}_{x} and R^{-}_{x} relations.

Proposition 13 *If* $\mathfrak{F} = (M, \{R^{+}_{x}, R^{-}_{x}, R^{+,u}_{x}, R^{-,u}_{x} : x \in [0, \infty]\})$ *is a well-structured real line continuous temporal frame, then for every* $w \neq u \in M$*, either* $R^{+,\mu}_{x}(w, u)$ *holds for exactly one* $x \in (0, \infty)$ *and* $R^{-,\mu}_{x}(w, u)$ *holds for no* $x \in [0, \infty]$*, or* $R^{-,\mu}_{x}(w, u)$ *holds for exactly one* $x \in (0, \infty)$ *and* $R^{+,\mu}_{x}(w, u)$ *holds for no* $x \in [0, \infty]$.

[8] For arbitrary $\varphi \in W^{\pm,\mu}_{[0,\infty]}$ and $x, y \in [0, \infty)$,

1.	$\langle + \rangle^{\mu}_{x+y} \langle - \rangle^{\mu}_{y} \varphi \rightarrow [+]^{\mu}_{x+y} \langle - \rangle^{\mu}_{y} \varphi$	(Instance of F^{μ}_{x+y} (+) on $\langle - \rangle^{\mu}_{y} \varphi$)
2.	$\langle - \rangle^{\mu}_{y} \varphi \rightarrow [-]^{\mu}_{y} \varphi$	(Instance of F^{μ}_{y} (−))
3.	$[+]^{\mu}_{x+y} \langle - \rangle^{\mu}_{y} \varphi \rightarrow [+]^{\mu}_{x+y} [-]^{\mu}_{y} \varphi$	(Nec with $[+]^{\mu}_{x+y}$, modal dist, and prop logic on 2)
4.	$[+]^{\mu}_{x+y} [-]^{\mu}_{y} \varphi \rightarrow [+]^{\mu}_{x} [+]^{\mu}_{y} [-]^{\mu}_{y} \varphi$	(Instance of $\mathrm{A4}^{\mu}$ on $[-]^{\mu}_{y} \varphi$)
5.	$[+]^{\mu}_{y} [-]^{\mu}_{y} \varphi \rightarrow \langle + \rangle^{\mu}_{y} [-]^{\mu}_{y} \varphi$	(Instance of D^{μ}_{y} (+) on $[-]^{\mu}_{y} \varphi$)
6.	$\langle + \rangle^{\mu}_{y} [-]^{\mu}_{y} \varphi \rightarrow \varphi$	(Instance of contraposition of μ-y-converse)
7.	$[+]^{\mu}_{y} [-]^{\mu}_{y} \varphi \rightarrow \varphi$	(Propositional logic on 5 and 6)
8.	$[+]^{\mu}_{x} [+]^{\mu}_{y} [-]^{\mu}_{y} \varphi \rightarrow [+]^{\mu}_{x} \varphi$	(Nec with $[+]^{\mu}_{x}$, modal dist, and prop logic on 7)
9.	$[+]^{\mu}_{x} \varphi \rightarrow \langle + \rangle^{\mu}_{x} \varphi$	(Instance of D^{μ}_{x} (+))
10.	$\langle + \rangle^{\mu}_{x+y} \langle - \rangle^{\mu}_{y} \varphi \rightarrow \langle + \rangle^{\mu}_{x} \varphi$	(Propositional logic on 1, 3, 4, 8, and 9)

Proof Fix a frame $\mathfrak{F} = (M, \{R_x^+, R_x^-, R_x^{+,u}, R_x^{-,u} : x \in [0, \infty]\})$, and worlds $w \neq u \in M$. By co-reflexivity in $R_0^{+,\mu}$ and in $R_0^{-,\mu}$, neither $R_0^{+,\mu}(w, u)$ nor $R_0^{-,\mu}(w, u)$ hold. By finite μ-accessibility, neither $R_\infty^{+,\mu}(w, u)$ nor $R_\infty^{-,\mu}(w, u)$ hold. The proposition then follows immediately from Proposition 12. □

7.5 Real Lines as Real Line Continuous Temporal Frames

The real number line can be understood as the pair $(\mathbb{R}, d)$, where the directed distance function $d : \mathbb{R}^2 \to \mathbb{R}$ is such that, for all $w, u \in \mathbb{R}$, $d(w, u) := u - w$. To conform to our indexing convention, we must restrict the range of the distance function to $[0, \infty)$. Therefore, define the *standard real line* as the triple $(\mathbb{R}, d^+, d^-)$, where $d^+(w, u) := u - w$ is defined only on $w \leq u \in \mathbb{R}$, and $d^-(w, u) := w - u$ is defined only on $w \geq u \in \mathbb{R}$. For our purposes, a *real line* is a triple that is isomorphic to the standard real line. Note that, on this definition, real lines are bimodal fuzzy frames.

Lemma 12 *(a) If a frame* $(M, \{R_x^+, R_x^-, R_x^{+,u}, R_x^{-,u} : x \in [0, \infty]\})$ *is a well-structured real line continuous temporal frame, then under the bimodal classical–fuzzy map,* (M, R^{+*}, R^{-*}) *is a real line. (b) If a bimodal fuzzy frame* (M, R^{+*}, R^{-*}) *is a real line, then under the μ-extended bimodal fuzzy–classical map,* $\mathfrak{F} = (M, \{R_x^+, R_x^-, R_x^{+,u}, R_x^{-,u} : x \in [0, \infty]\})$ *is a well-structured real line continuous temporal frame.*

Proof of (a) Fix a well-structured real line continuous temporal frame $\mathfrak{F} = (M, \{R_x^+, R_x^-, R_x^{+,u}, R_x^{-,u} : x \in [0, \infty]\})$, which maps to $\mathfrak{F}^* = (M, R^{+*}, R^{-*})$ using the bimodal classical–fuzzy map. We will construct a bijection $E : M \to \mathbb{R}$, and then show that E is an isomorphism from $\mathfrak{F}^*$ into the standard real line.

Via the axiom of choice, fix $w_0 \in M$. Define $E(w_0) := 0$. By Proposition 13, for every world $w \neq w_0 \in M$, there is some unique $x \in (0, \infty)$ such that exactly one of $R_x^{+,\mu}(w_0, w)$ or $R_x^{-,\mu}(w_0, w)$ holds. If $R_x^{+,\mu}(w_0, w)$ holds, then define $E(w) := x$. If $R_x^{-,\mu}(w_0, w)$ holds, then define $E(w) := -x$. By μ-$[0, \infty)$-seriality and μ-$[0, \infty]$-functionality, E is a bijection.

To show that E is an isomorphism, we prove that for every world $w, u \in M$, $R^{+*}(E^{-1}(w), E^{-1}(u)) = u - w$ and is defined only when $u \geq w$. The proof for R^{-*} is symmetric.

For all $w, u, v \in M$, using the bimodal classical–fuzzy map, note the following properties of the bimodal fuzzy accessibility relations:

- By reflexivity in $R_0^{+,\mu}$ and in $R_0^{-,\mu}$, $R^{+*}(w, w) = R^{-*}(w, w) = 0$.
- By μ-$\pm$-antisymmetry, $R^{+*}(w, u) = -R^{-*}(u, w)$.
- By additively transitive accessibility, $R^{+*}(w, v) = R^{+*}(w, u) + R^{+*}(u, v)$.
- By subtractively transitive accessibility, $R^{+*}(w, v) = R^{+*}(w, u) + R^{-*}(u, v)$.

That $R^{+*}(E^{-1}(w), E^{-1}(u))$ is defined only when $u \geq w$ follows immediately from the definition of E and μ-$\pm$-antisymmetry. So fix $w \leq u \in M$. Using the definition of E and the properties of the bimodal fuzzy accessibility relation, we examine three cases: $w \leq u < 0$, $w \leq 0 \leq u$, and $0 < w \leq u$.

When $w \leq u < 0$, using subtractively transitive accessibility and μ-$\pm$-antisymmetry:

$$\begin{aligned} R^{+*}(E^{-1}(w), E^{-1}(u)) &= R^{+*}(E^{-1}(w), E^{-1}(0)) + R^{-*}(E^{-1}(0), E^{-1}(u)) \\ &= -R^{-*}(w_0, E^{-1}(w)) + R^{-*}(w_0, E^{-1}(u)) \\ &= -w + u = u - w. \end{aligned}$$

When $w \leq 0 \leq u$, using additively transitive accessibility and μ-$\pm$-antisymmetry, along with reflexivity:

$$\begin{aligned} R^{+*}(E^{-1}(w), E^{-1}(u)) &= R^{+*}(E^{-1}(w), E^{-1}(0)) + R^{+*}(E^{-1}(0), E^{-1}(u)) \\ &= -R^{-*}(w_0, E^{-1}(w)) + R^{+*}(w_0, E^{-1}(u)) \\ &= -w + u = u - w. \end{aligned}$$

When $0 < w \leq u$, using μ-$\pm$-antisymmetry and subtractive transitivity:

$$\begin{aligned} R^{+*}(E^{-1}(w), E^{-1}(u)) &= -R^{-*}(E^{-1}(u), E^{-1}(w)) \\ &= R^{+*}(E^{-1}(0), E^{-1}(u)) - R^{+*}(E^{-1}(0), E^{-1}(w)) \\ &= R^{+*}(w_0, E^{-1}(u)) - R^{+*}(w_0, E^{-1}(w)) \\ &= u - w. \end{aligned}$$

□ *(a)*

Proof of (b) It is straightforward to check that real lines satisfy C_{RL}. □ *(b)*

As above, C_{RL} can be considered the canonical directed continuous modal logic for real lines.

Theorem 12 (Canonicity of C_{RL} for real lines) *A well-structured μ-extended continuous temporal frame satisfies C_{RL} if and only if it is a real line.*

Proof Follows directly from Corollary 13 and Lemma 12. □

Remark C^{μ}_{TL} could alternatively be extended to an A-continuous temporal logic, where A is any subset of $\mathbb{R}$ that is algebraically closed under $+$ and $-$, such as $\mathbb{Z}$ or $\mathbb{Q}$. To do so, extend C^{μ}_{TL} with schemata μ-$A \cap [0, \infty)$-seriality, μ-$A \cap [0, \infty]$-functionality, and no endpoints. Such a logic is canonical, by similar proofs to the above, in the class of frames isomorphic to $\left(A, d^+, d^-\right)$, the corresponding subset of the standard real line.

Appendix A: Plantability Proofs

Recall the definition of a plantable property from Section 3.3:

Definition 5 Let Π be a frame property. Let $\mathfrak{F} = (M, \{R_x : x \in [0, \infty]\})$ be an arbitrary frame that has Π. Let $(\mathfrak{M}, a)$ be an arbitrary pointed model extending $\mathfrak{F}$. Let $\left(\mathfrak{M}^\dagger, \langle a \rangle\right)$ be its planted model, with $\mathfrak{M}^\dagger = \left(M^\dagger, \{R^\dagger_x : x \in [0, \infty]\}, V^\dagger\right)$.

Property Π is **plantable** if and only if there exists a set of accessibility relations $\{R'_x : x \in [0, \infty]\}$ such that:

1. for all $x \in [0, \infty]$, $R'_x \supseteq R^\dagger_x$;
2. $\big(M^\dagger, \{R'_x : x \in [0, \infty]\}\big)$ has Π;
3. $\big(M^\dagger, \{R'_x : x \in [0, \infty]\}\big)$ is well-founded; and
4. for all $w^\dagger, u^\dagger \in M^\dagger$ corresponding to w and u, respectively, and for all $x \in [0, \infty]$, $R'_x(w^\dagger, u^\dagger)$ holds only if $R_x(w, u)$ holds.

Below we show that various properties considered in this paper are plantable.

A.1 Proof of Lemma 2

We will show that upwardly closed accessibility is plantable.

Let $(\mathfrak{M}, a)$ be an arbitrary pointed model with upwardly closed accessibility, with $\mathfrak{M} = (M, \{R_x : x \in [0, \infty]\}, V)$. Let $\big(\mathfrak{M}^\dagger, \langle a\rangle\big)$ be its planted model, with $\mathfrak{M}^\dagger = (M^\dagger, \{R^\dagger_x : x \in [0, \infty]\}, V^\dagger)$. Define $\mathfrak{M}' := \big(M^\dagger, \{R'_x : x \in [0, \infty]\}, V^\dagger\big)$, where $\{R'_x : x \in [0, \infty]\}$ is constructed as follows.

- Definition of R'_x : For $w^\dagger, u^\dagger \in M^\dagger$ and $x, y \in [0, \infty]$,

$$R'_x := \{(w^\dagger, u^\dagger) : R^\dagger_y(w^\dagger, u^\dagger) \text{ holds for some } y \leq x\}.$$

In other words, the x-accessibility relations of $\mathfrak{M}'$ are defined to be the upwardly closed accessibility relations of $\mathfrak{M}^\dagger$.

By construction, $\{R'_x : x \in [0, \infty]\}$ satisfies the first three conditions of being a plantable model. For the fourth condition, fix arbitrary $w^\dagger, u^\dagger \in M^\dagger$, corresponding to $w, u \in M$, respectively. Suppose that $R'_x(w^\dagger, u^\dagger)$ holds for some $x \in [0, \infty]$. By definition, there is some $y \leq x$ such that $R^\dagger_y(w^\dagger, u^\dagger)$ holds. Since $(\mathfrak{M}, a)$ is bisimilar to $(\mathfrak{M}^\dagger, \langle a\rangle)$, $R_y(w, u)$ holds. Since $(\mathfrak{M}, a)$ has upwardly closed accessibility, $R_x(w, u)$ holds.

A.2 Proof of Lemma 3

We will show that upwardly closed accessibility, reflexivity, symmetry, and additively transitive accessibility are jointly plantable.

Let $(\mathfrak{M}, a)$ be an arbitrary pointed model that has upwardly closed accessibility, reflexivity, symmetry, and additively transitive accessibility, with $\mathfrak{M} = (M, \{R_x : x \in [0, \infty]\}, V)$. Let $(\mathfrak{M}^\dagger, \langle a\rangle)$ be its planted model, with $\mathfrak{M}^\dagger = (M^\dagger, \{R^\dagger_x : x \in [0, \infty]\}, V^\dagger)$. Define $\mathfrak{M}' := (M^\dagger, \{R'_x : x \in [0, \infty]\}, V^\dagger)$, where $\{R'_x : x \in [0, \infty]\}$ is constructed via transfinite recursion as follows.

- Definition of R'_x:

$$R'_x := \bigcup_{\alpha \in \mathrm{ORD}} R'^{\alpha}_x.$$

- Base case: For $w^\dagger, u^\dagger \in M^\dagger$ and $x, y \in [0, \infty]$,

$$R'^0_x := \Big\{(w^\dagger, u^\dagger) : R^\dagger_y(w^\dagger, u^\dagger) \text{ or } R^\dagger_y(u^\dagger, w^\dagger) \text{ holds for some } y \leq x\Big\}.$$

- Successor case: For $w^\dagger, u^\dagger, v^\dagger \in M^\dagger$ and $x, y \in [0, \infty]$,

$$R'^{\alpha+1}_{x+y} := \left\{(w^\dagger, v^\dagger) : R'^{\alpha}_x(w^\dagger, u^\dagger) \text{ and } R'^{\alpha}_y(u^\dagger, v^\dagger) \text{ hold for some } u^\dagger\right\}.$$

- Limit case:

$$R'^{\alpha}_x := \bigcup_{\beta<\alpha} R'^{\beta}_x.$$

In other words, the x-accessibility relations of $\mathfrak{M}'$ are defined to be the upwardly closed and symmetrized accessibility relations of $\mathfrak{M}^\dagger$, successively made additively transitive (which preserves upwardly closed accessibility and symmetry).

By construction, $R'_x \supseteq R^\dagger_x$ for all $x \in [0, \infty]$, satisfying the first condition of being a plantable property. We show that $\{R'_x : x \in [0, \infty]\}$ satisfies the remaining three conditions.

Lemma 13 $\mathfrak{M}'$ *has all of upwardly closed accessibility, reflexivity, symmetry, and additively transitive accessibility.*

Proof To show that $\mathfrak{M}'$ has upwardly closed accessibility, we perform a simple proof by transfinite induction. For the base case, by construction, $\{R'^0_x : x \in [0, \infty]\}$ is upwardly closed. For the successor case, the inductive hypothesis is that, for fixed $\alpha \in \text{ORD}$, $\{R'^{\alpha}_x : x \in [0, \infty]\}$ is upwardly closed. Fix $w^\dagger, v^\dagger \in M^\dagger$ and $z' \geq z \in [0, \infty]$. Suppose that $R'^{\alpha+1}_z(w^\dagger, v^\dagger)$ holds. Then there is some $u^\dagger \in M^\dagger$ and $x, y \in [0, \infty]$ such that $R'^{\alpha}_x(w^\dagger, v^\dagger)$ and $R'^{\alpha}_y(u^\dagger, v^\dagger)$ hold and $x + y = z$. Since $\{R'^{\alpha}_x : x \in [0, \infty]\}$ is upwardly closed, there is an $x' \geq x, y' \geq y \in [0, \infty]$ such that $R'^{\alpha}_{x'}(w^\dagger, u^\dagger)$ and $R'^{\alpha}_{y'}(u^\dagger, v^\dagger)$ hold and $x' + y' = z'$. It follows from the definition of the successor case that $R'^{\alpha+1}_{z'}(w^\dagger, v^\dagger)$ holds. Therefore, $\{R'^{\alpha+1}_x : x \in [0, \infty]\}$ is upwardly closed. For the limit case, the proof is transparent.

To show that $\mathfrak{M}'$ is reflexive, note that $(\mathfrak{M}, a)$ is reflexive. By the definition of planted models, $(\mathfrak{M}^\dagger, \langle a \rangle)$ is thus reflexive in 0-accessibility. It follows that $\mathfrak{M}'$ is reflexive in 0-accessibility. Since $\mathfrak{M}'$ also has upwardly closed accessibility, $\mathfrak{M}'$ is reflexive.

To show that $\mathfrak{M}'$ is symmetric, we again proceed by transfinite induction. For the base case, by construction, $\{R'^0_x : x \in [0, \infty]\}$ is symmetric. For the successor case, the inductive hypothesis is that, for fixed $\alpha \in \text{ORD}$, $\{R'^{\alpha}_x : x \in [0, \infty]\}$ is symmetric. Fix $w^\dagger, v^\dagger \in M^\dagger$ and $z \in [0, \infty]$. Suppose that $R'^{\alpha+1}_z(w^\dagger, v^\dagger)$ holds. Then there is some $u^\dagger \in M^\dagger$ and $x, y \in [0, \infty]$ such that $R'^{\alpha}_x(w^\dagger, u^\dagger)$ and $R'^{\alpha}_y(u^\dagger, v^\dagger)$ hold and $x + y = z$. Since $\{R'^{\alpha}_x : x \in [0, \infty]\}$ is symmetric, $R'^{\alpha}_x(u^\dagger, w^\dagger)$ and $R'^{\alpha}_y(v^\dagger, u^\dagger)$ also hold. It follows from the definition of the successor case that $R'^{\alpha+1}_{x+y}(v^\dagger, w^\dagger)$ holds, which means $R'^{\alpha+1}_z(v^\dagger, w^\dagger)$ holds. Therefore, $\{R'^{\alpha+1}_x : x \in [0, \infty]\}$ is symmetric. For the limit case, the proof is transparent.

To show that $\mathfrak{M}'$ has additively transitive accessibility, fix $w^\dagger, u^\dagger, v^\dagger \in M^\dagger$ and $x, y \in [0, \infty]$. Suppose that $R'_x(w^\dagger, u^\dagger)$ and $R'_y(u^\dagger, v^\dagger)$ hold. By the definition of R'_x, it follows that for some $\alpha, \beta \in \text{ORD}$, $R'^{\alpha}_x(w^\dagger, u^\dagger)$ and $R'^{\beta}_y(u^\dagger, v^\dagger)$ hold. Fix one such α and β, and let γ be the larger of the two. By the definitions of the successor case and the limit case, it follows from reflexivity in 0-accessibility that if

$R'^{\alpha}_x(w^\dagger, u^\dagger)$ holds, $R'^{\gamma}_x(w^\dagger, u^\dagger)$ holds, and if $R'^{\beta}_y(u^\dagger, v^\dagger)$ holds, $R'^{\gamma}_y(u^\dagger, v^\dagger)$ holds. Thus, by the definition of the successor case, $R'^{\gamma+1}_{x+y}(w^\dagger, v^\dagger)$ holds. From this it follows by the definition of R'_x that $R'_{x+y}(w^\dagger, v^\dagger)$ holds. $\mathfrak{M}'$ therefore has additively transitive accessibility. □

Lemma 14 $\mathfrak{M}'$ *is well-founded.*

Proof $(\mathfrak{M}^\dagger, \langle a\rangle)$ is well-founded. Here is the concern for $\mathfrak{M}'$. Planted models ensure well-foundedness by expanding one world's continuum-many accessibility relations to a single world into one world's single accessibility relation to continuum-many worlds. But the construction of R'_x allows one world to access another through continuum-many paths. We must ensure that these paths don't generate a set of accessibility relations with no minimum.

Definition 14 For all $w^\dagger, v^\dagger \in M^\dagger$, a **direct path from $w^\dagger$ to $v^\dagger$** is a path from $w^\dagger$ to $v^\dagger$ such that:

1. for each element of the path $u^\dagger_1$ before $v^\dagger$, the next element $u^\dagger_2$ is such that, for some $x \in [0, \infty]$, either $R^\dagger_x(u^\dagger_1, u^\dagger_2)$ holds or $R^\dagger_x(u^\dagger_2, u^\dagger_1)$ holds; and
2. no element appears in the path more than once.

The planted model $(\mathfrak{M}^\dagger, \langle a\rangle)$ is just the tree-unraveling of $(\mathfrak{M}, a)$, save for self-0-accessibility. Since, by definition, no direct path has a world that appears more than once, self-0-accessibility cannot connect two elements of a direct path. So for the construction of direct paths, $(\mathfrak{M}^\dagger, \langle a\rangle)$ is a rooted tree-like structure. From this it follows that, for all $w^\dagger, v^\dagger \in M^\dagger$, there is exactly one direct path from $w^\dagger$ to $v^\dagger$, and the path has a finite number of elements. We will henceforth call it *the* direct path from $w^\dagger$ to $v^\dagger$.

Definition 15 For all $w^\dagger, v^\dagger \in M^\dagger$, the **length of the direct path from $w^\dagger$ to $v^\dagger$** is the sum of the indices of the accessibility relations connecting each pair of successive elements of the path.

More formally, let $\langle u^\dagger_1, u^\dagger_2, \ldots, u^\dagger_n\rangle$ be the direct path from $w^\dagger$ to $v^\dagger$. Define $\{x_i : 1 \le i \le n\}$ as follows.

$$x_i := \begin{cases} x \text{ such that } R^\dagger_x(u^\dagger_i, u^\dagger_{i+1}) \text{ or } R^\dagger_x(u^\dagger_{i+1}, u^\dagger_i) \text{ holds} & \text{if } i < n; \\ 0 & \text{if } i = n. \end{cases}$$

The length of the path is then:

$$\lambda(w^\dagger, v^\dagger) := \sum_{i=1}^{n} x_i.$$

Consider arbitrary worlds $w^\dagger, u^\dagger, v^\dagger \in M^\dagger$, and the direct paths from $w^\dagger$ to $u^\dagger$, from $u^\dagger$ to $v^\dagger$, and from $w^\dagger$ to $v^\dagger$. Construct a possibly indirect path from $w^\dagger$ to $v^\dagger$ by concatenating the direct paths from $w^\dagger$ to $u^\dagger$ and from $u^\dagger$ to $v^\dagger$, merging the last element of the former with the first element of the latter. The direct path from $w^\dagger$ to

$v^\dagger$ is a subpath of the possibly indirect path from $w^\dagger$ to $v^\dagger$, for otherwise there would be more than one direct path from $w^\dagger$ to $v^\dagger$. Therefore, for all $w^\dagger, u^\dagger, v^\dagger \in M^\dagger$,

$$\lambda(w^\dagger, v^\dagger) \leq \lambda(w^\dagger, u^\dagger) + \lambda(u^\dagger, v^\dagger).$$

To prove that $\mathfrak{M}'$ is well-founded, we will prove that, for all $w^\dagger, v^\dagger \in M^\dagger$, $\lambda(w^\dagger, v^\dagger)$ is the minimum x such that $R'_x(w^\dagger, v^\dagger)$ holds. Since $\mathfrak{M}'$ is reflexive in 0-accessibility and has additively transitive accessibility, for all $w^\dagger, v^\dagger \in M^\dagger$, $R'_{\lambda(w^\dagger, v^\dagger)}(w^\dagger, v^\dagger)$ holds. We thus need to show that, for all x such that $R'_x(w^\dagger, v^\dagger)$ holds, $x \geq \lambda(w^\dagger, v^\dagger)$. We show this by performing a proof by transfinite induction.

For the base case, fix $w^\dagger, v^\dagger \in M^\dagger$ and $x \in [0, \infty]$. Suppose that $R'^0_x(w^\dagger, v^\dagger)$ holds. By construction, there is a $y \leq x$ such that either $R^\dagger_y(w^\dagger, v^\dagger)$ holds or $R^\dagger_y(v^\dagger, w^\dagger)$ holds. Since $(\mathfrak{M}^\dagger, \langle a \rangle)$ is a rooted tree-like structure, it follows that the direct path between $w^\dagger$ and $v^\dagger$ is $\langle w^\dagger, v^\dagger \rangle$, which has length $\lambda(w^\dagger, v^\dagger) = y$. Therefore, $x \geq \lambda(w^\dagger, v^\dagger)$.

For the successor case, the inductive hypothesis is that, for fixed $\alpha \in$ ORD: for all $w^\dagger, v^\dagger \in M^\dagger$ and $x \in [0, \infty]$, if $R'^\alpha_x(w^\dagger, v^\dagger)$ holds, then $x \geq \lambda(w^\dagger, v^\dagger)$. Now fix $w^\dagger, v^\dagger \in M^\dagger$ and $z \in [0, \infty]$. Suppose that $R'^{\alpha+1}_z(w^\dagger v^\dagger)$ holds. Then there is some $u^\dagger \in M^\dagger$ and $x, y \in [0, \infty]$ such that $R'^\alpha_x(w^\dagger, u^\dagger)$ and $R'^\alpha_y(u^\dagger, v^\dagger)$ hold and $x + y = z$. By hypothesis, $x \geq \lambda(w^\dagger, u^\dagger)$ and $y \geq \lambda(u^\dagger, v^\dagger)$. Therefore, $z = x + y \geq \lambda(w^\dagger, u^\dagger) + \lambda(u^\dagger, v^\dagger) \geq \lambda(w^\dagger, v^\dagger)$.

For the limit case, the proof is straightforward. □

Lemma 15 *For all $w^\dagger, v^\dagger \in M^\dagger$ corresponding to $w, v \in M$, respectively, and for all $x \in [0, \infty]$, if $R'_x(w^\dagger, v^\dagger)$ holds, then $R_x(w, v)$ holds.*

Proof We perform a proof by transfinite induction.

For the base case, fix $w^\dagger, v^\dagger \in M^\dagger$, corresponding to $w, v \in M$, respectively, and fix $x \in [0, \infty]$. Suppose that $R'^0_x(w^\dagger, v^\dagger)$ holds. By construction, there is a $y \leq x$ such that either $R^\dagger_y(w^\dagger, v^\dagger)$ holds or $R^\dagger_y(v^\dagger, w^\dagger)$ holds. Since $(\mathfrak{M}^\dagger, \langle a \rangle)$ is bisimilar to $(\mathfrak{M}, a)$, either $R_y(w, v)$ or $R_y(v, w)$ holds. Since $\mathfrak{M}$ is symmetric, if $R_y(v, w)$ holds, then $R_y(w, v)$ holds. Thus, $R_y(w, v)$ holds. Since $\mathfrak{M}$ has upwardly closed accessibility, $R_x(w, v)$ holds.

For the successor case, the inductive hypothesis is that, for fixed $\alpha \in$ ORD: for all $w^\dagger, v^\dagger \in M^\dagger$ corresponding to $w, v \in M$, respectively, and all $x \in [0, \infty]$, if $R'^\alpha_x(w^\dagger, v^\dagger)$ holds, then $R_x(w, v)$ holds. Now fix $w^\dagger, v^\dagger \in M^\dagger$ corresponding to $w, v \in M$, respectively, and fix $z \in [0, \infty]$. Suppose that $R'^{\alpha+1}_z(w^\dagger, v^\dagger)$ holds. Then there is some $u^\dagger \in M^\dagger$ corresponding to $u \in M$ and $x, y \in [0, \infty]$ such that $R'^\alpha_x(w^\dagger, u^\dagger)$ and $R'^\alpha_y(u^\dagger, v^\dagger)$ hold and $x + y = z$. By hypothesis, $R_x(w, u)$ and $R_y(u, v)$ hold. Since $\mathfrak{M}$ has additively transitive accessibility, $R_{x+y}(w, v)$ holds, which means $R_z(w, v)$ holds.

For the limit case, the proof is straightforward. □

A.3 Proof of Lemma 5

We will show that upwardly closed accessibility, reflexivity, symmetry, additively transitive accessibility, and D-bounded accessibility are jointly plantable for each

$D \in [0, \infty]$. The proof is the same as for Lemma 3 in Appendix A.2, except as follows.

Fix $D \in [0, \infty]$. Stipulate that $(\mathfrak{M}, a)$ also has D-boundedness. Define $\mathfrak{M}' := (M^\dagger, \{R_x^D : x \in [0, \infty]\}, V^\dagger)$, with:

- Definition of R_x^D: For $w^\dagger, u^\dagger \in M^\dagger$ and $x, y \in [0, \infty]$,

$$R_x^D := \{(w^\dagger, u^\dagger) : R'_x(w^\dagger, u^\dagger) \text{ holds, or } R'_y(w^\dagger, u^\dagger) \text{ holds for some } y \geq x \geq D\}.$$

In other words, the accessibility relations of $\mathfrak{M}'$ are defined to be the same as in Lemma 3 in Appendix A.2, except that they are D-bounded.

By construction, $R_x^D \supseteq R_x^\dagger$ for all $x \in [0, \infty]$. Also by construction, $\mathfrak{M}'$ will have D-bounded accessibility, preserving upwardly closed accessibility, reflexivity, symmetry, and additive transitivity. Again by construction, since R'_x is well-founded, R_x^D is well-founded. So the first three conditions of being a plantable property are satisfied.

For the fourth condition, fix $w^\dagger, u^\dagger, \in M^\dagger$, corresponding to $w, u \in M$, respectively, and fix $x \in [0, \infty]$. Suppose $R_x^D(w^\dagger, u^\dagger)$ holds. Then either $R'_x(w^\dagger, u^\dagger)$ holds or there is a $y \geq x \geq D$ such that $R'_y(w^\dagger, u^\dagger)$ holds. In case $R'_x(w^\dagger, u^\dagger)$ holds, then by Lemma 15, $R_x(w, u)$ holds. In case there is a $y \geq x \geq D$ such that $R'_y(w^\dagger, u^\dagger)$ holds, then by Lemma 15, $R_y(w, u)$. Since $\mathfrak{M}$ has upwardly closed accessibility, $R_\infty(w, u)$ holds. Since $\mathfrak{M}$ has D-bounded accessibility, $R_D(w, u)$ holds. Since $\mathfrak{M}$ has upwardly closed accessibility, $R_x(w, u)$ holds.

A.4 Proof of Lemma 9

We will show that upwardly closed accessibility, reflexivity, symmetry, additively transitive accessibility, D-bounded accessibility, and coreflexive 0-accessibility are jointly plantable for each $D \in [0, \infty]$. The proof is the same as for Lemma 5 in Appendix A.3, except as follows.

Stipulate that $(\mathfrak{M}, a)$ is coreflexive in 0-accessibility. By the definition of planted models, $(\mathfrak{M}^\dagger, \langle a \rangle)$ is also coreflexive in 0-accessibility. From this it follows that $\mathfrak{M}'$ is coreflexive in 0-accessibility.

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