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Paraconsistent Metatheory: New Proofs with Old Tools

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Abstract

This paper is a step toward showing what is achievable using non-classical metatheory—particularly, a substructural paraconsistent framework. What standard results, or analogues thereof, from the classical metatheory of first order logic(s) can be obtained? We reconstruct some of the originals proofs for Completeness, Löwenheim-Skolem and Compactness theorems in the context of a substructural logic with the naive comprehension schema. The main result is that paraconsistent metatheory can 're-capture' versions of standard theorems, given suitable restrictions and background assumptions; but the shift to non-classical logic may recast the meanings of these apparently 'absolute' theorems.

Keywords Paraconsistent logic · Inconsistent mathematics · Substructural logic · Non-classical metatheory · Completeness theorems

1 Introduction

What can be done in a properly *non-classical metatheory*—particularly, a *substructural paraconsistent* framework?[1] Metatheory (or sometimes, in older sources,

[1]For discussions of paraconsistent metatheory, see [35], [15, 47], or [39].

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metamathematics) is the study of mathematical systems themselves, and in particular of formal properties of logic(s).[2] For example, given some proposed formal logic, often both a model theoretic interpretation and a proof system are presented, and *soundness* and *completeness* theorems guarantee that these two bits of formalism relate in the right way; that sort of metatheoretic result is the *sine qua non* for showing that any would-be logic is legitimate. But facts in a metatheory are themselves proven by valid arguments. While classical logic has been the dominant theory of validity for the past century, it has always been disputed, and there are today many other well-motivated and well-understood non-classical logics [33]. A natural question arises: when giving valid proofs in metatheory—'valid' according to what logic?

1.1 Metamathematics, Classical and Non-Classical

Here are three famous theorems of *classical* metatheory, what we call the 'Big Three', where we will focus our attention:

Completeness (1929) If an argument is semantically valid, then it is proof-theoretically valid.
Löwenheim-Skolem (1922) If a theory has a model with an uncountably infinite domain, then it has a model with a countable domain.
Compactness (1930) If every finite subset of a set of sentences is satisfiable, then the whole set of sentences is satisfiable.

Gödel proved completeness in his doctoral dissertation [16]. He used a highly syntactic argument, improving earlier techniques from Löwenheim [22] and Skolem [43]. Together, these theorems are now understood to *characterize* classical first order logic, via Lindström's theorem [21]. They also show some *limitations* of classical first order logic; Skolem, for example, thought he had found a *paradox*; cf. [23]. In any case, whatever one makes of the 'Big Three', they are nowadays presented and taught as absolute mathematical facts—core truths about logic, valid simpliciter.

Notably, the Big Three are all (usually) informally glossed as *conditionals*, as above. Put this way, these can look not only absolute but rather arresting at first meeting. Conditionals, though, as anyone casually acquainted with logical research (classical and non-classical alike) knows, are highly charged. There have always been doubts, going back to Łukasiewicz in 1910 and Lewis in 1912,[3] about the *material conditional* $p \supset q := \neg p \vee q$ as a formalization of implication, due to the fact that various paradoxical-sounding arguments come out as valid using it.[4] So in the standard theorems of metatheory, what is the 'if / then'? An examination of *proofs* of the Big Three may reveal that some more mundane 'material' fact has been argued, which happens to be logically equivalent to the more exciting conditional statements. For example, '*if every finite subset of a set of sentences is satisfiable, then the whole*

[2] In this we follow Gödel, taking a broader stance than Kleene in the classic [18], where only work with strictly finitary methods is allowed in metamathematics. Kleene [18, p.423] in fact claims that "Gödel's completeness theorem (...) cannot belong to metamathematics".

[3] Even ancient doubt, as reported by Sextus Empiricus [10, Book II, 115-118].

[4] E.g. '$\neg(p \supset q)$ therefore p' is valid but has the instance 'It's false that if ghosts exist then materialism is true, so, ghosts exist' has a true premise but false conclusion. See [41, ch.1] for a barrage of further counterexamples.

set is satisfiable' verges on seeming miraculous; but it holds because the contrapositive, '*if a whole set of sentences is inconsistent, then it is inconsistent in some finite part*' holds, and this because of the disjunction '*either a whole set of sentences is consistent, or some finite part of it is inconsistent*' holds. This last is, perhaps, the most stripped-down way of putting the compactness theorem, and it verges on seeming obvious. Of course, the reader will want to point out, these are all ways of saying the same thing—the same, anyway, in classical logic. But, we suggest, maybe not in *non*-classical logic.

The proper statement and logical status of metatheorems is no idle curiosity. At the end of a widely-used textbook on non-classical logics, Graham Priest writes:

> It may fairly be asked what logic I have been using to specify and reason about ... the various logics we have been dealing with. The procedures employed have not been formal ones, of course. Like most mathematics, matters have been left at an informal level. They could be formalized in...classical logic. But to someone, such as an intuitionist or paraconsistent logician, who takes such reasoning not to be correct, at least in part, things cannot be left like this. The classical ladder must, so to speak, be kicked away [33, p.585].

Priest's preferred (paraconsistent) logic LP is provided a model theory and a proof theory and he proves the completeness of LP proof methods with respect to their semantics [33, theorem 8.7.9, p.157]. But as Priest says above, adherents of a logic are within their rights to expect *proofs* to hold up *in their preferred logic*. And for, say, someone committed to the paraconsistent logic LP as a basis for reasoning, proofs using *ex contradictione quodlibet* ($p, \neg p \vDash q$), *disjunctive syllogism* ($p \vee q, \neg p \vDash q$), *contraposition* ($p \vDash q$ therefore $\neg q \vDash \neg p$), and other principles are *not valid* proofs, due to constraints imposed by e.g. the Liar [3, ch.1] and Russell paradoxes [32, ch.1, 2], [33, ch.7.7.3]. And this is to say nothing of *substructural* constraints imposed e.g. by versions Curry's paradox, where *contraction* ($p, p \vDash q$ therefore $p \vDash q$) is not (meta)valid [5, ch.7].[5] The standard proofs, however, for completeness and related results *do* use all these principles. So, for an adherent of (substructural) LP, it is worth asking: what is the status of these famous meta-theorems? Are the 'Big Three' absolute mathematical truths, or contextual truths relative to some specific logical background?

In 1947 Henkin gave the now-standard proof of Gödel's completeness theorem, via a contrapositive argument: *if there is no proof from A to B, then there is a counterexample making A true but B false*. As is well known, he proves it by building a maximum consistent set of sentences $\mathcal{H}$ and showing that there are some things not provable from $\mathcal{H}$—on the assumption that nothing is both provable and not. Henkin's method is now standard for proving completeness for a wide class of logics. But Henkin's proof cannot, it seems, be repeated paraconsistently.[6] As a matter of pure (paraconsistent) logic, one simply can't assume for reductio that the proof relation at issue is consistent, i.e. that $\Gamma \vdash A$ and $\Gamma \nvdash A$ together are *impossible*; indeed,

[5] Alternatively, one may go substructural in response to the paradoxes by keeping contraction but dropping *transitivity* [37] or even *reflexivity* [13].

[6] For example, in the meta-theoretic reasoning involved in the Henkin proof of completeness for propositional LP in [31], Lemma 2 (the existence lemma for a suitable deductively closed, prime theory) relies on reductio arguments not available using LP.

the possibility that the proof relation may be *inconsistent* is one of the founding motivations for Priest's LP approach [30], [32, ch.3, 17].

If Henkin's argument is out of reach for a paraconsistentist, whither completeness? We are faced with an apparent dilemma. On the one hand, an adherent of LP can believe their logic is complete, but then must admit that LP is not *independently viable*, insofar as it on its own is not up to the task of providing a logic for proving true mathematical results.[7] On the other hand, an LPer can deny (or refuse to accept) that their logic is complete after all, but in doing so they must say something *radically* at odds with classical logic, taking the stance of a mathematical *revisionist*—something that Priest, at least, has explicitly avoided [32, p.221].

We submit that this dilemma is a false one, and look for an alternative strategy. Firstly, we approach this problem from the direction of committed *inconsistent mathematics*[8] and in particular a 'purist' approach that does not appeal to classical logic or model theory—a step toward independent inconsistent *metamathematics*. Secondly, we use as 'low-tech' an argument as possible, going back to Gödel's original completeness proof from his dissertation (and so avoiding some of the higher-power machinery needed for Henkin's more abstract proof; cf. [2]). The proof is *direct*, proving a (classical) equivalent of the same fact without a detour through its (classically equivalent) contrapositive version. Together, we suggest that these strategies—starting from scratch, so to speak, avoiding as much as possible "theft over honest toil," and using some old tools to do it—are ways forward for a paraconsistent mathematics program.

1.2 Targets

A proof of the completeness of *propositional* LP (more precisely, a substructural version of it), also by a low-tech direct argument, is in [47]. The methods there, though, do not extend to quantifiers, leaving open the mathematically more substantial question of the completeness of first order LP (from a purely paraconsistent viewpoint). Answering that is our main task in this paper; then we consider the closely-related Löwenheim-Skolem and compactness properties. So, with the plan of stripping back any classical 'special effects' to see what bare, mutually-agreed-upon mathematical facts underly the Big Three, let us set targets by fixing definitions to work with. The idea is to focus on the basic forms of the statements that are proved in Gödel's thesis; these are, quite explicitly, material conditionals.

A key concept in Gödel's original completeness proof is the following, a refinement of the more common notion of prenex normal form:[9]

[7] Of course, taking this horn is not the end of the story, qua classicality and recapture. For ways a paraconsistentist can proceed see [3, ch.5], [4], [34].

[8] A field of mathematics that emerged in the second half of the 20th century, which uses paraconsistent logic to study abstract structures and describe them with non-trivial theories that include contradictions [25, 26, 40].

[9] Strictly speaking, Gödel worked with a different but equivalent notion of what we call SNFs. More pressingly, since we've admitted the possibility of inconsistency, there may be concerns about this definition or others in this paper, e.g. how do we know that 'preceedes' behaves consistently? This is addressed, to the extent that it can be addressed, with Principle 2 below; for discussion of the possibility of an inconsistent proof relation see [32, pp.237-243].

Definition 1 A formula is said to be in Skolem normal form (SNF) iff all its quantifiers appear at the beginning, and, furthermore, all existential quantifiers precede all universal quantifiers, or, in other words, an $\exists^*\forall^*$-formula (where the number of existential or universal quantifiers may be 0) [17, p. 85].

This definition immediately raises a challenge. In Gödel's original argument, he shows (as in Hilbert and Ackermann [17, p. 88]) that any formula can be put into Skolem normal form; or more carefully, for any formula A there is a formula A' in SNF such that A is a theorem if and only if A' is. But this is highly sensitive to the logical principles available, and could be seen as more of an artifact of the classical tendency to show (too) many things are equivalent, rather than any deep truth about logical forms. Paraconsistently, it appears to be out of reach.[10] Compare this situation to the intuitionistic case, where a completeness proof with respect to the Tarskian semantics by fully intuitionistic methods is only available for restricted classes of formulas (since in intuitionistic logic there is no prenex-normal form theorem, let alone Skolem normal forms) [19, 20]. It is therefore more neutral to state the Big Three explicitly with the hypothesis involving Skolem normal forms.

The statement of the type of completeness we aim for uses notions (like satisfiability and validity) that will be properly defined and internalized below.[11]

Definition 2 A logic L is *Gödel complete* iff for any formula A in SNF, either $\not\models A$ or $\vdash A$.

This notion of completeness is well-motivated historically: it is a close variant (see theorem 4 below) of the one found at the heart of the main result of Gödel's famous article [16, Thm. II]. Similarly, we have the material version of Löwenheim-Skolem, with the SNF assumption made explicit:

[10]The proof is in [17]. Another standard reference on this is [7, p. 224–227], which we will follow. The strategy generally consists in obtaining a prenex-normal form for the formula A (which we can also do in a paraconsistent setting) and cleverly performing some manipulations, including substitutions of predicates, to obtain a desired A'. Some of the final manipulations demand some form of disjunctive syllogism, which we do not have available. In particular, at some point we need to show that if A is of the form

$$\exists x_1, \dots x_n \forall y B(x_1, \dots x_n, y),$$

where $B(x_1, \dots x_n, y)$ might have any number of quantifiers, A is a theorem (or valid) iff the formula A'

$$\exists x_1, \dots x_n (\exists y (B(x_1, \dots x_n, y) \& \neg H(x_1, \dots x_n, y)) \vee \forall z H(x_1, \dots x_n, z))$$

is a theorem (or valid), where H is a new relation symbol. Already the direction "(A' is a theorem) $\Rightarrow$ (A is a theorem)" is problematic: the idea of substituting B for H and eliminating the contradictory formula $\exists y(B(x_1, \dots x_n, y) \& \neg B(x_1, \dots x_n, y))$ from the result is an application of disjunctive syllogism.

[11]From the classical point of view, another, perhaps less spectacular, result of Gödel is that the validity problem is decidable for the class of $\exists^2\forall^*$-formulas since the fragment $\forall^2\exists^*$ of the language of first-order logic without equality has the finite model property. In fact, $\exists^2\forall^*$ is best possible with respect to decidability of the validity problem and, of course, the full $\exists^*\forall^*$ fragment is undecidable by Church's theorem. An interesting open problem would be to investigate the situation in a paraconsistent setting.

Definition 3 A logic L has the *Löwenheim-Skolem property* iff, for every formula A in SNF, either A is true in every domain, or else it is falsified in some denumerable domain.

And compactness:[12]

Definition 4 A logic L is *countably compact* iff, for every denumerable set X of formulas in negated SNF, either some finite subset of X is not jointly satisfiable, or else X is jointly satisfiable.

Again the property being defined is in its 'material' form: not 'if p then q' but rather 'either not p or q'. (E.g. Definition 4 says, materially: every finite subset of X is jointly satisfiable $\supset$ X is jointly satisfiable.) This is the form of the target theorems; we return to the issue of their meanings in Section 5 at the end. The logic in question will be, again, a version of first order $\texttt{LP}$, the details of which we now spell out.[13]

2 Logic

This section presents a proof system for doing metatheory, for 'talking about' logical languages and structures: evaluation, interpretation, and validity. It is a substructural paraconsistent logic, taking as a base *substructural* $\texttt{LP}$ (that is, $\texttt{LP}$ but without structural contraction), with quantifiers $\forall$, $\exists$, identity, the Church constant $\bot$ (which stands for an absurd statement, discussed below), a conditional operator $\Rightarrow$ that obeys modus ponens, and substitution abiding identity $=$. We call this logic $\texttt{subLPQ}^{\bot}_{\Rightarrow}$. Our aim is to prove the Big Three (or something like them) about the object logic $\texttt{subLPQ}^{\bot}$, using the stronger system $\texttt{subLPQ}^{\bot}_{\Rightarrow}$ plus some mathematical axioms.[14]

2.1 Proof Theory

Here is the logic presented as a Gentzen system, taking cues from [12]; cf. [29, p.1026]. The introduction of this system is being conducted in the meta-metatheory—which we consider to be a substructural paraconsistent framework, too.[15] The language of $\texttt{subLPQ}^{\bot}$ consists of the constants $\&$, $\vee$, $\neg$, $\exists$, $\forall$, and $\bot$,

[12]In the literature on *reverse mathematics*, it is known that (classically) countable compactness is equivalent to the so called Weak König's Lemma (which is provable in Zermelo-Fraenkel set theory without choice): every infinite tree of finite sequences of 0s and 1s has a path [42, Thm. IV 3.3]. This is established in a subtheory of second-order Peano Arithmetic known as RCA_0. What the situation looks like in our case is an interesting open problem.

[13]The question of whether a paraconsistent metatheory can prove the completeness of *classical* first order logic is highly interesting but for another day.

[14]This leaves open the completeness of $\texttt{subLPQ}^{\bot}_{\Rightarrow}$, and the 'holy grail' (as a referee puts it) of a system that could establish its own completeness without recourse to anything stronger. Gödel also weighed in on that question, though paraconsistency puts more options back on the table. See [32, ch.3].

[15]As the great relevant/paraconsistent logician R.K. Meyer puts it, "I am not going to fight with the C[lassical]-partisan about what goes on in the 'metalanguage.' ... [T]hat would just give us another formal system to talk about...but eventually the escalator ride has got to stop. ... The solution, already presented, is not to ride escalators" [24, p.160]).

variables $x, y, z, ...$, constants $a, b, c, ...$, predicate symbols of any finite arity, and brackets. Well-formed expressions $A, B, C, ...$ are defined in the usual way (see Principles 3, 4 below).

A *sequent* is of the form $\Gamma \vdash \varphi$, with Γ a multiset. The following are *initial sequents*:

$$\frac{A \vdash A}{}$$

$$\frac{\vdash A \vee \neg A}{} \qquad \frac{\bot \vdash A}{} \qquad \frac{\vdash \neg\bot}{}$$

$$\frac{\forall x(A\&B) \vdash \forall x A \& \forall x B}{} \qquad \frac{\forall x(A \vee B(x)) \vdash A \vee \forall x B(x)}{}$$

Left and right introduction rules for connectives are as follows. For (additive) disjunction,

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,(R\vee) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,(R\vee) \qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\,(L\vee)$$

For (multiplicative) conjunction,

$$\frac{\Gamma \vdash B \qquad \Delta \vdash C}{\Gamma, \Delta \vdash B\&C}\,(R\&) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A\&B \vdash C}\,(L\&)$$

For (de Morgan) negation,[16]

$$\frac{\Gamma, A \vdash B}{\Gamma, \neg\neg A \vdash B} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A} \qquad \frac{\Gamma \vdash \neg A}{\Gamma \vdash \neg(A\&B)} \qquad \frac{\Gamma \vdash \neg B}{\Gamma \vdash \neg(A\&B)}$$

$$\frac{\Gamma, \neg A \vdash C \qquad \Gamma, \neg B \vdash C}{\Gamma, \neg(A\&B) \vdash C}$$

$$\frac{\Gamma \vdash \neg A \qquad \Delta \vdash \neg B}{\Gamma, \Delta \vdash \neg(A \vee B)} \qquad \frac{\Gamma, \neg A, \neg B \vdash C}{\Gamma, \neg(A \vee B) \vdash C}$$

which we will refer to as (R de Morgan) or (L de Morgan) depending on whether the rule introduces on the right or the left. For quantifiers,

$$\frac{\Gamma, A^x_t \vdash B}{\Gamma, \forall x A \vdash B}\,(L\forall) \quad t \text{ any term} \qquad \frac{\Gamma \vdash A^x_y}{\Gamma \vdash \forall x A}\,(R\forall) \quad y \text{ not free in } \Gamma$$

$$\frac{\Gamma \vdash A^x_t}{\Gamma \vdash \exists x A}\,(R\exists) \quad t \text{ any term} \qquad \frac{\Gamma, A^x_y \vdash B}{\Gamma, \exists x A \vdash B}\,(L\exists) \quad y \text{ not free in } B, \Gamma$$

$$\frac{\Gamma \vdash \neg A^x_t}{\Gamma \vdash \neg\forall x A}\,(R\neg\forall) \quad t \text{ any term} \qquad \frac{\Gamma, \neg A^x_y \vdash B}{\Gamma, \neg\forall x A \vdash B}\,(L\neg\forall) \quad y \text{ not free in } B, \Gamma$$

$$\frac{\Gamma, \neg A^x_t \vdash B}{\Gamma, \neg\exists x A \vdash B}\,(L\neg\exists) \quad t \text{ any term} \qquad \frac{\Gamma \vdash \neg A^x_y}{\Gamma \vdash \neg\exists x A}\,(R\neg\exists) \quad y \text{ not free in } \Gamma$$

[16] It is a little outré to mix additive disjunction with multiplicative conjunction, but not unprecedented; see [38]. It is more unusual to connect them via de Morgan laws, but not incoherent; see [1]. On a similar note, the issue of additive versus multiplicative *quantifiers* eventually needs to be addressed (for a start, see [49, p.509]); but there is not space here.

The logic has the following structural rules,

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\ (Weakening) \qquad \frac{\Gamma \vdash B \qquad \Delta, B \vdash C}{\Gamma, \Delta \vdash C}\ (Cut)$$

noting the lack of contraction. Exchange—where if $A, B \vdash C$ then $B, A \vdash C$—follows automatically from the order-insensitivity of multisets.

A sequent is *derivable* iff it is an initial sequent, or follows from an initial sequent by rules. An argument from Γ to A is *valid* iff $\Gamma \vdash A$ is a derivable sequent. A *theorem* $\vdash A$ is a derivable sequent with nothing on the left.

That's $\texttt{subLPQ}^{\perp}$. To get $\texttt{subLPQ}^{\perp}_{\Rightarrow}$, add $\Rightarrow$ to the language and the rules:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ (R{\Rightarrow}) \qquad \frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \Rightarrow B \vdash C}\ (L{\Rightarrow})$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash \neg B}{\Gamma, \Delta \vdash \neg(A \Rightarrow B)}\ (R\neg \Rightarrow)$$

giving a decent working (non-material) conditional.[17] Adding $=$ to the language of $\texttt{subLPQ}^{\perp}_{\Rightarrow}$, too, it is governed by the following initial sequents:

$$\overline{\vdash x = x} \qquad \overline{x = y \vdash A(x) \Rightarrow A(y)}$$

where substitution holds for any A.

2.2 Set Theory

The purpose of this perhaps baroque system is to add the highly natural *naive comprehension* schema to it. For this we add to our language the set membership symbol $\in$ and the term-forming operator $\{\cdot : \cdot\}$, and to our logic add the axioms:

Principle 1 (Comprehension) $x \in \{z : A(z)\} \Leftrightarrow A(x), x \notin \{z : A(z)\} \Leftrightarrow \neg A(x)$

This makes the system inconsistent, but the logic is prepared to handle this.[18]

From comprehension, it follows immediately that

$$\forall z(x \in z \Rightarrow y \in z) \vdash x = y$$

since $x \in \{u : u = x\}$. Combined with our logical assumption that identity substitutes, which may now be expressed $x = y \vdash \forall z(x \in z \Rightarrow y \in z)$, this makes $=$ an equivalence relation.[19]

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[17] There is no rule for introducing $\neg(A \Rightarrow B)$ on the left, to avoid falling back into a material conditional. See Section 3 below.

[18] Without explosion (EFQ) or contraction, the Russell contradiction is derivable but inert, and Curry's paradox is avoided. There are other problematic pieces of reasoning to be careful of, for example, [6, 27]; to avoid these, we do not have an axiom of *extensionality*, which is integral to several more advanced paradoxes. There is a proof that the *logic* itself is robustly contraction free [1]. But there is not yet known a full non-triviality proof for this system once comprehension is added. Such a proof would be a relative-consistency proof carried out in a classical metatheory, which may be problematic for some principled reasons. See [46, ch.3].

[19] Symmetry: $x = y, x \in \{u : u = x\} \vdash y = x$. Transitive: $\forall u(x \in u \Rightarrow y \in u), \forall u(y \in u \Rightarrow z \in u) \vdash \forall u(x \in u \Rightarrow z \in u)$.

In terms of the language, further notation needed below is taken to be defined in terms of $\in$, as in

$$X \subseteq Y := \forall z((z \in X \Rightarrow z \in Y)\&(z \notin Y \Rightarrow z \notin X))$$
$$\langle x, y \rangle := \{\{x\}, \{x, y\}\}$$
$$X \times Y := \{\langle x, y \rangle : x \in X \;\&\; y \in Y\}$$

The law of ordered pairs, $\langle x, y \rangle = \langle x', y' \rangle \Leftrightarrow x = x' \;\&\; y = y'$ is derivable (by the same proof as in [6, p.356]). Then a *function* $f : X \longrightarrow Y$ is a set $f \subseteq X \times Y$ such that $\forall z(\langle z, y \rangle \in f \;\&\; \langle z, x \rangle \in f) \Rightarrow y = x$; and so forth. Along these lines, we can introduce, for example, the following abbreviations:

$$\mathrm{Func}(f) := (\forall u \in f)(\exists y, z(u = \langle y, z \rangle))\&(\forall x, y, z(\langle z, y \rangle \in f \;\&\; \langle z, x \rangle \in f) \Rightarrow y = x)$$
$$\mathrm{Dom}(f) := \{x : \exists y(\langle x, y \rangle \in f)\}$$
$$\mathrm{Range}(f) := \{y : \exists x(\langle x, y \rangle \in f)\}$$

We can express $f : X \longrightarrow Y$ by writing $\mathrm{Func}(f)$ & $\mathrm{Dom}(f) = X$ & $\mathrm{Range}(f) \subseteq Y$. With each function with a finite domain, there is an associated ordered n-tuple $\overline{f} = \langle f(1), \ldots, f(n) \rangle$.[20]

Beyond notation, ideally, one would use set theory as a foundation for other mathematical theories; but that is a story for another day. For present purposes, the following additional postulates simply record how much strength is necessary to carry off our results, as a rough kind of reverse mathematics. For steps towards deriving these principles substructurally, see [36, ch.11], [1, sec.8], [45, p.88], [46, ch.5, 6].

Principle 2 (Arithmetic) There is a set $\mathbb{Z}^+ = \{1, ...\}$ of positive integers for which basic properties of arithmetic hold. But $1 = 2$ does *not* hold, on pain of $\bot$. In general, for any $i \in \mathbb{Z}^+$, $i = i + 1$ is not provable, although other contradictions may be.

As a consequence, we have

Principle 3 (Mathematical Induction) For any set X, if $1 \in X$, and whenever $k \in X$ s.t. $k \in \mathbb{Z}^+$ also $k + 1 \in X$, then $\forall y(y \in \mathbb{Z}^+ \Rightarrow y \in X)$.

Informally here X stands for some property, which we want to obtain for all the integers. More generally, we can work inductively over the initial segment of ordinals. Let $V = \{x : \neg\bot\}$ (the universe of sets).

[20] In a paraconsistent context, there are significant complications that can arise around the addition of function symbols to the language [28], [46, p.174–178]. The language of $\mathtt{subLPQ}^{\bot}_{\Rightarrow}$ with its conditional is stronger than ‘stock’ $\mathtt{LP}$ and so avoids some of these worries, but there is more work to be done in the paraconsistent theory of functions. The ‘functional’ properties here are minimal.

Principle 4 (Recursion) Let $h : \mathbb{Z}^+ \times V \longrightarrow V$ be a 'class function'. Then there exists a unique function $f : \mathbb{Z}^+ \longrightarrow V$ such that, for every $n \in \mathbb{Z}^+$, $f(n) = h(n, f \upharpoonright n)$, where $f \upharpoonright n$ is the restriction of f to n.

Principle 5 (Countable Choice) A countable union of finite sets is countable.

This provides a substantial basis for undertaking a mathematical study of logic itself. It is a modification of the apparatus used in [47].

This presentation of our framework brings out one general aspect of how we will tackle these proofs, and also why our solution is only a partial one. We help ourselves to *restricted quantification* e.g. over members of a function, writing '$\forall x \in f$', rather than $\forall x$(if x is in f then...). For proofs, we will need to write e.g. $(\forall n \in \mathbb{Z}^+)(A(n))$ or $\exists \mathfrak{B} A(\mathfrak{B})$ to mean 'all natural numbers are A' or 'some structure is A', respectively. The final component of our framework, then, recording the requirements for proving completeness, are primitive restricted quantifiers: obeying dualities,

$$(\forall x A(x))(\neg B(x)) \vdash \neg(\exists x A(x))(B(x)) \quad \neg(\exists x A(x))(B(x)) \vdash (\forall x A(x))(\neg B(x))$$
$$(\exists x A(x))(\neg B(x)) \vdash \neg(\forall x A(x))(B(x)) \quad \neg(\forall x A(x))(B(x)) \vdash (\exists x A(x))(\neg B(x))$$

a kind of 'modus ponens',

$$A(a), (\forall x A(x))(B(x)) \vdash B(a)$$

and at least the rules:

$$\frac{\Gamma, A^x_y, B^x_y \vdash C}{\Gamma, (\exists x A(x))(B(x)) \vdash C}\ y \text{ not free in } \Gamma \qquad \frac{\Gamma, A^x_y \vdash B^x_y}{\Gamma \vdash (\forall x A(x))(B(x))}\ y \text{ not free in } \Gamma$$

These are used, for instance, in the proof of Theorem 2, dealing with Cases 1 and 2 in the proof of Theorem 4, and elsewhere. Until restricted quantification is given a full treatment, our result here is a *partial* solution.[21] The next section opens with a discussion of just how useful restricted quantification is, focusing on the definition of validity.

2.3 Propositional Semantics

With this, we can turn around and *interpret* parts of our logical calculus. That is, the theory just presented can account for the meaning of its own terms, say for the propositional part of the language. Let's record this here for intuitive interest and for use in more involved encoding later. Let t, f be two objects, like 1, 2 which by principle 2 are distinct. Focusing on just a propositional language, with `PROP` a set of

[21] The issue of restricted quantification is extremely difficult and is, to our knowledge, as yet unsolved. But research does not advance by waiting for everything to be finished before proceeding. We here suspend judgment on the issue to solve several *other* problems. See [34, §8], [11].

propositional atoms, then a relation $v \subseteq \texttt{PROP} \times \{\mathsf{t}, \mathsf{f}\}$ is a *valuation* (or *an assignment of truth-values*) iff (where $v[p] = \{x \mid \langle p, x\rangle \in v\}$):

$$\mathsf{t} \in v[p] \;\vee\; \mathsf{f} \in v[p]$$

$$\mathsf{t} \in v[p] \;\Leftrightarrow\; \mathsf{f} \notin v[p]$$
$$\mathsf{f} \in v[p] \;\Leftrightarrow\; \mathsf{t} \notin v[p]$$

Given a valuation v, we want to extend it recursively to a relation v' assigning truth values to all formulas of a propositional language in the following manner:[22]

$$\mathsf{t} \in v'[\neg A] \;\Leftrightarrow\; \mathsf{f} \in v'[A]$$
$$\mathsf{f} \in v'[\neg A] \;\Leftrightarrow\; \mathsf{t} \in v'[A]$$

$$\mathsf{t} \in v'[A \,\&\, B] \Leftrightarrow \mathsf{t} \in v'[A] \,\&\, \mathsf{t} \in v'[B] \qquad \mathsf{t} \in v'[A \vee B] \Leftrightarrow \mathsf{t} \in v'[A] \vee \mathsf{t} \in v'[B]$$
$$\mathsf{f} \in v'[A \,\&\, B] \;\Leftrightarrow\; \mathsf{f} \in v'[A] \vee \mathsf{f} \in v'[B] \qquad \mathsf{f} \in v'[A \vee B] \Leftrightarrow \mathsf{f} \in v'[A] \,\&\, \mathsf{f} \in v'[B]$$

$$\mathsf{t} \in v'[\top] \;\Leftrightarrow\; \top \qquad \mathsf{t} \notin v'[\top] \;\Leftrightarrow\; \bot$$
$$\mathsf{f} \in v'[\top] \;\Leftrightarrow\; \bot \qquad \mathsf{f} \notin v'[\top] \;\Leftrightarrow\; \top$$

A proposition A is a logical law iff $\mathsf{t} \in v(A)$ for every valuation v.

Notably, there is at least one proposition that is true and only true on pain of triviality, ruling out any 'trivial' models in which every formula is both true and false (as can happen in basic LP [35]). Otherwise, completeness can be made almost trivially true, if there is an evaluation that makes everything both true and false: the existence of a 'universal counterexample' makes studying the relationship between proofs and counterexamples vacuous [46, ch.10]. Having $\bot$ in the language makes proving completeness a meaningful exercise; cf. theorem 4 below.

3 Validity: Expressing and Internalizing

To do metamathematics requires representing the critical notions within the system itself. We must interpret 'interpretations'. The work we are about to do will indicate how validity may be 'internalized' in a paraconsistent system. But the way it is expressed brings out an important issue that arises repeatedly in shifting from classical formalisms to non-classical ones. This is to do with the relationship between validity and *counterexamples*, which is at the heart of any soundness and completeness theorems.

The problem is already visible at the level of a conditional operator. Because the material conditional is sub-optimal, non-classicists propose an alternative implication connective, $\Rightarrow$, that is supposed to fare better. For a 'gap' theorist (who does not accept LEM),

- if $p \Rightarrow q$ then $p \supset q$

[22]The recursive extension v' can be obtained by applying Principle 4 and defining a notion of *complexity degree* for the formulas of the propositional language (in symbols, $\mathrm{dg}(A)$ for a given formula A), i.e., propositional variables have degree 0, $\top$ has degree 1, a formula $\neg A$ has degree $2 + \mathrm{dg}(A)$, and both $A \vee B$ and $A \& B$ have degree $3 + \mathrm{dg}(A) + \mathrm{dg}(B)$. Then $v'[A]$ can be given a recursive definition on $\mathrm{dg}(A)$ using the above conditions (strictly speaking we obtain $v'[\mathrm{dg}(A)]$).

will break down (else the (true) instance $p \Rightarrow p$ will deliver $p \vee \neg p$) while for the 'glut' theorist (who accepts failures of ex falso),

- if $p \supset q$ then $p \Rightarrow q$

will break down (else the (true) instance $p \& \neg p \supset q$ will deliver real explosion). Both of these end up challenging the direct identification, made so explicitly in classical logic, between the *truth* of an implication $p \supset q$ and the *falsity* of any counterexample $p \& \neg q$. Non-classically, just because we don't have an implication doesn't mean we do have a counterexample; just because we don't have a counterexample doesn't mean we have an implication. And while that is by design, it leads to serious difficulties in 'recapturing' classical results—*especially* classical results about conditionality itself and its grown-up cousin, *validity*. For a completeness theorem is fundamentally about validity, and yet in its classical formulation, the theorem (along with soundness) amounts to identifying valid arguments with absences of counterexamples, and vice versa.

How does this play out? The standard definitions of validity and invalidity, if uncritically transcribed into our formalism, would leave room for a gap. An argument from Γ to φ is *valid* iff for every valuation v, if $\mathsf{t} \in v(\psi)$ for all $\psi \in \Gamma$ then $\mathsf{t} \in v(\varphi)$,

$$\forall v(\mathsf{t} \in v(\psi_0) \& ... \& \mathsf{t} \in v(\psi_n) \Rightarrow \mathsf{t} \in v(\varphi))$$

An argument is *invalid* iff there is a valuation that provides a counterexample to the argument,

$$\exists v(\mathsf{t} \in v(\psi_0) \& ... \& \mathsf{t} \in v(\psi_n) \& \mathsf{t} \notin v(\varphi))$$

But in the paraconsistent case, it is not true that $\neg\exists x(\varphi(x) \& \neg\psi(x))$ implies that $\forall x(\varphi(x) \Rightarrow \psi(x))$; and it is not true that $\neg\forall x(\varphi(x) \Rightarrow \psi(x))$ implies $\exists x(\varphi(x) \& \neg\psi(x))$. Some arguments may have no counterexample, so not be invalid, yet be without proof of validity. That is, on this way of writing things, just because an argument is *not valid* does not mean that it is *invalid*.

This raises some profound questions about the meaning of general validity, which are not our main purpose today. For the purposes of today's exercise, we observe that completeness proofs, even direct ones, *do* trade in the duality between validity and counterexamples. Our aim is to find a way to prove completeness. So to avoid 'validity-gaps', let's not work with the general notion of validity, but (as in [47]) just with *tautologies*:

- A proposition φ is *valid*, $\models \varphi$, iff $\forall v(\mathsf{t} \in v(\varphi))$.
- A proposition φ is *invalid*, $\not\models \varphi$, iff $\exists v(\mathsf{t} \notin v(\varphi))$.

From quantifier duality, these are now interlinked in the classical way: tautologies are *identified* with those propositions that have no counterexample. This is where appealing to restricted quantification is so important. But non-classicality is just below the surface: for all that has been said, there would be nothing incoherent about a proposition being both valid and invalid.

Once the basic idea of the semantics has been understood at the propositional level (as above), the task is to introduce the universal and existential quantifiers $\forall$, $\exists$. The first step in that direction is as usual to introduce a more sophisticated notion of an interpretation, including a domain of objects and relations on it. Our *object language* will consist of the so called *pure predicate calculus* [7, 18], where there is no equality symbol and the only terms are the individual variables.

Definition 5 An *interpretation* $\mathfrak{A}$ is a pair $\langle D, I\rangle$ consisting of a non-empty domain D (i.e., $\exists x(x \in D)$) and a map I assigning to each individual variable an element of D and to each relation symbol R^n of the vocabulary τ an n-ary relation $R^n_I \subseteq D^n \times \{\mathsf{t}, \mathsf{f}\}$ such that

$$\langle\langle a_1, \ldots, a_n\rangle, \mathsf{t}\rangle \in R^n_I \;\Leftrightarrow\; \langle\langle a_1, \ldots, a_n\rangle, \mathsf{f}\rangle \notin R^n_I$$
$$\langle\langle a_1, \ldots, a_n\rangle, \mathsf{f}\rangle \in R^n_I \;\Leftrightarrow\; \langle\langle a_1, \ldots, a_n\rangle, \mathsf{t}\rangle \notin R^n_I$$

For any interpretation $\mathfrak{A}$ of the logical vocabulary τ, we will define the relation of satisfaction $\models$, which holds between $\mathfrak{A}$ and formulas of the language. This is typically done by some sort of recursion but in fact using some tricks originally due to Dana Scott it can be done without any lengthly recursion-theoretic methods [9, p. 91]. We can start by identifying formulas with particular sequences of set-theoretic objects (their "Gödel set"). For relations,

$$\ulcorner R^n_k x_{i_1}, \ldots, x_{i_n}\urcorner \text{ is } \langle 1, k, n, \langle i_1, \ldots, i_n\rangle\rangle$$

where n is the arity of the relation (working in a language with a single countable sequence of variables and x_i represented by $i \in \mathbb{Z}^+$). For the connectives,

$\ulcorner A \vee B\urcorner$	is	$\langle 2, \ulcorner A\urcorner, \ulcorner B\urcorner\rangle$	$\ulcorner \exists v_i A\urcorner$	is	$\langle 5, i, \ulcorner A\urcorner\rangle$
$\ulcorner A \& B\urcorner$	is	$\langle 3, \ulcorner A\urcorner, \ulcorner B\urcorner\rangle$	$\ulcorner \forall v_i A\urcorner$	is	$\langle 6, i, \ulcorner A\urcorner\rangle$
$\ulcorner \neg A\urcorner$	is	$\langle 4, \ulcorner A\urcorner\rangle$	$\ulcorner \bot\urcorner$	is	$\langle 7\rangle$

For $\forall$ and $\exists$, we say that v_i—represented by i—is *bound* by the respective quantifier. Then we can introduce a precise meta-mathematical definition of formula in terms of Gödel sets:

Definition 6 A relation $Fm(u, s, n)$ says that $u = \ulcorner M\urcorner$ for some formula M, and s is a function which describes the construction of u as the Gödel set of M in n steps (the strict definition is just like in [9, p. 91]).

It is important to observe that no two formulas can have the same Gödel set or else $i = i + 1$ for some $i \in \mathbb{Z}^+$, which by Principle 2 gives $\bot$. Furthermore, we will define the notion of a variable x being *free* in a formula M as: either M has no quantifiers on pain of $\bot$, or x is the bound variable of a quantifier from M implies $\bot$. For instance, in $\forall x_1(R^2_1 x_1 x_2)$, the Gödel set is $\langle 6, 1, \langle 1, 1, 2, \langle 1, 2\rangle\rangle\rangle$, so if x_2 were the variable bound by $\forall$ (i.e. x_1), $\langle 6, 1, \langle 1, 1, 2, \langle 1, 2\rangle\rangle\rangle = \langle 6, 1, \langle 1, 1, 2, \langle 1, 1\rangle\rangle\rangle$ and then $1 = 2$, which gives $\bot$. For another example take the formula $R^4_6 x_5 x_1 x_1 x_1$, which is represented by $\langle 1, 6, 4, \langle 5, 1, 1, 1\rangle\rangle$, and ask: is x_1 free in this formula? If $R^4_6 x_5 x_1 x_1 x_1$ had a quantifier that would mean it was two different formulas at the same time, which again leads to $\bot$.

We are now ready to define the satisfaction relation $\models$:

Definition 7 Let M be a formula with Gödel set u, and let $\mathfrak{A} = \langle D, I\rangle$ be an interpretation. Define the relation $\models$ between $\mathfrak{A}$, M (more precisely, u), and a set b (in symbols, $\mathfrak{A} \models M[b]$) as

$$\exists t, s, n, r(n, r \in \mathbb{Z}^+ \&(Fm(u, s, n))\&(Func(t))\&(Dom(t) = n + 1)$$
$$\&(b \in t(n))\&(\forall k < n + 1\ S(k, t, r, s, \mathfrak{A})))$$

where $S(k, t, r, s, \mathfrak{A})$ (the 'compounded' satisfaction of sub-fomulas of M) is the disjunction of the following statements:

(i) $\exists o, p \in \mathbb{Z}^+ \exists f : o + 1 \longrightarrow \mathbb{Z}^+ ((s(k) = \langle 1, p, o, \overline{f}\rangle)\&(t(k) = \{a \in D^r \mid \langle\langle a_{f(1)}, \ldots, a_{f(o)}\rangle, \mathrm{t}\rangle \in I(R^o_p)\}))$, i.e., k is the Gödel set of an atomic formula and $t(k)$ is the set of sequences of elements that satisfy it: $\mathfrak{A} \models R^o_p x_{f(1)}, \ldots, x_{f(o)}[a]$ iff $\langle\langle a_{f(1)}, \ldots, a_{f(o)}\rangle, \mathrm{t}\rangle \in I(R^o_p)$,

(ii) $\exists l, m < k((s(k) = \langle 2, s(l), s(m)\rangle)\&(t(k) = t(l) \cup t(m)))$, i.e., k is the Gödel set of a disjunction of the formulas represented by the Gödel sets l and $m < k$, and $t(k)$ is the union of the sets $t(l)$ and $t(m)$ of sequences satisfying the respective disjuncts: $\mathfrak{A} \models (A \vee B)[a]$ iff $\mathfrak{A} \models A[a]$ or $\mathfrak{A} \models B[a]$,

(iii) $\exists l, m < k((s(k) = \langle 3, s(l), s(m)\rangle)\&(t(k) = t(l) \cap t(m)))$,

(iv) $\exists l < k((s(k) = \langle 4, s(l)\rangle)\&(t(k) = \{a \in D^r \mid a \notin t(l)\}))$,

(v) $((s(k) = \langle 7\rangle)\&(t(k) = \{D^r \mid \bot\}))$

(vi) $\exists i \in \mathbb{Z}^+ \exists l < k((s(k) = \langle 5, i, s(l)\rangle)\&(t(k) = \{a \in D^r \mid (\exists x \in D)(a(i/x) \in t(l))\}))$, where $a(i/x) = \langle d_1, \ldots, x, \ldots d_r\rangle$ if $a = \langle d_1, \ldots, d_i, \ldots d_r\rangle$.

(vii) $\exists i \in \mathbb{Z}^+ \exists l < k((s(k) = \langle 6, i, s(l)\rangle)\&(t(k) = \{a \in D^r \mid (\forall x \in D)(a(i/x) \in t(l))\}))$, where $a(i/x) = \langle d_1, \ldots, x, \ldots d_r\rangle$ if $a = \langle d_1, \ldots, d_i, \ldots d_r\rangle$.

Think of (vi) as saying that for all $a \in D^r$, $\mathfrak{A} \models \exists x_i B[a]$ iff there is $x \in D$ s.t. $\mathfrak{A} \models B[a(i/x)]$. Think of (vii) as saying that for all $a \in D^r$, $\mathfrak{A} \models \forall x_i B[a]$ iff for all $x \in D$, $\mathfrak{A} \models B[a(i/x)]$. Importantly, think of r as the maximum of all numbers appearing in the Gödel set u; that way, it suffices to consider r-sequences of elements in the satisfaction relation.

Definition 8 Define finitary logical consequence,[23] in symbols, $N_1, N_2, \ldots, N_m \vDash M$, as

$$(\forall\mathfrak{A})((\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m) \Rightarrow (\mathfrak{A} \models M)),$$

i.e., any interpretation $\mathfrak{A}$ that satisfies $N_1\&N_2\&\ldots\&N_m$ also satisfies M.

Definition 9 Define logical validity (in symbols, $\vDash M$) as $\emptyset \vDash M$, where $\emptyset$ is the empty set of premises, and the conjunction of its members is defined as $\top$. So $\emptyset \vDash M$ amounts to

$$\forall\mathfrak{A}(\mathfrak{A} \models M),$$

i.e., any interpretation $\mathfrak{A}$ satisfies M.

Now we can show that the theory is sound. The proof is conducted in mathematical English, saving more detailed Gentzen-style derivations for the more substantial Big Three.

[23] The premises may be thought of a finite multiset. Repeat occurrences of a premise are tracked by & in the definition of $\vDash$. Cf. multiset consequence as studied in [8].

Lemma 1 *For any* $n, r \in \mathbb{Z}^+$, $n \leqslant r$, $\mathfrak{A}$ *an interpretation,* $\overline{a}$ *and* $\overline{b}$ *sequences of length* r *of elements from* $\mathfrak{A}$ *identical on their first* n *values, and* M *a formula with free variables in the list* $x_1, \ldots, x_n$*, we have that* $\mathfrak{A} \models M[\overline{a}]$ *if and only if* $\mathfrak{A} \models M[\overline{b}]$.

Proof Induction on complexity of M. □

Theorem 2 (Soundness of $\mathtt{SubLPQ}^{\perp}$) *If* $M_1, M_2, \ldots, M_m \vdash A$ *is a provable sequent then* $M_1, M_2, \ldots, M_m \vDash A$*. In particular, every theorem is valid, i.e.,* $\vdash A$ *implies that* $\vDash A$*.*

Proof By induction on the length of the proof of $N_1, N_2, \ldots, N_m \vdash M$. The following illustrates the (tedious) use of Definition 7 to show soundness of some initial sequents in the basis case of the induction. Such details are then elided in the inductive step and the rest of the paper.[24]

BASIS: We need to show that the initial sequents have the property.

- $A \vdash A$ is an initial sequent. But $\forall\mathfrak{A}((\mathfrak{A} \models A) \Rightarrow (\mathfrak{A} \models A))$ is an instance of a valid form in $\mathtt{subLPQ}^{\perp}_{\Rightarrow}$, so $A \vDash A$.
- $\perp \vdash A$ is an initial sequent. Take an arbitrary interpretation $\mathfrak{A}$ and element $b \in D$ such that $\mathfrak{A} \models \perp[b]$. By Definition 7, there exists t, s, r such that $b \in t(1)$ and $S(1, t, r, s, \mathfrak{A})$. The latter is the formula $s(1) = \langle 7 \rangle \,\&\, t(1) = \{D^r | \perp\}$, so $b \in t(1)$ implies that $b \in \{D | \perp\}$. By Comprehension, $b \in \{D^r | \perp\}$ implies $\perp$ (in the meta-theory). Since $\perp$ implies everything, we get for free that $\mathfrak{A} \models A[b]$ for any A.
- In the following proof we replace contractions by using the following pattern of reasoning: if $A \vdash C$ and $B \vdash D$ then $A \,\&\, B \vdash C \,\&\, D$. This is derivable from applying the right and left conjunction rules. So, on with the induction: $\forall x(B \,\&\, C) \vdash \forall x B \,\&\, \forall x C$ is an initial sequent. We need to show that $\forall x(B \,\&\, C) \models (\forall x B \,\&\, \forall x C)$. Take an arbitrary interpretation $\mathfrak{A}$ and element $b \in D$ such that $\mathfrak{A} \models \forall x(B \,\&\, C)[b]$. By Definition 7 (vii), there exists $i, n \in \mathbb{Z}^+$, a function t and an $l < n$ such that $s(l) = \ulcorner B \,\&\, C \urcorner$ and $b \in \{a \in D | (\forall x \in D)(a(i/x) \in t(l)\}$, so $(\forall x \in D)(b(i/x) \in t(l))$. Take an arbitrary $c \in D$, then $b(i/c) \in t(l) = t(o) \cap t(p)$, for some $o, p < l$ such that $s(o) = \ulcorner B \urcorner$, $s(p) = \ulcorner C \urcorner$ and $s(l) = \langle 3, s(o), s(p) \rangle$. Thus, $b(i/c) \in t(o)$ and $b(i/c) \in t(p)$. Because c was chosen arbitrarily, we get that $(\forall x \in D)(b(i/x) \in t(o))$ and $(\forall x \in D)(b(i/x) \in t(p))$, so $b \in \{a \in D | (\forall x \in D)(a(i/x) \in t(o))\}$ and $b \in \{a \in D | (\forall x \in D)(a(i/x) \in t(p))\}$. Now, let s' and t' be functions such that $s'(x) = s(x)$ and $t'(x) = t(x)$ for every $x < l$. Let $s'(o + 1) = \langle 6, i, s(o) \rangle$

[24] E.g. a standard looking step like

$$\mathfrak{A} \models \forall x A \Rightarrow \mathfrak{A} \models A^x_t$$

is rough shorthand for

$$\mathfrak{A} \models \forall x A[b] \Rightarrow \exists i \in \mathbb{Z}^+ \exists l < k((s(k) = \langle 6, i, s(l) \rangle) \& (b \in \{a \in D \mid (\forall x \in A)(a(i/x) \in t(l))\}))$$

which may be unpacked using Definition 7.

and $s'(p+1) = \langle 6, i, s(p)\rangle$, and let $t'(p+1) = t'(o+1) = \{b\}$. We get that there exists t', s', n', r' such that $b \in t'(n') = t'(o') \cap t'(p')$ for some o', p' with $s'(o') = \ulcorner\forall x B\urcorner$ and $s'(p') = \ulcorner\forall x C\urcorner$, and finally $s'(n') = \langle 3, s'(o'), s'(p')\rangle$. Therefore, $\mathfrak{A} \models (\forall x B \,\&\, \forall x C)[b]$.

The other initial sequents are left as exercises.

INDUCTIVE STEP: Here we have various rules to check the property for.

Case (L$\forall$): Assume that $\Gamma, A^x_t \vDash B$ by inductive hypothesis to show that $\Gamma, \forall x A \vDash B$. Let Γ be $N_1, N_2, \ldots, N_m$. So our assumption consists in that

$$(\forall\mathfrak{A})((\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m\&A^x_t) \Rightarrow (\mathfrak{A} \models B)),$$

where t is a term (i.e. a given variable). We then need to show that

$$(\forall\mathfrak{A})((\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m\&\forall x A) \Rightarrow (\mathfrak{A} \models B)).$$

By the principles of restricted quantifiers, we can take an arbitrary $\mathfrak{A}$ to show that

$$(\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m\&\forall x A) \Rightarrow (\mathfrak{A} \models B),$$

which would follow by a conditional proof. So assume then that $\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m\&\forall x A$, so $\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m$ and $\mathfrak{A} \models \forall x A$, and from the latter, again by restricted quantifiers, $\mathfrak{A} \models A^x_t$. But then we have by conjunction principles (without contraction!) that $\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m$ and $\mathfrak{A} \models \forall x A$ gives that $\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m$ and $\mathfrak{A} \models A^x_t$, and the latter implies that $\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m\&A^x_t$. Applying the inductive hypothesis, $\mathfrak{A} \models B$, and we are done.

Case (L$\neg\forall$): Assume for inductive hypothesis that $\Gamma, \neg A^x_y \vDash B$ to show that $\Gamma, \neg\forall x A \vDash B$. Let Γ be $N_1, N_2, \ldots, N_m$. By assumption:

$$(\forall\mathfrak{A})((\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m\&\neg A^x_y) \Rightarrow (\mathfrak{A} \models B)),$$

where y is not free in Γ. We want to show that

$$(\forall\mathfrak{A})((\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m\&\neg\forall x A) \Rightarrow (\mathfrak{A} \models B)).$$

By the principles of restricted quantifiers, we can take an arbitrary $\mathfrak{A}$ to show that

$$(\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m\&\neg\forall x A) \Rightarrow (\mathfrak{A} \models B).$$

We do a conditional proof: assume that $\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m\&\neg\forall x A$, so $\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m$ and $\mathfrak{A} \models \neg\forall x A$. From the latter, by restricted quantifiers principles, for some $a \in A$, $\mathfrak{A} \not\models A[a]$, so $\mathfrak{A} \not\models A^x_y[a]$, i.e., $\mathfrak{A} \models \neg A^x_y[a]$. But then $\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m\&\neg A^x_y[a]$ (using Lemma 1), and, by inductive hypothesis, $\mathfrak{A} \models B$.

Case (R$\forall$): Assume for inductive hypothesis that $\Gamma \vDash A^x_y$ to show that $\Gamma \vDash \forall x A$. Let Γ be $N_1, N_2, \ldots, N_m$. The assumption consists in that

$$(\forall\mathfrak{A})((\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m) \Rightarrow (\mathfrak{A} \models A^x_y)),$$

where y is not free in Γ. We want to show that

$$(\forall\mathfrak{A})((\mathfrak{A} \models N_1\&N_2\&\ldots\&N_m) \Rightarrow (\mathfrak{A} \models \forall x A).$$

Suppose that for arbitrary $\mathfrak{A}$ we have that $\mathfrak{A} \models N_1 \& N_2 \& \ldots \& N_m$, and, in particular, we have this for any assignment of an element a from $\mathfrak{A}$ as the value of the variable y (since y is not free in Γ), so $\mathfrak{A} \models A[a]$ for any such a by inductive hypothesis, which gives us that $\mathfrak{A} \models \forall x A$ as desired.

Other cases are omitted. □

4 Proving the Big Three

Now for our main order of business. Our strategy will be to carefully follow and modify the completeness argument from [7, Thm. 440], identifying the logical and mathematical resources required for our version of the proof. The argument is originally due to Gödel [16] (with a few modifications) but the presentation in [7] (which corrects the first textbook proof given in [17]) stands out for its clarity. Furthermore, we will be interested in other applications of the same proof method (and its dual version). Gödel's approach originates with Löwenheim [22] and Skolem [43]: his proof rehashes the original argument of the Löwenheim-Skolem theorem. Appropriately, then, each of the Big Three will be established below using variations on the same argument.

In this section, we let A be a formula in SNF,

$$\exists a_1, \ldots, a_m \forall b_1, \ldots, b_n M,$$

where M is the quantifierless matrix of A in the free variables $a_1, \ldots, a_m, b_1, \ldots, b_n$ (now allowing a, b, c, ... to be variables rather than constants). Recall that $\vdash A$ means the sequent $\emptyset \vdash A$ is provable in the system $\texttt{subLPQ}^{\perp}$.

4.1 Gödel's Completeness

The general strategy of this kind of proof is to reduce first-order problems about $\vdash A$ and $\nvDash A$ to propositional problems. For each A we obtain a list of propositional formulas (i.e., no quantifiers involved) $\langle C_k \mid k \in \mathbb{Z}^+ \rangle$ such that (1) if any C_k were provable, A would be too, and that, furthermore, (2) if none of the C_k were valid, then A would not be valid either, and, in particular, that it would have a *countermodel* in the domain $\mathbb{Z}^+$. The trick will be to make the formulas $\langle C_k \mid k \in \mathbb{Z}^+ \rangle$ be all the possible instantiations of the quantifiers $\exists a_1, \ldots, a_m \forall b_1, \ldots, b_n$ in the domain $\mathbb{Z}^+$.

Since the first stream of quantifiers is a block of m existentials, begin by finding a way to organize all the possible values for these variables. Hence, for each m, define an ordering $<^*$ on all the m-tuples of positive integers according to increasing sums, and lexicographically within each group having the same index sum. More precisely,

$$\begin{array}{ll} \langle i_1, \ldots, i_m \rangle <^* \langle j_1, \ldots, j_m \rangle & \text{if } \; i_1 + i_2 + \cdots + i_m < j_1 + j_2 + \cdots + j_m \\ & \text{or } \; i_1 + i_2 + \cdots + i_m = j_1 + j_2 + \cdots + j_m \\ & \text{and } \; \exists k < m, i_1 = j_1, \ldots, i_k = j_k, i_{k+1} < j_{k+1}. \end{array}$$

Then for the sequences of m-tuples, our ordering should start:

$$\langle 1, 1, \ldots, 1 \rangle, \langle 1, 1, \ldots, 2 \rangle, \langle 1, 1, \ldots, 2, 1 \rangle, \langle 1, 1, \ldots, 2, 1, 1 \rangle, \ldots$$

The notation $[kl]$ is for the lth integer in the kth m-tuple according to the ordering $<^*$. No integer in the kth m-tuple is greater than k on pain of absurdity: it would imply that $i = i + 1$ for some integer i, which by Principle 2 cannot be.

Next, for any $k \geqslant 1$, define B_k (using vertical notation to show substitutions, i.e. $M\binom{a}{x}$ for $M(a/x)$) as

$$M\begin{pmatrix} a_1 & a_2 & \ldots & a_m & b_1 & b_2 & \ldots & b_n \\ x_{[k1]} & x_{[k2]} & \ldots & x_{[km]} & x_{(k-1)n+2} & x_{(k-1)n+3} & \ldots & x_{kn+1} \end{pmatrix}.$$

Each B_k is a substitution instance of the quantifierless matrix M. By the way we have defined things, the lists of variables $x_{(k-1)n+2}, x_{(k-1)n+3}, \ldots, x_{kn+1}$ are pairwise different for different k. The numbering is chosen so that the variables that are substituted for the bs are not the same as the variables that are substituted for the as, and also that they are all different among themselves.

Furthermore, C_k will be defined as

$$B_1 \vee B_2 \vee \cdots \vee B_k,$$

and, finally, we let D_k be

$$\forall x_1, \ldots, x_{kn+1} C_k.$$

To simplify notation where possible, runs a_1, ..., a_m will be denoted $\overline{a_m}$.

Lemma 3 *The sequent $D_k \vdash A$ is provable for all k.*

Proof By our coding into Gödel sets, if y is one of

$$a_1, a_2, \ldots, a_m, b_1, \ldots, b_n$$

and z one of $x_1, x_2, x_3, \ldots$, then $y = z$ implies triviality ($\perp$). The proof of the Lemma proceeds by Principle 3 on the number k.

Basis: $k = 1$. The sequent

$$\forall x_1 \forall b_1, \ldots, b_n M(\overline{a_m}/x_1) \vdash \exists a_1, \ldots, a_m \forall b_1, \ldots, b_n M(\overline{a_m}/x_1)$$

is derivable by one application of (L$\forall$) followed by m applications of (R$\exists$).

Inductive Step: $k + 1$. Assume the claim for k, i.e., that the following is provable:

$$D_k \vdash A.$$

Then we establish it for $k + 1$ (i.e., $D_{k+1} \vdash A$).

First, by multiple applications of (Cut) and the initial sequent (Distribution of $\forall$ over $\vee$),

$$\forall x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1}(C_k \vee B_{k+1}) \vdash C_k \vee \forall x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1} B_{k+1}$$

is a provable sequent. C_{k+1} is by definition $C_k \vee B_{k+1}$ and D_{k+1} is $\forall x_1, \ldots, x_{(k+1)n+1} C_{k+1}$. Using these facts with multiple applications of (L$\forall$) and (Cut) the following sequent is provable:

$$D_{k+1} \vdash C_k \vee \forall x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1} B_{k+1}.$$

By multiple applications of the following derivation (on the hypothesis that $(B^y_x)^x_y = B$—making sure that x is free for substitution in B—and observing that $(B^y_x)^x_x = B^y_x$) and (Cut),

$$\dfrac{\dfrac{A \vdash A}{A \vdash A \vee \forall y B}\,(R\vee) \qquad \dfrac{\dfrac{\dfrac{\dfrac{B^y_x \vdash B^y_x}{(B^y_x)^x_x \vdash B^y_x}}{\forall x(B^y_x) \vdash B^y_x}\,(L\forall)}{\forall x(B^y_x) \vdash \forall y B}\,R\forall}{\forall x(B^y_x) \vdash A \vee \forall y B}\,(R\vee)}{A \vee \forall x(B^y_x) \vdash A \vee \forall y B}\,(L\vee)$$

we obtain the provability of the sequent:

$$C_k \vee \forall x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1} B_{k+1} \vdash C_k \vee \forall \overline{b_n} M(\overline{a_m}/\overline{x_{[(k+1)m]}})$$

Once more by (Cut),

$$D_{k+1} \vdash C_k \vee \forall \overline{b_n} M(\overline{a_m}/\overline{x_{[(k+1)m]}}).$$

Now, by multiple applications of (R∃),

$$\forall \overline{b_n} M(\overline{a_m}/\overline{x_{[(k+1)m]}}) \vdash A.$$

So we have the derivation

$$\dfrac{\dfrac{C_k \vdash C_k}{C_k \vdash C_k \vee A}\,(R\vee) \qquad \dfrac{\forall \overline{b_n} M(\overline{a_m}/\overline{x_{[(k+1)m]}}) \vdash A}{\forall \overline{b_n} M(\overline{a_m}/\overline{x_{[(k+1)m]}}) \vdash C_k \vee A}\,(R\vee)}{C_k \vee \forall \overline{b_n} M(\overline{a_m}/\overline{x_{[(k+1)m]}}) \vdash C_k \vee A}\,(L\vee)$$

and then $D_{k+1} \vdash C_k \vee A$ is provable by (Cut) since we also have that $D_{k+1} \vdash C_k \vee \forall \overline{b_n} M(\overline{a_m}/\overline{x_{[(k+1)m]}})$. By (R∀), the initial sequent (Distribution of ∀ over ∨) and (Cut),

$$D_{k+1} \vdash D_k \vee A.$$

By inductive hypothesis (which is used only once!), we know that

$$D_k \vdash A,$$

so we obtain the derivation:

$$\dfrac{D_{k+1} \vdash D_k \vee A \qquad \dfrac{D_k \vdash A \qquad A \vdash A}{D_k \vee A \vdash A}\,(L\vee)}{D_{k+1} \vdash A.}\,(Cut)$$

□

Now for (Gödel) completeness:

Theorem 4 (Gödel Completeness) *Either* $\not\models A$ *or* $\vdash A$.

Proof The argument follows an application of (Law of Excluded Middle) and (L∨) in the meta-theory:

Case (1): There is a $k \in \mathbb{Z}^+$ such that C_k is valid $((\exists k \in \mathbb{Z}^+)(\vDash C_k))$.
Case (2): There is no $k \in \mathbb{Z}^+$ such that C_k is valid $(\neg(\exists k \in \mathbb{Z}^+)(\vDash C_k)$, i.e., $(\forall k \in \mathbb{Z}^+)(\nvDash C_k))$.

From Case (1), and, by (Kalmar's Completeness of Propositional $\texttt{SubLP}^{\perp}$) [47, Theorem 4], $\vdash C_k$, and, by (R$\forall$), $\vdash D_k$. From Lemma 3 and (Cut), $\vdash A$, as desired.

Our next goal is to show that $\nvDash A$ when Case (2) holds. From Case (2), for each C_k, we can find some assignment of truth-values to the atomic subformulas of C_k such that the truth-value of C_k, when calculated by the truth tables for the propositional connectives, is at least f (i.e., a *falsifying assignment*). The collection of all falsifying assignments for C_k will be denoted by S_k.

By Principle 5, we let $E_1, E_2, E_3, \ldots$ be an enumeration, without repetitions, of the atomic formulas in $C_1, C_2, C_3, \ldots$, in the order that they appear in the latter enumeration (each C_i contains finitely many atomic formulas). Our strategy will consist in making a "master" assignment of truth-values, ϕ, to $E_1, E_2, E_3, \ldots$ from which we can define a first-order interpretation $\mathfrak{A}$, with the natural numbers $\mathbb{Z}^+$ as domain, where $\mathfrak{A} \not\models A$.

We proceed by Principle 4 on n to define the values of E_n by ϕ. The basis is $n = 1$. By (Law of Excluded Middle), either

(i) some infinite subset of $\bigcup_{k\in\mathbb{Z}^+} S_k$ is s.t. each of its elements gives E_1 at least the value t, or
(ii) no infinite subset of $\bigcup_{k\in\mathbb{Z}^+} S_k$ is s.t. each of its elements gives E_1 at least the value t.

Note that if (ii), every infinite subset of the enumeration $\bigcup_{k\in\mathbb{Z}^+} S_k$ is such that at least one of its elements gives E_1 at least the value f (since it does not give it the value t (though perhaps it also does!)). So we define $\langle E_1, \mathsf{x}\rangle \in \phi$ (where $\mathsf{x} = \mathsf{t}$ or $\mathsf{x} = \mathsf{f}$) by cases as follows:

$$E_1\phi \begin{cases} \mathsf{t} \text{ if (i) holds,} \\ \mathsf{f} \text{ if (ii) holds.} \end{cases}$$

Conditions (i) and (ii) might hold simultaneously, in which case our definition assigns both t and f to E_1 by ϕ.

Next we assign truth-values to E_{n+1} after we have assigned them to $E_1, \ldots, E_n$ as follows. We consider the subset $S^{E_1,\ldots,E_n}$ of $\bigcup_{k\in\mathbb{Z}^+} S_k$ of all assignments that give $E_1, \ldots, E_n$ the same truth-values as they have according to the master assignment ϕ. Then, by (Law of Excluded Middle), either

(i) some infinite subset of $S^{E_1,\ldots,E_n}$ is s.t. each of its elements gives E_{n+1} at least the value t, or
(ii) no infinite subset of $S^{E_1,\ldots,E_n}$ is s.t. each of its elements gives E_{n+1} at least the value t.

So we set again:

$$E_{n+1}\phi \begin{cases} \mathsf{t} \text{ if (i) holds,} \\ \mathsf{f} \text{ if (ii) holds.} \end{cases}$$

We show by Principle 3 that for any k, C_k gets value at least f according to ϕ. The basis of the induction is $n = 1$, so we need to show the claim for C_1, which by construction is only the formula B_1. All the atomic formulas in B_1 must appear in some section of the list $E_1, E_2, E_3, \ldots,$ say in, $E_1, E_2, \ldots, E_l$. Observe that any two assignments that give the same truth-values to a list of atomic formulas will give the same truth-values to any $\mathtt{SubLP}^{\perp}$-formula. An induction on formula complexity suffices to establish this. Now, when assigning truth-values to E_l according to ϕ, we restricted attention to only those falsifying assignments of B_1 that had the same truth-values assigned to $E_1, E_2, \ldots, E_{l-1}$ as ϕ had. Indeed, for such assignments either E_l would get t in infinitely many of them, or it would get f in all infinite subsets of them.[25] Hence, in either case, B_1 must take at least the value f according to ϕ. Now, for the inductive step, assume that C_k gets value at least f according to ϕ, and we show that C_{k+1} also does. Since C_{k+1} is defined as $C_k \vee B_{k+1}$, by the truth conditions for $\vee$ and the inductive hypothesis (used only once!), it suffices to show that B_{k+1} also gets value at least f according to ϕ. This follows as in the basis of the induction.

Now, to construct the interpretation $\mathfrak{A}$, with the natural numbers $\mathbb{Z}^+$ as domain using ϕ we proceed as follows. We define the interpretation function I of $\mathfrak{A}$, which assigns to each relation symbol R^n of τ a relation $\mathbb{Z}^{+n} \times \{\mathsf{t}, \mathsf{f}\}$. We let $I(R^n)$ be

$$\{\langle\langle u_1, \ldots, u_n\rangle, \mathsf{x}\rangle \mid \mathsf{x} \in \{\mathsf{t}, \mathsf{f}\}, Rx_{u_1}, \ldots, x_{u_n}\phi\mathsf{x}, \langle u_1, \ldots, u_n\rangle \in \mathbb{Z}^{+n}\}$$

and, for any variable x_u, $I(x_u) = u$. The interpretation I simply uses ϕ as a guide to compute the truth-value of the claim 'tuple $\langle u_1, \ldots, u_n\rangle$ stands in the relation R^n'

For any k, in this model $\mathfrak{A}$, B_k gets truth-value f, which means that for any k-tuple $\langle[k1], [k2], \ldots, [km]\rangle$,

$$\mathfrak{A} \not\models \forall b_1, \ldots, b_n B_k^{x_{(k-1)n+2}/b_1, x_{(k-1)n+3}/b_2, \ldots, x_{[kn+1]}/b_n}[[k1], [k2], \ldots, [km]],$$

i.e.,

$$\mathfrak{A} \not\models \forall b_1, \ldots, b_n M^{a_1/x_{[k1]}, a_2/x_{[k2]}, \ldots, a_m/x_{[km]}}[[k1], [k2], \ldots, [km]].$$

The latter implies that

$$\mathfrak{A} \not\models \exists a_1, \ldots, a_m \forall b_1, \ldots, b_n M,$$

as desired. □

An erudite reader might point out that in [16, theorem II], the formulation of completeness is actually as follows:

Definition 10 A logic L is *Gödel complete*$_0$ if for any formula A which is the negation of a formula in SNF, either A is *refutable* ($\vdash \neg A$) or *satisfiable* (it has a model).

So, *prima facie*, it appears like our formulation is not faithful to the name it bears. However, since $\neg\neg A$ and A are interdeducible classically as well as in $\mathtt{subLPQ}^{\perp}$, Gödel completeness and Gödel completeness$_0$ are, in fact, equivalent:

[25] If E_l getting both t and f in infinitely many such assignments would be trivial, then there would be infinitely many such assignments where it would get simply t (it would be trivial that they get f) or infinitely many such assignments where it would get simply f (it would be trivial that they get t).

Lemma 5 *A logic L is Gödel complete$_0$ $\Leftrightarrow$ L is Gödel complete.*

Proof To prove this, one only needs conditional proof and proof by cases. In the first direction, assume that every A in SNF is either a theorem or invalid. Consider $\neg B$, the negation of a formula B in SNF. By Definition 1, either $\not\models B$ or $\vdash B$. In the latter case $\vdash \neg\neg B$, by double-negation introduction, so $\neg B$ is refutable. In the former case, by definition and using the behaviour of our restricted quantifiers, $\neg B$ is satisfiable. Conversely, assume that every A which is the negation of a formula in SNF is either refutable or satisfiable. Now take an arbitrary B in SNF, then $\neg B$ is the negation of a formula in SNF. By Definition 2, either $\vdash \neg\neg B$ (so $\vdash B$) or $\neg B$ is satisfiable ($\not\models B$). □

Corollary 6 *subLPQ$^{\perp}$ is Gödel complete$_0$.*

4.2 Löwenheim-Skolem Theorem

Modifying the argument for completeness leads to an independent proof of a Löwenheim-Skolem theorem. This result could be easily obtained as a corollary of Theorem 4 together with soundness; however, the proof below has the virtue of introducing a simpler case, used to establish Theorem 9, which cannot be obtained as a consequence of Theorem 4, even in the classical case. Recall that Theorem 4, classically, is sometimes called "weak completeness", and various logics have this property without being compact.

Theorem 7 (Löwenheim-Skolem Theorem) *Either (i) A is not true in some enumerable domain or (ii) A is true in every non-empty domain.*

Proof This follows using meta-theoretic (L$\vee$) and (Law of Excluded Middle), following the proof technique of Theorem 4. For any $k \in \mathbb{Z}^+$, A^k will denote the conjunction (&) of k-many copies of A, whereas $\bigvee_k A$ will be the disjunction ($\vee$) of k-many copies of A.

By (Law of Excluded Middle), either

Case (1): for some k, $\bigvee_k A$ is true in every non-empty domain; or
Case (2): for every k, $\bigvee_k A$ is not true in some non-empty domain.

Since

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad A \vdash A}{A \vee A \vdash A}(L\vee)}{\vdots} \qquad A \vdash A}{\bigvee_{k-1} A \vdash A}(L\vee) \qquad A \vdash A}{(\bigvee_{k-1} A) \vee A \vdash A}(L\vee)}{}$$

we have that $\bigvee_k A \vdash A$ is provable, and, hence, when (1) holds, then A is true in every non-empty domain by soundness. If (2) holds, on the other hand, if we can establish that

(∗) for every k, $(\neg A)^k \vdash \neg D_k$ is provable,

then we would have essentially Case (2) of Theorem 4, and we can construct the interpretation showing that A is not true in $\mathbb{Z}^+$ as in the proof of that theorem. So all our work consists in establishing that (∗) holds. As before, the proof proceeds by Principle 3 on the number k.

First, observe that

$$\neg A \vdash \forall a_1, \ldots, a_m \exists b_1, \ldots, b_n \neg M$$

is provable by first alternating applications of (R∃) and (L¬∀) n times, starting with the sequent $\neg M \vdash \neg M$, and then alternating applications of (L¬∃) and (R∀) m times.

Second observe that

$$\forall a_1, \ldots, a_m \exists b_1, \ldots, b_n \neg M \vdash \neg A$$

is provable by a dual procedure, first alternating applications of (R¬∀) and L∃, and then (L∀) and (R¬∃). Moreover, the sequents

$$\neg D_k \vdash \exists x_1, \ldots, x_{kn+1} \mathop{\&}_{i=1}^{k} (\neg B_i)$$

and

$$\exists x_1, \ldots, x_{kn+1} \mathop{\&}_{i=1}^{k} (\neg B_i) \vdash \neg D_k$$

are also both provable by similar methods. Now, for the proof of (∗):

Basis: $k = 1$. We observe that

$$\neg A \vdash \exists x_1 \exists \overline{b_n} (\neg M(\overline{a_m}/x_1))$$

is provable by m applications of (L∀) to the sequent $\exists \overline{b_n}(\neg M(\overline{a_m}/x_1) \vdash \exists \overline{b_n}(\neg M(\overline{a_m}/x_1)$, followed by one application of (R∃).

Inductive Step: $k + 1$. Assume the claim for k, i.e.,

$$(\neg A)^k \vdash \neg D_k$$

is provable and we establish it for $k + 1$ ($(\neg A)^{k+1} \vdash \neg D_{k+1}$ is provable). First, applying the derivations (here we use A and B schematically where no confusion should arise),

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B(y) \vdash B(y)}{A, B(y) \vdash A \& B(y))}\,(R\&)}{A, B(y) \vdash \exists x (A \& B(x))}\,(R\exists)}{A, \exists x B(x) \vdash \exists x (A \& B(x))}\,(L\exists)}{A \& \exists x B(x) \vdash \exists x (A \& B(x))}\,(L\&)$$

and

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad \dfrac{B(y) \vdash B(y)}{B(y) \vdash \exists x\, B(x)}\,(R\exists)}{A, B(y) \vdash A\&\exists x\, B(x)}\,(R\&)}{A\&B(y) \vdash A\&\exists x\, B(x)}\,(L\&)}{\exists x(A\&B(x)) \vdash A\&\exists x\, B(x)}\,(L\exists)$$

and various applications of (Cut), the following sequent is provable:

$$(\neg C_k)\&\exists x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1}(\neg B_{k+1})$$
$$\vdash \exists x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1}((\neg C_k)\&(\neg B_{k+1})).$$

We also have the derivation

$$\dfrac{\dfrac{\dfrac{\vdots}{\dfrac{\dfrac{\dfrac{\dfrac{\neg C_k \vdash \neg C_k \qquad \neg B_{k+1} \vdash \neg B_{k+1}}{(\neg C_k), (\neg B_{k+1}) \vdash \neg(C_k \vee B_{k+1})}\,\text{(R de Morgan)}}{(\neg C_k)\&(\neg B_{k+1}) \vdash \neg(C_k \vee B_{k+1})}\,(L\&)}{(\neg C_k)\&(\neg B_{k+1}) \vdash \neg C_{k+1}}\,\text{Def of } C_{k+1}}{\vdots}\,(R\exists)}}{(\neg C_k)\&(\neg B_{k+1}) \vdash \exists x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1}\neg C_{k+1}}\,(R\exists)}{\vdots}\,(L\exists)}{\exists x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1}((\neg C_k)\&(\neg B_{k+1})) \vdash \exists x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1}\neg C_{k+1}}\,(L\exists)$$

It is a similar exercise to derive the sequent

$$\exists x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1}\neg C_{k+1} \vdash \neg D_{k+1}.$$

By another application of (Cut),

$$\exists x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1}((\neg C_k)\&(\neg B_{k+1})) \vdash \neg D_{k+1}$$

is provable. Hence, by (Cut) again,

$$(\neg C_k)\&\exists x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1}(\neg B_{k+1}) \vdash \neg D_{k+1}.$$

is also provable. Again very much as in the proof of completeness, by multiple applications of the following derivation (on the hypothesis that $(B^y_x)^x_y = B$—making sure that x is free for substitution in B—and observing that $(B^y_x)^x_x = B^y_x$) and (Cut),

$$\dfrac{\dfrac{A \vdash A \qquad \dfrac{\dfrac{B^y_x \vdash B^y_x}{B^y_x \vdash \exists y B}\,(R\exists)}{\exists x(B^y_x) \vdash \exists y B}\,(L\exists)}{A, \exists x(B^y_x) \vdash A\&\exists y B}\,(R\&)}{A\&\exists x(B^y_x) \vdash A\&\exists y B}\,(L\&)$$

we obtain the provability of the following sequent:

$$(\neg C_k)\&\exists\overline{b_n}(\neg M(\overline{a_m}/\overline{x_{(k+1)i}})) \vdash (\neg C_k)\&\exists x_{kn+2}, x_{kn+3}, \ldots, x_{(k+1)n+1}(\neg B_{k+1}).$$

All we have done here is change bound variables. Another application of (Cut) returns

$$(\neg C_k)\&\exists\overline{b_n}(\neg M(\overline{a_m}/\overline{x_{(k+1)i}})) \vdash \neg D_{k+1}.$$

By multiple applications of (L∀), $\forall\overline{a_m}\exists\overline{b_n}(\neg M) \vdash \exists\overline{b_n}(\neg M(\overline{a_m}/\overline{x_{(k+1)i}}))$ is provable. Using (Cut), $\neg A \vdash \exists\overline{b_n}(\neg M(\overline{a_m}/\overline{x_{(k+1)i}}))$ is provable. But then we have the following derivation:

$$\frac{\neg C_k \vdash \neg C_k \qquad \neg A \vdash \exists\overline{b_n}(\neg M(\overline{a_m}/\overline{x_{(k+1)i}}))}{(\neg C_k), \neg A \vdash (\neg C_k)\&\exists\overline{b_n}(\neg M(\overline{a_m}/\overline{x_{(k+1)i}}))}\ (R\&)$$

and then, by (Cut),

$$(\neg C_k), \neg A \vdash \neg D_{k+1}$$

is provable. We have the following derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{(\neg C_k), \neg A \vdash \neg D_{k+1}}{(\neg\forall x_{kn+1} C_k), \neg A \vdash \neg D_{k+1}}\ (L\neg\forall)}{\vdots}}{\cfrac{(\neg\forall x_2, \ldots, x_{kn+1} C_k), \neg A \vdash \neg D_{k+1}}{(\neg\forall x_1, \ldots, x_{kn+1} C_k), \neg A \vdash \neg D_{k+1}}\ (L\neg\forall)}}{\cfrac{(\neg D_k), \neg A \vdash \neg D_{k+1}}{(\neg D_k)\&\neg A \vdash \neg D_{k+1}}\ (L\&)}\ \text{Def of } D_k$$

Our inductive hypothesis (which will be used only once!) is

$$\neg A^k \vdash \neg D_k,$$

which we can use in our final derivation:

$$\cfrac{\cfrac{\neg A^k \vdash (\neg D_k) \qquad \neg A \vdash \neg A}{\neg A^k, \neg A \vdash (\neg D_k)\&\neg A}\ (R\&)}{\neg A^k\&\neg A \vdash (\neg D_k)\&\neg A}\ (L\&)$$

Finally, by (Cut),

$$\neg A^{k+1} \vdash \neg D_{k+1}$$

is provable. □

4.3 Compactness

To show compactness, we introduce an equivalent notion, that of *dual* compactness. The following definition comes from [14], plus the restriction to formulas in Skolem normal form:

Definition 11 A logic L is *dually countably compact* iff, for every denumerable set X of formulas in SNF, either some finite subset of X is not jointly falsifiable, or else X is jointly falsifiable.

Lemma 8 *Dual compactness is equivalent (in the sense of* $\Leftrightarrow$*) to compactness as stated in Definition* 4.

Proof Suppose that L is dually countably compact. Let X be a countable set of formulas in negated SNF. Take $X^* = \{A \mid \neg A \in X\}$. By assumption, either (1) some finite subset of X^* is not jointly falsifiable, or else (2) X^* is jointly falsifiable. If (1) holds, then some finite subset of X is not jointly satisfiable, and if (2) holds, then X is not jointly satisfiable. On the other hand, suppose that L is countably compact. Let X be a countable set of formulas in SNF. Take $X' = \{\neg A \mid A \in X\}$. By assumption, either (1) some finite subset of X' is not jointly satisfiable or else (2) X' is jointly satisfiable. If (1), then some finite subset of X is not jointly falsifiable, whereas if (2), X is jointly falsifiable. □

The (dual) compactness theorem for countable sets of sentences can be obtained by the same method for establishing the Löwenheim-Skolem result. [26]

Theorem 9 *Let X be a set of formulas enumerated with positive integers. For every denumerable set X of formulas in SNF, either some finite subset of X is not jointly falsifiable, or X is jointly falsifiable.*

Proof Start by displaying the formulas in X, i.e. those listed in the sequence $A_1, A_2, A_3, \ldots,$ as:

$$\begin{gathered}
\exists a_{1_1}, \ldots, a_{m_1} \forall b_{1_1}, \ldots, b_{n_1} M_1 \\
\exists a_{1_2}, \ldots, a_{o_2} \forall b_{1_2}, \ldots, b_{p_2} M_2 \\
\vdots \\
\exists a_{1_j}, \ldots, a_{q_j} \forall b_{1_j}, \ldots, b_{r_j} M_j \\
\vdots
\end{gathered}$$

[26] The logical resources required for obtaining completeness (Theorem 4) are a bit less than those needed for Theorem 9 below. Theorem 4 does not require as many duality principles between ∃ and ∀ as Theorem 9 do. So if we had been only interested in Theorem 4 some of that duality could have been dropped.

We may always assume that none of the formulas in X use the same variables on pain of triviality. Next we define, for any $k, j \geqslant 1$, B_{k_j} as

$$M_j \begin{pmatrix} a_{1_j} & a_{2_j} & \dots & a_{q_j} & b_{1_j} & b_{2_j} & \dots & b_{r_j} \\ x_{[k1]} & x_{[k2]} & \dots & x_{[kq_j]} & x_{(k-1)r_j+2} & x_{(k-1)r_j+3} & \dots & x_{kr_j+1} \end{pmatrix}$$

Furthermore, C_k will be defined inductively as

$$C_{k-1} \vee (B_{k_1} \vee B_{k-1_2} \vee \dots \vee B_{1_k}).$$

We then obtain a sequence of formulas following the pattern:

$$C_1 := M_1 \begin{pmatrix} a_{1_1} & a_{2_1} & \dots & a_{m_1} & b_{1_1} & b_{2_1} & \dots & b_{n_1} \\ x_1 & x_1 & \dots & x_1 & x_2 & x_3 & \dots & x_{n_1+1} \end{pmatrix}$$

$$C_2 := C_1 \vee (M_1 \begin{pmatrix} a_{1_1} & a_{2_1} & \dots & a_{m_1} & b_{1_1} & b_{2_1} & \dots & b_{n_1} \\ x_1 & x_1 & \dots & x_2 & x_{n_1+2} & x_{n_1+3} & \dots & x_{2n_1+1} \end{pmatrix}$$

$$\vee M_2 \begin{pmatrix} a_{1_2} & a_{2_2} & \dots & a_{o_2} & b_{1_2} & b_{2_2} & \dots & b_{p_2} \\ x_1 & x_1 & \dots & x_1 & x_2 & x_3 & \dots & x_{p_2+1} \end{pmatrix})$$

$$\vdots$$

Finally, we let D_k be the universal closure of the formula C_k. By metatheoretic (Law of Excluded Middle), either

(1) for some k, and every non-empty domain,

$$\neg A_1 \& \dots \& (\neg A_1 \& \dots \& \neg A_{k-1}) \& (\neg A_1 \& \dots \& \neg A_k)$$

is falsified, or

(2) for every k,

$$\neg A_1 \& \dots \& (\neg A_1 \& \dots \& \neg A_{k-1}) \& (\neg A_1 \& \dots \& \neg A_k)$$

is satisfiable in some non-empty domain.

We argue by meta-theoretic (L$\vee$). If (1) holds, some finite subset of X is not jointly falsifiable (in every model, one of its members is true, but as usual this doesn't preclude the possibility that it is also false), and there is nothing to prove by meta-theoretic (R$\vee$). So we focus on showing that if (2) holds, X is jointly falsifiable. The idea is to extend the construction technique in Theorem 7 to handle now a denumerable list of formulas simultaneously.

Hence, we wish to establish that

(**) for every k, $\neg A_1 \& \dots \& (\neg A_1 \& \dots \& \neg A_{k-1}) \& (\neg A_1 \& \dots \& \neg A_k) \vdash \neg D_k$.

is provable. Once more, the proof proceeds by Principle 3 on the number k.

Basis: $k = 1$. Exactly as in the proof of the corresponding claim in the Löwenheim-Skolem theorem, we obtain that

$$\neg A_1 \vdash \exists x_1 \exists \overline{b_n} (\neg M(\overline{a_{m_1}}/x_1))$$

which, by the sequent obtained one paragraph before in the mentioned proof, change of variables and (Cut), gives that

$$\neg A_1 \vdash \neg D_1$$

is provable.

INDUCTIVE STEP: $k+1$. Next assume the claim for k, i.e.,

$$\neg A_1 \& \ldots \& (\neg A_1 \& \ldots \& \neg A_{k-1}) \& (\neg A_1 \& \ldots \& \neg A_k) \vdash \neg D_k,$$

is provable and we establish it for $k+1$:

$$\neg A_1 \& \ldots \& (\neg A_1 \& \ldots \& \neg A_k) \& (\neg A_1 \& \ldots \& \neg A_{k+1}) \vdash \neg D_{k+1}$$

is provable.

Reasoning almost exactly as in the corresponding part of the proof of the Löwenheim-Skolem theorem,

$$\begin{aligned}(\neg A_1 \& \ldots \& \neg A_{k+1}) \vdash\ & (\exists b_{1_1}, b_2, \ldots, b_{n_1} \neg M_1 \left(\begin{smallmatrix} a_{1_1} & a_{2_1} & \cdots & a_{m_1} \\ x_{[k1]} & x_{[k2]} & \ldots & x_{[k+1m_1]} \end{smallmatrix}\right)) \\ & \& \ldots \& (\exists b_{1_{k+1}}, b_{2_{k+1}}, \ldots, b_{r_{k+1}} \neg M_{k+1} \left(\begin{smallmatrix} a_{1_{k+1}} & a_{2_{k+1}} & \cdots & a_{q_{k+1}} \\ x_1 & x_1 & \ldots & x_1 \end{smallmatrix}\right))\end{aligned}$$

is provable, and then,

$$\begin{aligned}(\neg C_k) \& (\neg A_1 \& \ldots \& \neg A_{k+1}) \vdash\ & (\neg C_k) \& (\exists b_{1_1}, b_2, \ldots, b_{n_1} \neg M_1 \left(\begin{smallmatrix} a_{1_1} & a_{2_1} & \cdots & a_{m_1} \\ x_{[k1]} & x_{[k2]} & \ldots & x_{[k+1m_1]} \end{smallmatrix}\right)) \\ & \& \ldots \& (\exists b_{1_{k+1}}, b_{2_{k+1}}, \ldots, b_{r_{k+1}} \neg M_{k+1} \left(\begin{smallmatrix} a_{1_{k+1}} & a_{2_{k+1}} & \cdots & a_{q_{k+1}} \\ x_1 & x_1 & \ldots & x_1 \end{smallmatrix}\right))\end{aligned}$$

is also provable. Thus

$$(\neg C_k) \& (\neg A_1 \& \ldots \& \neg A_{k+1}) \vdash \neg D_{k+1}$$

is provable, so

$$(\neg D_k) \& (\neg A_1 \& \ldots \& \neg A_{k+1}) \vdash \neg D_{k+1}$$

is provable. By inductive hypothesis (applied only once!), we know that

$$\neg A_1 \& \ldots \& (\neg A_1 \& \ldots \& \neg A_{k-1}) \& (\neg A_1 \& \ldots \& \neg A_k) \vdash \neg D_k,$$

is provable and, since

$$(\neg A_1 \& \ldots \& \neg A_{k+1}) \vdash \neg A_{k+1}$$

is provable, we obtain the sequent

$$\neg A_1 \& \ldots \& (\neg A_1 \& \ldots \& \neg A_k) \& (\neg A_1 \& \ldots \& \neg A_{k+1}) \vdash (\neg D_k) \& (\neg A_{k+1}).$$

Finally, by (Cut),

$$\neg A_1 \& \ldots \& (\neg A_1 \& \ldots \& \neg A_k) \& (\neg A_1 \& \ldots \& \neg A_{k+1}) \vdash \neg D_{k+1}$$

is provable. □

5 Discussion

This paper has specific results, and a general one. The specific results are (qualified) versions of the Big Three for quantified substructural LP. The general result is that a substructural paraconsistent framework has the potential to do its own metatheoretic 'heavy lifting'. The paraconsistent logician can, given some assumptions, 'recapture' important theorems that are also proved using classical logic. This can be done *without* the assumption of global consistency.

This is important in itself, as many paraconsistent attempts to recapture results up to now have been not encouraging.[27] The approach we use suggests that some version of a recapture is still possible, and *explains* why some previous efforts fell short. Previous efforts attempted to take highly-developed *models* of various theories, fashioned after decades of group effort, and to tweak those high-tech models to try to get all classical results 'for free'. But, obviously, this is not how the classical results were obtained. Classical results were obtained through a lengthy process of reasoning from basic building blocks and first principles; only later did more abstract (and more powerful) notions get developed. A real 're'-capture for a non-classical approach will need to do the same thing—not try to skip to the end of a long process done by others based on other (disputed) methods and ideas, but rather to *capture* results (maybe new or different ones) afresh: new proofs with old tools.

What does all of this 'honest toil' prove? One objection might be that we didn't really prove completeness, on two counts. First, our theorems are only for the fragment of the language that can be brought into Skolem normal form. Second, completeness is a conditional—'if A is a tautology, then it is derivable'—whereas we proved a disjunction, and in a paraconsistent setting, that's weaker. Our broad reply on both counts is that what we proved is *classically equivalent to the classical theorem*. But this reply raises as many questions as it answers. To unpack it, let's view it from three different perspectives.

One fairly simple view is that, from the classical standpoint, our restricted theorems and the classical versions thereof *are* the same. On this view, we agree on the completeness of e.g. `LP`, because we agree that, for all the sentences in SNF, either they are invalid, or else provable. The difference lies in what *further* commitments one makes, e.g that all sentences have an SNF, or what further reading one wants to give statements of the form 'not p, or else q'. Going on to say that, if a sentence is not provable, it has a counterexample, or that any sentence with no counterexample is provable, is from our perspective going on to say significantly *more*. Löwenheim and Gödel gave arguments that clearly show the material versions, and we have too. So we do say something at odds with the *interpretation* of the Big Three: they are just disjunctions. But that is only a meaningful distinction from our non-classical standpoint.

Another view is that things are more complicated and nuanced than the simpler view suggests. Contra the simple view, it could be fair to say that, from either the classical or paraconsistent standpoint, just because our results are 'classically equivalent' does not change the fact that our results are only partial. After all, we admit that we do not think that our theorems are conditionals or that they apply to the entire language, and everyone else knows that we think so. The above response requires a kind of stereotyped Quinean inability to understand the meaning of non-classical

[27] If only there were some way to cite all the work towards this end that was never finished or published. Cf. [32, pp.221–2].

language, and only to be able to parse anything classically (see [44]); but in the current pluralistic, anti-exceptionalist climate perhaps this is an unrepresentative view, maybe even of Quine. Perhaps both classical and paraconsistent logicians can reason 'counter-logically' and see that, from the other party's perspective, something is missing. We know that our theorems do not say the same thing, *to us*, as the original Big Three say to the classicist—and the classicist can 'see' this [48]. When it comes to the meaning of what we've proved, we aren't trying to prove things in a way that the classicist finds acceptable; they can do that for themself just fine. So, what have we done?[28]

We've tried to answer an initial challenge to the non-classicist, which was: *can you do what the classicist can do?* And we've put this paper forward as a candidate for that challenge. The classical theorems are extensional statements about SNF formulas and we've at least approximated those. Proving intensional conditional statements would be, from our perspective, doing a great deal *more* than Gödel's proof achieves, showing something stronger that might not even be true. If it turns out that what we've done here wasn't what was asked for—perhaps we've met the letter but not the spirit of the challenge?—then at least we hope the effort provides more concrete evidence to examine, for advancing a more informed and subtle discussion in future. A more complicated appraisal of the situation indeed raises hard questions about how discussion and communication between committed logical 'partisans' can proceed, and what—if anything—the absolute or logic independent meaning of mathematical theorems might be. If our efforts here bring out some new aspects of the *question* a little more vividly, that in itself is progress.

A third, neutral sort of view—if there is such a thing—is that this exercise simply shows what *assumptions* are required for the Big Three, and what Gödel's proof techniques establish. They show that the Big Three hold in material form for the fragment of the language that can be put into SNF. In summary, in $\texttt{subLPQ}^{\perp}_{\Rightarrow}$,

Completeness of $\texttt{subLPQ}^{\perp}$ Either A is not semantically valid, or it is proof-theoretically valid, for all A in SNF.

Compactness of $\texttt{subLPQ}^{\perp}$ Either some finite subset of X is not jointly falsifiable, or X is jointly falsifiable, for every deumerable set X of formulas in SNF.

Löwenheim-Skolem property for $\texttt{subLPQ}^{\perp}$ Either A is valid, or it is falsifiable in an enumerable domain, for all A in SNF.

This paper shows how substructural paraconsistency can prove metatheorems that are the same or similar to those of classical logic, suggesting that the framework is *independently viable* without being (too) *mathematically revisionary*. From a purely paraconsistent standpoint, the completeness theorem is true, in a weak sense—the one Gödel proved.

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