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Neighbourhood Semantics for Quantified Relevant Logics

Andrew Tedder[1] · **Nicholas Ferenz**[2]

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Abstract

The Mares-Goldblatt semantics for quantified relevant logics have been developed for first-order extensions of **R**, and a range of other relevant logics and modal extensions thereof. All such work has taken place in the the ternary relation semantic framework, most famously developed by Sylvan (née Routley) and Meyer. In this paper, the Mares-Goldblatt technique for the interpretation of quantifiers is adapted to the more general *neighbourhood* semantic framework, developed by Sylvan, Meyer, and, more recently, Goble. This more algebraic semantics allows one to characterise a still wider range of logics, and provides the grist for some new results. To showcase this, we show, using some non-augmented models, that some quantified relevant logics are not conservatively extended by connectives the addition of which do conservatively extend the associated propositional logics, namely fusion and the dual implication. We close by proposing some further uses to which the neighbourhood Mares-Goldblatt semantics may be put.

Keywords Relevant logic · Quantified nonclassical logic · Neighbourhood semantics · Substructural logic

1 Introduction

There have been a number of proposals for enriching the relational semantics of relevant logics to interpret quantifiers, such as [6, 7, 13, 20] and [8, Ch. 13]. These

✉ Andrew Tedder
ajtedder.at@gmail.com

Nicholas Ferenz
ferenz@ualberta.ca

1 Institute of Computer Science, Czech Academy of Sciences, Pod Vodárenskou věží 271/2, 182 07 Praha 8, Czech Republic

2 Department of Philosophy, University of Alberta, 2-40 Assiniboia Hall, Edmonton, T6G 2E7, AB, Canada

all have their virtues, but one which strikes a nice balance between flexibility and naturalness, and which seems particularly generalisable beyond the usual relational semantic framework, is that developed by Mares and Goldblatt [21]. This semantics starts from a *general* frame version of the ternary relation semantics most famously developed by Sylvan (née Routley) and Meyer (see [1, Section 48.3] or [28, Ch. 4] for detailed presentations). This involves adding to a ternary relation frame a set of *admissible propositions*, by which formulas are interpreted. The usual problem with providing semantics for quantified relevant logics, that one cannot simply use a constant domain and a "Tarski-style" interpretation of the quantifiers, for which see [14] (or [1, Section 52] for a summary), is thus avoided, as one can restrict the interpretations of the quantifiers by explicit appeal to the admissible propositions. While Mares and Goldblatt only consider quantified extensions of **R**, it is possible to generalise their work to quantified extensions of other logics which admit of a ternary relation semantic treatment along the same lines as **R** – this has been done by Ferenz [10, 12] (who also considers quantified extensions of relevant logics with *modal* operators), by Goldblatt and Kane, in interpreting propositional quantifiers [17], and by Standefer [31] and Ferenz [11], who consider different ways of adding identity. The general theory is developed further by Goldblatt in his [16].

It is, however, possible to generalise this semantic framework further, by considering a yet more general treatment of the propositional parts of the logics in question – particularly, that of *neighbourhood* ternary relation semantics. A version of this was, to our knowledge, first developed by Sylvan and Meyer [26, 27], and further work in this semantic framework has been done by Lavers [19], Goble [15], Standefer [30], and Tedder [32]. Many of these, following Sylvan and Meyer, worked with a framework where one also includes a set of admissible propositions (the others opting for something more akin to neighbourhood semantics as usually studied in modal logic [24], where any set of points forms a proposition). This generalisation of the original ternary relation framework allows one to characterise sets of models for which many weaker logics are complete – as will be relevant to our purposes here, the usual way in which relational models of modal logics of the "Kripke" variety can be seen as special cases of neighbourhood models does naturally translate over to the relevant logic setting. Indeed, in that setting, it can be seen that the difference comes in precisely at the closure of the set of propositions in a model under intensional operators other than the arrow. Indeed, this fact is close to the heart of Gaggle theory, another generalisation of the ternary relation semantics developed by Dunn [9] and his collaborators, including Bimbó [3]. The reason for this is that the additional intensional operators are those which form a "complete gaggle" with implication.

In this paper, we'll generalise the Mares-Goldblatt machinery, used to interpret the quantifiers, to apply to the framework of neighbourhood ternary relation semantics. In so doing, we'll prove completeness for a collection of relevant logics weaker than those to which the M-G machinery has been previously applied. Along the way, we'll investigate the relationship of the quantifiers and the additional intensional operators, namely the fusion $\circ$ and the dual implication $\leftarrow$. In particular, we'll show, using some simple neighbourhood models, that certain quantified relevant logics are not conservatively extended by the addition of $\circ$ or $\leftarrow$ (though the propositional logics of which they are quantified extensions are conservatively extended by them).

2 Preliminaries

2.1 Languages

We'll deal here with logics in a few languages – a basic logical language, and then a couple of extensions by further connectives. For the most part, the basic language will be our focus.

The logical part of every language includes $Var = \{x_i\}_{i \in \omega}$, a denumerable set of individual variables, as well as the connectives $t, \neg, \wedge, \vee, \rightarrow$ (of arities 0,1,2,2,2, respectively) and quantifiers $\forall, \exists$. In some cases, the set of logical connectives will be expanded to include $\circ, \leftarrow$ (both arity 2) as well. Whatever the case may be, a language signature is composed of $Pred = \{F, F_0, F_1, \dots\}$, a set of of predicate letters, $Con = \{c, c_0, c_1, \dots\}$, a set of name constants, and $\mathbb{P} = \{p, p_0, p_1, \dots\}$, a set of propositional atoms. We fix $Term = Var \cup Con$, and define the set of formulas to the be the smallest set satisfying the following conditions:

- All elements of $\mathbb{P}$ are formulas, as is t.
- If $P \in Pred$, of arity $n + 1$, and $\tau_0, \dots, \tau_n \in Term$, then $P(\tau_0, \dots, \tau_n)$ is a formula.
- If A, B are formulas, then so are $A \wedge B$, $A \vee B$, and $A \rightarrow B$. (If $\circ, \leftarrow$ are among the connectives, then so are $A \circ B$, $A \leftarrow B$.)
- If A is a formula and $x \in Var$, then $\forall x A$, $\exists x A$ are both formulas.

As here, we'll use the first few capital letters of the Latin alphabet as variables over formulas. We'll write that an occurrence of $x \in Var$ is free in the formula A when it is not bound by any quantifier (all other variables in A are bound). Furthermore, we'll call $\tau \in Term$ "substitutable for x in A" when no variable clashes result from substituting τ for x. As a notational convention, $\rightarrow$ is assumed to bind least strongly of all the connectives, and $A \leftrightarrow B$ is defined as $(A \rightarrow B) \wedge (B \rightarrow A)$.

We'll write the basic language, that without $\leftarrow, \circ$, as $\mathcal{L}$. Extensions thereto by these additional connectives will be notated $\mathcal{L}_\Phi$ where $\Phi \subseteq \{\leftarrow, \circ\}$. In the next section, we'll occasionally use various $\mathcal{L}_\Phi$ to refer to the *propositional* languages, and not their first-order extensions, but no confusion should arise from this, as once we start to talk about first-order logics, and languages, we won't revert to talk of the purely propositional parts.

2.2 Propositional Logics

The basic logic, **F**, of the neighbourhood semantic framework we'll employ here is axiomatised below.[1] $\Rrightarrow$ is a separator for rules of *proof*, so $A_0, \dots, A_n \Rrightarrow B$ should be understood to mean "if $A_0, \dots, A_n$ are all theorems, then so is B":

(Id) $A \rightarrow A$
($\wedge$E) $A \wedge B \rightarrow A, A \wedge B \rightarrow B$
($\vee$I) $A \rightarrow A \vee B, B \rightarrow A \vee B$

[1] Goble [15] calls the system **F** by the name Min.

$$
\begin{array}{rl}
\text{(Dist)} & A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C) \\
\text{(DeM)} & \neg(A \wedge B) \rightarrow \neg A \vee \neg B \\
(t) & t
\end{array}
$$

$$
\begin{array}{rl}
\text{(rMP)} & A \rightarrow B, A \Rrightarrow B \\
\text{(rWB)} & A \rightarrow B, B \rightarrow C \Rrightarrow A \rightarrow C \\
\text{(rCong)} & A' \leftrightarrow A, B' \leftrightarrow B \Rrightarrow (A \rightarrow B) \rightarrow (A' \rightarrow B') \\
\text{(rAdj)} & A, B \Rrightarrow A \wedge B \\
(\text{r}\wedge\text{I}) & A \rightarrow B, A \rightarrow C \Rrightarrow A \rightarrow B \wedge C \\
(\text{r}\vee\text{E}) & A \rightarrow C, B \rightarrow C \Rrightarrow A \vee B \rightarrow C \\
\text{(rCont)} & A \rightarrow B \Rrightarrow \neg B \rightarrow \neg A \\
(\text{r}t) & A \Lleftarrow\Rrightarrow t \rightarrow A
\end{array}
$$

The concept of an axiomatic derivation is defined as usual – a sequence of formulas each of which is either an instance of an axiom or follows from other formulas in the sequence by one of the rules. All the logics we'll deal with here, propositional or quantified, will be considered just as sets of formulas (theorems) – i.e., in the terminology of [18] they'll be presented as FMLA systems. One may also be interested in more general formulations, such as FMLA-FMLA, SET-FMLA, or something else, but given the lingering unclarity about what is the appropriate account of *logical consequence* for relevant logics (see [1, p. 169] for some discussion), we'll leave that, and so questions of strong completeness, for consideration elsewhere.

A neighbourhood **F**-frame is a tuple $\langle W, N, R, *, Prop\rangle$ composed out of elements of the following kind:[2]

- $\varnothing \neq N \subseteq W$
- $R \subseteq W \times \mathscr{P}(W) \times \mathscr{P}(W)$
- $* : W \longrightarrow W$
- $Prop \subseteq \mathscr{P}(W)$

Intuitively, W is a set of situations (worlds, set-ups,. . .) and N is a subset of these (the *regular*, *normal*, or *logical* situations), alongside R which interprets the conditional (and, sometimes, $\circ$ and $\leftarrow$), and $*$ which interprets the negation. Finally, we fix a set of *admissible propositions*, which shall be used to interpret formulas of the language. Out of these elements, we define, for $X, Y \subseteq W$, the following subsets of W (alongside $X \cap Y$, $X \cup Y$ defined as usual):

- $\neg X = \{\alpha \in W \mid \alpha^* \notin X\}$
- $X \rightarrow Y = \{\alpha \mid R\alpha XY\}$

[2]It should be noted that what we are defining here are a special class of neighbourhood frames, where $Prop$ is made explicit, and allowed to be a strict subset of (W). This means we are using what Pacuit [24] calls *general neighbourhood frames*, and this is done to more smoothly transition into giving a Mares-Goldblatt treatment of the quantifiers, which makes use of general frames. From now on, a 'neighbourhood **L**-frame', for a logic **L** extending **F**, will just be referred to as an '**L**-frame', or when no confusion will arise just a "frame".

Finally, an **F**-frame is required to satisfy the following constraints:

(c0) $N \in Prop$; $X, Y \in Prop$ only if $\neg X$, $X \cap Y$, $X \cup Y$, $X \to Y \in Prop$
(c1) $X \subseteq Y$ iff $N \subseteq X \to Y$, for any $X, Y \in Prop$

In order to obtain a model M from a frame, for a *propositional* language, add a function $M : \mathbb{P} \longrightarrow Prop$, and define $[\![\cdot]\!]^M : \mathcal{L} \longrightarrow Prop$ as follows:

- $[\![p]\!]^M = M(p)$
- $[\![t]\!]^M = N$
- $[\![\neg A]\!]^M = \neg([\![A]\!]^M)$
- $[\![A \wedge B]\!]^M = [\![A]\!]^M \cap [\![B]\!]^M$
- $[\![A \vee B]\!]^M = [\![A]\!]^M \cup [\![B]\!]^M$
- $[\![A \to B]\!]^M = [\![A]\!]^M \to [\![B]\!]^M$

Let $\vDash_M A$ hold iff $N \subseteq [\![A]\!]^M$, and $\vDash_{\mathbf{L}} A$ iff $\vDash_M A$ holds for every M built on an **L**-frame.

In order to obtain sets of frames, and models, for logics expanding **F**, we need to consider some additional frame constraints. We'll be considering a handful of such extensions, for which the salient constraints are below (where X, Y, Z are variable over $Prop$).[3]

(DNE) $\alpha^{**} = \alpha$ (i.e. $\neg\neg X = X$)
(rB) $X \subseteq Y$ only if $Z \to X \subseteq Z \to Y$
(rB′) $X \subseteq Y$ only if $Y \to Z \subseteq X \to Z$
(∧I) $(X \to Y) \cap (X \to Z) \subseteq X \to (Y \cap Z)$
(∨E) $(X \to Z) \cap (Y \to Z) \subseteq (X \cup Y) \to Z$
(Cont) $X \to Y \subseteq \neg Y \to \neg X$
(CM) $X \to \neg X \subseteq \neg X$
(WB) $(X \to Y) \cap (Y \to Z) \subseteq X \to Z$
(B) $X \to Y \subseteq (Z \to X) \to (Z \to Y)$
(B′) $X \to Y \subseteq (Y \to X) \to (X \to Z)$
(W) $X \to (X \to Y) \subseteq X \to Y$
(CII) $N \to Y \subseteq Y$
(C) $X \to (Y \to Z) \subseteq Y \to (X \to Z)$
(Mingle) $X \subseteq X \to X$

To discover axioms/rules appropriate to each constraint, simply rewrite the above expressions, replacing each instance of "only if" with one of $\Rrightarrow$, elements of $Prop$ by formulas, $\cap$ by $\wedge$, $\cup$ by $\vee$, and $\subseteq$ by $\to$ (inserting parentheses as necessary). Under this translation scheme, we have that (rB) becomes $A \to B \Rrightarrow (C \to A) \to (C \to B)$ and (∧I), $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$, for example.

[3] Those of the following clauses whose names are rendered as strings of sans-serif upper-case letters from the Latin alphabet are so named to reflect the fact that most of the principles are related, in a more or less direct way, to the implication type schemata of combinators bearing the same names – see [4] for a treatment of combinatory logic which highlights the salient points.

In terms of these extra axioms and rules, we can compile at least the following propositional logics, including most of the usual relevant suspects – the axiomatisations here are more or less standard, and one can find nice natural deduction versions of some of these systems in [5].[4]

FDE is **F** plus (DNE)	**TW** is **B** plus (B), (B′)
BB is **FDE** plus (rB), (rB′)	**T** is **TW** plus (W), (CM)
BM is **BB** plus (∧I), (∨E) minus (DNE)	**E** is **T** plus (CII)
B is **BB** plus (∧I), (∨E)	**RW** is **TW** plus (C)
BJ is **B** plus (WB)	**R** is **T** plus (C)
DW is **B** plus (Cont)	**RM** is **R** plus (Mingle)

To augment this naming convention, let $\mathbf{L}_{(A_1),\ldots,(A_n)}$ be **L** extended by the axioms/rules $(A_1), \ldots, (A_n)$. So, for example, $\mathbf{BJ}=\mathbf{B}_{(\mathsf{WB})}$.

Alongside the principles listed above, we'll be concerned with logics including the further connectives $\circ$ and $\leftarrow$. These are required to obey the following rules:

$$(\mathrm{r}\circ) \quad A \to (B \to C) \Longleftrightarrow (A \circ B) \to C$$
$$(\mathrm{r}\leftarrow) \quad A \to (B \to C) \Longleftrightarrow B \to (C \leftarrow A)$$

Together $\leftarrow, \circ, \to$ form a *residuated triple*, or, in the language of [3], a *complete gaggle*. It is a well-known fact that any of our logics which obey these rules also prove some of the above axioms.

Fact 2.2.1 *Suppose that* **L** *extends* **F**. *If* **L** *obeys (r$\circ$), then* **L** *proves (∧I), and if it obeys (r$\leftarrow$), then it proves (∨E).*

The salient derivations will have the same form as some derivations to be presented below, in Facts 2.3.3 and 2.3.4, so we'll leave working them out to the skeptical (and impatient) reader.[5]

2.3 Adding $\circ$ and $\leftarrow$

One of the topics of interest in this paper, besides proving completeness results for first-order extensions of the logics introduced in Section 2.2, will be with the relationship between neighbourhood Mares-Goldblatt models and relational Mares-Goldblatt models, and this relation is, it will turn out, intimately related to the inclusion of

[4]The choice of the name **FDE** for '**F** plus (DNE)' relies on the "first degree" part of "first degree entailment." As defined here, **FDE** does have higher-degree theorems, for instance $(A \to B) \to (A \to B)$, but the important part is that if $A \to B$ is a first degree formula, then it is provable in **F** plus (DNE) just in case it is the FMLA-FMLA sequent $\langle A, B\rangle$ is valid in **FDE**, as presented, for instance, in [2]. To see this, note that using (DNE), we can obtain a short proofs of $\neg(A \wedge B) \to \neg A \vee \neg B$ and $\neg A \wedge \neg B \to \neg(A \vee B)$, against the background of **F**, and further note that the converses of these are already derivable in **F** using (rCont).

[5]Related discussion can be found in [19].

$\circ$, $\leftarrow$, so let us detour, a bit, into the behaviour of the connectives and how they cut the difference between standard relational models and neighbourhood models.[6]

In order to model logics with $\circ$ or $\leftarrow$, we first need to ensure that $Prop$ is closed under some new operations, and furthermore that the models obey a constraint appropriate to ensuring that the above rules are satisfied. The trick here is well known from the algebraic semantics for substructural logics, and the results simply variations on well known facts therefrom. Let us, first, pick out the following new operations, under which we'll require $Prop$ to be closed:

- $X \circ Y = \bigcap\{Z \mid X \subseteq Y \to Z\}$
- $Y \leftarrow X = \bigcup\{Z \mid X \subseteq Z \to Y\}$

and furthermore, we'll enforce the following frame constraints:

$$\text{(c}\circ\text{)} \quad \text{For any } \{X_i\}_{i\in I} \cup \{Y\} \subseteq Prop, \bigcap_{i\in I}(Y \to X_i) = Y \to \bigcap_{i\in I} X_i.$$

$$\text{(c}\leftarrow\text{)} \quad \text{For any } \{X_i\}_{i\in I} \cup \{Y\} \subseteq Prop, \bigcap_{i\in I}(X_i \to Y) = \bigcup_{i\in I} X_i \to Y.$$

Fact 2.3.1 *Let M be a model in which $Prop$ is closed under $\circ$, obeying (c$\circ$). Then M satisfies (r$\circ$).*

Proof First, note that if M satisfies (c$\circ$), then it satisfies (rB). For suppose that $X \subseteq Y$, i.e. $X = X \cap Y$. It follows that $Z \to X = Z \to (X \cap Y)$, and so, by (c$\circ$), $Z \to X = (Z \to X) \cap (Z \to Y)$, and thus $Z \to X \subseteq Z \to Y$.

Now, in order to show that the set of formulas satisfied by M is closed under (r$\circ$), it suffices to show that for any $X, Y, Z \in Prop$, $X \subseteq Y \to Z$ iff $X \circ Y \subseteq Z$. The left-to-right direction of this equivalence is immediate from the definition (since if $X \subseteq Y \to Z$ then $Z \in \{Z' \mid X \subseteq Y \to Z'\}$), so let us consider the converse.

Suppose that $Z \supseteq X \circ Y = \bigcap\{Z' \mid X \subseteq Y \to Z'\}$ Then $Y \to (X \circ Y) \subseteq Y \to Z$, by (rB). It suffices to show that $X \subseteq Y \to (X \circ Y)$. For this, let $\{Z_i\}_{i\in I} = \{Z' \mid X \subseteq Y \to Z'\}$, and note that $X \subseteq \bigcap_{i\in I}(Y \to Z_i)$. By (c$\circ$), $\bigcap_{i\in I}(Y \to Z_i) \subseteq Y \to \bigcap_{i\in I} Z_i$, from which it follows that $X \subseteq Y \to \bigcap_{i\in I} Z_i = X \circ Y$. Thus $X \subseteq Y \to Z$. □

We have an analogous fact for $\leftarrow$:

Fact 2.3.2 *Let M be a model in which $Prop$ is closed under $\leftarrow$, obeying (c$\leftarrow$). Then M satisfies (r$\leftarrow$).*

Proof First, note that if M satisfies (c$\leftarrow$) then it also satisfies (rB$'$). For if $X \subseteq Y$ then $Y = X \cup Y$ and so $Y \to Z = (X \cup Y) \to Z$ and so, by (c$\leftarrow$), $Y \to Z = (X \to Z) \cap (Y \to Z)$, and so $Y \to Z \subset X \to Z$.

With this in hand, the argument that $X \subseteq Y \to Z$ holds iff $Y \subseteq Z \leftarrow X$ goes similarly to that above, so we'll just consider the left-to-right direction. To that end,

[6] As a matter of fact, the interesting relations between relational and neighbourhood models mostly arises in the propositional setting, so most of what we'll have to say about this comes in this section, though these points will resurface later.

suppose that $Y \subseteq Z \leftarrow X = \bigcup\{Y' \mid X \subseteq Y' \rightarrow Z\}$. It follows that $(Z \leftarrow X) \rightarrow Z \subseteq Y \rightarrow Z$, by (rB$'$), so it would be sufficient to show that $X \subseteq (Z \leftarrow X) \rightarrow Z$. To show this, let $\{Y_i\}_{i \in I} = \{Y' \mid X \subseteq Y' \rightarrow Z\}$, and note that $X \subseteq \bigcap_{i \in I}(Y_i \rightarrow Z)$, and so, by (c$\leftarrow$), $X \subseteq \bigcup_{i \in I} Y_i \rightarrow Z = (Z \leftarrow X) \rightarrow Z$, as desired. Thus $X \subseteq Y \rightarrow Z$. □

Furthermore, we can go the other way around.

Fact 2.3.3 *If M, with $Prop$ closed under $\leftarrow$, satisfies (rB$'$) and the condition that*

$$(\leftarrow\text{-res.}) X \subseteq Y \rightarrow Z \text{ iff } Y \subseteq Z \leftarrow X$$

then it satisfies (c$\leftarrow$).

Proof First, note the following short 'derivation':

1. $\bigcap_{i \in I}(Y_i \rightarrow X) \subseteq Y_i \rightarrow X$ — fact about $\bigcap$
2. $Y_i \subseteq X \leftarrow \bigcap_{i \in I}(Y_i \rightarrow X)$ — ($\leftarrow$-res.)
3. $\bigcup_{i \in I} Y_i \subseteq X \leftarrow \bigcap_{i \in I}(Y_i \rightarrow X)$ — fact about $\bigcup$
4. $\bigcap_{i \in I}(Y_i \rightarrow X) \subseteq \bigcup_{i \in I} Y_i \rightarrow X$ — ($\leftarrow$-res.)

For the other half of the inclusion, just note that $Y_i \subseteq \bigcup_{i \in I} Y_i$, and so, by (rB$'$), $\bigcup_{i \in I} Y_i \rightarrow X \subseteq Y_i \rightarrow X$ holds for every $i \in I$. □

The argument for the following is similar:

Fact 2.3.4 *If M, with $Prop$ closed under $\circ$, satisfies (rB) and the condition that*

$$(\circ\text{-res.}) X \subseteq Y \rightarrow Z \text{ iff } X \circ Y \subseteq Z$$

then it satisfies (c$\circ$).

Proof Note the following 'derivation':

1. $\bigcap_{i \in I}(X \rightarrow Y_i) \subseteq X \rightarrow Y_i$ — fact about $\bigcap$
2. $\bigcap_{i \in I}(X \rightarrow Y_i) \circ X \subseteq Y_i$ — ($\circ$-res.)
3. $\bigcap_{i \in I}(X \rightarrow Y_i) \circ X \subseteq \bigcap_{i \in I} Y_i$ — fact about $\bigcap$
4. $\bigcap_{i \in I}(X \rightarrow Y_i) \subseteq X \rightarrow \bigcap_{i \in I} Y_i$ — ($\circ$-res.)

For the converse, note that $X \rightarrow \bigcap_{i \in I} Y_i \subseteq X \rightarrow Y_i$ holds for each $i \in I$, by (rB$'$), and so $X \rightarrow \bigcap_{i \in I} Y_i \subseteq \bigcap_{i \in I}(X \rightarrow Y_i)$. □

These results are salient for the fact that they indicate the way in which the inclusion of $\circ$, $\leftarrow$ (against the background of logics including (rB) and (rB$'$)) splits the difference between neighbourhood semantics and the usual ternary relation semantics. To see this, note that (c∘) and (c←) are closely related to the *augmentation* constraints given by Goble [15, pp.502–503]. To state these as Goble does, and in a way which makes obvious the connection to homonymous conditions in the neighbourhood semantics for modal logics (see [24]), we need to define $C_\alpha(X) = \{Y \mid \alpha \in X \to Y\}$ and $A_\alpha(X) = \{Y \mid \alpha \in Y \to X\}$, and furthermore assume that $Prop$ is closed under these. With these definitions and assumptions, Goble's augmentation constraints are as follows:

(aug 1) For any $Y \in Prop$, $\bigcap C_\alpha(X) \subseteq Y$ iff $\alpha \in X \to Y$
(aug 2) For any $X, Y \in Prop$, $\forall b \in X(\alpha \in \{b\} \to Y)$ iff $\alpha \in X \to Y$

Goble shows that, in the presence of (rB) and (rB′), these are equivalent to the following:

(aug 1') $\bigcap C_\alpha(X) \in C_\alpha(X)$ (i.e. $\alpha \in X \to \bigcap C_\alpha(X)$)
(aug 2') $\bigcup A_\alpha(X) \in A_\alpha(X)$ (i.e. $\alpha \in \bigcup A_\alpha(X) \to X$)

In fact, in that context, these are, respectively, equivalent to (c∘) and (c←). The first part of this can be proved in short order.

Fact 2.3.5 *Let M be a neighbourhood model. Then we have:*

- *If M satisfies (c∘), its $Prop$ closed under ∘, then it satisfies (aug 1').*
- *If M satisfies (c←), its $Prop$ closed under ←, then it satisfies (aug 2').*

Proof First, it is immediate that $\alpha \in \bigcap_{Y \in C_\alpha(X)} (X \to Y)$, and thus $\alpha \in X \to \bigcap_{Y \in C_\alpha(X)} Y$, by (c∘). Since $\bigcap C_\alpha(X) = \bigcap_{Y \in C_\alpha(X)} Y$, it follows that $\alpha \in X \to \bigcap C_\alpha(X)$, as desired.

Second, it is immediate that $\alpha \in \bigcap_{Y \in A_\alpha(X)} (Y \to X)$, from which it follows that $\alpha \in \bigcup_{Y \in A_\alpha(X)} Y \to X = \bigcup A_\alpha(X) \to X$, by (c←). □

The other part is only slightly more involved:

Fact 2.3.6 *Given any neigbourhood model M:*

- *If M satisfies (aug 1') and (rB), and its $Prop$ is closed under ∘, then it satisfies (c∘).*
- *If M satisfies (aug 2') and (rB′), and its $Prop$ is closed under ←, then it satisfies (c←).*

Proof First, suppose that (aug.1') holds, and let $\{Y_i\}_{i \in I} \subseteq (Prop)$. If $\alpha \in \bigcap_{i \in I}(X \to Y_i)$, then $\bigcap C_\alpha(X) \subseteq \bigcap_{i \in I} Y_i$, and thus, since $\alpha \in X \to C_\alpha(X)$, by (rB), $\alpha \in X \to \bigcap_{i \in I} Y_i$. For the converse, note that $\alpha \in \bigcap_{Y \in C_\alpha(X)} (X \to Y)$, and thus $\alpha \in X \to \bigcap_{Y \in C_\alpha(X)} Y$. Since $\bigcap C_\alpha(X) = \bigcap_{Y \in C_\alpha(X)} Y$, it follows that $\alpha \in X \to \bigcap C_\alpha(X)$.

Second, suppose that (aug.2') holds, and let $\{Y_i\}_{i\in I} \subseteq (Prop)$. If $\alpha \in \bigcap_{i\in I}(Y_i \to X)$, then $Y_i \in A_\alpha(X)$ for every Y_i, and thus $\bigcup_{i\in I} Y_i \subseteq \bigcup A_\alpha(X)$, and thus, by (rB′), $\alpha \in \bigcup_{i\in I} Y_i \to X$. □

So, in the context of logics all of whose neighbourhood models satisfy (rB) and (rB′), models are augmented just in case they satisfy (c∘) and (c←) (supposing they have *Prop*'s which are closed by the operations in question). This is especially salient because, as Goble notes, augmented neighbourhood models are, in a sense, just ternary relation models of the usual variety. So in those logics, the line of demarcation between the neighbourhood and the usual models just is the admission of ∘ and ←, along with the requirement that these form a complete gaggle with →, and it serves to explain why the admissibility of these connectives is cooked into logics whose neighbourhood models are all equivalent to ternary relation models. Indeed, this is one of the core projects of Gaggle theory, if we substitute "algebraic" for "neighbourhood" models.[7]

As we've seen, any logic with ∘ and ← is like this, but it is worth pausing a moment to note what some of these look like. The most famous relevant logic, **R**, is noteworthy in having both of these connectives definable. First, in the presence of (C) the formula $B \leftarrow A$ is simply a notational variant of $A \to B$, as this axiom, added to an extension of **F**, implies that the ← rule $A \to (B \to C) \Longleftarrow\!\!\!\Longrightarrow B \to (A \to C)$ is derivable. So any logic extending $\mathbf{F}_{(r\mathsf{B}'),(\mathsf{C})}$ will be characterised by a class of neighbourhood models satisfying (aug 2), so long as *Prop* is closed under the appropriate operations.

Furthermore, if such an extension also has (rB), (DNE), and (Cont), then we can define $A \circ B$ as $\neg(A \to \neg B)$, for note, then, that the following are all interderivable:

1. $A \to (B \to C)$
2. $A \to (\neg C \to \neg B)$
3. $\neg C \to (A \to \neg B)$
4. $\neg(A \to \neg B) \to C$

Hence, both connectives are definable in **RW**. This serves to ensure that any extension of **RW** will, of necessity, have only augmented neighbourhood models. However, as we'll see, the introduction of quantifiers allows one to prove that some extensions of weaker relevant logics can have non-augmented neighbourhood models. Indeed, these logics also turn out *not* to be conservatively extended by ∘ and ←, while, for some of them, their propositional parts are known to be so extended. This result provides one reason why neighbourhood models for quantified relevant logics are of interest. Before we can get there, though, we need to get some machinery to interpret quantifiers under our belts.

[7]Mathematically speaking, the difference between general neighbourhood models and algebraic models is so fine as to make little difference, though perhaps there is something to be said about the difference philosophically.

2.4 The Mares-Goldblatt Framework

With the neighbourhood semantics machinery now out in the open, the remaining piece of the puzzle is the interpretation of the quantifiers via the Mares-Goldblatt machinery. To present this, we'll need to briefly discuss the standard ternary relation semantic framework. To that end, a TR frame is a tuple $\langle W, N, *, R\rangle$ where $W, N, *$ are defined as before, and $R \subseteq W^3$.

The difference comes in with the treatment of the defined order: $\alpha \leq \beta$ holds iff there is a $\gamma \in N$ s.t. $R\gamma\alpha\beta$. A great deal of complexity is brought in by the need to use an order, and so we'll discuss it as little as possible. The important point is that, in building models on such a frame, we'll need to interpret formulas as *upwardly closed sets* of elements of W – i.e. we want to ensure that every formula is interpreted as a member of $\mathscr{P}(W)^{\uparrow} = \{X \subseteq W \mid \alpha \leq \beta \text{ and } \alpha \in X \text{ imply } \beta \in X\}$.[8] Added to this, in order to interpret the quantifiers, are the following frame elements (in our case, $Prop$ is not new, but is, unlike in the neighbourhood setting to be developed momentarily, defined in terms of $\mathscr{P}(W)^{\uparrow}$, rather than $\mathscr{P}(W)$):

- $D \neq \varnothing$
- $Prop \subseteq \mathscr{P}(W)^{\uparrow}$
- $PropFun \subseteq \{\phi \mid \phi : D^{\omega} \longrightarrow Prop\}$

These are, intuitively, a domain, D, a set of "admissible propositions", $Prop$, and a set of *propositional functions*, $PropFun$, which take assignments to individual variables, i.e. elements of D^{ω} (the stipulation that Var be denumerable is required since ω is being used to assign values to variables), to admissible propositions. We then need constraints appropriate to ensure that $PropFun$ and $Prop$ are closed under enough operations to ensure that the resulting model assigns elements of $PropFun$ to every open formula, and elements of $Prop$ to every closed formula, when we take atomic propositions to these sets.

The first of these closure constraints are straightforward, but the last two, those for the quantifiers, are more complicated. We'll first state them, and then go into some detail about how to understand them.

- There is an $\phi_N \in PropFun$ s.t. for all $f \in D^{\omega}$, $\phi_n f = N$
- $\phi f \otimes \psi f = (\phi \otimes \psi) f$ for and $\phi, \psi \in PropFun$, $f \in D^{\omega}$, and $\otimes \in \{\cap, \cup, \rightarrow\}$[9]
- $\neg(\phi f) = (\neg\phi) f$ for all $f \in D^{\omega}$
- for all $n \in \omega$, $\phi \in PropFun$, there is an $\forall_n \phi \in PropFun$ s.t.
$$(\forall_n \phi) f = \bigsqcap_{j \in D} \phi(f[j/n]) = \bigcup\{X \in Prop \mid X \subseteq \bigcap_{j \in D} \phi(f[j/n])\}$$

where $f[j/n] \in D^{\omega}$ has values $\langle f0, \ldots, f(n-1), j, f(n+1), \ldots\rangle$ (i.e. this is one of the "x-variants" of f, that assigning j to the nth variable, x_n),

[8] To do this, we need to ensure that N is upwardly closed (i.e. if $\alpha \in N$ and $\alpha \leq \beta$, then $\beta \in N$), that $*$ is an inversion w.r.t. $\leq$ (i.e. if $\alpha \leq \beta$ then $\beta^* \leq \alpha^*$) and that R has some *toncicity* properties w.r.t. $\leq$ (i.e that if $R\alpha\beta\gamma$, $\alpha' \leq \alpha$, $\beta' \leq \beta$, and $\gamma \leq \gamma'$, then $R\alpha'\beta'\gamma'$). It is also usually assumed that $*$ is an involution, but we'll flag this wherever it matters.

[9] Also when $\otimes \in \{\circ, \leftarrow\}$, where appropriate.

- for all $n \in \omega$, $\phi \in PropFun$, there is an $\exists_n \phi \in PropFun$ s.t.
$$(\exists_n \phi) f = \bigsqcup_{j \in D} \phi(f[j/n]) = \bigcap\{X \in Prop \mid \bigcup_{j \in D} \phi(f[j/n]) \subseteq X\}$$

The idea behind these last two bullet points is that we ensure that $PropFun$ includes elements, $\forall_n \phi$, $\exists_n \phi$, which, given a variable assignment, take us to *restrictions* of the intersections and unions, respectively, of the collections of ϕ's instances. Goldblatt [16] has a brief, helpful explanation of how to understand the operation $\sqcap$, so we'll quote it at length:

> Now the conjunction of a collection $\{X_i \mid i \in I\}$ of admissible propositions is to be an *admissible* X that (i) entails all of the $X_i's$ and (ii) does no more than that. Here (ii) means that X is weaker than, i.e. is entailed by, any other admissible proposition that entails all of the X_i's. In other words:
>
> (i) $X \subseteq X_i$ for all $i \in I$
> (ii) if $Z \in Prop$ and $Z \subseteq X_i$ for all $i \in I$, then $Z \subseteq X$
>
> So the conjunction is to be the *weakest admissible* proposition that entails all of the X_i's. It will be a subset of $\bigcap_{i \in I} X_i$ by (i), but need not be equal to $\bigcap_{i \in I} X_i$ because the latter may not be admissible. In general this conjunction will be the weakest (=largest) admissible subset of $\bigcap_{i \in I} X_i$, and will be denoted by $\bigsqcap_{i \in I} X_i$ when it exists.
>
> [16, p. 17]

The interpretation of $\bigsqcup$ is analogous. Given $\{X_i \mid i \in I\} \subseteq Prop$, $\bigsqcup_{i \in I} X_i$ (i') is entailed by every X_i and (ii') entails any other element of $Prop$ that is entailed by every X_i.

This maneuver is what allows us to avoid the well-known failures of completeness for mainstream quantified relevant logics w.r.t. their constant domain models (see [14]). With this treatment, one can, a bit surreptitiously, sneak in the varying domains by treating with restricted intersections and unions, while nonetheless having just one fixed D for every point in W.[10]

Mares and Goldblatt concern themselves with two logics, **QR** and **RQ**, both quantified extensions of **R**, but the machinery is quite flexible and can be more broadly applied, including to logics with modal operators. Rather than going into that here, we'll just detail some of the quantifier extensions we'll be interested in. First, every logic will be extended by the following axioms and rules:

($\forall$E)	$\forall x A \rightarrow A$
($\exists$I)	$A \rightarrow \exists x A$
(r$\forall$I)	$A \rightarrow B \Rrightarrow A \rightarrow \forall x B$ (x not free in A)
(r$\exists$E)	$B \rightarrow A \Rrightarrow \exists x B \rightarrow A$ (x not free in A)

[10]There are some interpretive difficulties with this semantic framework, of a similar kind that haunt any proposed semantics for relevant logics, but we won't attempt to provide an account here. One proposed reading is given by Mares [22].

Given a propositional logic **L**, call that quantified extension resulting just from the addition of these axioms and rules $\mathbf{QL}^-$. So $\mathbf{QF}^-$ extends **F** by (∀E), (∀I), (r∀I), and (r∃E). Note that, by the fact that we included t we also have as derivable the rule (rGen) $A \Rrightarrow \forall x A$. Furthermore, note that if **L** has (DNE), then the following further axioms are derivable:

$$\text{(dual 1)} \quad \forall x A \leftrightarrow \neg\exists x \neg A$$
$$\text{(dual 2)} \quad \exists x A \leftrightarrow \neg\forall x \neg A$$

We can derive (dual 1) by the following pair of derivations (and those for (dual 2) are left to the interested reader):

1.	$\forall x A \rightarrow A$	(∀E)
2.	$\neg A \rightarrow \neg\forall x A$	(rCont)
3.	$\exists x \neg A \rightarrow \neg\forall x A$	(r∃E)
4.	$\neg\neg\forall x A \rightarrow \neg\exists x \neg A$	(rCont)
5.	$\forall x A \rightarrow \neg\exists x \neg A$	(DNE)

1.	$\neg A \rightarrow \exists x \neg A$	(∃I)
2.	$\neg\exists x \neg A \rightarrow \neg\neg A$	(rCont)
3.	$\neg\exists x \neg A \rightarrow A$	(DNE)
4.	$\neg\exists x \neg A \rightarrow \forall x A$	(r∀I)

Hence we won't include these as axioms, but just as consequences of (DNE) in the various $\mathbf{QL}^-$.[11]

To remove the minus sign from the name of the logic, add as axioms all instances of the following formulas, where x is not free in A:

$$\text{(∀I)} \quad \forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall x B)$$
$$\text{(∃E)} \quad \forall x(B \rightarrow A) \rightarrow (\exists x B \rightarrow A)$$

Finally, to turn $\mathbf{QL}^-$ into $\mathbf{LQ}^-$, or **QL** into **LQ**, add instances of the following formula, where x is not free in A:

$$\text{(EC)} \quad \forall x(A \vee B) \rightarrow A \vee \forall x B$$

As discussed in [21, p. 177], we can appeal to some of these axioms and rules that substitute particular members of Con for the variable – let $A[c/x]$ is the formula A with c uniformly substituted for x. Then the following are all derivable in $\mathbf{QF}^-$ (and hence in all its extensions) when c is substitutable for x in A and x is not free in B – note that only one of these, (r∀I$_{Con}$), is proved by Mares and Goldblatt, but the others are also derivable along the same lines:

$$(\forall \text{E}_{Con}) \quad \forall x A \rightarrow A[c/x]$$
$$(\exists \text{I}_{Con}) \quad A[c/x] \rightarrow \exists x A$$
$$(\text{r}\forall \text{I}_{Con}) \quad B \rightarrow A[c/x] \Rrightarrow B \rightarrow \forall x A$$
$$(\text{r}\exists \text{E}_{Con}) \quad A[c/x] \rightarrow B \Rrightarrow \exists x A \rightarrow B$$

Later we'll discuss what needs to be added to the definition of a model in order to accommodate the various added axioms. For now, note that if we have ∘ (←) then we

[11] It is, perhaps, interesting to consider systems which don't include (DEM) but do satisfy these two axioms, but we won't go into that question here.

can derive (∀I) from (r∀I) ((∃E) from (r∃E)) using reasoning which is now familiar (from the 'derivations' in Section 2.3):

1.	$\forall x(A \to B) \to (A \to B)$	(∀E)	1.	$\forall x(B \to A) \to (B \to A)$	(∀E)
2.	$\forall x(A \to B) \circ A \to B$	(r∘)	2.	$B \to (A \leftarrow \forall x(A \to B))$	(r←)
3.	$\forall x(A \to B) \circ A \to \forall x B$	(∀I)	3.	$\exists x B \to (A \leftarrow \forall x(A \to B))$	(∃E)
4.	$\forall x(A \to B) \to (A \to \forall x B)$	(r∘)	4.	$\forall x(A \to B) \to (\exists x A \to B)$	(r←)

So we have some collapses – for instance, $\mathbf{QR}^-$ = **QR** and $\mathbf{RQ}^-$ = **RQ**, as ∘, ← are definable in **R**. Finally, note that if **L** is a quantified logic with (Cont), then (∀I) and (∃E) are interderivable – in logics without (Cont), one might consider investigating systems with just one or the other of these quantifier axioms, analogously to studying extensions of **BB** with just one or the other of (∧I), (∨E) as in [32], but we'll leave that to the side for now.

The task, now that our throat clearing is done with, is to combine the neighbourhood and Mares-Goldblatt machineries into a general semantic framework encompassing all the logics which can be put together out of the parts considered thus far.

3 Neighbourhood M-G Semantics

Now the task is just to glue together the M-G quantifier-interpreting machinery and the neighbourhood machinery for the 'propositional parts' of quantified relevant logics. We'll start with a definition of frames and models, and proofs of soundness and completeness, for $\mathbf{QF}^-$, before moving on to consider extensions thereof.

Definition 3.0.1 $\langle W, N, R, *, Prop, D, PropFun\rangle$ is a $\mathbf{QF}^-$-frame when:

- $\langle W, N, R, *, Prop\rangle$ is an **F**-frame
- $D \neq \varnothing$
- $PropFun \subseteq \{\phi \mid \phi : D^\omega \longrightarrow Prop\}$

such that:

(c0) $N \in Prop$; $X, Y \in Prop$ only if $\neg X, X \cap Y, X \cup Y, X \to Y \in Prop$ (also, $X \circ Y$, $X \leftarrow Y \in Prop$, when appropriate, with similar remaks for (c0.0))

(c0.0) There is a $\phi_N \in PropFun$ s.t. for all $f \in D^\omega$, $\phi_N f = N \in Prop$. Futhermore, if $\phi, \psi \in PropFun$, then $\neg\phi, \phi \cap \psi, \phi \cup \psi, \phi \to \psi \in PropFun$ where, for all $f \in D^\omega$:

- $(\neg\phi)f = \neg(\phi f)$ for all $f \in D^\omega$
- $(\phi \otimes \psi)f = \phi f \otimes \psi f$ for all $f \in D^\omega$ and $\otimes \in \{\cap, \cup, \to\}$

(c0.1) If $n \in \omega$ and $\phi \in PropFun$, then $\forall_n\phi, \exists_n\phi \in PropFun$, where these are defined, for an argument $f \in D^\omega$, in terms of $\bigsqcap, \bigsqcup$ as above.

(c1) for all $X, Y \in Prop$, $X \subseteq Y$ iff $N \subseteq X \rightarrow Y$

A model M is obtained from a $\mathbf{QF}^{-}$-frame by adding an interpretation M which evaluates the non-logical parts of the language (including variables, when this is paired with an $f \in D^{\omega}$):

- for $c \in Con$, $M(c) \in D$, and for any $f \in D^{\omega}$, $M_f(c) = M(c)$
- for $x_n \in Var$, $M_f(x_n) = fn$
- for $P \in Pred$ of arity n, $M(P) : D^n \longrightarrow Prop$
- for $p \in \mathbb{P}$, $M(p) \in Prop$

From M we define $[\![\cdot]\!]^M : \mathcal{L} \times D^{\omega} \longrightarrow (W)$, satisfying the following constraints (as a convention, $([\![A]\!]^M)f = [\![A]\!]^M_f$):

- $[\![P(t_1, \ldots, t_n)]\!]^M_f = M(P)(M_f(t_1), \ldots, M_f(t_n))$
- $[\![t]\!]^M_f = \phi_N f$
- $[\![\neg A]\!]^M_f = \neg([\![A]\!]^M_f)$
- $[\![A \wedge B]\!]^M_f = [\![A]\!]^M_f \cap [\![B]\!]^M_f$
- $[\![A \vee B]\!]^M_f = [\![A]\!]^M_f \cup [\![B]\!]^M_f$
- $[\![A \rightarrow B]\!]^M_f = [\![A]\!]^M_f \rightarrow [\![B]\!]^M_f$
- $[\![\forall x_n A]\!]^M_f = (\forall_n [\![A]\!]^M)f$
- $[\![\exists x_n A]\!]^M_f = (\exists_n [\![A]\!]^M)f$

Finally, $\vDash^M_f A$ (i.e. "A is satisfied at M, f") holds iff $N \subseteq [\![A]\!]^M_f$. Next, $\vDash^M A$ ("A is true in M") holds iff for all $f \in D^{\omega}$, $\vDash^M_f A$. Finally, $\vDash_{\mathbf{QF}^-} A$ iff $\vDash^M A$ holds for every $\mathbf{QF}^-$ model M. (Similar definitions apply for other logics to be considered.)

As a notational convention, let "$f \sim_x f'$" be shorthand for "f is an x-variant of f'", and note the following fact, provable as in [21].

Fact 3.0.2 *For any M and f s.t. $f \sim_x f'$, $[\![A]\!]^M_f = [\![A]\!]^M_{f'}$ when x isn't free in A.*

With this fact in hand, the following theorem is, as usual, pretty straightforward:

Theorem 3.0.3 (Soundness) *If $\vdash_{QF^-} A$ then $\vDash_{QF^-} A$*

Proof The proof is standard, by induction on the length of the derivation of A. In the case of axioms, given (c1), and the fact that every $\mathbf{QL}^-$ axiom is of the form $A \rightarrow B$, we just need to show that for every M, f, $[\![A]\!]^M_f \subseteq [\![B]\!]^M_f$. For rules, we just need to show that the appropriate *quasi*-inequations hold. For instance, to cover (rCont), we need to show that if that if $[\![A]\!]^M_f \subseteq [\![B]\!]^M_f$ then $[\![\neg B]\!]^M_f \subseteq [\![\neg A]\!]^M_f$, for (rAdj), we need to show that if $N \subseteq [\![A]\!]^M_f$ and $N \subseteq [\![B]\!]^M_f$ then $N \subseteq [\![A \wedge B]\!]^M_f$, and for (r$\wedge$I) we need to show that if $N \subseteq [\![(A \rightarrow B) \wedge (A \rightarrow C)]\!]^M_f$ then $N \subseteq [\![A \rightarrow B \wedge C]\!]^M_f$. In the case of the propositional axioms, and those rules governing $\wedge$, $\vee$, this follows immediately from the fact that $\langle Prop, \subseteq \rangle$ is a distributive lattice. For instance, for (r$\vee$E) it is enough to note that $N \subseteq (X \rightarrow Z) \cap (Y \rightarrow Z)$ holds iff $N \subseteq X \rightarrow Z$ and $N \subseteq Y \rightarrow Z$, and so, by (c1), $X \subseteq Z$ and $Y \subseteq Z$, from which it follows that $X \cup Y \rightarrow Z$, and so, by (c1), $N \subseteq (X \cup Y) \rightarrow Z$. Given this, since all propositions will be evaluated to elements of $Prop$, all instances of the rule will preserve truth in the model. The other axioms and rules can be verified similarly. The negation principles are immediate from the properties of $*$. With this in mind, let's consider the quantifier principles.

Case ($\forall$E): We want to show that $[\![\forall x_n A]\!]^M_f \subseteq [\![A]\!]^M_f$ holds for every M, f. But note that $[\![\forall x_n A]\!]^M_f = \bigcup\{X \in Prop \mid X \subseteq \bigcap_{j \in D} [\![A]\!]^M_{f[j/n]}\} \subseteq [\![A]\!]^M_{f[j/n]}$ holds for every $j \in D$, including fn.

Case (r$\forall$I): Suppose that $\vdash_{\mathbf{QF}^-} A \rightarrow B$, where x is not free in A. Thus, for any M, f, $[\![A]\!]^M_f \subseteq [\![B]\!]^M_f$, and furthermore for any $f' \sim_x f$, $[\![A]\!]^M_{f'} = [\![A]\!]^M_f$. It follows that $[\![A]\!]^M_f \subseteq \bigcap_{j \in D} [\![B]\!]^M_{f[j/n]}$, and so $[\![A]\!]^M_f \subseteq (\forall_n [\![B]\!]^M) f = [\![\forall x_n B]\!]^M_f$ follows from the definition of $\forall_n$.

Case (r$\exists$E): Given that for any M, f, $[\![B]\!]^M_f \subseteq [\![A]\!]^M_f$ and that $[\![A]\!]^M_{f'} = [\![A]\!]^M_f$ holds for any $f' \sim_x f$, we have that $\bigcup_{j \in D} [\![B]\!]^M_{f[j/n]} \subseteq [\![A]\!]^M_f$, from which it follows that $[\![\exists x_n B]\!]^M_f \subseteq [\![A]\!]^M_f$ by the definition of $\exists_n$.

Case (rGen): Suppose that $N \subseteq [\![A]\!]^M_f$ holds for every M, f. It is immediate, then, that for every M, $N \subseteq \bigcap_{j \in D} [\![A]\!]^M_{f[j/n]}$, and thus $N \subseteq [\![\forall x_n A]\!]^M_f$, since $N \in Prop$. $\square$

3.1 Completeness of QF$^-$

As per usual, the more difficult argument is completeness, though much of the fiddlier work needed here can be taken over, with minor variations, from the proofs given in [21]. We'll present the broad strokes of the argument, going into details where they are new, but where the moves are the same as, or substantially similar to, those made in the Mares-Goldblatt argument, we'll merely provide sketches. Also, for now, we'll concern ourselves only with **QF**$^-$ over the the base language, not including $\leftarrow, \circ$. The proof proceeds, as usual, by the construction of a canonical model.

Definition 3.1.1 The canonical model of **QF**$^-$, $M^{\mathbf{QF}^-}$, is composed out of the following elements:

- $W^{\mathbf{QF}^-}$ is the set of *prime theories* of **QF**$^-$; i.e. those $\alpha \subseteq \mathcal{L}$ s.t. (theoryness) if $\vdash_{\mathbf{QF}^-} A \rightarrow B$ and $A \in \alpha$ then $B \in \alpha$, and if $A, B \in \alpha$ then $A \wedge B \in \alpha$ and (primeness) if $A \vee B \in \alpha$, then either $A \in \alpha$ or $B \in \alpha$.
- $N^{\mathbf{QF}^-} = \{\alpha \in W^{\mathbf{QF}^-} \mid \text{if} \vdash_{\mathbf{QF}^-} A \text{ then } A \in \alpha\}$

Given any closed formula A, let $[\![A]\!]^{\mathbf{QF}^-} = \{\alpha \in W^{\mathbf{QF}^-} \mid A \in \alpha\}$. We need this to go on to define:

- $R^{\mathbf{QF}^-}\alpha XY$ holds iff for some closed formulas B, C we have $B \to C \in \alpha$, $X = [\![B]\!]^{\mathbf{QF}^-}$, and $Y = [\![C]\!]^{\mathbf{QF}^-}$
- $\alpha^{*^{\mathbf{QF}^-}} = \{A : \neg A \notin \alpha\}$
- $D^{\mathbf{QF}^-} = Con$
- $Prop^{\mathbf{QF}^-} = \{X \subseteq W^{\mathbf{QF}^-} \mid \text{for some closed formula } A, X = [\![A]\!]^{\mathbf{QF}^-}\}$

Given any formula A, let A^f be A with every free variable x_n replaced by $fx_n \in Con$. Furthermore, Let $\phi_A : (D^{\mathbf{QF}^-})^\omega \longrightarrow (W)$ be defined by $\phi_A f = [\![A^f]\!]^{\mathbf{QF}^-}$. With this, we can go on to define:

- $PropFun^{\mathbf{QF}^-} = \{\phi_A : A \text{ a formula}\}$
- $M_f(c) = c \in Con$
- $M_f(x_n) = fx_n \in Con$
- $M^{\mathbf{QF}^-}(P)(M_f^{\mathbf{QF}^-}(t_1), \ldots, M_f^{\mathbf{QF}^-}(t_n)) = [\![P(t_1, \ldots, t_n)]\!]^{\mathbf{QF}^-}$

As usual, the procedure is to first, verify that $M^{\mathbf{QF}^-}$ is a model of $\mathbf{QF}^-$, and then to show that if $\nvdash_{\mathbf{QF}^-} A$ then $M^{\mathbf{QF}^-} \nvDash A$. The work needed for the former falls naturally into two parts, dealing with the 'propositional' stuff, which mainly concerns $Prop^{\mathbf{QF}^-}$, and then dealing with the 'quantifier' stuff, which mainly concerns $PropFun^{\mathbf{QF}^-}$. First, however, we'll state a pair of key lemmas:

Lemma 3.1.2 (Pair Extension) *Fix Γ, Δ, sets of formulas, and $\mathbf{L}$ an extension of $\mathbf{QF}^-$. If there are no finite sets $\{A_i\}_{i\in I} \subseteq \Gamma$, $\{B_j\}_{j\in J} \subseteq \Delta$ s.t. $\vdash_{\mathbf{L}} \bigwedge_{i\in I} A_i \to \bigvee_{j\in J} B_j$, then there is a prime theory $\alpha \supseteq \Gamma$ s.t. $\alpha \cap \Delta = \varnothing$.*

Proof See [25, 5.1–5.2] for details (note that we don't require theories to be closed under the constraint "if $A[\tau/x] \in \alpha$ for every term τ, then $\forall x A \in \alpha$" or the dual "if $\exists x A \in \alpha$ then for some term τ, $A[\tau/x] \in \alpha$"). □

Lemma 3.1.3 *For any closed formulas A, B, and logic $\mathbf{L}$ extending $\mathbf{QF}^-$:*
$\vdash_{\mathbf{L}} A \to B$ iff $[\![A]\!]^{\mathbf{L}} \subseteq [\![B]\!]^{\mathbf{L}}$
$\vdash_{\mathbf{L}} A \leftrightarrow B$ iff $[\![A]\!]^{\mathbf{L}} = [\![B]\!]^{\mathbf{L}}$

Proof See [15, Lemma 1.12, Corollary 1.13] for details. □

With these, it is now a routine matter of checking that $M^{\mathbf{QF}^-}$ satisfies all the needed constraints. First, those concerning the propositional part:

Lemma 3.1.4 *$M^{\mathbf{QF}^-}$ satisfies (c0) and (c1).*

Proof For (c0), note first that $N^{\mathbf{QF}^-} = [\![t]\!]^{\mathbf{QF}^-} \in Prop^{\mathbf{QF}^-}$, and next note that whenever $X, Y \in Prop^{\mathbf{QF}^-}$, there are closed formulas A, B s.t. $X = [\![A]\!]^{\mathbf{QF}^-}$ and $Y = [\![B]\!]^{\mathbf{QF}^-}$ (for the rest of this proof, we'll continue to use A, B as the formulas 'defining' X and Y, respectively). We then proceed by cases, showing that there are formulas witnessing that $\neg X, X \to Y \ldots \in Prop^{\mathbf{QF}^-}$ - in fact, we prove that $[\![\cdot]\!]^{\mathbf{QF}^-}$ is a homomorphism over the propositional connectives, which suffices. The arguments that $X \cap Y = [\![A \wedge B]\!]^{\mathbf{QF}^-}$ and $X \cup Y = [\![A \vee B]\!]^{\mathbf{QF}^-}$ are straightforward, and left to the skeptical reader, leaving the interesting cases.

Case $\neg$: We want to show that $\neg X = [\![\neg A]\!]^{\mathbf{QF}^-}$, for which note:

$$\begin{aligned} \neg X &= \{\alpha \in W^{\mathbf{QF}^-} \mid \alpha^{*^{\mathbf{QF}^-}} \notin X\} \\ &= \{\alpha \in W^{\mathbf{QF}^-} \mid A \notin \alpha^{*^{\mathbf{QF}^-}}\} \\ &= \{\alpha \in W^{\mathbf{QF}^-} \mid \neg A \in \alpha\} = [\![\neg A]\!]^{\mathbf{QF}^-} \end{aligned}$$

Case $\to$: We want to show that $X \to Y = [\![A \to B]\!]^{\mathbf{QF}^-}$, for which the following series of identities suffices; note the use of Lemma 3.1.3 to get between the second and third:

$$\begin{aligned} X \to Y &= \{\alpha \mid R^{\mathbf{QF}^-} \alpha XY\} \\ &= \{\alpha \mid \text{for some } C, D(X = [\![C]\!]^{\mathbf{QF}^-}, Y = [\![D]\!]^{\mathbf{QF}^-}, \text{ and } C \to D \in \alpha)\} \\ &= \{\alpha \mid \text{for some } C, D(\vdash_{\mathbf{QF}^-} A \leftrightarrow C, \vdash_{\mathbf{QF}^-} B \leftrightarrow D, \text{ and } C \to D \in \alpha)\} \\ &= \{\alpha \mid A \to B \in \alpha\} = [\![A \to B]\!]^{\mathbf{QF}^-} \end{aligned}$$

For (c1), the result follows by appeal to Lemma 3.1.3 and the definition of $N^{\mathbf{QF}^-}$, that $N^{\mathbf{QF}^-} \subseteq X \to Y$ iff $\vdash_{\mathbf{QF}^-} A \to B$ iff $[\![A]\!]^{\mathbf{QF}^-} \subseteq [\![B]\!]^{\mathbf{QF}^-}$ iff $X \subseteq Y$. □

Now we turn to the quantified part:

Lemma 3.1.5 *M^{QF^-} satisfies (c0.0).*

Proof First, note that $N = \phi_N = \phi_t$, $t^f = t$ for every f, so $[\![t]\!]^{\mathbf{QF}^-} = [\![t^f]\!]^{\mathbf{QF}^-}$. Next, note that $\neg(A^f)$ is syntactically identical to $(\neg A)^f$, as is $A^f \otimes B^f$ to $(A \otimes B)^f$ for $\otimes \in \{\wedge, \vee, \to\}$, and thus $\phi_{\neg A} = \neg\phi_A$, $\phi_{A\to B} = \phi_A \to \phi_B$, $\phi_{A\wedge B} = \phi_A \cap \phi_B$, and $\phi_{A\vee B} = \phi_A \cup \phi_B$ are all in $PropFun^{\mathbf{QF}^-}$, and furthermore variable assignments commute over each, i.e. $(\neg\phi_A)f = \neg(\phi_A f)$ holds for every f (and similarly for the other connectives). □

Lemma 3.1.6 *If $\forall x A$ is closed, then $[\![\forall x A]\!]^{QF^-} = \bigsqcap_{c\in Con} [\![A[c/x]]\!]^{QF^-}$.*

Proof $[\![\forall x A]\!]^{\mathbf{QF}^-} \subseteq [\![A[c/x]]\!]^{\mathbf{QF}^-}$ holds for each $c \in Con$ given ($\forall E_{Con}$) and the definition of $W^{\mathbf{QF}^-}$, and thus since $[\![\forall x A]\!]^{\mathbf{QF}^-} \in Prop^{\mathbf{QF}^-}$, we have that $[\![\forall x A]\!]^{\mathbf{QF}^-} \subseteq \bigsqcap_{c\in Con} [\![A[c/x]]\!]^{\mathbf{QF}^-}$.

For the converse, suppose that $\alpha \in \bigsqcap_{c\in Con} [\![A[c/x]]\!]^{\mathbf{QF}^-}$. Then, by the definition of $\bigsqcap$, there is a formula B s.t. $\alpha \in [\![B]\!]^{\mathbf{QF}^-} \subseteq \bigcap_{c\in Con} [\![A[c/x]]\!]^{\mathbf{QF}^-}$. Pick a $c \in Con$ which occurs in neither A nor B, and suppose that $\nvdash_{\mathbf{QF}^-} B \to A[c/x]$. Thus, using the pair extension lemma, we can find a prime theory β s.t. $A[c/x] \notin \beta$ and $B \in \beta$, which contradicts the assumption that $[\![B]\!]^{\mathbf{QF}^-} \subseteq \bigcap_{c \in Con} [\![A[c/x]]\!]^{\mathbf{QF}^-}$. It follows that $\vdash_{\mathbf{QF}^-} B \to A[c/x]$ and therefore $\vdash_{\mathbf{QF}^-} B \to \forall x A$ follows, using (r∀I$_{Con}$), and thus $\alpha \in [\![\forall x A]\!]^{\mathbf{QF}^-}$, as desired. □

Corollary 3.1.7 *M^{QF^-} satisfies (c0.1).*

Proof It suffices to show that for any formula A and $n \in \omega$, $\phi_{\forall x_n A} = \forall_n \phi_A$.

Let $A^{f/n}$ be that formula which substitutes into A the value $f x_i$ for every $i \neq n$. Then we have, given the l lemma, that:

$$(\forall_n \phi_A) f = \bigsqcap_{c \in Con} \phi_A(f[c/n]) = \bigsqcap_{c \in Con} [\![A^{f[c/n]}]\!]^{\mathbf{QF}^-} = \bigsqcap_{c \in Con} [\![A^{f/n}[c/x_n]]\!]^{\mathbf{QF}^-}$$
$$= [\![\forall x_n A^{f/n}]\!]^{\mathbf{QF}^-} = [\![(\forall x_n A)^f]\!]^{\mathbf{QF}^-} = \phi_{\forall x_n A} f.$$ □

Theorem 3.1.8 (Completeness) *If $\vDash_{QF^-} A$ then $\vdash_{QF^-} A$.*

Proof If $\nvdash_{\mathbf{QF}^-} A$, then we can construct an $\alpha \in N^{\mathbf{QF}^-}$ s.t. $A \notin \alpha$ by noting that, by the supposition, $\{B \mid \vdash_{\mathbf{QF}^-} B\}$ and $\{A\}$ are independent – the pair extension lemma does the rest, generating the desired α. Therefore, $\nvDash_{M^{\mathbf{QF}^-}} A$, and thus $\nvDash_{\mathbf{QF}^-} A$, as desired. □

Corollary 3.1.9 (Adequacy) $\vdash_{QF^-} A$ *iff* $\vDash_{QF^-} A$

This suffices for the simplest logic, which leaves the extensions, to which we now turn.

4 Extending the Adequacy Theorem

As before, it is simplest to split the work into propositional extensions and quantified extensions, and for the latter, to cleave off ∘ and ← for special treatment.

4.1 Propositional Expansions of QF⁻ in the Basic Language

We need to address both the soundness and completeness directions.

Fact 4.1.1 *If M satisfies any constraint in the above list, then it satisfies the axiom/obeys the rule one obtains from the translation procedure described in Section 2.2.*

Proof We'll consider two examples, one axioms and one rule – the others follow the same pattern (many of the needed arguments are given in detail in [15, 26]).

Case: If M satisfies (rB), then $\vDash_M A \to B$ only if $\vDash_M (C \to A) \to (C \to B)$, for any $A, B, C \in \mathcal{L}$. If $\vDash_M A \to B$, then $N \subseteq [\![A \to B]\!]^M = [\![A]\!]^M \to [\![B]\!]^M$, and thus $[\![A]\!]^M \subseteq [\![B]\!]^M$. Thus, by (rB), for any $Z \in Prop^M$, $Z \to [\![A]\!]^M \subseteq Z \to [\![B]\!]^M$, so this holds when $Z = [\![C]\!]^M$, for any $C \in \mathcal{L}$, and thus $N \subseteq [\![C \to A]\!]^M \to [\![C \to B]\!]^M$, and so $\vDash_M (C \to A) \to (C \to B)$, as desired.

Case: If M satisfies (B′), then $\vDash_M (A \to B) \to ((B \to C) \to (A \to C))$ holds for any $A, B, C \in \mathcal{L}$. Fixing $X = [\![A]\!]^M$, $Y = [\![B]\!]^M$, and $Z = [\![C]\!]^M$, the supposition entails that $X \to Y \subseteq (Y \to Z) \to (X \to Z)$, which entails that $N \subseteq [\![A \to B]\!]^M \to [\![(B \to C) \to (A \to C)]\!]^M$, from which the desired result is immediate. □

We must also consider (dual 1) and (dual 2):

Fact 4.1.2 *The axiom (dual 1) and (dual 2) are valid in all neighbourhood models whose frames satisfy the constraint (DNE).*

Proof Note that with (DNE), we have, for any $\{X_i \mid i \in I\} \subseteq (W)$, both of the following identites:

$$\bigcap_{i \in I} X_i = \neg \bigcap_{i \in I} \neg X_i \qquad \bigcup_{i \in I} X_i = \neg \bigcup_{i \in I} \neg X_i$$

and it is immediate from this, and the definitions of $\sqcap$ and $\sqcup$, that:

$$\bigsqcap_{i \in I} X_i = \neg \bigsqcap_{i \in I} \neg X_i \qquad \bigsqcup_{i \in I} X_i = \neg \bigsqcup_{i \in I} \neg X_i$$

and any model whose frame satisfies these constraints validates all instances of (dual 1) and (dual 2). □

This leaves the completeness direction, for which the following suffices: – letting $M^{M^{\mathbf{L}}}$ be defined analogously to $M^{\mathbf{QF}^-}$, for $\mathbf{L}$ extending $\mathbf{QF}^-$.

Fact 4.1.3 *If* $\mathbf{L}$*, extending* $\mathbf{QF}^-$*, contains one of the propositional axioms, then* $M^{\mathbf{L}}$ *satisfies the associated condition.*

Proof As before, we just consider the cases in turn, for which we give two illustrative examples.

Case: If $\mathbf{L}$ is closed under (rB′), then $M^{\mathbf{L}}$ is s.t. for any $X, Y, Z \in Prop^{\mathbf{L}}$, we have $X \subseteq Y$ only if $Y \to Z \subseteq X \to Y$. By the definition of $M^{\mathbf{L}}$, there are closed formulas A, B, C s.t. $[\![A]\!]^{\mathbf{L}} = X$, $[\![B]\!]^{\mathbf{L}} = Y$, and $[\![C]\!]^{\mathbf{L}} = Z$ (and we'll continue with this convention linking X, Y, Z and A, B, C in the rest of the cases). From the assumption that $X \subseteq Y$ it follows that $\vdash_{\mathbf{L}} A \to B$, and thus $\vdash_{\mathbf{L}} (B \to C) \to (A \to C)$. So $[\![B \to C]\!]^{\mathbf{L}} \subseteq [\![A \to C]\!]^{\mathbf{L}}$, and thus $Y \to Z \subseteq X \to Z$.

Case: Suppose $\mathbf{L}$ proves (B), and fix $X, Y, Z \in Prop^{\mathbf{L}}$. Since $\vdash_{\mathbf{L}} (A \to B) \to ((C \to A) \to (C \to B))$, it follows that $[\![A \to B]\!]^{\mathbf{L}} \subseteq [\![(C \to A) \to (C \to B)]\!]^{\mathbf{L}}$, and thus $X \to Y \subseteq (Z \to X) \to (Z \to Y)$, as desired. □

One may ask, why not just give proofs of frame definability? The reason, basically, is that while the canonical $Prop$'s are required to be defined to contain only the 'truth sets' of formulas in the language, this is not required in all models, given the way we've set things up. In the generic definition of "model", $Prop$ is required to contain *all*, but not necessarily *only*, the truth sets of formulas. For this reason, it is not obvious that just because some logic contains an axiom, that *all* propositions satisfy the associated constraint – all those propositions which are the truth sets of formulas do, but it's not clear that the others have to.[12]

4.2 Quantifier Extensions of QF$^-$

We'll consider only those axioms introduced above not already included in **QF**$^-$, namely (∀I), (∃E), and (EC). The last of these is extensively treated in [21, Sections 10–11], so we'll just summarise their results. The frame condition which does the trick for (EC) is that, for every $\phi \in PropFun$, $X, Y \in Prop$, $n \in \omega$, and $f \in D^\omega$:

$$\text{(EC)} \quad X/Y \subseteq \bigcap_{j \in D} \phi(f[j/n]) \text{ only if } X/Y \subseteq (\forall_n \phi) f$$

Soundness and completeness (i.e. if **L** contains the axiom (EC), then $M^{\mathbf{L}}$ satisfies the frame-constraint bearing that name) are both provable in the same way as by Mares and Goldblatt. Note also that in any extension of **QFDE**$^-$, where we have all the DeMorgan laws available, we also get the dual of (EC):

$$\text{(dEC)} \quad A \wedge \exists x B \to \exists x(A \wedge B) \ (x \text{ not free in } A)$$

There is more to be said concerning the interaction between the quantifiers and the lattice connectives in this setting (for instance, as discussed in [23]), but for now we'll focus on the interaction between the quantifiers and the conditional. To that end, the constraints for (∀I) and (∃E), though, as could be anticipated, one winds up with constraints which simply mirror the axioms.[13] Namely, the constraints are as follows, stated for any $\phi, \psi \in PropFun$, $f \in D^\omega$, $n \in \omega$:

$$\text{(∀I)} \quad \text{if, for all } f' \sim_x f, \phi f = \phi f', (\forall_n(\phi \to \psi)) f \subseteq \phi f \to (\forall_n \psi) f$$
$$\text{(∃E)} \quad \text{if, for all } f' \sim_x f, \phi f = \phi f', (\forall_n(\psi \to \phi)) f \subseteq (\exists_n \psi) f \to \phi f$$

Lemma 4.2.1 *If M satsfies the constraint (∀I) then it validates the axiom (∀I).*

Proof Fix a model M, and $f \in D^\omega$, and suppose that $\alpha \in [\![\forall x(A \to B)]\!]^M_f$ holds, and that x is not free in A. Thus $[\![A]\!]^M_f = [\![A]\!]^M_{f'}$ holds for every $f' \sim_x f$, and fix $\phi = [\![A]\!]^M$, $\psi = [\![B]\!]^M \in PropFun$. By definition, $\alpha \in (\forall_n(\phi \to \psi))f$, and since $\phi f = \phi f'$ for all $f' \sim_x f$, it follows that $\alpha \in \phi f \to (\forall_n \psi) f$, and thus $\alpha \in [\![A]\!]^M_f \to [\![\forall x B]\!]^M_f = [\![A \to \forall x B]\!]^M_f$, as desired. □

[12] In Goble's presentation, where $Prop = \mathscr{P}(W)$ (actually, he just leaves "$Prop$" tacit), another problem arises in certain cases, for instance (rB) and (rB′), which requires him to slightly tweak the natural constraint to something more general. [15, p. 496] The route we take here is simpler, in one sense, in allowing us to appeal to the natural constraints and in fitting nicely into the Mares-Goldblatt approach for standard ternary relation semantics, but it does invite this added problem for defining frames via formulas.

[13] Noted by Sylvan and Meyer [27], this feature of the semantics has been criticised, for instance by Smiley [29, p. 246], who claimed that in virtue of this property, which he likened to "garbage in, garbage out", the resulting modeling has "little prospect of establishing the correctness of an axiom system." We won't try to respond to this charge here, as doing so adequately would take us somewhat afield of the technical aim of this paper, so we'll just note it as a philosophical challenge.

Lemma 4.2.2 *If M satisfies the constraint (∃E) then it validates the axiom (∃E)*

Proof Similar to the above. □

So much for the soundness direction, let us turn to the completeness direction. As before, we'll prove the salient fact just for the canonical model – as before, when plugged into the proof for $\mathbf{QF}^-$, this is enough to give the desired result for any logic including this axiom.

Lemma 4.2.3 *If* $\mathbf{L}$ *proves (∀I), then* $M^{\mathbf{L}}$ *satisfies the condition that, if for arbitrary* $\phi_A, \phi_B \in PropFun^{\mathbf{L}}$, *with* $\phi_A f = \phi_A f'$ *for any* $f' \sim_x f$, *then for any* $f \in Con^{\omega}$:
$(\forall_n(\phi_A \to \phi_B))f \subseteq \phi_A f \to (\forall_n \phi_B) f$

Proof What we need to show is that $M^{\mathbf{L}}$ satisfies the constraint, equivalent to that stated in the lemma, that when $\phi_A f = \phi_A f'$ holds for every $f' \sim_x f$:
$[\![\forall x(A \to B)]\!]^{\mathbf{L}} \subseteq [\![A^f]\!]^{\mathbf{L}} \to [\![\forall x B]\!]^M$

This is immediate, for if $\phi_A f = \phi_A f'$ then $[\![A]\!]^{\mathbf{L}} = [\![A^f]\!]^{\mathbf{L}}$, and so x is not free in A. So, if $\alpha \in (\forall_n(\phi_A \to \phi_B))f = [\![\forall x_n(A \to B)^f]\!]^{\mathbf{L}}$, from which it follows, by (∀I), that $\alpha \in [\![(A \to \forall x B)^f]\!]^{\mathbf{L}}$ and thus, noting that $(A \to \forall x B)^f$ is syntactically identical to $A^f \to (\forall x B)^f$, $\alpha \in [\![A^f \to (\forall x B)^f]\!]^{\mathbf{L}} = [\![A^f]\!]^{\mathbf{L}} \to [\![(\forall x B)^f]\!]^{\mathbf{L}} = \phi_A f \to (\forall_n \phi_B) f$. □

The argument for the following follows the same pattern.

Lemma 4.2.4 *If* $\mathbf{L}$ *contains (∃E), then* $M^{\mathbf{L}}$ *satisfies the condition:*
$(\forall_n(\phi_A \to \phi_B)f \subseteq (\exists_n \phi_A) f \to \phi_B f$

4.3 Incorporating ∘ and ←

Something akin to a soundness theorem has already been shown w.r.t. models satisfying (c∘) or (c←) – i.e., it's shown that if $Prop^M$ is closed under the operations ∘/←, and M satisfies the constraint (c∘)/(c←), then the set of formulas true in M are closed under (r∘)/(r←).

For the completeness direction, we'll just focus on logics with (rB) and (rB′), noting the, well-known, fact that if $\mathbf{L}$, in the language including ∘/← is closed under (r∘)/(r←), then we have that $[\![\cdot]\!]^{\mathbf{L}}$ is a homomorphism w.r.t. ∘/← as defined earlier. Note that this also ensures that $Prop^{\mathbf{L}}$ is closed under these operations.

Fact 4.3.1 *Let* $\mathbf{L}$ *be a logic extending* $\boldsymbol{QF}^-$ *in the language including* ∘. *Then if* $X = [\![A]\!]^{\mathbf{L}}$ *and* $Y = [\![B]\!]^{\mathbf{L}}$, *it follows that* $[\![A \circ B]\!]^{\mathbf{L}} = X \circ Y = \bigcap\{Z \in Prop^{\mathbf{L}} \mid X \subseteq Y \to Z\}$.

Proof Suppose that $\alpha \in X \circ Y$, from which it follows that for any $Z \in Prop^{\mathbf{L}}$ s.t. $X \subseteq Y \to Z$, $\alpha \in Z$. That is, for any $C \in \mathcal{L}$ s.t. $\vdash_{\mathbf{L}} A \to (B \to C)$, $\alpha \in [\![C]\!]^{\mathbf{L}}$. But note that $\vdash_{\mathbf{L}} A \to (B \to (A \circ B))$, from which it follows that $A \circ B \in \alpha$, i.e. $\alpha \in [\![A \circ B]\!]^{\mathbf{L}}$.

For the converse, suppose that $A \circ B \in \alpha$. Suppose that $Z \in Prop^{\mathbf{L}}$ and $X \subseteq Y \to Z$. Then there is a $C \in \mathcal{L}$ s.t. $\vdash_{\mathbf{L}} A \to (B \to C)$, and thus $\vdash_{\mathbf{L}} (A \circ B) \to C$, from which it follows that $C \in \alpha$, and so $\alpha \in Z$. □

The proof of the following is similar.

Fact 4.3.2 *Let* $\mathbf{L}$ *be a logic extending* $\boldsymbol{QF}^-$ *in the language including* $\leftarrow$. *Then if* $X = [\![A]\!]^{\boldsymbol{L}}$ *and* $Y = [\![B]\!]^{\boldsymbol{L}}$, *it follows that* $[\![B \leftarrow A]\!]^{\boldsymbol{L}} = Y \leftarrow X = \bigcup\{Z \in Prop^{\boldsymbol{L}} \mid X \subseteq Z \to Y\}$.

With these, let us prove a less-than-fully general completeness fact:

Theorem 4.3.3 *If* $\mathbf{L}$ *is a logic with* $\circ$ *[*$\leftarrow$*] which obeys (rB) and (r*$\circ$*) [(rB′) and (r*$\leftarrow$*)], then* $M^{\boldsymbol{L}}$ *satisfies (c*$\circ$*) [(c*$\leftarrow$*)].*

Corollary 4.3.4 *If* $\mathbf{L}$ *is a logic with* $\circ$ *and* $\leftarrow$, *which obeys (rB), (rB′), (r*$\circ$*), and (r*$\leftarrow$*), then* $M^{\boldsymbol{L}}$ *is augmented.*

While there is more to be said about $\circ$ and $\leftarrow$, this provides us at least a certain completeness property, and one which highlights the completeness of certain logics with these connectives for augmented models – indeed, for all logics which are characterisable in the usual ternary relation semantics framework. Given that, we'll leave this here to be developed further in future work.

5 Putting Neighbourhood Models to Work

So we have developed an adequate model-theoretic characterisation of a wide range of quantified relevant logics using neighbourhood models. One reason to be interested in this is that it accommodates a wider range of logics than are accommodatable using the Mares-Goldblatt enrichment of the usual ternary relation framework, since it provides semantics for quantified extensions of propositional logics weaker than **B**. Another is that while this semantics has a clear algebraic flavour, it is still not too far removed from the ternary relation semantics which are, for various reasons, the most commonly used semantics for relevant logics. It is similar to that semantics while, nonetheless, allowing us to pull apart distinctions which are not available in that semantics.

For instance, in building non-augmented neighbourhood models we can obtain results not available in the usual setting. Here we'll give two very simple neighbourhood model constructions, built on the natural numbers. Using these we'll show that, in certain weak quantified relevant logics, (∀I) and (∃E) are independent of *all* the other quantifier axioms and rules considered thus far. Given that these are provable with the use of $\circ$ and $\leftarrow$, these models also provide proofs that certain weak quantified relevant logics are not conservatively extended by these connectives. While a subtler model construction may work for stronger logics, we'll just focus on weak logics, and the simple model construction here – as a proof of concept, this will do the trick. It should be noted that we get conservative extensions for free in any quantified logics extending **RW**, where both $\circ$ and $\leftarrow$ are definable.[14]

The idea behind the model construction is to use the Tarskian definition for the universal quantifier, i.e.

$$(\forall_n \phi) f = \bigcap_{f' \sim_x f} (\phi f')$$

and to fix the behaviour of $\to$ so that it satisfies (rB) and (rB′), and permits 'finite augmentation', i.e. that it verifies ($\wedge$I) and ($\vee$E), but does not permit full augmentation, so that we don't have one of the following, when $|I| \geq \aleph_0$:

$$\bigcap_{i \in I}(X \to Y_i) \subseteq X \to \bigcap_{i \in I} Y_i$$

$$\bigcap_{i \in I}(Y_i \to X) \subseteq \bigcup_{i \in I} Y_i \to X$$

5.1 The First Model

We start with the following, where R_α, for $\alpha \in W$, is the set of tuples $\langle X, Y \rangle$ s.t. $R\alpha XY$.

- $W = D = \omega$
- $N = \{0\}$
- $* = \{\langle n, n \rangle \mid n \in \omega\}$
- $Prop = \mathscr{P}(W)$
- $PropFun = \{\phi \mid \phi : D^\omega \longrightarrow Prop\}$
- $M(F)n = \{i \mid n < i\}$ (we'll write this set $[n)$)
- $M(p) = \{0\}$
- $R_0 = \{\langle X, Y \rangle \mid X \subseteq Y\}$
- $R_1 = \{\langle X, Y \rangle \mid X \subseteq \{0\}$ and for some $n \in \omega$, $[n) \subseteq Y\}$
- $R_n = \varnothing$ for $n > 1$

By the construction, we have that $\bigcap_{n \in \omega} M(F)n = \varnothing$. Furthermore, since this is a full frame, $(\forall_n \phi) f = \bigcap_{f' \sim_x f} \phi f'$, and so $[\![\forall x Fx]\!]_f = \varnothing$ for any $f \in D^\omega$.

By the construction, we have that for any n, $1 \in \{0\} \to M(F)n$, and thus that $1 \in [\![p]\!] \to [\![Fx]\!]_f$ for any $f \in D^\omega$, and thus that $1 \in \bigcap_{f' \sim_x f} [\![p \to Fx]\!]_{f'} = [\![\forall x(p \to Fx)]\!]_f$ or any f. Yet, $1 \notin \{0\} \to \varnothing$, and thus $1 \notin [\![p \to \forall x Fx]\!]_f$ for any f. Thus $[\![\forall x(p \to Fx)]\!]_f \not\subseteq [\![p \to \forall x Fx]\!]_f$ and thus $0 \notin [\![\forall x(p \to Fx) \to (p \to \forall x Fx)]\!]_f$. So this is a countermodel to ($\forall$I). Now what kind of countermodel is it?

Well, since the frame is full, it is trivial that we have (c0) and (c1), given the definition of R. So it is at least a model of QF^-. Furthermore, by the definition of $*$, we have that $\neg$ is Boolean (in a *first*-degree fragment way – that is, if $\neg A \vee B$ is a

[14] Thanks to a referee for stressing this.

theorem of classical logic in the language $\vee, \neg$, then this model validates $A \to B$). So, it is at least a model of **QFDE**$^-$ (among the logics we consider here).

Let's check some frame constraints to see how $\to$ behaves.

($\wedge$I) If $(X \to Y) \cap (X \to Z) \neq \varnothing$, then one of 0,1 is a member of it. If 0 is, then $X \subseteq Y$ and $X \subseteq Z$, in which case $X \subseteq Y \cap Z$, and so $0 \in X \to (Y \cap Z)$ as desired. If $1 \in (X \to Y) \cap (X \to Z)$ then $X \subseteq \{0\}$ and there are $n, m \in \omega$ s.t. $[n) \subseteq Y$ and $[m) \subseteq Z$, in which case $[n) \cap [m) \subseteq Y \cap Z$. Note, then, that either $n \leq m$ or $m \leq n$. If the former, then $[m) \subseteq [n)$, and so $[m) \cap [n) = [n)$, in which case $[n) \subseteq Y \cap Z$, and so $1 \in X \to (Y \cap Z)$. The case where $m \leq n$ is similar.

($\vee$E) If $(X \to Z) \cap (Y \to Z) \neq \varnothing$, then one of 0,1 is a member of it. If 0, then we have that $0 \in (X \cup Y) \to Z$ just in virtue of properties of $\subseteq$. For 1, suppose that $X, Y \subseteq \{0\}$ and Z is as needed. Then $X \cup Y \subseteq \{0\}$, and so $1 \in (X \cup Y) \to Z$.

(rB) Suppose that $X \subseteq Y$. Note that $Z \to X \subseteq \{0, 1\}$. If $Z \to X = \varnothing$, then we have $Z \to X \subseteq Z \to Y$ immediately, so suppose otherwise. If $0 \in Z \to X$, then $Z \subseteq X$, so $Z \subseteq Y$, and $0 \in Z \to Y$. If $1 \in Z \to X$, then $Z \subseteq \{0\}$ and for some $n \in \omega$, $[n) \subseteq X$, from which it follows that $[n) \subseteq Y$, so $1 \in Z \to Y$. In any case, $Z \to X \subseteq Z \to Y$.

(rB′) Suppose that $X \subseteq Y$, and note that $Y \to Z \subseteq \{0, 1\}$. If $Y \to Z$, then we're done. If $0 \in Y \to Z$ then $Y \supseteq Z$, so $0 \in X \to Z$. If $1 \in Y \to Z$, then $Y \subseteq \{0\}$, in which case $X \subseteq \{0\}$, and Z is as needed, and thus $1 \in X \to Z$. In any case, $Y \to Z \subseteq X \to Z$.

So far, we know that this thing is a model of **QB**$^-$. However, since we know that *Prop* is closed under Boolean negation, we do satisfy (EC), so we know it's a model of **BQ**$^-$ enriched with such a negation (something that would, following the usual naming conventions, be called **CBQ**$^-$, but since we're not focused on Boolean negation here, and the names of logics employed are decorated enough, we'll ignore the fact that we have Boolean negations from here on).[15] However, it *does not* validate (Cont); e.g. $1 \in \{0\} \to [2)$ but $1 \notin \{0, 1\} \to W/\{0\}$. It is a model of some other axioms, particularly two variations on contraction:

(WB) Suppose that $(X \to Y) \cap (Y \to Z) \neq \varnothing$. If 0 is a member, then we immediately have that $0 \in X \to Z$ since $\supseteq$ is an order. If $1 \in Y \to Z$, then $Y \subseteq \{0\}$, in which case there is no $n \in \omega$ s.t. $Y \supseteq [n)$, and so there is no $X \in Prop$ s.t. $1 \in X \to Y$. In any case, then, $(X \to Y) \cap (Y \to Z) \subseteq X \to Z$.

(W) It suffices to show that $X \to (X \to Y) = \varnothing$ holds for any $X, Y \in Prop$. So suppose otherwise, then either 0 or 1 is a member of $X \to (X \to Y)$. If 1, then $X \subseteq \{0\}$ and for some $n \in \omega$, $[n) \subseteq X \to Y$. But note that $X \to Y \subseteq \{0, 1\}$ always holds, and so there can be no such $n \in \omega$. So $1 \notin X \to (X \to Y)$. Next, note that for $0 \in X \to (X \to Y)$ it must be that $X \subseteq X \to Y$. But if this were the case, then $1 \in X$ implies $1 \in X \to Y$. But note that if $1 \in X$ then $X \nsubseteq \{0\}$ and so

[15] As a related note, both of the theorems in this section (and their corollaries), since they employ the Tarski-style truth conditions for the quantifiers, actually concern the *constant domain* extensions of the logics in question. However, given that, to our knowledge, the constant domain versions of these logics are not axiomatised (neither are the constant domain versions of their stronger neighbours), we'll focus on their 'usual' axiomatic presentation, even though this is strictly weaker than the systems the theorems actually concern.

$1 \notin X \to Y$. Thus $X \not\subseteq X \to Y$ for any X, Y, and so $0 \notin X \to (X \to Y)$. Thus $X \to (X \to Y) = \varnothing$, and so $X \to (X \to Y) \subseteq X \to Y$ holds vacuously.

Since the logic extending **B** with *both* of these last two axioms does not, to our knowledge, have a conventional name, we'll use the, somewhat ugly, name $\mathbf{BJQ}^{-}_{(\mathsf{W})}$ to pick out the resulting extension of $\mathbf{BQ}^{-}$.

Theorem 5.1.1 *$BJQ^{-}_{(W)}$ does not prove (∀I).*

Corollary 5.1.2 *$BJQ^{-}_{(W)}$ is not conservatively extended by* $\circ$.

5.2 The Second Model

To counterexample (∃E), let's keep the same construction from before, changing only the definition of R_1. Note that $\bigcup_{n \in \omega} M(F)n = W$. Let us set:

$R_1 = \{\langle X, Y\rangle \mid X \subseteq [n) \text{ for some } n \in \omega \text{ and } 0 \in Y\}$

Note that this implies that $1 \in \bigcap_{n\in\omega}([n) \to \{0\}) = \bigcap_{f \in D^\omega}([\![Fx]\!]_f \to [\![p]\!])$ and thus $1 \in [\![\forall x(Fx \to p)]\!]_f$ holds for any f. However, $1 \notin W \to \{0\}$, and so $1 \notin [\![\exists x Fx \to p]\!]_f$ for any f. Thus $0 \notin [\![\forall x(Fx \to p) \to (\exists x Fx \to p)]\!]_f$.

As before, it remains to see what this is now a model of, and the only new things are the frame constraints, but with this new case for 1. Thus, we'll only focus on the new things.

($\wedge$I) Suppose that $1 \in (X \to Y) \cap (X \to Z)$, for some $n \in \omega$, $X \subseteq [n)$ and $0 \in Y \cap Z$. It follows that $1 \in X \to (Y \cap Z)$.

($\vee$E) Suppose that $1 \in (X \to Z) \cap (Y \to Z)$, and that $0 \in Z$ and that there are $n_1, n_2 \in \omega$ s.t. $X \subseteq [n_1)$ and $Y \subseteq [n_2)$. Then $X \cup Y \subseteq [n_1) \cup [n_2)$. Furthermore, either $n_1 \leq n_2$ or $n_2 \leq n_1$. If the former holds, then $[n_2) \subseteq [n_1)$ and so $[n_2) = [n_1) \cup [n_2)$, and thus $X \cup Y \subseteq [n_2)$. It follows, then, that $1 \in (X \cup Y) \to Z$.

(rB) Suppose that $X \subseteq Y$, $1 \in Z \to X$, $Z \subseteq [n)$, and $0 \in X$. It follows that $0 \in Y$, so $1 \in Z \to Y$.

(rB′) Suppose that $X \subseteq Y$, $1 \in Y \to Z$, $Y \subseteq [n)$, and $0 \in Z$. But then $X \subseteq [n)$ so $1 \in X \to Z$.

(WB) Suppose that $1 \in (X \to Y) \cap (Y \to Z)$, and that both of these implications hold at 1 because of the interesting clause in the definition of R_1. Then for some $n_1, n_2 \in \omega$, $X \subseteq [n_1)$, $Y \subseteq [n_2)$, and $0 \in Y \cap Z$. It follows immediately that $1 \in X \to Z$.

Note that this construction doesn't satisfy (W); note that $\{1\} \subseteq \{1\} \to \{0\}$ but $\{1\} \not\subseteq \{0\}$, so that $0 \in \{1\} \to (\{1\} \to \{0\})$ but $0 \notin \{1\} \to \{0\}$. (Incidentally, it doesn't satisfy (WI) either). So we'll state the following theorem for a more nicely named system than in the case of (∀I):

Theorem 5.2.1 *BJQ^{-} does not prove (∃E).*

Corollary 5.2.2 *BJQ^{-} is not conservatively extended by* $\leftarrow$.

6 Conclusion

We'll conclude by noting a striking fact, which seems to be to suggest an avenue of future work in this area. In particular, while we've noted that the inclusion of $\circ$, $\leftarrow$ in logics extending $\mathbf{QF}^{-}_{(r\mathsf{B}),(r\mathsf{B}')}$ enforces augmentation, and enforces ($\forall$I) and ($\exists$E) to hold, it seems possible to build models of (some) logics extending $\mathbf{QF}_{(r\mathsf{B}),(r\mathsf{B}')}$ which are not augmented, and hence which don't support $\circ$ or $\leftarrow$. This makes it seem that we might find extensions of this logic which are not conservatively extended by these connectives. Such results would be particularly interesting, and would employ the distinctively 'M-G' machinery more than do the failure of conservative extension results we've given here.

Neighbourhood semantics is something of a waypoint between relational semantics and algebraic semantics, and this project is, similarly, something of a waypoint between taking the Mares-Goldblatt machinery, and adapting it, more generally, to the algebraic semantics available for various relevant logics. Nonetheless, it is an interesting waypoint, which allows for just enough additional generality beyond the relational framework to start posing and answering some questions which, though answerable using general algebraic methods, are particularly interesting when seen from the perspective of relational semantics. This makes neighbourhoods a rather natural waypoint, and these results suggest that further model constructions may prove interesting for deepening our understanding of quantified extensions to relevant logics – a topic which, while understudied for some time, seems, at the time of writing, to have come up for re-evaluation as interesting and fruitful.

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