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Logic and Majority Voting

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Abstract

To investigate the relationship between logical reasoning and majority voting, we introduce logic with groups Lg in the style of Gentzen's sequent calculus, where every sequent is indexed by a group of individuals. We also introduce the set-theoretical semantics of Lg, where every formula is interpreted as a certain closed set of groups whose members accept that formula. We present the cut-elimination theorem, and the soundness and semantic completeness theorems of Lg. Then, introducing an inference rule representing majority voting to Lg, we introduce logic with majority voting Lv. Formalizing the discursive paradox in judgment aggregation theory, we show that Lv is inconsistent. Based on the premise-based and conclusion-based approaches to avoid the paradox, we introduce logic with majority voting for axioms Lva, where majority voting is applied only to non-logical axioms as premises to construct a proof in Lg, and logic with majority voting for conclusions Lvc, where majority voting is applied only to the conclusion of a proof in Lg. We show that both Lva and Lvc are syntactically complete and consistent, and we construct collective judgments based on the provability in Lva and Lvc, respectively. Then, we discuss how these systems avoid the discursive paradox.

Keywords Majority voting · Judgment aggregation · Proof theory

1 Introduction

Majority voting is one of the most commonly used methods in group decision-making. It is a simple and effective method, and it seems to be convincing, to certain extent. However, in the 18th century, Condorcet showed that majority voting may be inconsistent with logical reasoning. Although the Condorcet's paradox is a paradox in social choice theory, where logical reasoning is represented by the transitivity of

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the preference relation, it can be generalized to a paradox for general propositions and reasoning thereof. Dietrich and List [9] investigated the following example.

Example 1 (Discursive paradox [9]) Three individuals 1, 2, and 3 make a collective judgment on the following three propositions P, Q, $P \to Q$.

P: Carbon dioxide emissions are above the threshold x.
Q: There will be global warming.
$P \to Q$: If carbon dioxide emissions are above the threshold x, then there will be global warming.

Each individual's judgment is summarized in the following table.

	P	$P \to Q$	Q
Individual 1	T	T	T
Individual 2	F	T	F
Individual 3	T	F	F
Majority	T	T	F

In the above table, T means acceptance of the proposition, and F means rejection of it. Thus, Individual 1 accepts all three propositions, Individual 2 accepts only $P \to Q$, and Individual 3 accepts only P. We here assume that "Individual i rejects φ" is equivalent to "i accepts $\neg\varphi$" for any proposition φ. In this situation, the majority accepts P and $P \to Q$, and rejects Q (i.e., accepts $\neg Q$). However, this collective judgment is inconsistent as a whole from the viewpoint of standard logic, where P and $P \to Q$ imply Q, although every individual makes a consistent judgment.

Both this discursive paradox and the original Condorcet's paradox show that majority voting may produce an inconsistent collective judgment. Recognizing this paradox, Arrow [1] explored possible methods of preference aggregation and established the impossibility theorem: There exists no aggregation procedure that satisfies certain reasonable conditions, including not being inconsistent with logic, without being dictatorial. The impossibility theorem has been generalized and investigated in the framework of judgment aggregation theory. See [1, 2] for Arrow's impossibility theorem and investigation on the conditions of the theorem. See, e.g., [9, 16, 19–21] for the generalized impossibility theorem in judgment aggregation theory.

In this article, instead of an investigation of Arrow's impossibility result, we give a further analysis of the relationship between majority voting and logical reasoning. Majority voting itself has also been studied extensively, and many variants thereof, such as quota rules [10] and scoring rules [8], have been investigated, cf. [32]. Various procedures to construct consistent collective judgments using majority voting have also been proposed, cf. [16, 19, 32]. Among them, we investigate, from a proof-theoretical viewpoint, the well-known restriction on majority voting; the premise-based approach and the conclusion-based approach. In these approaches, majority voting is used only for predetermined premises and conclusions, respectively. See, for example, [11, 16, 33].

To this purpose, in Section 2, we introduce logic with groups Lg, which gives a basis for our logic with majority voting. We introduce the syntax of Lg in the style of

Gentzen's sequent calculus, where every sequent is indexed by a group of individuals. If we ignore the indexes of sequents, then all inference rules, other than non-logical axioms, of Lg are the rules of the usual sequent calculus for classical logic. Non-logical axioms of Lg are atoms or their negation with groups whose members accept them. Thus, the non-logical axioms in Lg are not formulas considered to be true or accepted by all members but just the starting points for construct a proof. We also introduce the semantics for Lg, based on the idea of the phase semantics of linear logic [14, 29]. In our semantics, every formula is interpreted as certain set of groups whose members accept the formula. Our semantics also can be regarded as a kind of Kripke semantics by considering every group as a possible world. We investigate well-established logical theorems for Lg, that is, the cut-elimination theorem and the semantic completeness theorem.

In Section 3, we extend Lg to logic with majority voting Lv by introducing an inference rule representing majority voting. We seek to determine collective judgment by constructing a proof for the formula in question. That is, every collectively accepted formula is a formula that is provable in our logic with majority voting. Thus, our approach can be called the proof-based approach, where every proof can be considered to support the accepted formula. However, as shown by the discursive paradox, Lv itself is inconsistent, and we cannot adopt Lv as a logical system for constructing collective judgments. Thus, based on the premise-based and conclusion-based approaches in the literature of judgment aggregation theory, we introduce systems of logic with majority voting for axioms Lv*a*, where majority voting can only be applied to non-logical axioms of Lg as premises, and logic with majority voting for conclusions Lv*c*, where majority voting can only be applied to the conclusion of a proof in Lg. We define corrective judgments based on Lv*a* and Lv*c* by constructing proofs in the respective systems. Our approach, based on Lv*a*, may be considered to be a particular case of the premise-based approach, where predetermined premises are only atoms or their negation as non-logical axioms. By contrast, our approach, based on Lv*c*, may be considered to be an extension of the usual conclusion-based approach, where we first construct a proof and then apply majority voting, instead of immediately voting for the predetermined conclusion. We show that collective judgments based on Lv*a* and Lv*c* are complete and consistent.

2 Logic with Groups Lg

In this section, we introduce our logic with groups Lg, which is the underlying logic for our logic with majority voting. In Section 2.1, we introduce basic concepts in judgment aggregation theory. In Section 2.2, we introduce sequent calculus for Lg, and we investigate some syntactic properties of Lg in Section 2.3. We further investigate the cut-elimination theorem of Lg, and we show the consistency of Lg in Section 2.4. In Section 2.5, we introduce the semantics of Lg and prove the soundness theorem. In Section 2.6, we prove the completeness theorem for Lg with respect to our semantics.

2.1 Judgment Aggregation

In this article, we introduce propositional logic as the underlying logic for judgment aggregation theory. See, e.g., [9, 16, 19, 20] for judgment aggregation theory.

Definition 1 (Formulas) Formulas, denoted by $\varphi, \psi, \sigma, \ldots$, are defined inductively as follows.

$$\varphi, \psi ::= \quad P \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \varphi \vee \psi \mid \neg\varphi$$

where atoms are denoted by $P, Q, R, \ldots$. We call every atom and its negation **literals**.

Definition 2 (Judgment aggregation)

- We denote n individuals by $1, 2, 3, \ldots, n$, and the set of all individuals by N. We assume that, when not specified explicitly, the cardinality $|N|$ is always n with $n \geq 3$. To ensure that majority voting always works we further assume that n is odd.
- An **agenda** $\mathcal{A}$ consists of formulas that are
  1. closed under the negation: $\varphi \in \mathcal{A}$ implies $\neg\varphi \in \mathcal{A}$, and
  2. closed under atoms: for any $\varphi \in \mathcal{A}$, every atom P constitutes φ is contained in $\mathcal{A}$.

  In an agenda $\mathcal{A}$, we identify $\neg\neg\varphi$ and φ.
- Each **judgment set** $J_i \subseteq \mathcal{A}$ for $i \in N$ is a set of formulas accepted by the member i of N such that:
  1. J_i contains exactly one of φ or $\neg\varphi$ for every $\varphi \in \mathcal{A}$, and
  2. J_i is consistent.
- A sequence $(J_1, J_2, \ldots, J_n)$ of judgment sets of all $i \in N$ is called a **profile**. A profile is denoted by $\mathbf{J}, \mathbf{J}^*, \ldots$.
- An **aggregation rule** or **aggregation function** F is a function from the set of profiles to the set of judgment sets.

  F defines a **collective judgment** $F(\mathbf{J})$ based on the profile $\mathbf{J}$ (i.e., individuals' judgments).

The second condition of the agenda, i.e., the closure under atoms, is not a standard condition, and it makes possible to introduce atoms and their negation as non-logical axioms in our system. This condition is introduced, for example, in [7, 23, 31] to investigate conditions on the impossibility theorem. In particular, [7, 23] suggest that by restricting the independence condition or the unanimity condition to atoms and their negation, a consistent collective judgement is obtained with an appropriate majority voting rule.

Example 2 (Agenda and judgment set) The agenda of Example 1 is $\mathcal{A} = \{P, \neg P, Q, \neg Q, P \to Q, \neg(P \to Q)\}$, and the judgment sets are $J_1 = \{P, Q, P \to Q\}$, $J_2 = \{\neg P, \neg Q, P \to Q\}$, and $J_3 = \{P, \neg Q, \neg(P \to Q)\}$.

By $\mathcal{G}$, we denote the set of all groups over N, i.e., $\mathcal{G} = \mathcal{P}(N)$, where $\mathcal{P}(N)$ is the power set of N. By $\mathcal{G}(\varphi)$, we denote all groups that accept φ; i.e., $\mathcal{G}(\varphi) = \{\alpha \mid \varphi \in J_i \text{ for any } i \in \alpha\}$.

2.2 Sequent Calculus for Lg

We introduce our logic with groups Lg in the style of the sequent calculus of Gentzen [13]. In the sequent calculus, the basic component is a sequence of formulas called a *sequent* instead of a formula. A sequent has the form $\varphi_1, \ldots, \varphi_k \vdash \psi_1, \ldots, \psi_l$, which can be identified with the formula $\varphi_1 \wedge \cdots \wedge \varphi_k \to \psi_1 \vee \cdots \vee \psi_l$. Although $\varphi_1, \ldots, \varphi_k$ or $\psi_1, \ldots, \psi_l$ is normally defined as a "sequence" of formulas, we define it as a "multiset" of formulas, that is, a finite sequence, modulo the ordering of occurrences of formulas. For example, we identify the following two sequents: $\varphi, \varphi, \psi \vdash \sigma, \delta$ and $\varphi, \psi, \varphi \vdash \delta, \sigma$. Although the one is obtained from the other normally by an inference rule called *exchange*-rule, we do not include the rule in our calculus by considering multisets of formulas. See [24, 25, 37] for the sequent calculus.

In our Lg, every sequent is indexed by a group $\alpha, \beta, \ldots$ of N. We define our sequent calculus for Lg under the given agenda $\mathcal{A}$ and profile $\mathbf{J}$.

Definition 3 (**Sequent**) Multisets of formulas separated by the symbol $\vdash_\alpha$ with $\alpha \in \mathcal{G}$ of the following form is called a **sequent**.

$$\varphi_1, \ldots, \varphi_k \vdash_\alpha \psi_1, \ldots, \psi_l$$

We call the multiset on the left of the $\vdash_\alpha$ the **antecedent**, and the multiset on the right is the **succedent** of the sequent. (Subsets of) antecedent or succedent in a sequent are collectively called the **context**, and are denoted by a Greek capital letter $\Gamma, \Delta, \Sigma, \Lambda, \ldots$.

The above sequent means that the members of group α accept "ψ_1 or $\cdots$ or ψ_l is a logical consequence of φ_1 and $\cdots$ and φ_k." Both the antecedent and succedent of a sequent may be empty, and a sequent $\varphi_1, \ldots, \varphi_k \vdash_\alpha$ with the empty succedent means that the members of α accept "$\varphi_1, \ldots, \varphi_k$ imply a contradiction." When both antecedent and succedent are empty, the sequent $\vdash_\alpha$ means that the members of α are in contradiction.

Inference rules of our sequent calculus have the following forms:

$$\frac{\Sigma \vdash_\beta \Lambda}{\Gamma \vdash_\alpha \Delta}\mathit{rule} \qquad \text{or} \qquad \frac{\Sigma \vdash_\beta \Lambda \qquad \Pi \vdash_\gamma \Theta}{\Gamma \vdash_\alpha \Delta}\mathit{rule}$$

The above expression means that we can infer the lower sequent $\Gamma \vdash_\alpha \Delta$ by the *rule* from the upper sequents $\Sigma \vdash_\beta \Lambda$ and $\Pi \vdash_\gamma \Theta$.

Definition 4 (**Inference rules of** Lg) Let an agenda $\mathcal{A}$ and a profile **J** be given. Then, inference rules of Lg are divided into three groups: (1) the axioms; (2) the logical rules for $\wedge, \rightarrow, \neg, \vee$, which are directly related to logical connectives in question; (3) the structural rules, which are not directly related to logical connectives.

Axioms

- Logical axioms are the following form of sequents for any atom P:

$$P \vdash_N P$$

- Non logical axioms are the following forms of sequents for every atom P:
  - $\vdash_\alpha P$ where $\alpha = \{i \in N \mid P \in J_i\}$
  - $P \vdash_\alpha$ where $\alpha = \{i \in N \mid \neg P \in J_i\}$
  - $\vdash_\emptyset$

Logical rules

- $\wedge$-rules

$$\frac{\varphi_1, \Gamma \vdash_\alpha \Delta}{\varphi_1 \wedge \varphi_2, \Gamma \vdash_\alpha \Delta} \wedge L1 \qquad \frac{\varphi_2, \Gamma \vdash_\alpha \Delta}{\varphi_1 \wedge \varphi_2, \Gamma \vdash_\alpha \Delta} \wedge L2 \qquad \frac{\Gamma \vdash_\alpha \Delta, \varphi \qquad \Sigma \vdash_\beta \Lambda, \psi}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda, \varphi \wedge \psi} \wedge R$$

- $\rightarrow$-rules

$$\frac{\Gamma \vdash_\alpha \Delta, \varphi \qquad \psi, \Sigma \vdash_\beta \Lambda}{\varphi \rightarrow \psi, \Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda} \rightarrow L \qquad \frac{\varphi, \Gamma \vdash_\alpha \Delta, \psi}{\Gamma \vdash_\alpha \Delta, \varphi \rightarrow \psi} \rightarrow R$$

- $\neg$-rules

$$\frac{\Gamma \vdash_\alpha \Delta, \varphi}{\neg\varphi, \Gamma \vdash_\alpha \Delta} \neg L \qquad \frac{\varphi, \Gamma \vdash_\alpha \Delta}{\Gamma \vdash_\alpha \Delta, \neg\varphi} \neg R$$

- $\vee$-rules

$$\frac{\Gamma \vdash_\alpha \Delta, \varphi_1}{\Gamma \vdash_\alpha \Delta, \varphi_1 \vee \varphi_2} \vee R \qquad \frac{\Gamma \vdash_\alpha \Delta, \varphi_2}{\Gamma \vdash_\alpha \Delta, \varphi_1 \vee \varphi_2} \vee R \qquad \frac{\varphi, \Gamma \vdash_\alpha \Delta \qquad \psi, \Gamma \vdash_\alpha \Delta}{\varphi \vee \psi, \Gamma \vdash_\alpha \Delta} \vee L$$

Structural rules

- w (Weakening)- and c (Contraction)-rules

$$\frac{\Gamma \vdash_\alpha \Delta}{\varphi, \Gamma \vdash_\alpha \Delta} wL \qquad \frac{\Gamma \vdash_\alpha \Delta}{\Gamma \vdash_\alpha \Delta, \varphi} wR \qquad \frac{\varphi, \varphi, \Gamma \vdash_\alpha \Delta}{\varphi, \Gamma \vdash_\alpha \Delta} cL \qquad \frac{\Gamma \vdash_\alpha \Delta, \varphi, \varphi}{\Gamma \vdash_\alpha \Delta, \varphi} cR$$

- cut-rule

$$\frac{\Gamma \vdash_\alpha \Delta, \varphi \qquad \varphi, \Sigma \vdash_\beta \Lambda}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda} cut$$

- mer (Merge)-rule

$$\frac{\Gamma \vdash_\alpha \Delta \qquad \Sigma \vdash_\beta \Lambda}{\Gamma, \Sigma \vdash_{\alpha\cup\beta} \Delta, \Lambda} mer$$

- sub (Subgroup)-rule: When $\beta \subseteq \alpha$,

$$\frac{\Gamma \vdash_\alpha \Delta}{\Gamma \vdash_\beta \Delta} sub$$

Our non-logical axioms are not formulas considered to be true or accepted by all members but are starting points to construct a proof. As it is the case in the usual sequent calculus, we do not need to restrict logical axioms to consist only of atomic formulas, cf. Proposition 1. However, our restriction on non-logical axioms to consist only of atomic formulas is essential to prove the cut-elimination theorem, because the theorem does not generally hold in a system with non-logical axioms consisting of compound formulas. Cf. e.g., [5, 6, 26, 37].

We also introduce the empty sequent with the empty group $\vdash_{\emptyset}$ as our non-logical axiom. This axiom is introduced mainly because of sub-rule and of the set-theoretical properties of $\emptyset$. See Proposition 4.

If we ignore inessential context Γ of $\wedge L1$-rule, the rule means that when the members of α accept "φ_1 implies Δ," they also accept "$\varphi_1 \wedge \varphi_2$ implies Δ." $\wedge L2$-rule is similar. $\wedge R$-rule means that when the members of α accept "Γ implies φ" and the members of β accept "Σ implies ψ," the common members $\alpha \cap \beta$ accept "Γ and Σ imply $\varphi \wedge \psi$."

As for $\vee L$-rule, we require group α to be shared in the lower sequent and the upper sequents. We may formulate the rule by making the intersection of groups $\alpha \cap \beta$ as the other two premise rules. However, if we formulate $\vee L$-rule by unifying two different groups $\alpha \cup \beta$, then such a rule is shown to be unsound with respect to our semantics. See Remark 3.

mer-rule and sub-rule are original structural rules of this article. mer-rule means that when the members of α accept "Γ implies Δ" and the members of β accept "Σ implies Λ," every member belongs to α or β; i.e., $\alpha \cup \beta$ accepts "Γ and Σ implies Δ or Λ." sub-rule means that when the members of α accept "Γ implies Δ," the members of subgroup β of α also accept it.

In our sequent calculus, a proof is a tree consisting of applications of inference rules whose leaves are logical or non-logical axioms as seen in the following Example 3. See [37] for a formal definition. A proof is denoted by $\pi, \pi_1, \pi_2, \ldots$. In our proof, when $\alpha = \{1, 2, 3\}$, by abbreviating the brackets { and } as well as the comma, we express $\Gamma \vdash_{\alpha} \Delta$ as $\Gamma \vdash_{123} \Delta$.

Example 3 (Proof in Lg) Let $N = \{1, 2, 3\}$, and $P \vdash_3$ and $Q \vdash_2$ be non-logical axioms.

$$\dfrac{\dfrac{\dfrac{P \vdash_3}{P \wedge Q \vdash_3}\wedge L1}{\vdash_3 \neg(P \wedge Q)}\neg R \quad \dfrac{\dfrac{Q \vdash_2}{P \wedge Q \vdash_2}\wedge L2}{\vdash_2 \neg(P \wedge Q)}\neg R}{\dfrac{\vdash_{23} \neg(P \wedge Q), \neg(P \wedge Q)}{\vdash_{23} \neg(P \wedge Q)}cR}mer$$

We refer the above proof as "a proof of $\vdash_{23} \neg(P \wedge Q)$," which is the lowermost sequent called the **end-sequent**, or the **conclusion**, of the proof. We consider non-logical axioms as the **premises** of the proof. In what follows, to avoid notational complexity in the proof, we omit the names of rules such as $\wedge L1$ and $\neg R$, above. However, we do indicate the names of mer- and sub-rules, as they are original rules in this article. Some repeated applications of inference rules are expressed by a double line, as follows.

$$\frac{\Sigma \vdash_{\beta} \Lambda}{\Gamma \vdash_{\alpha} \Delta}$$

Definition 5 (**Provability**) When there exists a proof of $\Gamma \vdash_{\alpha} \Delta$, we say that $\Gamma \vdash_{\alpha} \Delta$ is provable.

In particular, when $\Gamma \vdash_N \Delta$ is provable, it is a logical consequence accepted by all members of N.

Definition 6 (**Consistency**) When $\vdash_N$ is provable in a system, we say the system is **inconsistent**. Otherwise, the system is **consistent**.

Note that when $\vdash_N$ is provable in a system, i.e., in an inconsistent system, any sequent $\Gamma \vdash_{\alpha} \Delta$ is provable by applying w-rule and sub-rule.

Remark 1 (Mingle) Our mer-rule has essentially the same form as the rule called $mingle$ (cf. e.g., [18, 28, 36]). However, $mingle$ is introduced in a different context, such as substructural logics, where structural rules are restricted to analyze usual classical or intuitionistic logic. Thus, in this article, we call our rule mer-rule.

2.3 Some Properties of Lg

Let us investigate some syntactic properties of Lg.

Although we restrict our logical axioms to consist only of atomic formulas, this holds for any complex formula.

Proposition 1 *$\varphi \vdash_N \varphi$ is provable for any formula φ.*

Proof By induction on φ. For example, when $\varphi \equiv \varphi_1 \wedge \varphi_2$, the sequent $\varphi_1 \wedge \varphi_2 \vdash_N \varphi_1 \wedge \varphi_2$ is provable by the induction hypotheses for $\varphi_1 \vdash_N \varphi_1$ and $\varphi_2 \vdash_N \varphi_2$ as follows.

$$\cfrac{\cfrac{\cfrac{\varphi_1 \vdash_N \varphi_1}{\varphi_1 \wedge \varphi_2 \vdash_N \varphi_1} \quad \cfrac{\varphi_2 \vdash_N \varphi_2}{\varphi_1 \wedge \varphi_2 \vdash_N \varphi_2}}{\varphi_1 \wedge \varphi_2, \varphi_1 \wedge \varphi_2 \vdash_{N \cap N} \varphi_1 \wedge \varphi_2}}{\varphi_1 \wedge \varphi_2 \vdash_N \varphi_1 \wedge \varphi_2}$$

□

Although we formulate our mer-rule by merging groups α and β as well as contexts, we can formulate it by restricting contexts to be shared in the upper sequents.

Proposition 2 (mer-rule) *The following mer'-rule is equivalent to our mer-rule.*

$$\frac{\Gamma \vdash_{\alpha} \Delta \quad \Gamma \vdash_{\beta} \Delta}{\Gamma \vdash_{\alpha \cup \beta} \Delta}\ mer'$$

Proof mer'-rule is simulated by mer-rule as follows.

$$\dfrac{\dfrac{\Gamma \vdash_\alpha \Delta \quad \Gamma \vdash_\beta \Delta}{\Gamma, \Gamma \vdash_{\alpha\cup\beta} \Delta, \Delta}\, mer}{\Gamma \vdash_{\alpha\cup\beta} \Delta}\, c$$

The above double line expresses several applications of c-rule.

Conversely, mer-rule is simulated by mer'-rule as follows.

$$\dfrac{\dfrac{\Gamma \vdash_\alpha \Delta}{\Gamma, \Sigma \vdash_\alpha \Delta, \Lambda}\, w \quad \dfrac{\Sigma \vdash_\beta \Lambda}{\Gamma, \Sigma \vdash_\beta \Delta, \Lambda}\, w}{\Gamma, \Sigma \vdash_{\alpha\cup\beta} \Delta, \Lambda}\, mer'$$

□

It is well-known that the same applies to other rules $\wedge R$, $\to L$, $\vee L$ with the use of the structural rules w- and c-rules. That is, these rules are equivalent to the following rules, respectively. See [24, 37].

$$\dfrac{\Gamma \vdash_\alpha \Delta, \varphi \quad \Gamma \vdash_\beta \Delta, \psi}{\Gamma \vdash_{\alpha\cap\beta} \Delta, \varphi \wedge \psi}\, \wedge R' \qquad \dfrac{\Gamma \vdash_\alpha \Delta, \varphi \quad \psi, \Gamma \vdash_\beta \Delta}{\varphi \to \psi, \Gamma \vdash_{\alpha\cap\beta} \Delta}\, \to L' \qquad \dfrac{\varphi, \Gamma \vdash_\alpha \Delta \quad \psi, \Sigma \vdash_\alpha \Lambda}{\varphi \vee \psi, \Gamma, \Sigma \vdash_\alpha \Delta, \Lambda}\, \vee L'$$

In what follows, thus, we sometimes use the above rules interchangeably.

One of the remarkable rules in Lg is mer-rule, which makes it possible to merge given groups. For example, assume $\vdash_\alpha \varphi$ and $\vdash_{-\alpha} \psi$ are provable, where $-\alpha$ is the complement of α, and hence, we have $\alpha \cup -\alpha = N$. Then, $\vdash_N \varphi \vee \psi$ is provable by using mer-rule as follows.

$$\dfrac{\dfrac{\dfrac{\vdash_\alpha \varphi}{\vdash_\alpha \varphi \vee \psi}\, \vee R \quad \dfrac{\vdash_{-\alpha} \psi}{\vdash_{-\alpha} \varphi \vee \psi}\, \vee R}{\vdash_N \varphi \vee \psi, \varphi \vee \psi}\, mer}{\vdash_N \varphi \vee \psi}\, cR$$

Thus, without logical axioms, a sequent with the whole group N may be provable.

The following proposition says that we can rearrange the order of application of sub-rule.

Proposition 3 (sub-rule) *Applications of sub-rule are permutable.*

Proof We show some cases, and other cases are similar. We first show that sub-rule can be moved upward. The following proofs on the left are transformed into the proofs on the right, with the same end-sequents.

$$\dfrac{\dfrac{\begin{matrix}\vdots\, \pi_1 \\ \Gamma \vdash_\alpha \Delta, \varphi\end{matrix} \quad \begin{matrix}\vdots\, \pi_2 \\ \Sigma \vdash_\beta \Lambda, \psi\end{matrix}}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda, \varphi \wedge \psi}\, \wedge R}{\Gamma, \Sigma \vdash_\gamma \Delta, \Lambda, \varphi \wedge \psi}\, sub \quad \rhd \quad \dfrac{\dfrac{\begin{matrix}\vdots\, \pi_1 \\ \Gamma \vdash_\alpha \Delta, \varphi\end{matrix}}{\Gamma \vdash_\gamma \Delta, \varphi}\, sub \quad \dfrac{\begin{matrix}\vdots\, \pi_2 \\ \Sigma \vdash_\beta \Lambda, \psi\end{matrix}}{\Sigma \vdash_\gamma \Lambda, \psi}\, sub}{\Gamma, \Sigma \vdash_\gamma \Delta, \Lambda, \varphi \wedge \psi}\, \wedge R$$

where $\gamma \subseteq \alpha \cap \beta \subseteq \alpha, \beta$.

$$\dfrac{\dfrac{\overset{\vdots\ \pi_1}{\Gamma \vdash_\alpha \Delta} \quad \overset{\vdots\ \pi_2}{\Sigma \vdash_\beta \Lambda}}{\Gamma, \Sigma \vdash_{\alpha\cup\beta} \Delta, \Lambda}\ mer}{\Gamma, \Sigma \vdash_\gamma \Delta, \Lambda}\ sub \quad \rhd \quad \dfrac{\dfrac{\overset{\vdots\ \pi_1}{\Gamma \vdash_\alpha \Delta}}{\Gamma \vdash_{\alpha\cap\gamma} \Delta}\ sub \quad \dfrac{\overset{\vdots\ \pi_2}{\Sigma \vdash_\beta \Lambda}}{\Sigma \vdash_{\beta\cap\gamma} \Lambda}\ sub}{\Gamma, \Sigma \vdash_\gamma \Delta, \Lambda}\ mer$$

where $(\alpha \cap \gamma) \cup (\beta \cap \gamma) = (\alpha \cup \beta) \cap \gamma = \gamma$ because $\gamma \subseteq \alpha \cup \beta$.

Conversely, sub-rule can be moved downward as follows.

$$\dfrac{\dfrac{\overset{\vdots\ \pi_1}{\Gamma \vdash_\alpha \Delta, \varphi}}{\Gamma \vdash_{\alpha'} \Delta, \varphi}\ sub \quad \overset{\vdots\ \pi_2}{\varphi, \Sigma \vdash_\beta \Lambda}}{\Gamma, \Sigma \vdash_{\alpha'\cap\beta} \Delta, \Lambda}\ cut \quad \rhd \quad \dfrac{\dfrac{\overset{\vdots\ \pi_1}{\Gamma \vdash_\alpha \Delta, \varphi} \quad \overset{\vdots\ \pi_2}{\varphi, \Sigma \vdash_\beta \Lambda}}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda}\ cut}{\Gamma, \Sigma \vdash_{\alpha'\cap\beta} \Delta, \Lambda}\ sub$$

$$\dfrac{\dfrac{\overset{\vdots\ \pi_1}{\Gamma \vdash_\alpha \Delta}}{\Gamma \vdash_{\alpha'} \Delta}\ sub \quad \overset{\vdots\ \pi_2}{\Sigma \vdash_\beta \Lambda}}{\Gamma, \Sigma \vdash_{\alpha'\cup\beta} \Delta, \Lambda}\ mer \quad \rhd \quad \dfrac{\dfrac{\overset{\vdots\ \pi_1}{\Gamma \vdash_\alpha \Delta} \quad \overset{\vdots\ \pi_2}{\Sigma \vdash_\beta \Lambda}}{\Gamma, \Sigma \vdash_{\alpha\cup\beta} \Delta, \Lambda}\ mer}{\Gamma, \Sigma \vdash_{\alpha'\cup\beta} \Delta, \Lambda}\ sub$$

□

Thus, applications of sub-rule in a proof can be collected in either the upper part or the lower part of the proof. Hence, applications of sub-rule are inessential for investigation of the structure of a proof.

Proposition 4 ($\vdash_\emptyset$) *Any sequent $\Gamma \vdash_\emptyset \Delta$ with the empty group $\emptyset$ is provable in* Lg.

Proof Starting from the axiom $\vdash_\emptyset$, we can obtain any sequent with the empty group by applying wR- and wL-rules as follows.

$$\dfrac{\dfrac{\vdash_\emptyset}{\vdash_\emptyset \Delta}\ wR}{\Gamma \vdash_\emptyset \Delta}\ wL$$

□

Because any sequent $\Gamma \vdash_\emptyset \Delta$ is provable with the empty group $\emptyset$, it is difficult to give an informal interpretation of the sequent. Although we may exclude the empty group by restricting sub-rule with $\beta \neq \emptyset$, this makes our syntax and semantics much more complicated, as well as the cut-elimination (see Remark 2) and completeness theorems (see Remark 6). Thus, we keep the empty group and the non-logical axiom $\vdash_\emptyset$ in this article.

When a sequent $\Gamma \vdash \Delta$ is provable in the usual classical logical system, by replacing every sequent $\Sigma \vdash \Lambda$ in the proof to $\Sigma \vdash_N \Lambda$, we obtain a proof of $\Gamma \vdash_N \Delta$ in Lg. This is formally proved by induction on the length of given proof. Thus, by introducing the non-logical axioms, Lg can be considered as an extension of the usual classical logic.

Proposition 5 (**Classical logic**) *If $\Gamma \vdash \Delta$ is provable in classical logic, then $\Gamma \vdash_N \Delta$ is provable in* Lg.

2.4 Cut-elimination and Consistency of Lg

The cut-elimination theorem, more widely called the proof normalization theorem, is one of the most basic theorems in proof theory. It says that any proof is transformed into a normal proof, i.e., a cut-free proof in the sequent calculus, with the same conclusion. The theorem has various corollaries such as the consistency of the system, and makes various proof-theoretical analyses possible, such as the analysis of the structure of proofs. See [24, 25, 37]. Let us investigate the cut-elimination theorem of our Lg.

Proposition 6 (**Cut-elimination**) *If $\Gamma \vdash_\alpha \Delta$ is provable, then it is provable without cut-rule.*

To prove our cut-elimination theorem, the standard method of cut-elimination is applied. See, for example, [13, 15]. Instead of giving a detailed proof of the theorem, we here present an idea to prove the cut-elimination theorem, through which we show that the indexes of groups do not cause any trouble. In the following discussion, to make the idea of cut-elimination explicit, we exclude c-rule, as it requires more sophisticated method than the naive one explained in what follows. We describe this in the end of the explanation.
(1) Let us consider the following (part of) proof, where cut-rule is applied once.

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash_\alpha \Delta, \varphi\end{array} \quad \begin{array}{c}\vdots\ \pi_2\\ \varphi, \Sigma \vdash_\beta \Lambda\end{array}}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda}\ cut$$

When the last rule of π_1 or π_2 is not a rule for the cut-formula φ, by permuting the given cut-rule and the last rule of π_1 or π_2, we move the application of cut-rule upward until the last rules of π_1 and π_2 become the rules for the cut-formula φ. For example, let us examine the following cases.

($\wedge R$) When the last rule of π_2 is not a rule for φ, but $\wedge R$-rule introduces $\sigma_1 \wedge \sigma_2$ as in the following proof on the left, where we omit inessential contexts, this proof is transformed into the following proof on the right by permuting the cut-rule and the $\wedge R$-rule.

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash_\alpha \varphi\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \pi_{21}\\ \varphi, \Sigma_1 \vdash_{\beta_1} \sigma_1\end{array} \quad \begin{array}{c}\vdots\ \pi_{22}\\ \Sigma_2 \vdash_{\beta_2} \sigma_2\end{array}}{\varphi, \Sigma_1, \Sigma_2 \vdash_{\beta_1\cap\beta_2} \sigma_1 \wedge \sigma_2}\ \wedge R}{\Gamma, \Sigma_1, \Sigma_2 \vdash_{\alpha\cap\beta_1\cap\beta_2} \sigma_1 \wedge \sigma_2}\ cut \quad \rhd \quad \dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash_\alpha \varphi\end{array} \quad \begin{array}{c}\vdots\ \pi_{21}\\ \varphi, \Sigma_1 \vdash_{\beta_1} \sigma_1\end{array}}{\Gamma, \Sigma_1 \vdash_{\alpha\cap\beta_1} \sigma_1}\ cut \quad \begin{array}{c}\vdots\ \pi_{22}\\ \Sigma_2 \vdash_{\beta_2} \sigma_2\end{array}}{\Gamma, \Sigma_1, \Sigma_2 \vdash_{\alpha\cap\beta_1\cap\beta_2} \sigma_1 \wedge \sigma_2}\ \wedge R$$

The same transformation is applied to other rules than $\wedge R$-rule, when it is not a rule for the cut-formula φ. We further examine the cases of mer-rule and sub-rule, which are original rules in this article.

(mer) When the last rule of π_2 is mer-rule, as in the following proof on the left, we transform it into the following proof on the right.

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash_\alpha \varphi\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \pi_{21}\\ \varphi, \Sigma_1 \vdash_{\beta_1} \Lambda_1\end{array} \quad \begin{array}{c}\vdots\ \pi_{22}\\ \Sigma_2 \vdash_{\beta_2} \Lambda_2\end{array}}{\varphi, \Sigma_1, \Sigma_2 \vdash_{\beta_1\cup\beta_2} \Lambda_1, \Lambda_2}\,mer}{\Gamma, \Sigma_1, \Sigma_2 \vdash_{\alpha\cap(\beta_1\cup\beta_2)} \Lambda_1, \Lambda_2}\,cut \quad \rhd \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash_\alpha \varphi\end{array} \quad \begin{array}{c}\vdots\ \pi_{21}\\ \varphi, \Sigma_1 \vdash_{\beta_1} \Lambda_1\end{array}}{\Gamma, \Sigma_1 \vdash_{\alpha\cap\beta_1} \Lambda_1}\,cut \quad \begin{array}{c}\vdots\ \pi_{22}\\ \Sigma_2 \vdash_{\beta_2} \Lambda_2\end{array}}{\Gamma, \Sigma_1, \Sigma_2 \vdash_{(\alpha\cap\beta_1)\cup\beta_2} \Lambda_1, \Lambda_2}\,mer}{\Gamma, \Sigma_1, \Sigma_2 \vdash_{\alpha\cap(\beta_1\cup\beta_2)} \Lambda_1, \Lambda_2}\,sub$$

where $\alpha \cap (\beta_1 \cup \beta_2) \subseteq (\alpha \cup \beta_2) \cap (\beta_1 \cup \beta_2) = (\alpha \cap \beta_1) \cup \beta_2$.

(sub) When the last rule of π_2 is sub-rule, as in the following proof on the left, we transform it into the following proof on the right.

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash_\alpha \varphi\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \pi_{21}\\ \varphi, \Sigma \vdash_\gamma \Lambda\end{array}}{\varphi, \Sigma \vdash_\beta \Lambda}\,sub}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Lambda}\,cut \quad \rhd \quad \dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash_\alpha \varphi\end{array} \quad \begin{array}{c}\vdots\ \pi_{21}\\ \varphi, \Sigma \vdash_\gamma \Lambda\end{array}}{\Gamma, \Sigma \vdash_{\alpha\cap\gamma} \Lambda}\,cut}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Lambda}\,sub$$

where $\beta \subseteq \gamma$, and hence, $\alpha \cap \beta \subseteq \alpha \cap \gamma$.

Similarly for π_1.

(2) When both of the last rules of π_1 and π_2 introduce the cut-formula φ, by transforming the given proof, we reduce the complexity of the cut-formula.

($\wedge R$-$\wedge L$) For example, when the cut-formula is $\varphi_1 \wedge \varphi_2$, and the last rule of π_1 is $\wedge R$-rule and of π_2 is $\wedge L$-rule introducing $\varphi_1 \wedge \varphi_2$, we reduce the complexity of the cut-formula to φ_1 with the following transformation.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_{11}\\ \Gamma_1 \vdash_{\alpha_1} \varphi_1\end{array} \quad \begin{array}{c}\vdots\ \pi_{12}\\ \Gamma_2 \vdash_{\alpha_2} \varphi_2\end{array}}{\Gamma_1, \Gamma_2 \vdash_{\alpha_1\cap\alpha_2} \varphi_1 \wedge \varphi_2}\,\wedge R \quad \dfrac{\begin{array}{c}\vdots\ \pi_{21}\\ \varphi_1, \Sigma \vdash_\beta \Lambda\end{array}}{\varphi_1 \wedge \varphi_2, \Sigma \vdash_\beta \Lambda}\,\wedge L}{\Gamma_1, \Gamma_2, \Sigma \vdash_{\alpha_1\cap\alpha_2\cap\beta} \Lambda}\,cut \quad \rhd \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_{11}\\ \Gamma_1 \vdash_{\alpha_1} \varphi_1\end{array} \quad \begin{array}{c}\vdots\ \pi_{21}\\ \varphi_1, \Sigma \vdash_\beta \Lambda\end{array}}{\Gamma_1, \Sigma \vdash_{\alpha_1\cap\beta} \Lambda}\,cut}{\Gamma_1, \Gamma_2, \Sigma \vdash_{\alpha_1\cap\beta} \Lambda}\,wL}{\Gamma_1, \Gamma_2, \Sigma \vdash_{\alpha_1\cap\alpha_2\cap\beta} \Lambda}\,sub$$

The same transformation, reducing the complexity of the cut-formula, is applied to other combinations of rules than $\wedge R$-$\wedge L$ above.

(wL) When the cut-formula φ is introduced by wL-rule in π_2, we can eliminate the cut-rule with the following transformation.

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash_\alpha \Delta, \varphi\end{array} \quad \dfrac{\begin{array}{c}\vdots\ \pi_{21}\\ \Sigma \vdash_\beta \Lambda\end{array}}{\varphi, \Sigma \vdash_\beta \Lambda}\,wL}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda}\,cut \quad \rhd \quad \dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_{21}\\ \Sigma \vdash_\beta \Lambda\end{array}}{\Gamma, \Sigma \vdash_\beta \Delta, \Lambda}\,w}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda}\,sub$$

(3) When the given cut-formula is an atomic formula P, and the last rules of π_1 and π_2 are rules for P, we are able to eliminate the given cut-rule with the following transformation.

(*logical axiom*) When the last rule of π_2 is the logical axiom for P, by applying the following transformation, we obtain a proof without *cut*-rule.

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash_\alpha \Delta, P\end{array} \quad P \vdash_N P}{\Gamma \vdash_\alpha \Delta, P}\ cut \quad \rhd \quad \begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash_\alpha \Delta, P\end{array}$$

(*non-logical axiom*) When the last rule of π_2 is the non-logical axiom for P as in the following proof, we divide this case depending on the last rule of π_1.

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash_\alpha \Delta, P\end{array} \quad P \vdash_\beta}{\Gamma \vdash_{\alpha\cap\beta} \Delta}\ cut$$

• When the last rule of π_1 is a logical axiom for P, we are able to eliminate the *cut*-rule as follows.

$$\dfrac{P \vdash_\alpha P \quad P \vdash_\beta}{P \vdash_{\alpha\cap\beta}}\ cut \quad \rhd \quad \dfrac{P \vdash_\beta}{P \vdash_{\alpha\cap\beta}}\ sub$$

• When the last rule of π_1 is a non-logical axiom for P, we are able to eliminate the *cut*-rule as follows.

$$\dfrac{\vdash_\alpha P \quad P \vdash_\beta}{\vdash_{\alpha\cap\beta}}\ cut \quad \rhd \quad \vdash_{\alpha\cap\beta}$$

where $\alpha \cap \beta = \emptyset$ by the definition of $\mathbf{J}$, and hence $\vdash_{\alpha\cap\beta}$, i.e., $\vdash_\emptyset$ is the non-logical axiom.
• When the last rule of π_1 is wR, we are able to eliminate the *cut*-rule as follows.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_{11}\\ \Gamma \vdash_\alpha \Delta\end{array}}{\Gamma \vdash_\alpha \Delta, P}\ wR \quad P \vdash_\beta}{\Gamma \vdash_{\alpha\cap\beta} \Delta}\ cut \quad \rhd \quad \dfrac{\begin{array}{c}\vdots\ \pi_{11}\\ \Gamma \vdash_\alpha \Delta\end{array}}{\Gamma \vdash_{\alpha\cap\beta} \Delta}\ sub$$

In this way, by induction on the complexity of the cut-formula and on the distance from the place where the cut-formula is introduced, the cut-elimination theorem is proved.

Remark 2 ($\vdash_\emptyset$) Note that $\vdash_\emptyset$ is required to be a non-logical axiom in our proof of the cut-elimination theorem, when we eliminate *cut*-rule between two non-logical axioms $\vdash_\alpha P$ and $P \vdash_\beta$.

In the above explanation, we have excluded *c*-rule. For example, when the last rule of π_2 is cL-rule for the cut-formula φ as in the following proof, the above naive transformation does not work.

$$\dfrac{\dfrac{\vdots\,\pi_1}{\Gamma \vdash_{\alpha} \Delta, \varphi} \quad \dfrac{\dfrac{\vdots\,\pi_2}{\varphi, \varphi, \Sigma \vdash_{\beta} \Lambda}}{\varphi, \Sigma \vdash_{\beta} \Lambda}\,cL}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda}\,cut \quad \rhd \quad \dfrac{\dfrac{\dfrac{\vdots\,\pi_1}{\Gamma \vdash_{\alpha} \Delta, \varphi} \quad \dfrac{\dfrac{\vdots\,\pi_1}{\Gamma \vdash_{\alpha} \Delta, \varphi} \quad \dfrac{\vdots\,\pi_2}{\varphi, \varphi, \Sigma \vdash_{\beta} \Lambda}}{\varphi, \Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda}\,cut}{\Gamma, \Gamma, \Sigma \vdash_{\alpha\cap(\alpha\cap\beta)} \Delta, \Delta, \Lambda}\,cut}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda}\,c$$

This is because the *cut*-rule is duplicated without reducing the complexity of the cut-formula, and it is difficult to determine the distance between the lower *cut*-rule and the place where the cut-formula is introduced.

Thus, to deal with the case of *c*-rule, *cut*-rule is slightly generalized to the rule called *multicut*-, or *mix*-rule, which can eliminate multiple cut-formulas simultaneously. Then, the *multicut*-elimination theorem, which implies the cut-elimination, is proved by applying essentially the same transformations of given proofs as above. The same applies to our Lg, and see, for example, [13, 15] for a detailed proof. To deal with *c*-rule, there are other methods; for example, it is possible to modify the whole system so that *c*-rule is contained in other inference rules implicitly. See [37] for such approaches.

One of the main consequences of the cut-elimination theorem is the following subformula property.

Proposition 7 (**Subformula property**) *If $\Gamma \vdash_{\alpha} \Delta$ is provable, then there exists a proof of $\Gamma \vdash_{\alpha} \Delta$ that contains only the subformulas of formulas from Γ and Δ.*

Proof If $\Gamma \vdash_{\alpha} \Delta$ is provable, by the cut-elimination theorem, it is provable without *cut*-rule. Other than *cut*-rule, the upper sequents of every inference rule contain only subformulas of formulas contained in the lower sequent. □

By the subformula property, Lg is syntactically shown to be consistent.

Proposition 8 (**Consistency**) Lg *is consistent. That is, $\vdash_N$ is not provable in* Lg.

Proof If $\vdash_N$ is provable, by the subformula property, there exists a proof that does not contain any formula. However, this is impossible because all axioms, other than $\vdash_{\emptyset}$ that is not equivalent to $\vdash_N$, contain a formula. □

2.5 Semantics of Lg

Our semantics is constructed based on the idea of the phase semantics of linear logic, cf. [14, 29]. Girad explains the idea of phase semantics in [14] as follows. The semantic counterpart of a formula is a *fact*, and it is regarded as a set of tasks to verify the fact. These tasks can be seen as *phases* between a fact and its verification. This idea can be applied to our semantics of Lg. Our semantics is defined in terms of groups over N (instead of tasks). Every formula is interpreted as certain set of groups whose members accept the formula. A formula is true if the whole group N accepts the formula.

Let us first introduce the closure conditions that the interpretation of every formula should satisfy.

Definition 7 (**Closure condition**) $X \subseteq \mathcal{P}(N)$ is said to be:

1. $\subseteq$**-closed** if $\alpha \in X$ and $\beta \subseteq \alpha$ imply $\beta \in X$;
2. $\cup$**-closed** if $\alpha, \beta \in X$ implies $\alpha \cup \beta \in X$.

We next define a special set of groups denoted by $\perp$ as follows.

Definition 8 ($\perp$) We define $\perp \subseteq \mathcal{P}(N)$ so that it is $\subseteq$-closed and $\cup$-closed, and $N \notin \perp$.

$\perp$ is intended to represent the absurdity, and hence, to avoid an inconsistent model, where all formulas are true, we assume $N \notin \perp$ in the above definition. (Cf. Definition 12.)

Depending on concrete construction of $\perp$, different set-theoretical structures are induced, where formulas are interpreted. Cf. Example 4.

The set-theoretical operations corresponding to the connectives are defined as follows.

Definition 9 (**Operations**) For any $X, Y \subseteq \mathcal{P}(N)$, operations $\wedge, \rightarrow, \neg$ and $\vee$ are defined as follows.

- $X \wedge Y = \{\alpha \cap \beta \mid \alpha \in X, \beta \in Y\}$
- $X \rightarrow Y = \{\alpha \mid X \wedge \{\alpha\} \subseteq Y\}$
- $\neg X = X \rightarrow \perp = \{\alpha \mid X \wedge \{\alpha\} \subseteq \perp\}$
- $X \vee Y = \neg\neg(X \cup Y)$

We usually write $X \wedge \{\alpha\}$ as $X \wedge \alpha$ by abbreviating { and } for simplicity.

It is shown that $X \wedge Y$ and the usual intersection $X \cap Y$ are equivalent for any $\subseteq$-closed X and Y.

Lemma 1 ($\wedge$ and $\cap$) *$X \wedge Y = X \cap Y$ for any $\subseteq$-closed X and Y.*

Proof Let $\alpha \in X \wedge Y$. Then $\alpha = \alpha_1 \cap \alpha_2$ such that $\alpha_1 \in X$ and $\alpha_2 \in Y$ by definition. Because X and Y are $\subseteq$-closed, we have $\alpha_1 \cap \alpha_2 \in X$ and $\alpha_1 \cap \alpha_2 \in Y$, that is, $\alpha \in X \cap Y$.

Conversely, let $\alpha \in X \cap Y$. Then $\alpha \in X$ and $\alpha \in Y$, and hence, $\alpha = \alpha \cap \alpha \in X \wedge Y$. □

By the above lemma, the usual properties of $\cap$ also hold for $\wedge$ when we consider $\subseteq$-closed sets. In the following discussion, we apply such properties without explicitly referring to the above lemma.

Let us investigate some properties of $\neg$.

Lemma 2 *For any* $X, Y \subseteq \mathcal{P}(N)$,

1. $X \subseteq \neg\neg X$
2. $\neg\neg\neg X = \neg X$
3. *If* $X \subseteq Y$ *then* $\neg Y \subseteq \neg X$
4. $\bot \subseteq \neg X$
5. $\neg\neg\bot = \bot$

Proof (1) By the definition of $\neg$, we have $X \wedge \neg X \subseteq \bot$. Hence, again by the definition of $\neg$, we have $X \subseteq \neg\neg X$. (3) Let $\alpha \in \neg Y$, that is, $\alpha \wedge Y \subseteq \bot$. Because $X \subseteq Y$, we have $\alpha \wedge X \subseteq \bot$, that is $\alpha \in \neg X$. (2) is obtained from (1) and (3). (4) Because $\bot \subseteq \bot$, we have $\bot \wedge X \subseteq \bot$, that is, $\bot \subseteq \neg X$. (5) $\bot \subseteq \neg\neg\bot$ is obtained from (1). To show $\neg\neg\bot \subseteq \bot$, assume $\alpha \in \neg\neg\bot$, that is, $\alpha \wedge \neg\bot \subseteq \bot$. Because $N \in \neg\bot$, which is equivalent to $\bot \subseteq \bot$, we have $\alpha = \alpha \cap N \in \bot$. □

Definition 10 (**Closed set**) $X \subseteq \mathcal{P}(N)$ is said to be **$\neg\neg$-closed** if $\neg\neg X = X$. $X \subseteq \mathcal{P}(N)$ is said to be **closed** if it is (1) $\subseteq$-closed, (2) $\cup$-closed, and (3) $\neg\neg$-closed.

In particular, $\cup$-closedness is required to show the soundness of mer-rule.

Lemma 3

1. $X \wedge Y \subseteq Z$ *implies* $X \wedge \neg Z \subseteq \neg Y$.
2. $X \wedge Y \subseteq Z$ *implies* $\neg\neg X \wedge Y \subseteq Z$ *for any* $\neg\neg$*-closed* Z.
3. $X \vee Y = \neg X \to Y$ *for any* $\neg\neg$*-closed* Y.

Proof (1) Because $Z \wedge \neg Z \subseteq \bot$, $X \wedge Y \subseteq Z$ implies $(X \wedge Y) \wedge \neg Z \subseteq \bot$. Thus, by the definition of $\neg$, we have $X \wedge \neg Z \subseteq \neg Y$.
(2) From $X \wedge Y \subseteq X \wedge Y$, by applying (1) twice, we obtain $\neg\neg X \wedge Y \subseteq \neg\neg(X \wedge Y)$. On the other hand, from $X \wedge Y \subseteq Z$, we have $\neg\neg(X \wedge Y) \subseteq \neg\neg Z = Z$ because Z is $\neg\neg$-closed. Thus, we have $\neg\neg X \wedge Y \subseteq Z$.
(3) We first show $\neg\neg(\neg X \to Y) \subseteq \neg X \to Y$ for any $\neg\neg$-closed Y. From $\neg X \wedge (\neg X \to Y) \subseteq Y$, by applying (2), we obtain $\neg X \wedge \neg\neg(\neg X \to Y) \subseteq Y$, and hence, we have $\neg\neg(\neg X \to Y) \subseteq \neg X \to Y$. We now show $X \vee Y \subseteq \neg X \to Y$. By definition, we have $X \wedge \neg X \subseteq \bot$, and by Lemma 2(4), we have $\bot \subseteq \neg\neg Y = Y$. Thus, we have $X \wedge \neg X \subseteq Y$, and hence, we have $X \subseteq \neg X \to Y$. Thus, together with the fact $Y \wedge \neg X \subseteq Y$, which implies $Y \subseteq \neg X \to Y$, we obtain $X \cup Y \subseteq \neg X \to Y$. Hence, we have $X \vee Y = \neg\neg(X \cup Y) \subseteq \neg\neg(\neg X \to Y) \subseteq \neg X \to Y$. For the other direction, we show $\neg X \to Y \subseteq X \vee Y$. From $X \subseteq X \vee Y$, we have $\neg(X \vee Y) \subseteq \neg X$, which implies $\neg(X \vee Y) \wedge (\neg X \to Y) \subseteq \neg X \wedge (\neg X \to Y)$. Because $\neg X \wedge (\neg X \to Y) \subseteq Y \subseteq X \vee Y$, we have $\neg(X \vee Y) \wedge (\neg X \to Y) \subseteq X \vee Y$, and hence, $\neg(X \vee Y) \wedge \neg(X \vee Y) \wedge (\neg X \to Y) \subseteq \bot$. By the idempotency of $\wedge$, we have $\neg(X \vee Y) \wedge (\neg X \to Y) \subseteq \bot$, which implies $\neg X \to Y \subseteq \neg\neg(X \vee Y) = X \vee Y$. □

Lemma 4 (**Closed set**) *$\neg X$, $X \wedge Y$, $X \to Y$, $X \vee Y$ are all closed, for any closed X and Y.*

Proof To show that $\neg X$ is closed, (1) assume $\alpha \in \neg X$ and $\beta \subseteq \alpha$. By definition, $\alpha \in \neg X$ means $\alpha \wedge X \subseteq \bot$, that is, for any $\gamma \in X$, $\alpha \cap \gamma \in \bot$. Then, because $\beta \subseteq \alpha$, we have $\beta \cap \gamma \subseteq \alpha \cap \gamma \in \bot$. Because $\bot$ is $\subseteq$-closed, we have $\beta \cap \gamma \in \bot$ for any $\gamma \in X$, that is, $\beta \in \neg X$. (2) Assume $\alpha, \beta \in \neg X$. We show $\alpha \cup \beta \in \neg X$, that is, $(\alpha \cup \beta) \wedge X \subseteq \bot$. By definition, we have $\alpha \wedge X \subseteq \bot$ and $\beta \wedge X \subseteq \bot$. Hence, for any $\gamma \in X$, we have $\alpha \cap \gamma, \beta \cap \gamma \in \bot$. Because $\bot$ is $\cup$-closed, we have $(\alpha \cap \gamma) \cup (\beta \cap \gamma) = (\alpha \cup \beta) \cap \gamma \in \bot$. (3) The $\neg\neg$-closedness of $\neg X$ is obtained by Lemma 2.

To show $X \wedge Y$ is closed, (1) assume $\alpha \in X \wedge Y$ and $\beta \subseteq \alpha$. Then, $\alpha = \alpha_1 \cap \alpha_2$ such that $\alpha_1 \in X$ and $\alpha_2 \in Y$. Because $\beta \subseteq \alpha = \alpha_1 \cap \alpha_2$, we have $\beta \subseteq \alpha_1 \in X$ and $\beta \subseteq \alpha_2 \in Y$. Because X and Y are $\subseteq$-closed, we have $\beta \in X$ and $\beta \in Y$, and hence, we have $\beta = \beta \cap \beta \in X \wedge Y$. (2) Assume $\alpha, \beta \in X \wedge Y$. Then, we have $\alpha, \beta \in X$ and $\alpha, \beta \in Y$. Because X and Y are $\cup$-closed, we have $\alpha \cup \beta \in X$ and $\alpha \cup \beta \in Y$, which imply $\alpha \cup \beta \in X \wedge Y$. (3) To show the $\neg\neg$-closedness of $X \wedge Y$, we use the following calculation. $X \wedge Y \subseteq X$ implies $\neg\neg(X \wedge Y) \subseteq \neg\neg X$, where $\neg\neg X = X$ because X is $\neg\neg$-closed. Similarly, we have $\neg\neg(X \wedge Y) \subseteq Y$. Hence, we have $\neg\neg(X \wedge Y) \subseteq X \wedge Y$.

To show that $X \to Y$ is closed, (1) assume $\alpha \in X \to Y$ and $\beta \subseteq \alpha$. By $\alpha \in X \to Y$, for any $\gamma \in X$, we have $\gamma \cap \alpha \in Y$. Because $\beta \subseteq \alpha$, we have $\gamma \cap \beta \subseteq \gamma \cap \alpha \in Y$, and hence, we have $\gamma \cap \beta \in Y$ because Y is $\subseteq$-closed. Thus, $\beta \in X \to Y$. (2) Assume $\alpha, \beta \in X \to Y$. We show $\alpha \cup \beta \in X \to Y$, that is, for any $\gamma \in X$, $(\alpha \cup \beta) \cap \gamma \in Y$. $\alpha \in X \to Y$ and $\gamma \in X$ imply $\alpha \cap \gamma \in Y$, and similarly, $\beta \in X \to Y$ and $\gamma \in X$ imply $\beta \cap \gamma \in Y$. Because Y is $\cup$-closed, we have $(\alpha \cup \beta) \cap \gamma = (\alpha \cap \gamma) \cup (\beta \cap \gamma) \in Y$. (3) To show the $\neg\neg$-closedness of $X \to Y$, we calculate the following. By Lemma 3(2), $X \wedge (X \to Y) \subseteq Y$ implies $X \wedge \neg\neg(X \to Y) \subseteq \neg\neg Y$, where $\neg\neg Y = Y$ because Y is $\neg\neg$-closed. Thus, by the definition of $\to$, we have $\neg\neg(X \to Y) \subseteq X \to Y$.

Because $X \vee Y = \neg\neg(X \cup Y)$, (1) $\subseteq$-closedness and (2) $\cup$-closedness of $X \vee Y$ are obtained by the same way as those of $\neg X$ above. (3) The $\neg\neg$-closedness of $X \vee Y$ is immediate because it is defined as$\neg\neg(X \cup Y)$. □

Every formula is interpreted by a closed set.

Definition 11 (**Model**) Let agenda $\mathcal{A}$ and profile $\mathbf{J}$ be given. Let $\bot$ be fixed. Then, $*$ is an interpretation function from the set of formulas to the set of closed sets over N, defined as follows.

- $P^* = \neg\neg\mathcal{G}(P) = \neg\neg\{\alpha \mid P \in J_i \text{ for any } i \in \alpha\}$
- $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$
- $(\varphi \to \psi)^* = \varphi^* \to \psi^*$
- $(\neg\varphi)^* = \neg\varphi^* = \varphi^* \to \bot$
- $(\varphi \vee \psi)^* = \varphi^* \vee \psi^*$

We call a pair $(\bot, *)$ a **model**.

The intended interpretation of an atom P is $\mathcal{G}(P)$, i.e., the set of groups whose members accept P. However, $\mathcal{G}(P)$ itself is not closed, and hence, we define P^* by using the $\neg\neg$-closure. Cf. Remark 5.

The interpretation of any formula is shown to be a closed set by Lemma 4.

Lemma 5 (**Interpretation**) *φ^* is closed for any formula φ.*

Definition 12 (**Truth**)

- φ is **true** in a model $(\bot, *)$ if $N \in \varphi^*$.
- φ is **valid** if it is true in any model.

Example 4 (Model) Let $N = \{1, 2, 3\}$, $\bot = \{\emptyset\}$, $P^* = \mathcal{P}(\{1, 3\}) = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$, $Q^* = \mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$. Cf. Example 1. In this model, $\neg P^* = \{\emptyset, \{2\}\}$ and $\neg\neg P^* = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\} = P^*$. Note that $\neg P^*$ and the complement of P^* are not equivalent. Furthermore, $N \notin P^*$ and $N \notin \neg P^*$, and hence, neither P or $\neg P$ is true in this model. This model is applied in Examples 6 and 7.

Note that $\bot$ cannot be restricted to the above set $\{\emptyset\}$ to prove our completeness theorem of Lg. See our canonical model given in Definition 13.

For an interpretation of the sequent $\Gamma \vdash_\alpha \Delta$, where $\Gamma \equiv \varphi_1, \ldots, \varphi_k$ and $\Delta \equiv \psi_1, \ldots, \psi_l$, by Γ^* we denote $\varphi_1^* \wedge \cdots \wedge \varphi_k^*$, and by Δ^* we denote $\psi_1^* \vee \cdots \vee \psi_l^*$. To avoid the notational complexity, we simply write Γ^* and Δ^* without mentioning the corresponding connective $\wedge$ or $\vee$, which is clear from the context. When given sequent is of the form $\Gamma \vdash_\alpha$, the right-hand side of the sequent, i.e., the empty context is interpreted as $\bot$.

We implicitly use Lemma 3 to prove the soundness theorem, in particular to prove the soundness of the right rules.

Lemma 6 (**Soundness**) *If $\Gamma \vdash_\alpha \Delta$ is provable, then $\alpha \in \Gamma^* \to \Delta^*$ in any model* $(\bot, *)$.

Proof We show the lemma by induction on the length of the given proof as usual.
- When the given $\Gamma \vdash_\alpha \Delta$ is a logical axiom of the form $P \vdash_N P$, we have $N \in P^* \to P^*$, because it is equivalent to $P^* \subseteq P^*$.
- When the given $\Gamma \vdash_\alpha \Delta$ is a non-logical axiom of the form $\vdash_\emptyset$, we have $\emptyset \in \bot$ because $\bot$ is $\subseteq$-closed.
- When the given $\Gamma \vdash_\alpha \Delta$ is a non-logical axiom of the form $\vdash_\alpha P$, $\alpha \in P^*$ is obtained by the definition of the interpretation of atoms.
- When the given $\Gamma \vdash_\alpha \Delta$ is a non-logical axiom of the form $P \vdash_\alpha$, we show $\alpha \wedge P^* \subseteq \bot$. Note that $\mathcal{G}(\neg P) \wedge \mathcal{G}(P) = \{\emptyset\} \subseteq \bot$. Thus, $\mathcal{G}(\neg P) \wedge \neg\neg\mathcal{G}(P) \subseteq \bot$ by Lemma 3 (2), that is, $\mathcal{G}(\neg P) \wedge P^* \subseteq \bot$. Because $P \vdash_\alpha$ is a non-logical axiom, we have $\alpha \in \mathcal{G}(\neg P)$, and hence, we have $\alpha \wedge P^* \subseteq \bot$.

The induction step is divided into the following cases, depending on the last rule applied in the given proof. By Lemma 3, to show $X \subseteq Y \vee Z$ especially in the right rules, we show $X \wedge \neg Y \subseteq Z$.

- When $\dfrac{\Gamma \vdash_\alpha \Delta, \varphi \quad \Sigma \vdash_\beta \Lambda, \psi}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda, \varphi \wedge \psi} \wedge R$, we show $\Gamma^* \wedge \Sigma^* \wedge (\alpha \cap \beta) \wedge \neg\Delta^* \wedge \neg\Lambda^* \subseteq \varphi^* \wedge \psi^*$, that is, $\gamma \cap \sigma \cap (\alpha \cap \beta) \cap \delta \cap \lambda \in \varphi^* \wedge \psi^*$ for any $\gamma \in \Gamma^*$, $\sigma \in \Sigma^*$, $\delta \in \neg\Delta^*$, and $\lambda \in \neg\Lambda^*$. This is obtained by the induction hypotheses $\gamma \cap \alpha \cap \delta \in \varphi^*$ and $\sigma \cap \beta \cap \lambda \in \psi^*$, as well as by the definition of $\wedge$.
- When $\dfrac{\varphi, \Gamma \vdash_\alpha \Delta}{\varphi \wedge \psi, \Gamma \vdash_\alpha \Delta} \wedge L1$, we show $\tau \cap \gamma \cap \alpha \in \Delta^*$ for any $\tau \in \varphi^* \wedge \psi^*$, $\gamma \in \Gamma^*$. Because $\tau \in \varphi^* \wedge \psi^*$ implies $\tau \in \varphi^*$, by the induction hypothesis, we have $\tau \cap \gamma \cap \alpha \in \Delta^*$.
- When $\dfrac{\varphi, \Gamma \vdash_\alpha \Delta, \psi}{\Gamma \vdash_\alpha \Delta, \varphi \to \psi} \to R$, we show $\Gamma^* \wedge \alpha \wedge \neg\Delta^* \subseteq \varphi^* \to \psi^*$, which is immediately obtained from the induction hypothesis $\varphi^* \wedge \Gamma^* \wedge \alpha \wedge \neg\Delta^* \subseteq \psi^*$ by the definition of $\to$.
- When $\dfrac{\Gamma \vdash_\alpha \Delta, \varphi \quad \psi, \Sigma \vdash_\beta \Lambda}{\varphi \to \psi, \Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda} \to L$, we show $\tau \cap \gamma \cap \sigma \cap (\alpha \cap \beta) \cap \delta \in \Lambda^*$ for any $\tau \in \varphi^* \to \psi^*$, $\gamma \in \Gamma^*$, $\sigma \in \Sigma^*$, and $\delta \in \neg\Delta^*$. $\tau \in \varphi^* \to \psi^*$ implies $\tau \wedge \varphi^* \subseteq \psi^*$ and, because $\gamma \cap \alpha \cap \delta \in \varphi^*$ by the induction hypothesis, we have $\tau \cap \gamma \cap \alpha \cap \delta \in \psi^*$. Furthermore, because $\psi^* \wedge (\sigma \cap \beta) \subseteq \Lambda^*$ by the induction hypothesis, we have $\tau \cap \gamma \cap \alpha \cap \delta \cap \sigma \cap \beta \in \Lambda^*$.
- When $\dfrac{\Gamma \vdash_\alpha \Delta, \varphi}{\Gamma \vdash_\alpha \Delta, \varphi \vee \psi} \vee R$, we show, for any $\gamma \in \Gamma^*$ and $\delta \in \neg\Delta^*$, $\gamma \cap \alpha \cap \delta \in \varphi^* \vee \psi^*$. This is obtained from the induction hypothesis $\gamma \cap \alpha \cap \delta \in \varphi^*$ and the fact $\varphi^* \subseteq \varphi^* \vee \psi^*$.
- When $\dfrac{\varphi, \Gamma \vdash_\alpha \Delta \quad \psi, \Gamma \vdash_\alpha \Delta}{\varphi \vee \psi, \Gamma \vdash_\alpha \Delta} \vee L$, we first show $\tau \cap \gamma \cap \alpha \in \Delta^*$ for any $\tau \in \varphi^* \cup \psi^*$ and $\gamma \in \Gamma^*$. Whichever $\tau \in \varphi^*$ or $\tau \in \psi^*$, we obtain $\tau \cap \gamma \cap \alpha \in \Delta^*$ by the induction hypotheses. Thus, we obtain $(\varphi^* \cup \psi^*) \wedge \Gamma^* \wedge \alpha \subseteq \Delta^*$, which implies $\neg\neg(\varphi^* \cup \psi^*) \wedge \Gamma^* \wedge \alpha \subseteq \Delta^*$ by Lemma 3.
- When $\dfrac{\varphi, \Gamma \vdash_\alpha \Delta}{\Gamma \vdash_\alpha \Delta, \neg\varphi} \neg R$, we show $\Gamma^* \wedge \alpha \wedge \neg\Delta^* \subseteq \neg\varphi^*$, which is obtained from the induction hypothesis $\varphi^* \wedge \Gamma^* \wedge \alpha \subseteq \Delta^*$.
- When $\dfrac{\Gamma \vdash_\alpha \Delta, \varphi}{\neg\varphi, \Gamma \vdash_\alpha \Delta} \neg L$, we show $\neg\varphi^* \wedge \Gamma^* \wedge \alpha \subseteq \Delta^*$, which is immediately obtained from the induction hypothesis.
- When $\dfrac{\Gamma \vdash_\alpha \Delta}{\Gamma \vdash_\alpha \Delta, \varphi} wR$, we show $\gamma \cap \alpha \cap \delta \in \varphi^*$ for any $\gamma \in \Gamma^*$ and $\delta \in \neg\Delta^*$. This is obtained by the induction hypothesis $\gamma \cap \alpha \in \Delta^*$, which is equivalent to $\gamma \cap \alpha \wedge \neg\Delta^* \subseteq \bot$, and the fact $\bot \subseteq \varphi^*$ (Lemma 2 (4)).
- When $\dfrac{\Gamma \vdash_\alpha \Delta}{\varphi, \Gamma \vdash_\alpha \Delta} wL$, we show $\tau \cap \gamma \cap \alpha \in \Delta^*$ for any $\tau \in \varphi^*$ and $\gamma \in \Gamma^*$. This is obtained from $\tau \cap \gamma \cap \alpha \subseteq \gamma \cap \alpha \in \Delta^*$ by the induction hypothesis and by the $\subseteq$-closedness of Δ^*.
- When $\dfrac{\varphi, \varphi, \Gamma \vdash_\alpha \Delta}{\varphi, \Gamma \vdash_\alpha \Delta} cL$, we show $\varphi^* \wedge \Gamma^* \wedge \alpha \subseteq \Delta^*$, which is obtained from the fact $\varphi^* = \varphi^* \wedge \varphi^*$ and the induction hypothesis.
- When $\dfrac{\Gamma \vdash_\alpha \Delta, \varphi, \varphi}{\Gamma \vdash_\alpha \Delta, \varphi} cR$, we show $\Gamma^* \wedge \alpha \wedge \neg\Delta^* \subseteq \varphi^*$, which is obtained from the fact $\varphi^* = \varphi^* \vee \varphi^*$ and the induction hypothesis.

- When $\dfrac{\Gamma \vdash_{\alpha} \Delta, \varphi \quad \varphi, \Sigma \vdash_{\beta} \Lambda}{\Gamma, \Sigma \vdash_{\alpha \cap \beta} \Delta, \Lambda}$ *cut*, we show $\gamma \cap \sigma \cap (\alpha \cap \beta) \cap \delta \in \Lambda^*$ for any $\gamma \in \Gamma^*$, $\sigma \in \Sigma^*$, $\delta \in \neg\Delta^*$, which is obtained from the induction hypotheses $\gamma \cap \alpha \cap \delta \in \varphi^*$ and $\varphi^* \wedge (\sigma \cap \beta) \subseteq \Lambda^*$.
- When $\dfrac{\Gamma \vdash_{\alpha} \Delta}{\Gamma \vdash_{\beta} \Delta}$ *sub* with $\beta \subseteq \alpha$, we show $\gamma \cap \beta \in \Delta^*$ for any $\gamma \in \Gamma^*$. $\beta \subseteq \alpha$ implies $\gamma \cap \beta \subseteq \gamma \cap \alpha$. Then, by the induction hypothesis $\gamma \cap \alpha \in \Delta^*$ and by the $\subseteq$-closedness of Δ^*, we have $\gamma \cap \beta \in \Delta^*$.
- When $\dfrac{\Gamma \vdash_{\alpha} \Delta \quad \Sigma \vdash_{\beta} \Lambda}{\Gamma, \Sigma \vdash_{\alpha \cup \beta} \Delta, \Lambda}$ *mer*, we show $\gamma \cap \sigma \cap (\alpha \cup \beta) \in \Delta^* \vee \Lambda^*$. By the induction hypothesis, we have $\gamma \cap \alpha \in \Delta^* \subseteq \Delta^* \vee \Lambda^*$ and $\sigma \cap \beta \in \Lambda^* \subseteq \Delta^* \vee \Lambda^*$. Because $\Delta^* \vee \Lambda^*$ is $\cup$-closed, we have $(\gamma \cap \alpha) \cup (\sigma \cap \beta) \in \Delta^* \vee \Lambda^*$. Because $\gamma \cap \sigma \cap (\alpha \cup \beta) \subseteq (\gamma \cap \alpha) \cup (\sigma \cap \beta)$, by the $\subseteq$-closedness of $\Delta^* \vee \Lambda^*$, we have $\gamma \cap \sigma \cap (\alpha \cup \beta) \in \Delta^* \vee \Lambda^*$. □

As a particular case of the above lemma, when $\alpha = N$, we obtain the following soundness theorem.

Theorem 1 (**Soundness**) *If $\varphi_1, \ldots, \varphi_k \vdash_N \psi_1, \ldots, \psi_l$ is provable, then $\varphi_1 \wedge \cdots \wedge \varphi_k \rightarrow \psi_1 \vee \cdots \vee \psi_l$ is valid.*

The consistency of Lg is also obtained semantically as a corollary of the soundness theorem.

Corollary 1 (**Consistency**) Lg *is consistent. That is, $\vdash_N$ is not provable in* Lg.

Proof If $\vdash_N$ is provable, then we have $N \in \bot$ in any model $(\bot, *)$ by the soundness theorem. However, this is not the case by the definition of $\bot$. Therefore, $\vdash_N$ is not provable in Lg. □

Remark 3 ($\vee L$) Let us consider the following form of the left-rule for $\vee$:

$$\frac{\varphi, \Gamma \vdash_{\alpha} \Delta \quad \psi, \Sigma \vdash_{\beta} \Lambda}{\varphi \vee \psi, \Gamma, \Sigma \vdash_{\alpha \cup \beta} \Delta, \Lambda} \vee L'$$

Although this $\vee L'$ has the dual form of our $\wedge R$-rule, the rule is unsound. This is because $(\varphi^* \vee \psi^*) \wedge (\alpha \cup \beta) \subseteq \Delta^* \vee \Lambda^*$ does not generally follow from $\varphi^* \wedge \alpha \subseteq \Delta^*$ and $\psi^* \wedge \beta \subseteq \Lambda^*$, where we omit irrelevant contexts Γ, Σ. Furthermore, from the syntactic viewpoint, the inconsistency $\vdash_N$ may be provable with the above $\vee L'$-rule as follows. Let $\vdash_N P$, $P \vdash_{\emptyset}$, $\vdash_{\emptyset} Q$, $Q \vdash_N$ are non-logical axioms:

$$\frac{\dfrac{\vdash_N P}{\vdash_N P \vee Q} \vee R \qquad \dfrac{P \vdash_{\emptyset} \quad Q \vdash_N}{P \vee Q \vdash_N} \vee L'}{\vdash_N} cut$$

Remark 4 (Kripke model) Our model of Lg can be regarded as the usual Kripke model in the fragment without the disjunction. (Disjunction destroys the simple correspondence). Let us consider every group α which belongs to a closed set φ^* as a possible world. Let us consider the pair $(\mathcal{G}, \subseteq)$. The $\subseteq$-closedness corresponds to the monotonicity condition in Kripke semantics, although the order is reversed: $\alpha \in \varphi^*$ *and* $\beta \subseteq \alpha$ *imply* $\beta \in \varphi^*$. Then, for the interpretation of connectives $\wedge$ and $\rightarrow$ ($\neg\varphi$ is defined as $\varphi \rightarrow \bot$), we have the following correspondence: $\alpha \in \varphi^* \wedge \psi^*$ iff $\alpha \in \varphi^*$ and $\alpha \in \psi^*$, and $\alpha \in \varphi^* \rightarrow \psi^*$ *iff for all* $\beta \subseteq \alpha, \beta \in \varphi^*$ *implies* $\beta \in \psi^*$. Thus, $\alpha \in \varphi^*$ *in our model of* Lg *if and only if* $\alpha \models \varphi$ *in Kripke model.*

2.6 Semantic Completeness of Lg

To prove the semantic completeness theorem, we slightly extend the notion of the model by introducing indexes of contexts.

Definition 13 (**Canonical model**) We extend the set of groups $\mathcal{G}$ over N to the following set $\mathcal{G}_C$ by introducing indexes of contexts.

$$\mathcal{G}_C = \{\alpha_{\Gamma;\Delta} \mid \alpha \subseteq N, \text{ and } \Gamma, \Delta \text{ are sets of formulas } \}$$

The set-theoretical operations $\cap$ and $\cup$ are extended as follows.

$$\alpha_{\Gamma;\Delta} \cap \beta_{\Sigma;\Lambda} = (\alpha \cap \beta)_{\Gamma\Sigma;\Delta\Lambda} \quad \text{and} \quad \alpha_{\Gamma;\Delta} \cup \beta_{\Sigma;\Lambda} = (\alpha \cup \beta)_{\Gamma\Sigma;\Delta\Lambda}$$

where $\Gamma\Sigma$ (and $\Delta\Lambda$) is the abbreviation for the union $\Gamma \cup \Sigma$ (and $\Delta \cup \Lambda$).
The subset relation is also extended as follows.

$\alpha_{\Gamma;\Delta} \subseteq \beta_{\Sigma;\Lambda}$ if (1) $\alpha \subseteq \beta$ and $\Gamma = \Sigma$ and $\Delta = \Lambda$ or
(2) $\alpha = \beta$ and $\Gamma \supseteq \Sigma$ and $\Delta \supseteq \Lambda$,

We define $\bot$ as follows.

$$\bot = \{\alpha_{\Gamma;\Delta} \mid \Gamma \vdash_\alpha \Delta \text{ is provable in Lg}\}$$

In the canonical model, we further define the set of groups $[\![\varphi]\!]$ as follows.

$$[\![\varphi]\!] = \{\alpha_{\Gamma;\Delta} \mid \Gamma \vdash_\alpha \Delta, \varphi \text{ is provable in Lg}\}$$

Although, in terms of syntax, the context Γ is a multiset of formulas, in our canonical model, the context Γ is a set of formulas that makes $\alpha_{\Gamma;\Delta} \cap \alpha_{\Gamma;\Delta} = \alpha_{\Gamma;\Delta}$ hold.

Strictly speaking, our canonical model is not exactly a model of Lg because the domain of the canonical model is extended from the simple $\mathcal{P}(N)$ by the introduction of the indexes of contexts. If we define the notion of a general model by introducing a monoid for contexts from the beginning, or if we define it more abstractly as a certain algebraic structure, we can avoid the gap between general models and our canonical model. However, this gap exists mainly in a notational difference, and this approach introduces inessential complication or abstraction in the semantics of Lg. Thus, at the expense of technical rigor, we maintain our simple semantics of groups in this article.

We first show that $[\![\varphi]\!]$ is $\neg\neg$-closed.

Lemma 7

1. $\neg[\![\varphi]\!] = [\![\neg\varphi]\!]$
2. $\neg[\![\neg\varphi]\!] = [\![\varphi]\!]$

3. $\neg\neg[\![\varphi]\!] = [\![\varphi]\!]$

Proof (1) $\Rightarrow$) Let $\alpha_{\Gamma;\Delta} \in \neg[\![\varphi]\!]$, that is, $\alpha_{\Gamma;\Delta} \wedge [\![\varphi]\!] \subseteq \bot$. Because $N_{\varphi;\emptyset} \in [\![\varphi]\!]$, we have $\alpha_{\Gamma\varphi;\Delta} \in \bot$, that is, $\Gamma, \varphi \vdash_\alpha \Delta$ is provable in Lg. By applying $\neg R$-rule as follows, we have $\Gamma \vdash_\alpha \Delta, \neg\varphi$ is provable in Lg, that is, $\alpha_{\Gamma;\Delta} \in [\![\neg\varphi]\!]$.

$$\frac{\Gamma, \varphi \vdash_\alpha \Delta}{\Gamma \vdash_\alpha \Delta, \neg\varphi}\ \neg R$$

$\Leftarrow$) Let $\alpha_{\Gamma;\Delta} \in [\![\neg\varphi]\!]$, that is, $\Gamma \vdash_\alpha \Delta, \neg\varphi$ is provable in Lg. We show $\alpha_{\Gamma;\Delta} \in \neg[\![\varphi]\!]$, that is, $\alpha_{\Gamma;\Delta} \wedge [\![\varphi]\!] \subseteq \bot$. Assume $\beta_{\Sigma;\Lambda} \in [\![\varphi]\!]$, that is, $\Sigma \vdash_\beta \Lambda, \varphi$ is provable. Then, we have:

$$\frac{\Gamma \vdash_\alpha \Delta, \neg\varphi \quad \dfrac{\Sigma \vdash_\beta \Lambda, \varphi}{\neg\varphi, \Sigma \vdash_\beta \Lambda}\ \neg L}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda}\ cut$$

Thus, we have $(\alpha \cap \beta)_{\Gamma\Sigma;\Delta\Lambda} \in \bot$, and hence, we have $\alpha_{\Gamma;\Delta} \wedge [\![\varphi]\!] \subseteq \bot$.
(2) $\Rightarrow$) Let $\alpha_{\Gamma;\Delta} \in \neg[\![\neg\varphi]\!]$, that is, $\alpha_{\Gamma;\Delta} \wedge [\![\neg\varphi]\!] \subseteq \bot$. Because $N_{\neg\varphi;\emptyset} \in [\![\neg\varphi]\!]$, we have $\alpha_{\Gamma\neg\varphi;\Delta} \in \bot$. Hence, by the following proof, $\Gamma \vdash_\alpha \Delta, \varphi$ is provable, that is, $\alpha_{\Gamma;\Delta} \in [\![\varphi]\!]$.

$$\frac{\dfrac{\varphi \vdash_N \varphi}{\vdash_N \varphi, \neg\varphi}\ \neg R \quad \Gamma, \neg\varphi \vdash_\alpha \Delta}{\Gamma \vdash_\alpha \Delta, \varphi}\ cut$$

$\Leftarrow$) Let $\alpha_{\Gamma;\Delta} \in [\![\varphi]\!]$, that is, $\Gamma \vdash_\alpha \Delta, \varphi$ is provable. We show $\alpha_{\Gamma;\Delta} \in \neg[\![\neg\varphi]\!]$, that is, $\alpha_{\Gamma;\Delta} \wedge [\![\neg\varphi]\!] \subseteq \bot$. Assume $\beta_{\Sigma;\Lambda} \in [\![\neg\varphi]\!]$. Then by the following proof, we have $(\alpha \cap \beta)_{\Gamma\Sigma;\Delta\Lambda} \in \bot$.

$$\frac{\Sigma \vdash_\beta \Lambda, \neg\varphi \quad \dfrac{\Gamma \vdash_\alpha \Delta, \varphi}{\neg\varphi, \Gamma \vdash_\alpha \Delta}\ \neg L}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda}\ cut$$

(3) By (1), we have $\neg[\![\neg\varphi]\!] = \neg\neg[\![\varphi]\!]$. Thus, the claim is obtained by (2). □

Lemma 8 $[\![\varphi]\!]$ *is closed for any formula* φ.

Proof To show the $\subseteq$-closedness of $[\![\varphi]\!]$, assume $\alpha_{\Gamma;\Delta} \in [\![\varphi]\!]$ and $\beta_{\Sigma;\Lambda} \subseteq \alpha_{\Gamma;\Delta}$. We show $\beta_{\Sigma;\Lambda} \in [\![\varphi]\!]$, that is, $\Sigma \vdash_\beta \Lambda, \varphi$ is provable. We examine two cases depending on the condition on $\beta_{\Sigma;\Lambda} \subseteq \alpha_{\Gamma;\Delta}$. (1) When $\beta \subseteq \alpha$ and $\Sigma = \Gamma$ and $\Lambda = \Delta$, we obtain the claim by applying *sub*-rule as follows.

$$\frac{\Gamma \vdash_\alpha \Delta, \varphi}{\Gamma \vdash_\beta \Delta, \varphi}\ sub$$

(2) When $\beta = \alpha$ and $\Sigma \supseteq \Gamma$ and $\Lambda \supseteq \Delta$, we obtain the claim by applying wL- and wR-rules as follows.

$$\frac{\Gamma \vdash_\alpha \Delta, \varphi}{\Sigma \vdash_\alpha \Lambda, \varphi}\ w$$

To show $\cup$-closedness, assume $\alpha_{\Gamma;\Delta}, \beta_{\Sigma;\Lambda} \in [\![\varphi]\!]$. Then, $\alpha_{\Gamma;\Delta} \cup \beta_{\Sigma;\Lambda} \in [\![\varphi]\!]$ is obtained by applying *mer*-rule as follows.

$$\frac{\Gamma \vdash_\alpha \Delta \quad \Sigma \vdash_\beta \Lambda}{\Gamma, \Sigma \vdash_{\alpha\cup\beta} \Delta, \Lambda}\ mer$$

where $\alpha_{\Gamma;\Delta} \cup \beta_{\Sigma;\Lambda} = (\alpha \cup \beta)_{\Gamma\Sigma;\Delta\Lambda}$.

$\neg\neg$-closedness is shown in Lemma 7 (3). □

Note that, in our canonical model, $\mathcal{G}_C(P) = \{\alpha_{\emptyset;\emptyset} \mid \vdash_\alpha P \text{ is an axiom}\}$. Then, every atom P is interpreted in our canonical model as $P^* = \neg\neg\mathcal{G}_C(P)$ in the same way as in general models. We show that $P^* = \neg\neg\mathcal{G}_C(P) = [\![P]\!]$.

Lemma 9 $\neg\neg\mathcal{G}_C(P) = [\![P]\!]$ *for any atom* P.

Proof $\Rightarrow$) We have $\mathcal{G}_C(P) \subseteq [\![P]\!]$ by definition. Hence, we have $\neg\neg\mathcal{G}_C(P) \subseteq [\![P]\!]$ because $\neg\neg[\![P]\!] = [\![P]\!]$.

$\Leftarrow$) We first show $\neg\mathcal{G}_C(P) \subseteq \neg[\![P]\!]$. Assume $\alpha_{\Gamma;\Delta} \in \neg\mathcal{G}_C(P)$, that is, $\alpha_{\Gamma;\Delta} \wedge \mathcal{G}_C(P) \subseteq \bot$. We show $\alpha_{\Gamma;\Delta} \in \neg[\![P]\!]$, that is, $\alpha_{\Gamma;\Delta} \wedge [\![P]\!] \subseteq \bot$. Let $\beta_{\Sigma;\Lambda} \in [\![P]\!]$, that is, $\Sigma \vdash_\beta \Lambda, P$ is provable. In Lg, by the definition of individuals' judgments $\mathbf{J}$, there exists a group γ such that $\vdash_\gamma P$ and $P \vdash_{-\gamma}$ are non-logical axioms, where $-\gamma$ is the complement of γ. Thus, because $\gamma_{\emptyset;\emptyset} \in \mathcal{G}_C(P)$, by the assumption $\alpha_{\Gamma;\Delta} \wedge \mathcal{G}_C(P) \subseteq \bot$, $\Gamma \vdash_{\alpha\cap\gamma} \Delta$ is provable. Then, $\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda$ is provable as follows.

$$\cfrac{\cfrac{\Gamma \vdash_{\alpha\cap\gamma} \Delta \qquad \cfrac{\Sigma \vdash_\beta \Lambda, P \quad P \vdash_{-\gamma}}{\Sigma \vdash_{\beta\cap-\gamma} \Lambda}\ cut}{\Gamma, \Sigma \vdash_{(\alpha\cap\gamma)\cup(\beta\cap-\gamma)} \Delta, \Lambda}\ mer}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda}\ sub$$

where $\alpha \cap \beta \subseteq (\alpha \cap \gamma) \cup (\beta \cap -\gamma)$. Thus, we obtain $\neg\mathcal{G}_C(P) \subseteq \neg[\![P]\!]$, and hence, we have $[\![P]\!] = \neg\neg[\![P]\!] \subseteq \neg\neg\mathcal{G}_C(P)$. □

Remark 5 (Closure) Note that $\mathcal{G}_C(P)$ without the $\neg\neg$-closure is too weak to prove the completeness of Lg. This is mainly because $\mathcal{G}_C(P)$ is not closed under the provability, that is, even though $\Gamma \vdash_\alpha P$ is provable, P does not necessarily come from the non-logical axiom for P (i.e., $[\![P]\!] \neq \mathcal{G}_C(P)$). Thus, we are required to make $\mathcal{G}_C(P)$ be closed by using the $\neg\neg$-closure. Similarly for $\vee$.

The semantic completeness of Lg is obtained from the following main lemma.

Lemma 10 $\varphi^* = [\![\varphi]\!]$ *for any formula* φ.

Proof We show this lemma by the induction on φ.

- When $\varphi \equiv P$, we have $P^* = [\![P]\!]$ by Lemma 9.
- When $\varphi \equiv \neg\varphi_1$, $\neg\varphi_1^* = [\![\neg\varphi_1]\!]$ is obtained by Lemma 7 (1), where $\varphi_1^* = [\![\varphi_1]\!]$ by the induction hypothesis.

• When $\varphi \equiv \varphi_1 \wedge \varphi_2$, we first show $\varphi_1^* \wedge \varphi_2^* \subseteq [\![\varphi_1 \wedge \varphi_2]\!]$. Let $\alpha_{\Gamma;\Delta} \in \varphi_1^*$ and $\beta_{\Sigma;\Lambda} \in \varphi_2^*$. By the induction hypothesis, we have $\alpha_{\Gamma;\Delta} \in [\![\varphi_1]\!]$ and $\beta_{\Sigma;\Lambda} \in [\![\varphi_2]\!]$. Thus, we obtain $(\alpha \cap \beta)_{\Gamma\Sigma;\Delta\Lambda} \in [\![\varphi_1 \wedge \varphi_2]\!]$ by applying $\wedge R$-rule as follows.

$$\dfrac{\Gamma \vdash_\alpha \Delta, \varphi_1 \quad \Sigma \vdash_\beta \Lambda, \varphi_2}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda, \varphi_1 \wedge \varphi_2}\ \wedge R$$

Next, we show $[\![\varphi_1 \wedge \varphi_2]\!] \subseteq \varphi_1^* \wedge \varphi_2^*$. Let $\alpha_{\Gamma;\Delta} \in [\![\varphi_1 \wedge \varphi_2]\!]$. Then, $\Gamma \vdash_\alpha \Delta, \varphi_1 \wedge \varphi_2$ is provable. By using *cut*-rule, $\Gamma \vdash_\alpha \Delta, \varphi_1$ is provable, and $\Gamma \vdash_\alpha \Delta, \varphi_2$ is provable as follows.

$$\dfrac{\Gamma \vdash_\alpha \Delta, \varphi_1 \wedge \varphi_2 \qquad \dfrac{\varphi_i \vdash_N \varphi_i}{\varphi_1 \wedge \varphi_2 \vdash_N \varphi_i}\ \wedge L}{\Gamma \vdash_\alpha \Delta, \varphi_i}\ cut$$

where $i = 1, 2$. Thus, by the induction hypothesis, we have $\alpha_{\Gamma;\Delta} \in [\![\varphi_1]\!] = \varphi_1^*$ and $\alpha_{\Gamma;\Delta} \in [\![\varphi_2]\!] = \varphi_2^*$, which imply $\alpha_{\Gamma;\Delta} \cap \alpha_{\Gamma;\Delta} = \alpha_{\Gamma;\Delta} \in \varphi_1^* \wedge \varphi_2^*$.

• When $\varphi \equiv \varphi_1 \to \varphi_2$, we first show $\varphi_1^* \to \varphi_2^* \subseteq [\![\varphi_1 \to \varphi_2]\!]$. Let $\alpha_{\Gamma;\Delta} \in \varphi_1^* \to \varphi_2^*$. Then, we have $\varphi_1^* \wedge \alpha_{\Gamma;\Delta} \subseteq \varphi_2^*$. Note that we have $N_{\varphi_1;\emptyset} \in [\![\varphi_1]\!] = \varphi_1^*$ by the induction hypothesis. Thus, we have $N_{\varphi_1;\emptyset} \cap \alpha_{\Gamma;\Delta} \in \varphi_2^* = [\![\varphi_2]\!]$. Then, we obtain $\alpha_{\Gamma;\Delta} \in [\![\varphi_1 \to \varphi_2]\!]$ by applying $\to R$-rule as follows.

$$\dfrac{\varphi_1, \Gamma \vdash_{N\cap\alpha} \Delta, \varphi_2}{\Gamma \vdash_\alpha \Delta, \varphi_1 \to \varphi_2}\ \to R$$

We next show $[\![\varphi_1 \to \varphi_2]\!] \subseteq \varphi_1^* \to \varphi_2^*$. Let $\alpha_{\Gamma;\Delta} \in [\![\varphi_1 \to \varphi_2]\!]$. Then $\Gamma \vdash_\alpha \Delta, \varphi_1 \to \varphi_2$ is provable. To show $\varphi_1^* \wedge \alpha_{\Gamma;\Delta} \subseteq \varphi_2^*$, assume $\beta_{\Sigma;\Lambda} \in \varphi_1^*$. Then, we have $(\alpha \cap \beta)_{\Gamma\Sigma;\Delta\Lambda} \in \varphi_2^*$ by using the induction hypothesis as follows.

$$\dfrac{\Gamma \vdash_\alpha \Delta, \varphi_1 \to \varphi_2 \qquad \dfrac{\Sigma \vdash_\beta \Lambda, \varphi_1 \quad \varphi_2 \vdash_N \varphi_2}{\varphi_1 \to \varphi_2, \Sigma \vdash_\beta \Lambda, \varphi_2}\ \to L}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda, \varphi_2}\ cut$$

• When $\varphi \equiv \varphi_1 \vee \varphi_2$, we first show $\varphi_1^* \vee \varphi_2^* \subseteq [\![\varphi_1 \vee \varphi_2]\!]$. Assume $\alpha_{\Gamma;\Delta} \in \varphi_1^*$. Then, by the induction hypothesis, $\Gamma \vdash_\alpha \Delta, \varphi_1$ is provable, and hence, we have:

$$\dfrac{\Gamma \vdash_\alpha \Delta, \varphi_1}{\Gamma \vdash_\alpha \Delta, \varphi_1 \vee \varphi_2}\ \vee R$$

Thus, we have $\alpha_{\Gamma;\Delta} \in [\![\varphi_1 \vee \varphi_2]\!]$, and hence, we have $\varphi_1^* \subseteq [\![\varphi_1 \vee \varphi_2]\!]$. Similarly, we have $\varphi_2^* \subseteq [\![\varphi_1 \vee \varphi_2]\!]$. Therefore, we have $\neg\neg(\varphi_1^* \cup \varphi_2^*) \subseteq [\![\varphi_1 \vee \varphi_2]\!]$ by the $\neg\neg$-closedness of $[\![\varphi_1 \vee \varphi_2]\!]$.

We next show $[\![\varphi_1 \vee \varphi_2]\!] \subseteq \neg\neg(\varphi_1^* \cup \varphi_2^*)$, that is, $[\![\varphi_1 \vee \varphi_2]\!] \wedge \neg(\varphi_1^* \cup \varphi_2^*) \subseteq \bot$. Let $\alpha_{\Gamma;\Delta} \in [\![\varphi_1 \vee \varphi_2]\!]$ and $\beta_{\Sigma;\Lambda} \in \neg(\varphi_1^* \cup \varphi_2^*)$, that is, $\beta_{\Sigma;\Lambda} \wedge (\varphi_1^* \cup \varphi_2^*) \subseteq \bot$. We show $(\alpha \cap \beta)_{\Gamma\Sigma;\Delta\Lambda} \in \bot$. Because $N_{\varphi_1;\emptyset} \in [\![\varphi_1]\!] = \varphi_1^*$ and $N_{\varphi_2;\emptyset} \in [\![\varphi_2]\!] = \varphi_2^*$ by the induction hypotheses, we have $\beta_{\Sigma\varphi_1;\Lambda} \in \bot$ and $\beta_{\Sigma\varphi_2;\Lambda} \in \bot$, respectively. Thus, we have the following proof, which shows $(\alpha \cap \beta)_{\Gamma\Sigma;\Delta\Lambda} \in \bot$:

$$\dfrac{\Gamma \vdash_\alpha \Delta, \varphi_1 \vee \varphi_2 \qquad \dfrac{\Sigma, \varphi_1 \vdash_\beta \Lambda \quad \Sigma, \varphi_2 \vdash_\beta \Lambda}{\varphi_1 \vee \varphi_2, \Sigma \vdash_\beta \Lambda}\ \vee L}{\Gamma, \Sigma \vdash_{\alpha\cap\beta} \Delta, \Lambda}\ cut$$

□

In our canonical model, the whole group N, which is the unit element in general models, is indexed by contexts. We define the truth in the canonical model in terms of $N_{\emptyset;\emptyset}$.

Definition 14 (**True in canonical model**) φ is true in the canonical model of Lg if $N_{\emptyset;\emptyset} \in \varphi^*$.

Theorem 2 (**Semantic completeness of** Lg) *If φ is valid, then $\vdash_N \varphi$ is provable in* Lg.

Proof Assume that φ is valid. Then, in particular, φ is true in the canonical model. By Lemma 10, we have $N_{\emptyset;\emptyset} \in \varphi^* = [\![\varphi]\!]$, that is, $\vdash_N \varphi$ is provable in Lg. □

Remark 6 ($\vdash_\emptyset$) Note that the empty set $\emptyset$ is a subset of any set, and hence, $\emptyset$ is a member of any closed set. In particular, $\emptyset \in \bot$ always holds, and this means that $\vdash_\emptyset$ should be a non-logical axiom in our system.

3 Logic with Majority Voting Lv

In Section 3.1, we introduce our logic with majority voting Lv, which is shown to be inconsistent. We discuss how we avoid this inconsistency, and based on the well-studied premise-based and conclusion-based approaches, we introduce logic with majority voting for axioms Lva in Section 3.2, and logic with majority voting for conclusions Lvc in Section 3.3. We show that Lva and Lvc are both consistent and syntactically complete, and we discuss the discursive paradox in terms of Lva and Lvc.

3.1 Majority Voting

We introduce the system Lv by introducing an inference rule representing majority voting to Lg.

Definition 15 (Lv) Logic with majority voting Lv is obtained by introducing the following mv-rule to Lg.

mv-rule: When $|\alpha| > \frac{n}{2}$,

$$\frac{\Gamma \vdash_\alpha \Delta}{\Gamma \vdash_N \Delta}\ mv$$

Unfortunately, Lv may be inconsistent, which is shown by the discursive paradox of Example 1.

Example 5 (Discursive paradox in Lv) We have the following proof of $\vdash_N$ in Lv with the given **J** as in Example 1. Here, the non-logical axioms are as follows: $\vdash_{13} P$, $P \vdash_2$, $\vdash_1 Q$, $Q \vdash_{23}$.

$$\dfrac{\dfrac{\dfrac{Q \vdash_{23}}{\vdash_{23} \neg Q}}{\vdash_N \neg Q}\,mv \qquad \dfrac{\dfrac{\dfrac{\dfrac{P \vdash_2 \quad \vdash_1 Q}{P \vdash_{12} Q}\,mer}{\vdash_{12} P \to Q}}{\vdash_N P \to Q}\,mv \qquad \dfrac{\dfrac{\vdash_{13} P}{\vdash_N P}\,mv \qquad \dfrac{P \vdash_N P \qquad \dfrac{Q \vdash_N Q}{Q, \neg Q \vdash_N}}{P, P \to Q, \neg Q \vdash_N}}{P \to Q, \neg Q \vdash_N}\,cut}{\neg Q \vdash_N}\,cut}{\vdash_N}\,cut$$

The right upper part derives $P, P \to Q, \neg Q \vdash_N$, which means $P, P \to Q, \neg Q$ are inconsistent. Then, together with $\vdash_N P$, $\vdash_N P \to Q$, and $\vdash_N \neg Q$ (these are obtained by mv-rule), the inconsistency $\vdash_N$ is provable in Lv with the given **J**.

The discursive paradox shows that a collective judgment based on majority voting may be inconsistent from the viewpoint of the standard classical logic. There are various approaches that can be taken to avoid the paradox. Two of the most popular such approaches are the premise-based approach and the conclusion-based approach. See, for example, [11, 16, 33] for these approaches. In the premise-based approach, we first take majority voting on predetermined "premises," and then, we collectively accept the conclusions obtained by logical reasoning from the collectively accepted premises. In Example 1, if we regard P and $P \to Q$ as premises, then these are collectively accepted by majority voting, and Q, as the logical consequence thereof, is also accepted collectively. In the conclusion-based approach, every member conducts logical reasoning separately and implicitly, and then, we take majority voting on the predetermined "conclusions" to decide the collective judgment. In Example 1, if we regard Q as a conclusion, then it is rejected (and hence, $\neg Q$ is accepted) by majority voting.

Although consistent collective judgments are obtained by these approaches, there are difficulties thereof, cf. [10, 16]. In particular, we need to determine in advance which formulas are premises and which are conclusions, and what is collectively accepted depends on the choice of the premises and conclusions. For example, in Example 1, by the premise-based approach, if we fix P and $P \to Q$ as the premises, then the collective judgment is $\{P, P \to Q, Q\}$. By contrast, if we fix P and Q as the premises, then the collective judgment is $\{P, \neg Q, \neg(P \to Q)\}$. In the conclusion-based approach, a collective judgment is not complete with respect to a given agenda in general. Thus, [33] investigated a procedure for making a collective judgment complete by the conclusion-based approach.

In this article, we determine the collective judgment based on our logic with majority voting by constructing proofs. Because a proof is considered to provide support to a collectively accepted formula, we may call our approach a proof-based approach. As shown in Example 5, Lv may be inconsistent, and hence, we cannot adopt Lv itself as a logical system in our approach. Thus, based on the ideas of the premise-based and the conclusion-based approaches, we introduce *logic with majority voting for axioms* Lva, where mv-rule can be applied only to every non-logical axioms (logical axioms are already accepted by all members) to construct a proof in Lg, and *logic with majority voting for conclusions* Lvc, where mv-rule can be applied only to every conclusion of a proof in Lg.

Note that non-logical axioms are generally considered as "premises" in a proof, that is, they appear at the top of a proof, and they are distinguished from the antecedent in a sequent. Thus, our approach based on Lva is a particular case of the premise-based approach, where "premises" are our non-logical axioms restricted to literals. In our Lvc, we construct a proof in Lg (without mv-rule), and if $\vdash_\alpha \varphi$ is provable with $|\alpha| > \frac{n}{2}$, then we accept φ as a collectively accepted conclusion. Hence, Lvc is different from the usual conclusion-based approach, where individuals just vote the predetermined conclusions.

In the next sections, we introduce Lva and Lvc, respectively, and then, we investigate their properties. Lva may be introduced by restricting applications of mv-rule in Lv only to first steps; i.e., non-logical axioms in a proof. However, instead of introducing mv-rule explicitly, we introduce Lva by modifying non-logical axioms of Lg. Similarly, although Lvc may be introduced by restricting applications of mv-rule in Lv to only the last step in a proof, we introduce Lvc by modifying the notion of validity in Lg without introducing mv-rule explicitly. This makes Lva and Lvc to be particular systems of Lg, and it is possible to apply syntax and semantics of Lg directly to Lva and Lvc.

3.2 Lv for Axioms: Lva

We first investigate the logic with majority voting for axioms Lva. By contrast to Lg, the non-logical axioms of Lva are formulas accepted by all members, with the use of majority voting.

Definition 16 (**Non-logical axioms of** Lva) Lva is obtained from Lg by replacing the non-logical axioms to the following ones for every atom P:

- $\vdash_N P$ when there exists $\alpha \in \mathcal{G}(P)$ such that $|\alpha| > \frac{n}{2}$
- $P \vdash_N$ when there exists $\alpha \in \mathcal{G}(\neg P)$ such that $|\alpha| > \frac{n}{2}$
- $\vdash_\emptyset$

Thus, in terms of Lv, the mv-rule has been already applied to all the non-logical axioms in Lva.

Note that, when $\vdash_N P$ is an axiom with $\alpha \in \mathcal{G}(P)$ and $|\alpha| > \frac{n}{2}$, we do not adopt $P \vdash_{-\alpha}$, where $-\alpha$ is the complement of α, as a non-logical axiom, although it is in Lg. We consider that it makes no sense to keep $P \vdash_{-\alpha}$ as a non-logical axiom when P is collectively accepted by majority voting. This approach avoids another difficulty pointed out by Nehring [27], see Remark 7. Note also that $\vdash_\emptyset P$ and $P \vdash_\emptyset$ are provable from the axiom $\vdash_\emptyset$.

A model of Lva is obtained from that of Lg by changing the interpretation of atoms.

Definition 17 (**Model of** Lva) The interpretation of every atom P is defined as follows.

- If there exists $\alpha \in \mathcal{G}(P)$ such that $|\alpha| > \frac{n}{2}$, then $P^* = \mathcal{G} = \mathcal{P}(N)$.
  Otherwise, $\neg P^* = \mathcal{G} = \mathcal{P}(N)$, that is, $P^* = \bot$.

The semantic completeness of Lva is proved with the construction of a canonical model in exactly the same way as for Lg. Let us check the interpretation of the atoms. When there exists $\alpha \in \mathcal{G}(P)$ such that $|\alpha| > \frac{n}{2}$, the sequent $\vdash_N P$ is a non-logical axiom. Hence, by applying w-rule and sub-rule, $\Gamma \vdash_\alpha \Delta, P$ is provable in Lva for any contexts Γ and Δ, and for any group α:

$$\dfrac{\dfrac{\vdash_N P}{\Gamma \vdash_N \Delta, P}\ w}{\Gamma \vdash_\alpha \Delta, P}\ sub$$

Thus, any $\alpha_{\Gamma;\Delta}$ belongs to $[\![P]\!]$, that is, $[\![P]\!] = \mathcal{G}_C = P^*$ (cf. Lemma 9). When there exists no $\alpha \in \mathcal{G}(P)$ such that $|\alpha| > \frac{n}{2}$, the sequent $P \vdash_N$ is a non-logical axiom by the definition of **J**. In this case, we have $[\![P]\!] = \bot = P^*$ as follows. Let $\alpha_{\Gamma;\Delta} \in [\![P]\!]$. Then $\Gamma \vdash_\alpha \Delta, P$ is provable. Hence, by applying cut-rule, $\Gamma \vdash_\alpha \Delta$ is provable, that is, $\alpha_{\Gamma;\Delta} \in \bot$.

$$\dfrac{\Gamma \vdash_\alpha \Delta, P \quad P \vdash_N}{\Gamma \vdash_\alpha \Delta}\ cut$$

Hence, we have $[\![P]\!] \subseteq \bot$. The other direction $\bot \subseteq [\![P]\!]$ is obtained by applying w-rule. Therefore, we obtain the semantic completeness of Lva from that for Lg (Theorem 2).

Theorem 3 (**Semantic completeness of** Lva) Lva *is semantically complete with respect to the models of* Lva.

Nehring [27] pointed out the following difficulty of the premise-based approach.

Remark 7 (Difficulty in the premise-based approach) Let $N = \{1, 2, 3\}$, $J_1 = \{P, Q, \neg R, \neg((P \wedge Q) \wedge R)\}$, $J_2 = \{\neg P, Q, R, \neg((P \wedge Q) \wedge R)\}$, $J_3 = \{P, \neg Q, R, \neg((P \wedge Q) \wedge R)\}$. Let P, Q, R be premises and $(P \wedge Q) \wedge R$ be the conclusion.

	P	Q	R	$(P \wedge Q) \wedge R$
1	T	T	F	F
2	F	T	T	F
3	T	F	T	F
majority	T	T	T	F

Then, because P, Q, R are all accepted by the majority, from the logical viewpoint, $(P \wedge Q) \wedge R$ should be accepted, even though everyone rejects it. From the proof-theoretic viewpoint, this difficulty arises from the fact that $(P \wedge Q) \wedge R \vdash_{123}$ is provable in Lv as seen in the following proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{P \vdash_2}{P \wedge Q \vdash_2}}{(P \wedge Q) \wedge R \vdash_2} \quad \dfrac{\dfrac{Q \vdash_3}{P \wedge Q \vdash_3}}{(P \wedge Q) \wedge R \vdash_3}}{(P \wedge Q) \wedge R \vdash_{23}}\ mer \quad \dfrac{R \vdash_1}{(P \wedge Q) \wedge R \vdash_1}}{(P \wedge Q) \wedge R \vdash_{123}}\ mer$$

Because $N = \{1, 2, 3\}$, the provable sequent $(P \wedge Q) \wedge R \vdash_{123}$ in the above proof is equivalent to the sequent $(P \wedge Q) \wedge R \vdash_N$. Thus, by applying cut-rule to the above proof and the following proof, the inconsistency $\vdash_N$ is provable in Lv.

$$\dfrac{\dfrac{\dfrac{\vdash_{13}\ P}{\vdash_N\ P}\,mv \quad \dfrac{\vdash_{12}\ Q}{\vdash_N\ Q}\,mv}{\vdash_N\ P \wedge Q} \quad \dfrac{\vdash_{23}\ R}{\vdash_N\ R}\,mv}{\vdash_N\ (P \wedge Q) \wedge R}$$

Note that mv-rule is applied only to axioms in the above proof, and hence, the above difficulty cannot be avoided even by the usual premise-based approach.

However, in our Lva, $(P \wedge Q) \wedge R \vdash_{123}$ is not provable, because $P \vdash_2$, $Q \vdash_3$, $R \vdash_1$ are not non-logical axioms. In terms of semantics, that is, $\neg((P^* \wedge Q^*) \wedge R^*) \subseteq \bot$ because $N \in (P^* \wedge Q^*) \wedge R^*$.

In addition to semantic completeness, it is shown that Lva is syntactically complete.

Lemma 11 (**Syntactic completeness of** Lva) *In any model* $(\bot, *)$ *of* Lva, $N \in \varphi^*$ *or* $N \in \neg\varphi^*$ *for any formula* φ.

Proof We show that $N \notin \varphi^*$ implies $N \in \neg\varphi^*$ in a given model by induction on φ.

- When $\varphi \equiv P$, assume $N \notin P^*$. Then, by definition, $P^* \subseteq \bot$, that is $N \in \neg P^*$.
- When $\varphi \equiv \varphi_1 \wedge \varphi_2$, assume $N \notin \varphi_1^* \wedge \varphi_2^*$. We show $N \in \neg(\varphi_1 \wedge \varphi_2)^*$, that is, $\varphi_1^* \wedge \varphi_2^* \subseteq \bot$. $N \notin \varphi_1^* \wedge \varphi_2^*$ implies $N \notin \varphi_1^*$ or $N \notin \varphi_2^*$. When $N \notin \varphi_1^*$, by the induction hypothesis, we have $N \in \neg\varphi_1^*$, that is, $\varphi_1^* \subseteq \bot$, which implies $\varphi_1^* \wedge \varphi_2^* \subseteq \bot$. The same applies to the case $N \notin \varphi_2^*$, and hence, we obtain $\varphi_1^* \wedge \varphi_2^* \subseteq \bot$.
- When $\varphi \equiv \varphi_1 \to \varphi_2$, assume $N \notin \varphi_1^* \to \varphi_2^*$. Note that, because $N \in \varphi_1^* \to \varphi_2^*$ is equivalent to $\varphi_1^* \subseteq \varphi_2^*$, $N \notin \varphi_1^* \to \varphi_2^*$ means that there exists $\beta \in \varphi_1^*$ such that $\beta \notin \varphi_2^*$.

We first show $N \in \varphi_1^*$. If $N \notin \varphi_1^*$, then $N \in \neg\varphi_1^*$, i.e., $\varphi_1^* \subseteq \bot$ by the induction hypothesis. However, because $\bot \subseteq \varphi_2^*$, we obtain $\varphi_1^* \subseteq \varphi_2^*$, which contradicts the assumption $N \notin \varphi_1^* \to \varphi_2^*$. Hence, we have $N \in \varphi_1^*$.

$N \in \varphi_1^*$ implies $\varphi_1^* \to \varphi_2^* \subseteq \varphi_2^*$. This is because $\alpha \in \varphi_1^* \to \varphi_2^*$ means that $\alpha \wedge \varphi_1^* \subseteq \varphi_2^*$, which implies $\alpha \cap N = \alpha \in \varphi_2^*$ because $N \in \varphi_1^*$.

Next, we show $\varphi_2^* \subseteq \bot$. If $N \in \varphi_2^*$, then, because φ_2^* is $\subseteq$-closed, $\alpha \in \varphi_2^*$ for any $\alpha \subseteq N$, that is, $\varphi_1^* \subseteq \varphi_2^*$, which contradicts to the assumption $N \notin \varphi_1^* \to \varphi_2^*$. Thus, we have $N \notin \varphi_2^*$, and hence, by the induction hypothesis, we have $\varphi_2^* \subseteq \bot$.

Therefore, we have $\varphi_1^* \to \varphi_2^* \subseteq \varphi_2^* \subseteq \bot$, that is, $N \in \neg(\varphi_1 \to \varphi_2)^*$.

- When $\varphi \equiv \varphi_1 \vee \varphi_2$, assume $N \notin \varphi_1^* \vee \varphi_2^*$. If $N \in \varphi_1^*$, then $N \in \varphi_1^* \cup \varphi_2^* \subseteq \neg\neg(\varphi_1^* \cup \varphi_2^*)$, which is the contradiction. Thus, $N \notin \varphi_1^*$, and hence, by the induction hypothesis, $N \in \neg\varphi_1^*$, that is, $\varphi_1^* \subseteq \bot$. The same applies to φ_2, and we have $\varphi_2^* \subseteq \bot$. Therefore, we have $\varphi_1^* \cup \varphi_2^* \subseteq \bot$, which implies $\neg\neg(\varphi_1^* \cup \varphi_2^*) \subseteq \bot$ because $\bot$ is $\neg\neg$-closed. Thus, we have $N \in \neg(\varphi_1^* \vee \varphi_2^*)$. □

Theorem 4 (**Syntactic completeness of** Lva) *In* Lva, $\vdash_N \varphi$ *is provable or* $\vdash_N \neg\varphi$ *is provable for any formula* φ.

Proof By Lemma 11, $N \in \varphi^*$ or $N \in \neg\varphi^*$ holds in any model $(\bot, *)$. In particular, in the canonical model, we have $N_{\emptyset;\emptyset} \in \varphi^* = [\![\varphi]\!]$ or $N_{\emptyset;\emptyset} \in \neg\varphi^* = [\![\neg\varphi]\!]$, that is, $\vdash_N \varphi$ is provable or $\vdash_N \neg\varphi$ is provable in $\mathsf{Lv}a$. □

The aggregation function F based on $\mathsf{Lv}a$, whose completeness is obtained from Theorem 4 and consistency is obtained from Proposition 8 is defined as follows.

Proposition 9 (**Collective judgment with** $\mathsf{Lv}a$) *Given* $\mathcal{A}$ *and* $\mathbf{J}$*, the collective judgment* $F(\mathbf{J})$ *based on* $\mathsf{Lv}a$ *is defined as follows.*

$$F(\mathbf{J}) = \{\varphi \in \mathcal{A} \mid \vdash_N \varphi \textit{ is provable in } \mathsf{Lv}a\}$$

Then, F is complete and consistent.

Let us examine the discursive paradox given in Example 1 in our $\mathsf{Lv}a$.

Example 6 (Discursive paradox in $\mathsf{Lv}a$) In $\mathsf{Lv}a$, $P \to Q \vdash_N$ is provable as follows.

$$\dfrac{\dfrac{\vdash_{13} P}{\vdash_N P}\,mv \qquad \dfrac{Q \vdash_{23}}{Q \vdash_N}\,mv}{P \to Q \vdash_N}$$

Formally speaking, the application of mv-rule in $\mathsf{Lv}a$ is implicit, and the above proof starts from the non-logical axioms $\vdash_N P$ and $Q \vdash_N$. However, for the sake of clarity, we indicate them in the first steps of the above proof.

$\vdash_N P \to Q$ is not provable in $\mathsf{Lv}a$. Let $(\bot, *)$ be the model given in Example 4, where $P^* = \mathcal{P}(\{1, 3\})$ and $Q^* = \mathcal{P}(\{1\})$. $N \in P^* \to Q^*$ is equivalent to $P^* \subseteq Q^*$, which does not hold in the model. Hence, $N \notin P^* \to Q^*$.

Therefore, the collective judgment based on $\mathsf{Lv}a$ is $F(\mathbf{J}) = \{P, \neg Q, \neg(P \to Q)\}$.

Let us investigate the relationship between our semantics of $\mathsf{Lv}a$ and the usual semantics of classical logic.

Lemma 12 *In any model of* $\mathsf{Lv}a$*, the following holds.*

1. $N \in \neg\varphi^*$ *if and only if* $\varphi^* = \bot$
2. $N \in \varphi^* \wedge \psi^*$ *if and only if* $N \in \varphi^*$ *and* $N \in \psi^*$
3. $N \in \varphi^* \to \psi^*$ *if and only if* $N \notin \varphi^*$ *or* $N \in \psi^*$
4. $N \in \varphi^* \vee \psi^*$ *if and only if* $N \in \varphi^*$ *or* $N \in \psi^*$

Proof (1) is obtained from the definition of $\neg$. (2) is immediate because $\wedge$ is equivalent to $\cap$. For (3), assume $N \in \varphi^* \to \psi^*$, that is, $\varphi^* \subseteq \psi^*$. Then, $N \in \varphi^*$ implies $N \in \psi^*$, that is, $N \notin \varphi^*$ or $N \in \psi^*$. Conversely, when $N \notin \varphi^*$, by Lemma 11, we have $N \in \neg\varphi^*$. Hence, $\varphi^* \subseteq \bot \subseteq \psi^*$, that is, $N \in \varphi^* \to \psi^*$. When $N \in \psi^*$, because $\psi^* = \mathcal{P}(N)$ in this case, we have $\varphi^* \subseteq \psi^*$, that is, $N \in \varphi^* \to \psi^*$. Therefore, in either case, we have $N \in \varphi^* \to \psi^*$. For (4), assume $N \in \varphi^* \vee \psi^*$. If $N \notin \varphi^*$ and $N \notin \psi^*$, by Lemma 11, we have $N \in \neg\varphi^*$ and $N \in \neg\psi^*$, that is, $\varphi^* \subseteq \bot$ and $\psi^* \subseteq \bot$. Thus, we have $\varphi^* \cup \psi^* \subseteq \bot$, which implies $\varphi^* \vee \psi^* \subseteq \bot$, which contradicts to $N \in \varphi^* \vee \psi^*$. Hence, $N \in \varphi^*$ or $N \in \psi^*$. The converse is immediate. □

The above lemma implies that the interpretation of connectives in the semantics of L$\mathsf{v}a$ corresponds to that found in the usual semantics of classical logic, i.e., truth table semantics. Thus, we have the following proposition.

Proposition 10 (**Classical logic and** Lva) *Any collective judgment $F(\mathbf{J})$ based on* Lva *is consistent with respect to the semantics of classical logic.*

3.3 Lv for Conclusions: Lvc

Syntax of Lvc is the same as Lg, but the notion of truth in a model is changed as follows.

Definition 18 (**Model of** Lvc) In a model $(\bot, *)$ of Lvc, φ is true if there exists $\alpha \in \varphi^*$ such that $|\alpha| > \frac{n}{2}$.

The truth in the canonical model is also defined in terms of the majority group of the form $\alpha_{\emptyset;\emptyset}$. Thus, when φ is valid, it is true in the canonical model, that is, there exists $\alpha_{\emptyset;\emptyset} \in \varphi^* = [\![\varphi]\!]$ such that $|\alpha| > \frac{n}{2}$. Hence, $\vdash_\alpha \varphi$ is provable with $|\alpha| > \frac{n}{2}$. Thus, we obtain the semantic completeness of Lvc.

Theorem 5 (**Semantic completeness of** Lvc) Lvc *is semantically complete with respect to the models of* Lvc.

In Lvc, $\alpha \in \varphi^*$ does not, in general, imply $-\alpha \in \neg\varphi^*$. For example, in the model of Example 4, $\{1\} \in P^*$ but $\{2, 3\} \notin \neg P^*$. However, for the greatest group $g\alpha$, $g\alpha \in \varphi^*$ implies $-g\alpha \in \neg\varphi^*$. This observation implies the syntactic completeness of Lvc as follows.

Lemma 13 (**Syntactic completeness of** Lvc) *φ is true or $\neg\varphi$ is true in any model of* Lvc.

Proof Because φ^* is $\cup$-closed, there exists the greatest group $g\alpha$ in φ^*. For $g\alpha \in \varphi^*$, we show $-g\alpha \in \neg\varphi^*$, that is, for any $\beta \in \varphi^*$, $-g\alpha \cap \beta \in \bot$. Let $\beta \in \varphi^*$. Then, $\beta \subseteq g\alpha$ because $g\alpha$ is the greatest group, and hence, $-g\alpha \cap \beta = \emptyset \in \bot$ because $-g\alpha \cap g\alpha = \emptyset$.

Then, for the greatest $g\alpha \in \varphi^*$, we have:

- φ is true if $|g\alpha| > \frac{n}{2}$, and
- $\neg\varphi$ is true if $|g\alpha| < \frac{n}{2}$, because then, $|-g\alpha| > \frac{n}{2}$ and $-g\alpha \in \neg\varphi^*$.

□

From the above lemma, we obtain the syntactic completeness of Lvc by the same argument for Lva.

Theorem 6 (**Syntactic completeness of** Lvc) *In* Lvc*, there exists α with $|\alpha| > \frac{n}{2}$ such that $\vdash_\alpha \varphi$ is provable or $\vdash_\alpha \neg\varphi$ is provable for any formula φ.*

The aggregation function F based on $\mathsf{Lv}c$ is defined as follows, whose completeness is obtained from Theorem 6, and whose consistency is obtained from Proposition 8.

Proposition 11 (**Collective judgment with** $\mathsf{Lv}c$) *Given* $\mathcal{A}$ *and* $\mathbf{J}$*, the collective judgment* $F(\mathbf{J})$ *based on* $\mathsf{Lv}c$ *is defined as follows.*

$$F(\mathbf{J}) = \{\varphi \in \mathcal{A} \mid \vdash_\alpha \varphi \text{ is provable with } |\alpha| > \tfrac{n}{2} \text{ in } \mathsf{Lv}c\}$$

Then, F *is complete and consistent.*

Let us examine the discursive paradox of Example 1 in our $\mathsf{Lv}c$.

Example 7 (Discursive paradox in $\mathsf{Lv}c$) $\vdash_N P \to Q$ is provable in $\mathsf{Lv}c$ as follows.

$$\dfrac{\dfrac{\dfrac{P \vdash_2 \quad \vdash_1 Q}{P \vdash_{12} Q}\, mer}{\vdash_{12} P \to Q}}{\vdash_N P \to Q}\, mv$$

There is a model such that there exists no α such that $|\alpha| > \frac{n}{2}$ and $\alpha \in \neg(P \to Q)$. Let $(\bot, *)$ be a model given in Example 4, where $\bot = \{\emptyset\}$, $P^* = \mathcal{P}(\{1,3\})$ and $Q^* = \mathcal{P}(\{1\})$. Then, in this model, we have $P^* \to Q^* = \mathcal{P}(\{1,2\})$ and $\neg(P^* \to Q^*) = \mathcal{P}(\{3\})$.

Therefore, the collective judgment based on $\mathsf{Lv}c$ is $F(\mathbf{J}) = \{P, \neg Q, P \to Q\}$.

By contrast to the collective judgment based on $\mathsf{Lv}a$ (cf. Proposition 10), this collective judgment is "inconsistent" from the viewpoint of the standard "classical logic," but it is consistent with respect to our $\mathsf{Lv}c$.

In $\mathsf{Lv}c$, even though P and $P \to Q$ are both valid, Q may not be valid. In other words, even though there exists $\alpha \in P^*$ and $\beta \in (P \to Q)^*$ such that $|\alpha|, |\beta| > \frac{n}{2}$ and α and β are the greatest groups, it is not necessarily true that $|\alpha \cap \beta| > \frac{n}{2}$. (In the above Example 7, $P^* = \mathcal{P}(\{1,3\})$ and $P^* \to Q^* = \mathcal{P}(\{1,2\})$, but $Q^* = \mathcal{P}(\{1\})$. From the syntactic viewpoint, when $\vdash_N P$ and $\vdash_N P \to Q$ are provable by applying mv-rule at the last steps of their respective proofs, we cannot obtain $\vdash_N Q$ by combining those two proofs, because no rule is applicable after an application of mv-rule in $\mathsf{Lv}c$. (Note that $\vdash_N Q$ is provable, when $\vdash_N P$ and $\vdash_N P \to Q$ are both provable without mv-rule).

4 Conclusion and Future Work

To investigate the relationship between logic and majority voting, we introduced logic with groups Lg in the style of sequent calculus by augmenting the index of a group to every sequent. If we ignore the indexes of the groups, we obtain the usual sequent calculus of classical logic. In relation to groups, mer-rule is a remarkable inference rule that makes it possible to merge given groups. We showed that the cut-elimination theorem of Lg is proved by the same way as the usual transformation of given proofs

(Proposition 6). As a corollary of the cut-elimination theorem, we showed the consistency of Lg (Proposition 8). We further introduced set-theoretical semantics of Lg. Our semantics is based on the phase semantics of linear logic, and hence, the usual techniques of linear logic can be straightforwardly applied to our Lg. Every formula is interpreted as a closed set of groups whose members accept that formula. We proved the soundness (Theorem 1) and semantic completeness (Theorem 2) of Lg by applying essentially the same method as that for linear logic. Our simple semantics based on groups may be applied to an analysis of Arrow's impossibility theorem, which is proved by constructing an ultrafilter consisting of certain set of groups, cf. [16].

By introducing an inference rule representing majority voting to Lg, we introduced logic with majority voting Lv. By formalizing the discursive paradox, we showed that Lv is inconsistent, that is, $\vdash_N$ is provable in Lv with given individuals' judgments **J** (Example 5). Thus, we introduced logic with majority voting for axioms Lva and logic with majority voting for conclusions Lvc. Lva is defined by modifying non-logical axioms of Lg, and Lvc is defined by modifying the notion of validity in Lg. Hence, the syntax and semantics of Lg, as well as related theorems, are straightforwardly applied to these systems without dealing with the rule of majority voting directly. Based on these systems, we defined the collective judgment as the set of formulas provable in Lva and Lvc, respectively. We proved that both Lva and Lvc are syntactically complete (Theorems 11 and 6) and consistent. For Lva, we further showed that any collective judgment based on Lva is consistent with respect to the standard semantics of classical logic (Proposition 10). By contrast, a collective judgment based on Lvc may be inconsistent from the viewpoint of classical logic (Example 7), and hence, we may consider Lvc as a kind of non-classical logic. We leave a characterization of Lvc by using an existing logical system as our future work.

To make the construction of collective judgments based on Lva and Lvc effective, a procedure of automated proof-search or theorem proving in Lg is desirable. As shown in Proposition 3, our sub-rule does not cause trouble in the proof-search. By contrast, although it is convenient to move mer-rule to the upper parts of a proof, it is not simply permutable as seen in the following example.

$$\dfrac{\dfrac{\Gamma \vdash_\alpha \varphi \quad \psi \vdash_\beta \Delta}{\varphi \to \psi, \Gamma \vdash_{\alpha\cap\beta} \Delta} \to L \qquad \Sigma \vdash_\gamma \Lambda}{\varphi \to \psi, \Gamma, \Sigma \vdash_{(\alpha\cap\beta)\cup\gamma} \Delta, \Lambda}\ mer \quad \rhd \quad \dfrac{\Gamma \vdash_\alpha \varphi \quad \dfrac{\psi \vdash_\beta \Delta \quad \Sigma \vdash_\gamma \Lambda}{\psi, \Sigma \vdash_{\beta\cup\gamma} \Delta, \Lambda}\ mer}{\varphi \to \psi, \Gamma, \Sigma \vdash_{\alpha\cap(\beta\cup\gamma)} \Delta, \Lambda} \to L$$

To move the mer-rule upwards, if we transform the above proof on the left to that on the right, then we have $(\alpha \cap \beta) \cup \gamma \supseteq \alpha \cap (\beta \cup \gamma)$, and hence, the index of the group in the end-sequent is not retained. Thus, we need more sophisticated methods of proof-search, including our mer-rule. This investigation is left to future work.

Among various logical approaches to judgment aggregation, let us discuss ones directly related to our study. Porello [34] analysed the discursive paradox by using the sequent calculus of linear logic. In his analysis, a sequent has the form $\alpha \vdash \varphi$, where α is a set of individuals and φ is a formula, and hence, a set of individuals (appears in the antecedent of a sequent) and formulas (appears in the succedent) are mixed in a sequent. We avoid this difficulty by introducing a set of individuals as the index of a sequent.

Porello extended his analysis in [35], where he proposed to use different logics to evaluate the consistency of every individual's judgment (say, the standard classical logic) and the consistency of the collective judgment (say, linear logic). Endriss [12] also introduced similar framework by distinguishing rationality constraints (imposed on every individual's judgment) and feasibility constraints (imposed on the collective judgment). Our idea to use $\mathsf{Lv}a$ and $\mathsf{Lv}c$ is considered to be in their framework. We assume every individual's judgment set to be consistent with respect to the standard classical logic, and we evaluate the consistency of the collective judgment set with respect to $\mathsf{Lv}a$ and $\mathsf{Lv}c$, respectively.

In the literature of multiple agent systems, Belhadi et al. [3] introduced Multiple agent logic. Although the system is obtained from their possibilistic logic, the idea of their system is essentially the same as our Lg. Every formula has the form (φ, α), where φ is a formula and α is a set of individuals (agents), and the formula is read as "at least all the agents in α believe that φ is true." Although in our Lg, we introduce the index of a group to every sequent instead of every formula, the idea is the same. However, their system is essentially based on the resolution calculus, and their system lacks non-logical axioms. Without non-logical axioms, our Lg is nothing but the usual sequent calculus for classical logic, where every sequent is indexed by the group N. Thus, the introduction of non-logical axioms is essential in our Lg, and it makes "premises" in a proof explicit and our analysis on the premise-based approach smoother. Furthermore, in Multiple agent logic, there is no rule corresponding to our mer-rule, and hence, the group accepting a conclusion of a proof is the smallest group in the proof. Thus, it seems difficult to analyse the discursive paradox by using Multiple agent logic.

Our Lg is quite simple because it is obtained from the usual sequent calculus by augmenting groups of individuals as indexes to sequents. Hence, by replacing the basic sequent system to other systems such as intuitionistic and modal logic systems, we can extend our Lg in various ways. There are some modal logical systems related to our study. For example, to investigate the role of acceptance of a proposition by agents in institutional contexts, [4, 22] introduced a modal logic called Acceptance Logic. Furthermore, [17] introduced a sequent calculus for Acceptance Logic, and formalized the discursive paradox thereof. We may be able to introduce institutional contexts to our Lg, and then, investigate the relationship between Acceptance Logic and our Lg. Further, using modal logic, [30] provided a formalization and analysis on aggregation rules of consensus voting and dictatorship in addition to majority voting. One advantage of the proof-based, or syntactic, approach is that even if it is difficult to give a semantic counterpart of a non-deductive rule such as majority voting, we can include a syntactic inference rule relatively easily. Thus, applying our proof-theoretical approach, we can investigate other concrete rules beyond majority voting in future work.

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