



One-Step Modal Logics, Intuitionistic and Classical, Part 1

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Abstract

This paper and its sequel “look under the hood” of the usual sorts of proof-theoretic systems for certain well-known intuitionistic and classical propositional modal logics. Section 1 is preliminary. Of most importance: a marked formula will be the result of prefixing a formula in a propositional modal language with a step-marker, for this paper either **0** or **1**. Think of **1** as indicating the taking of “one step away from **0**.” Deductions will be constructed using marked formulas. Section 2 presents the model-theoretic concepts, based on those in [7], that guide the rest of this paper. Section 3 presents Natural Deduction systems IK and CK, formalizations of intuitionistic and classical one-step versions of K. In these systems, occurrences of step-markers allow deductions to display deductive structure that is covered over in familiar “no step” proof-theoretic systems for such logics. \Box and \Diamond are governed by Introduction and Elimination rules; the familiar K rule and Necessitation are derived (i.e. admissible) rules. CK will be the result of adding the **0**-version of the Rule of Excluded Middle to the rules which generate IK. Note: IK is the result of merely dropping that rule from those generating CK, without addition of further rules or axioms (as was needed in [7]). These proof-theoretic systems yield intuitionistic and classical consequence relations by the obvious definition. Section 4 provides some examples of what can be deduced in IK. Section 5 defines some proof-theoretic concepts that are used in Section 6 to prove the soundness of the consequence relation for IK (relative to the class of models defined in Section 2.) Section 7 proves its completeness (relative to that class). Section 8 extends these results to the consequence relation for CK. (Looking ahead: Part 2 will investigate one-step proof-theoretic systems formalizing intuitionistic and classical one-step versions of some familiar logics stronger than K.)

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Familiar proof-theoretic systems formalizing normal modal logics take as primitive the rule Necessitation, and either the K-axioms (if the systems are Frege-systems) or a K-rule (if they are natural deduction or sequent systems); for the modal logic K, these are the only modal proof-theoretic primitives. This paper will attend to natural deduction systems. The just-mentioned rules may be pictured as follows.

$$\frac{\mathcal{D}}{\frac{\varphi_{Nec}}{\Box\varphi}} \qquad \frac{\mathcal{D}_1 \quad \mathcal{D}_0}{\frac{\Box(\psi \supset \varphi) \quad \Box\psi_K}{\Box\varphi}}$$

Here \mathcal{D}_0 and \mathcal{D}_1 are any deductions in the relevant ND system with the indicated conclusions. But \mathcal{D} must be a deduction with no assumptions; this makes Necessitation a rule of proof rather than a rule of inference.

When contrasted with the primitive rules governing the familiar logical constants, these rules are anomalous in two respects.

- (A) A good formalization of a classical logic should transform into a formalization of its natural intuitionistic counterpart merely by removal of a “classicalizing” rule or axiom. But doing this to the usual formalizations of classical K yield proof-theoretic systems that are too weak; to obtain the natural intuitionistic version of K we need to follow Plotkin and Sterling’s lead in [7] by adding either axioms (as in the Frege-system that Plotkin and Sterling actually offered) or rules (if the system is a natural deduction or sequent system) that go beyond Necessitation and the K-rule, and that make use of \supset , \vee and \perp , in addition to \Box and \Diamond .
- (B) Necessitation differs from familiar introduction rules by virtue of not being a rule of inference. The K-rule doesn’t look anything like an elimination rule for \Box , at least if our paradigm elimination rules are those for $\&$, \supset , \vee and \perp .

These two anomalies might make one suspect that the familiar proof-theoretic systems for K (and for other normal modal logics) hide some important machinery “under the hood”. I endorse this suspicion. These anomalies result from formalizing K (and the other normal modal logics) in a “no step” format¹, which cuts down on notational clutter at the expense of explicit articulation of deductive structure.

This project is motivated by the thought that the senses of logical constants are constituted by at least some of the deductive rules that govern them. I propose those rules are exactly the introduction and elimination rules (or, in terms of sequent

¹That is, without the step-markers to be introduced below.

calculi of the usual sort, right-entry and left-entry rules).² This is not the place for an extended discussion of these matters.³

The status \Box and \Diamond as logical constants is obscured by proof-theoretic systems in which one cannot “see” their underlying introduction and elimination rules. “One step” proof-theoretic systems, of the sort to be introduced below, give deeper proof-theoretic insight into normal modal logics than is provided by “no step” systems: the primitive rules governing \Box and \Diamond for both intuitionistic and classical K will be proper introduction and elimination rules of inference; and classicalization will not render any intuitionistic rules redundant. Similarly for the most popular strengthenings of K.

1 Preliminaries

Most mathematical notation and terminology used below will be familiar. Natural numbers are finite von-Neumann ordinals. For $n \in \omega$ let $(n) = \{1, \dots, n\}$.

1.1 Notation

Let our logical lexicon $LL = \{\perp, \top, \supset, \&, \vee, \Box, \Diamond\}$. For what follows we will consider a countable set S of formula (i.e. propositional, sentential) constants. From LL generate the set Fml of formulas based on S in the usual way.

\neg and \leftrightarrow are defined in the usual way. Note: we could have taken \top to be defined, e.g. as $(\perp \supset \perp)$. But \top will do work in the primitive rules of the proof-theoretic systems to be discussed below, and this work will not make use of the internal structure

²For more on this idea, see [3]. This and related ideas have been developed in many places. Regarding \Box and \Diamond , see Andrew Parisi’s dissertation [5], which takes a proof-theoretic approach using hypersequents – quite different from the approach I take in this paper.

The motivating idea raises the question of what counts as an introduction or an elimination rule. This is not the place for an extended discussion, but let me say this. Determining the canonical “sense-fixing” rules for first-order \forall and \exists is a delicate matter. I think that they should be “free logic” rules, but there are still several ways to go with that. All ways put them in a “local holism” (in the sense of Christopher Peacocke, see [6]) with a primitive logical 1-place predicate for existence. (The more traditional existence-presupposing first-order logics are convenient but philosophically misleading simplifications of their free counterparts.)

In [8] (p. 15), Jason Turner gives an argument that one might think shows that “the” (better: any reasonable choice of) introduction and elimination rules for \exists do not determine its sense. Suffice to say: I think that that impression turns on a misunderstanding of what is required for introduction and elimination rules to be sense-determining.

³But I want to digress on one issue. In addition to the introduction and elimination rules governing \forall and \exists , their amenability to domain restrictions determined by contexts-of-use may be thought to be built into their senses. One could take this to be a friendly weakening of my motivating thought. And there are various “species” of necessity and possibility which can be expressed by \Box and \Diamond , according to contexts-of-use; and this too may be thought to be built into the senses of \Box and \Diamond . On the other hand, I think it better to construe the domain-restrictability of \forall and \exists as a matter of logical syntax: the determiners of natural languages that express universal and existential quantification are two-place, of type $((i), (i))$ rather than type $((i))$ (with the first place for a restrictor, which can in principle be null). Is a similar approach to the species of necessity and possibility optimal? This is an interesting question, not to be pursued here; but see [1].

that \top would have under any such definition; taking \top to be defined would make the rules in which \top plays a role more complex than they need to be. Furthermore there are infinitely many ways in which \top could be defined. Making a choice would make our choice of certain primitive rules arbitrary.⁴ This point does not cut against defining \neg and \leftrightarrow , since these will play no role our primitive rules.

1.2 Definitions

0 and **1** are our step-markers. A marked formula (hereafter an *m*-formula) is a symbol of the form $\mathbf{0}\varphi$ or $\mathbf{1}\varphi$ for $\varphi \in Fml$.

For $\Delta \subseteq Fml$ and $\mathbf{m} = \mathbf{0}$ or $\mathbf{1}$ let $\mathbf{m}\Delta = \{\mathbf{m}\varphi \mid \varphi \in \Delta\}$. Let $MFml =$ the set of *m*-formulas.

1.3 Heuristic Remark

In $\mathbf{m}\varphi$, the marker \mathbf{m} indicates a “mode of acceptance” of φ , one whose “cash value” lies in certain roles that φ can play in deductions. **0** marks acceptance as actually true. It is the only step in play in almost all work in logic and mathematics with which I am acquainted; because of this, that literature takes the “no step” approach; it ignores stepping. **1** indicates acceptance as true at a (modally) accessible possible world (or, if you prefer, state);⁵ one can think of **1** as indicating one step taken from the “world of evaluation” along an accessibility relation.⁶

When I started work on this paper, I thought of step-markers as indicating mood for deductions: the 0-step as indicative mood, the 1-step as subjunctive mood. I no longer think that this analogy is helpful; for more on this, see 3.10 below.

2 Model-Theoretic Semantics

Although the main point of this paper is proof-theoretic, the deductive system will be easier to understand if one has seen how the corresponding model-theory works.

2.1 Definitions

Following [7], let $F = \langle W, R, \sqsubseteq \rangle$ be an IK-frame iff: $R \subseteq W^2$, $\sqsubseteq \subseteq W^2$ is transitive and reflexive on W , and the following conditions are satisfied: (“right completeness”) for any u, u' and v , if $u \sqsubseteq u'$ and uRv then there is a v' so that $v \sqsubseteq v'$ and $u'Rv'$; (“left completeness”) for any u, v and v' , if $uRv \sqsubseteq v'$ then there is a

⁴If for a given logic there is a unique set of natural-deduction rules governing its logical constants that captures the senses of these constants, and thus that carves logical reality at its joints, then the choice of primitive rules should not involve arbitrary choices.

⁵For a different use of markers for formulas, see [2] (p. 172).

⁶A caveat: under a reflexive accessibility relation, one can take one step from the actual world and remain where one started.

u' so that $u \sqsubseteq u'Rv'$. R corresponds to modal accessibility as in familiar frame-models for “no step” modal languages; \sqsubseteq corresponds to cognitive accessibility as in familiar frame-models for intuitionistic and intermediate logics. In the following diagram-completion pictures, writing x above y indicates that $y \sqsubseteq x$, horizontal arrows represent R , and underlining indicates the existentially bound variable.

$$\begin{array}{cc}
 u' - \text{---} > \underline{v'} & \underline{u'} - \text{---} > v' \\
 u \text{---} > v & u \text{---} > v \\
 \text{right completeness} & \text{left completeness}
 \end{array}$$

Consider a IK-frame F as above. Let $W^F = W$.

Let u be a dead-end under F iff there is no v so that uRv . Let uR^+w iff for some $u \sqsubseteq v$ and vRw .

Let \mathcal{V} be a valuation on $W \times S$ iff $\mathcal{V} : W \times S \rightarrow 2$, and \mathcal{V} is persistent with respect to \sqsubseteq (i.e. for any $u, v \in W$ and $\varphi \in S$, if $u \sqsubseteq v$ then $\mathcal{V}(u, \varphi) \leq \mathcal{V}(v, \varphi)$).

$\mathcal{M} = \langle F, \mathcal{V} \rangle$ is a model with signature S iff F is a frame and \mathcal{V} is a valuation on $W \times S$. Set $W^{\mathcal{M}} = W^F$.

\mathcal{M} is an IK-model iff it is a model and F is an IK-frame.

2.2 Definitions

Consider any model \mathcal{M} as above.

For $u \in W^{\mathcal{M}}$, $\langle \mathcal{M}, u \rangle$ is a pointed model. (I will omit brackets where confusion is unlikely.)

We define the relation \models between pointed models and formulas by clauses most of which are familiar, but these deserve to be stated:

$\mathcal{M}, u \models (\varphi \supset \psi)$ iff: for any v , if $u \sqsubseteq v$ and $\mathcal{M}, v \models \varphi$ then $\mathcal{M}, v \models \psi$;

$\mathcal{M}, u \models \Box\varphi$ iff: for any u and w , if uR^+w then $\mathcal{M}, w \models \varphi$;

$\mathcal{M}, u \models \Diamond\varphi$ iff: for some v , uRv and $\mathcal{M}, v \models \varphi$.

As usual, for $\Delta \subseteq Fml$ let $\mathcal{M}, u \models \Delta$ iff for every $\delta \in \Delta$ $\mathcal{M}, u \models \delta$.

2.3 Persistence Lemma

Consider an IK-model \mathcal{M} . (1) For every $\varphi \in Fml$, if $\mathcal{M}, u \models \varphi$ and $u \sqsubseteq u'$ then $\mathcal{M}, u' \models \varphi$.

Proof: induction on the stages of the inductive definition of Fml . In the induction step, if φ is $\Box\psi$, use the transitivity of \sqsubseteq . If φ is $\Diamond\psi$, use the “right completeness” condition.

2.4 Definitions

Consider a model $\mathcal{M} = \langle F, \mathcal{V} \rangle$ and $u, v \in W$, $\varphi \in Fml$, and $\Gamma \subseteq MFml$.

$\mathcal{M}, u \Vdash m\varphi$ iff $m = \mathbf{0}$, and $\mathcal{M}, u \models \varphi$.

$\mathcal{M}, u, v \Vdash m\varphi$ iff uRv , and either $m = \mathbf{0}$ and $\mathcal{M}, u \models \varphi$, or $m = \mathbf{1}$ and $\mathcal{M}, v \models \varphi$.

$\mathcal{M}, u \Vdash \Gamma$ iff for every $\gamma \in \Gamma$ $\mathcal{M}, u \Vdash \gamma$.

$\mathcal{M}, u, v \Vdash \Gamma$ iff uRv , and for every $\gamma \in \Gamma$ $\mathcal{M}, u, v \Vdash \gamma$.

2.5 Heuristic Remark

Think of $\mathbf{1}$ as carrying a free variable that relative to u ranges over $\{v \mid uRv\}$. Since that set can be empty, our proof-theoretic systems will bear a structural similarity to systems tailored to free-logic semantics.

2.6 Definitions

Let $\langle \Gamma, \chi \rangle$ be an inference iff $\Gamma \subseteq MFml$ and $\chi \in MFml$.

Consider an inference $\langle \Gamma, \chi \rangle$. Given a model \mathcal{M} and $u \in W^{\mathcal{M}}$, $\langle \Gamma, \chi \rangle$ is \mathcal{M} -valid at u iff: (V1) if u is a dead-end and $\mathcal{M}, u \Vdash \Gamma$ (in which case $\Gamma \subseteq \mathbf{0}Fml$), then $\mathcal{M}, u \Vdash \chi$ (in which case $\chi \in \mathbf{0}Fml$); (V2) for every v , if $\mathcal{M}, u, v \models \Gamma$ then $\mathcal{M}, u, v \models \chi$. $\langle \Gamma, \chi \rangle$ is \mathcal{M} -valid iff it is \mathcal{M} -valid at every $u \in W^{\mathcal{M}}$.

A weaker property deserves some attention: let $\langle \Gamma, \chi \rangle$ is \mathcal{M} -valid⁻ at u iff: (V1⁻) if $\mathcal{M}, u \Vdash \Gamma$ then $\mathcal{M}, u \Vdash \chi$; (V2) is true. $\langle \Gamma, \chi \rangle$ is \mathcal{M} -valid⁻ iff it is \mathcal{M} -valid⁻ at every $u \in W^{\mathcal{M}}$.

$\langle \Gamma, \chi \rangle$ is IK-valid [IK-valid⁻], alternatively χ is an IK-consequence [IK-consequence⁻] of Γ , iff for every IK-model \mathcal{M} , $\langle \Gamma, \chi \rangle$ is \mathcal{M} -valid [\mathcal{M} -valid⁻].

For $\chi \in MFml$, χ is \mathcal{M} -valid [\mathcal{M} -valid⁻] iff $\langle \{\}, \chi \rangle$ is \mathcal{M} -valid [\mathcal{M} -valid⁻].

χ is IK-valid [IK-valid⁻] iff for every IK-model \mathcal{M} , φ is \mathcal{M} -valid [\mathcal{M} -valid⁻].

2.7 Consider any Model \mathcal{M}

2.7.1 Observations

Consider any $u \in W^{\mathcal{M}}$ and $\theta \in Fml$. (1) $\langle \Gamma, \mathbf{0}\theta \rangle$ is \mathcal{M} -valid⁻ at u iff it is \mathcal{M} -valid at u . (2) If $\langle \Gamma, \mathbf{1}\theta \rangle$ is \mathcal{M} -valid⁻ at u then it is \mathcal{M} -valid at u . (3) $\langle \{\mathbf{0}\diamond\top\}, \mathbf{1}\top \rangle$ is \mathcal{M} -valid⁻ at u iff u is a dead-end in $W^{\mathcal{M}}$. These observations show that for inferences IK-validity is slightly stronger than IK-validity⁻.

Proofs. For (1), left-to-right is straightforward. Assume the right-side. Assume that $\mathcal{M}, u \Vdash \Gamma$. If u is a dead-end in $W^{\mathcal{M}}$, $\mathcal{M}, u \Vdash \mathbf{0}\theta$. If u is not a dead-end, fix a v so that uRv ; so $\mathcal{M}, u, v \models \Gamma$; so $\mathcal{M}, u, v \models \mathbf{0}\theta$; so $\mathcal{M}, u \Vdash \mathbf{0}\theta$. The left-side follows. (2) is straightforward. For (3), assume the right-side. So $\mathcal{M}, u \not\models \mathbf{0}\diamond\top$; the left-side vacuously follows. Assume the left-side. Assume that u is not a dead-end. So $\mathcal{M}, u \models \diamond\top$; so $\mathcal{M}, u \Vdash \mathbf{0}\diamond\top$. By the left-side, $\mathcal{M}, u \Vdash \mathbf{1}\top$; so $\mathbf{1}\top \in \mathbf{0}Fml$, a contradiction. The right-side follows.

2.7.2 Observations

(1) If $\langle \Gamma, \mathbf{1}\theta \rangle$ is \mathcal{M} -valid and every member of $W^{\mathcal{M}}$ is a dead-end, then $\langle \Gamma, \mathbf{1}\theta \rangle$ is \mathcal{M} -valid⁻. (2) $\langle \{\mathbf{0}\diamond\top\}, \mathbf{1}\top \rangle$ is \mathcal{M} -valid, and so is IK-valid. (3) If some member of $W^{\mathcal{M}}$ is not a dead-end then $\langle \{\mathbf{0}\diamond\top\}, \mathbf{1}\top \rangle$ is not \mathcal{M} -valid⁻. (4) For any $\Sigma \subseteq Fml$ and $\theta \in Fml$, if $\langle \mathbf{0}\Sigma, \mathbf{1}\theta \rangle$ is \mathcal{M} -valid⁻ then so is $\langle \mathbf{0}\Sigma, \mathbf{0}\perp \rangle$.

Proofs. For (1), assume the if-clause. Consider any $u \in W^{\mathcal{M}}$. Assume that $\mathcal{M}, u \Vdash \Gamma$; so $\mathcal{M}, u \Vdash \mathbf{1}\theta$, a contradiction; so $\langle \Gamma, \mathbf{1}\theta \rangle$ vacuously satisfies (V1). For any v , $\mathcal{M}, u, v \not\Vdash \Gamma$ because not uRv ; so $\langle \Gamma, \mathbf{1}\theta \rangle$ vacuously satisfies (V2), yielding the then-clause. For (2): if u is a dead-end in $W^{\mathcal{M}}$ then $\mathcal{M}, u \not\Vdash \mathbf{0}\Diamond\top$, showing (V1) to be vacuously true of $\langle \{\mathbf{0}\Diamond\top\}, \mathbf{1}\top \rangle$. If u is not a dead-end, (V2) is true of $\langle \{\mathbf{0}\Diamond\top\}, \mathbf{1}\top \rangle$. Universally quantifying-out u , and then \mathcal{M} , (2) follows. (3) follows from 2.7.1.(3). Assume that $\langle \mathbf{0}\Sigma, \mathbf{1}\theta \rangle$ is \mathcal{M} -valid⁻. For any v , if $\mathcal{M}, u, v \Vdash \mathbf{0}\Sigma$ then $\mathcal{M}, u \Vdash \mathbf{0}\Sigma$; so by (V1⁻) $\mathcal{M}, u \Vdash \mathbf{1}\theta$; so $\mathbf{1}\theta \in \mathbf{0}Fml$, a contradiction. So $\mathcal{M}, u, v \not\Vdash \mathbf{0}\Sigma$ and $\mathcal{M}, u \not\Vdash \mathbf{0}\Sigma$. So $\langle \mathbf{0}\Sigma, \mathbf{0}\perp \rangle$ is vacuously \mathcal{M} -valid⁻, proving (4).

2.7.3 Observations

(1) If χ is \mathcal{M} -valid⁻ then $\chi \in \mathbf{0}Fml$. (2) If χ is \mathcal{M} -valid and $W^{\mathcal{M}}$ contains a dead-end then $\chi \in \mathbf{0}Fml$. (3) $\mathbf{1}\top$ is \mathcal{M} -valid iff $W^{\mathcal{M}}$ contains no dead-end. (4) χ is IK-valid⁻ iff χ is IK-valid.

Proofs. (1) is straightforward. Assume (2)'s if-clause; fix u to be a dead-end in $W^{\mathcal{M}}$. Since $\mathcal{M}, u \Vdash \chi$, $\chi \in \mathbf{0}Fml$, yielding (2). Assume that $\mathbf{1}\top$ is \mathcal{M} -valid. If $W^{\mathcal{M}}$ contains a dead-end, (2) yields a contradiction. Assume (3)'s right-side. For any $u \in W^{\mathcal{M}}$, fix a v so that uRv ; so $\mathcal{M}, u, v \Vdash \mathbf{1}\top$. So $\mathbf{1}\top$ is \mathcal{M} -valid. (3) follows. Assume that χ is IK-valid⁻. By (1) $\chi \in \mathbf{0}Fml$; so by 2.7.1.(1) χ is IK-valid. Assume that χ is IK-valid. Since there are models whose frames contain dead-ends, by 2.7.3.(2) $\chi \in \mathbf{0}Fml$; so by 2.7.1.(1) χ is IK-valid⁻. (4) follows.

2.8 Observation

$\mathbf{0}\perp$ and $\mathbf{1}\perp$ are IK-equivalent in this sense: $\langle \{\mathbf{0}\perp\}, \mathbf{1}\perp \rangle$ and $\langle \{\mathbf{1}\perp\}, \mathbf{0}\perp \rangle$ are IK-valid. So in a sense $\mathbf{0}\perp$ and $\mathbf{1}\perp$ are degenerate marked-formulas, since the markers “don’t matter”.⁷

3 The Proof-Theoretic System IK

3.1 Definitions

A string is a function with domain in ω . Let $\hat{\ }^$ be concatenation for strings. For any string s_1 , $s_0 \preceq s_1$ iff for some string s $s_1 = s_0\hat{\ }s$. Read \preceq as “is an initial segment of”.

T is a naked (aka bare) tree iff T is a non-empty set of strings of natural numbers closed under taking initial segments (i.e. for any $s \in T$, if $t \preceq s$ then $t \in T$) and T satisfies the “no left gaps” condition (i.e. for any s, m and n , if $s\hat{\ }[m] \in T$ and $n < m$ then $s\hat{\ }[n] \in T$). s is a leaf of a naked tree T iff $s \in T$ and for every s' , if $s' \in T$ and $s \preceq s'$ then $s = s'$. For such a T and $s \in T$, let $T_s = \{t \mid s\hat{\ }t \in T\}$.

⁷Had we allowed for “exploding” IK-models instead of our “non-exploding” ones in 2.6, we would have a narrower IK-consequence relation: e.g. $\langle \{\mathbf{1}\perp\}, \mathbf{0}\perp \rangle$ would not be IK-valid. This is not the place for a full treatment of exploding models.

3.2 Remarks

[] is a member, the “root”, of every naked tree. So {[]} is the smallest naked tree. The “no left gaps” condition insures that any two naked trees are identical if they are isomorphic with respect to \leq .⁸

3.3 Definition

A labeled tree is a function whose domain is a naked tree. For a labeled tree \mathcal{T} , let \mathcal{T}_s be the labeled tree with domain $dom(\mathcal{T})_s$ and such that $\mathcal{T}_s(t) = \mathcal{T}(s\hat{t})$ for each $t \in dom(\mathcal{T})_s$

3.4 Notation

For the rest of this paper, we will make use of a countably infinite set Var of variables (disjoint from the other sets just introduced, and neither step-markers is a variable).

Definition. A tagged m-formula (i.e. a primitive type-assignment) is a symbol of the form $v:\chi$ for $\chi \in MFml$; v is its tag. We will sometimes treat $v:\chi$ as if it were an ordered pair.

3.5 Definitions

C is a context iff C is a single-valued set of tagged m-formulas (i.e. for any v, χ and χ' , if $v:\chi, v:\chi' \in C$ then χ is χ'). So $dom(C) =$ the set of variables occurring on the left-side of members of C , and $ran(C) =$ the set of m-formulas occurring on the right-side of members of C . A set of contexts is coherent iff its union is a context.

3.6 Preparatory Remarks

In 3.8 below we will inductively define a type-assignment system \Rightarrow_{IK} . Informally, “ $C \Rightarrow_{IK} \mathcal{D}:\chi$ ” means: relative to context C , \mathcal{D} is an IK-deduction with conclusion χ . Through Section 7, ‘ \Rightarrow ’ and ‘deduction’ abbreviate ‘ \Rightarrow_{IK} ’ and ‘IK-deduction’.

As is usual, in “ $C \Rightarrow \mathcal{D}:\chi$ ” the mention of χ is redundant, since \mathcal{D} will uniquely determine χ . Our definition will be slightly unusual because it will require that if $C \Rightarrow \mathcal{D}:\chi$ and $C' \Rightarrow \mathcal{D}:\chi$ then $C = C'$; the idea is to keep contexts relative to which labeled trees are deductions as small as possible.

A deduction will be a labeled tree. Leaves of $dom(\mathcal{D})$ will be labeled by tagged m-formulas; non-leaves in $dom(\mathcal{D})$ will be labeled by either an m-formula or an ordered pair of an m-formula followed by a tagged m-formula or an ordered triple of an m-formula followed by two tagged m-formulas. In the latter two cases, the left-most

⁸The following multi-place operation on naked trees will briefly be convenient.

Definition. For $n \in \omega, n > 0$ and naked trees $T_{i \in (n)}$, let

$$\#(T_1, \dots, T_n) = \{[]\} \cup \{[i]s \mid s \in T_i, i \in (n)\}.$$

component is the m-formula label, and the other components indicate discharging of those tagged m-formulas at that string.

In what follows, distinct metavariables (e.g. ‘ v_0 ’ and ‘ v_1 ’) represent distinct variables. Let “ $C, v:\xi \Rightarrow D:\chi$ ” abbreviate this: $v \notin dom(C)$, and either $C \cup \{v:\xi\} \Rightarrow D:\chi$ or $C \Rightarrow D:\chi$. So that notation DOES NOT indicate that $v:\xi$ belongs to the context relative to which D is a deduction! Extend this notation in the obvious way to allow for $C, v_1:\xi_1, \dots, v_k:\xi_k \Rightarrow D:\chi$.

The induction clauses will be presented pictorially. Discharging of an occurrence of a tagged formula (or equivalently of the leaf it labels) will be indicated by putting that label within square-brackets. The discharge at a given string in $dom(D)$ of some tagged m-formula(s) will be indicated by superscripting the relevant tagged m-formula(s) being discharged, though to minimize clutter in practice, if the discharging is non-vacuous it is convenient to write only the tagging variable(s).

We will define the “dependency-set” $dpd(D)$ for D , informally the set of leaves of $dom(D)$ on which D “depends”, simultaneously with \Rightarrow .

3.7 Definitions

For $s \in dom(D)$, let s be open in D iff for every $s' \in dpd(D_s)$, $s \hat{=} s' \in dpd(D)$.⁹

Consider a context C . A barrier in D with with exception for C will be an anti-chain $\{s_1, \dots, s_m\} \subseteq dom(D)$ such that (i) for each $i \in (m)$ s_i is open in D , the formula-label of s_i in D is in $\mathbf{0}Fml$, there is no $t \in dpd(D)$ with $s_i \leq t$ and $D(t) \in C$, but (the non-vacuity condition) for some $t \in dpd(D)$ with $s_i \leq t$ there is a v and φ such that $D(t) = v:\mathbf{1}\varphi$, and (ii) for every $t \in dpd(D)$ with $D(t)$ of the form $v:\mathbf{1}\varphi$ for $v \notin dom(C)$ there is an $i \in (m)$ with $s_i \leq t$.¹⁰ If $C = \{\mu:\mathbf{1}\psi\}$, let $\{s_1, \dots, s_m\}$ be a barrier with exception for $\mu:\mathbf{1}\psi$ iff it is a barrier with exception for C .

3.8 Definition

Now, our much awaited definition of \Rightarrow . It has two base-clauses.

Assumption For $v \in Var$ and $\chi \in MFml$, $v:\chi \Rightarrow D:\chi$ for $D = \{\langle [], v:\chi \rangle\}$, i.e. D is $[]$ labelled by $v:\chi$. Also $dpd(D) = dom(D) = \{[]\}$.

$\mathbf{0}\top$ Introduction $\{\} \Rightarrow D:\mathbf{0}\top$ for $D = \{\langle [], \mathbf{0}\top \rangle\}$; D may be pictured thus.

$$\overline{\mathbf{0}\top}^{\mathbf{0}\top I}$$

Also $dpd(D) = \{[]\}$. The over-bar indicates that $\mathbf{0}\top$ is not an assumption in D ; one might think of it as being inferred from the empty set of assumptions.

Next, the induction clauses, starting with $\mathbf{0}$ -versions of all but one of the standard rules for the Boolean constants.

⁹Informally: for every leaf of D identical to or above s , if it was not discharged by the time we reach s (reading down from the leaves), then it is not discharged in D .

¹⁰Informally; members of a barrier “isolate” leaves whose m-formula labels are marked by $\mathbf{1}$, other than labels in the exception set. Nothing precludes a barrier from being empty.

0⊥ Elimination For $\chi \in MFml$, if $C \Rightarrow \mathcal{D}_0:0\perp$ then $C \Rightarrow \mathcal{D}:\chi$ for \mathcal{D} as pictured.¹¹

$$\frac{\mathcal{D}_0}{\frac{0\perp}{0\perp E} \chi}$$

Also, $dpd(\mathcal{D}) = \{[0]\hat{\ }s \mid s \in dpd(\mathcal{D}_0)\}$.

0⊃ Introduction If $C, v:0\psi \Rightarrow \mathcal{D}_0:0\psi$ then $C \Rightarrow \mathcal{D}:0(\psi \supset \varphi)$ for \mathcal{D} as pictured.

$$\frac{\frac{[v:0\psi]}{\mathcal{D}_0} \quad 0\psi}{0(\varphi \supset \psi) \supset I} v$$

Let $dpd(\mathcal{D}) = \{[0]\hat{\ }s \mid s \in dpd(\mathcal{D}_0) \text{ and } \mathcal{D}_0(s) \neq v:0\psi\}$.

0∨ Introduction For $i \in 2$, if $C \Rightarrow \mathcal{D}_0:0\varphi_i$ then $C \Rightarrow \mathcal{D}:0(\varphi_0 \vee \varphi_1)$ for \mathcal{D} as pictured.

$$\frac{\frac{\mathcal{D}_0}{0\varphi_i} \quad 0\varphi_i}{0(\varphi_0 \vee \varphi_1) \vee I}$$

Let $dpd(\mathcal{D}) = \{[0]\hat{\ }s \mid s \in dpd(\mathcal{D}_0)\}$.

0∨ Elimination If $C_2 \Rightarrow \mathcal{D}_2:0(\varphi_0 \vee \varphi_1)$, for both $i \in 2$ $C_i, v_i:0\varphi_i \Rightarrow \mathcal{D}_i:\chi$, $v_0, v_1 \in Var$ are distinct, and $\{C_0, C_1, C_2\}$ is coherent, then $\bigcup_{i \in 3} C_i \Rightarrow \mathcal{D}:\chi$ for \mathcal{D} as pictured.

$$\frac{\frac{\mathcal{D}_2}{0(\varphi_0 \vee \varphi_1)} \quad \frac{[v_0:0\varphi_0] \quad [v_1:0\varphi_1]}{\mathcal{D}_0 \quad \mathcal{D}_1} \quad \chi \quad \chi}{\chi \vee_0, v_1} 0\vee E$$

Let $dpd(\mathcal{D}) = \bigcup_{i \in 2} \{[i+1]\hat{\ }s \mid s \in dpd(\mathcal{D}_i), \mathcal{D}_i(s) \neq v_i:0\varphi_i\} \cup \{[0]\hat{\ }s \mid s \in dpd(\mathcal{D}_2)\}$.¹²

0& Introduction If for $i \in \{2\}$, $C_i \Rightarrow \mathcal{D}_i:0\varphi_i$ and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow \mathcal{D}:0(\varphi_0 \& \varphi_1)$ for \mathcal{D} as pictured.¹³

$$\frac{\frac{\mathcal{D}_0}{0\varphi_0} \quad \frac{\mathcal{D}_1}{0\varphi_1}}{0(\varphi_0 \& \varphi_1) \& I}$$

Let $dpd(\mathcal{D}) = \bigcup_{i \in 2} \{[i]\hat{\ }s \mid s \in dpd(\mathcal{D}_i)\}$.

0& Elimination For $i \in 2$, if $C \Rightarrow \mathcal{D}_0:0(\varphi_0 \& \varphi_1)$, then $C \Rightarrow \mathcal{D}:0\varphi_i$ for \mathcal{D} as pictured.

$$\frac{\frac{\mathcal{D}_0}{0(\varphi_0 \& \varphi_1)} \quad 0\varphi_i}{0\varphi_i} 0\& E$$

Let $dpd(\mathcal{D}) = \{[0]\hat{\ }s \mid s \in dpd(\mathcal{D}_0)\}$.

¹¹ An interpretive aid: $dom(\mathcal{D}) = \#(\mathcal{D}_0)$.

¹² Another interpretive aid: $dom(\mathcal{D}) = \#(\mathcal{D}_2, \mathcal{D}_0, \mathcal{D}_1)$.

¹³ Another interpretive aid: $dom(\mathcal{D}) = \#(\mathcal{D}_1, \mathcal{D}_0)$. Hopefully from now on the reader will not need these aids.

The next rule is really two rules, one for each choice of marker m .

$m \supset$ Elimination If $C_1 \Rightarrow \mathcal{D}_1:m(\psi \supset \varphi)$, $C_0 \Rightarrow \mathcal{D}_0:m\psi$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow \mathcal{D}:m\varphi$ for \mathcal{D} as pictured.

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_0}{\frac{m(\psi \supset \varphi) \quad m\psi}{m\varphi} m \supset E}$$

Let $dpd(\mathcal{D}) = \bigcup_{i \in 2} \{[1 - i]^s \mid s \in dom(\mathcal{D}_i)\}$.

The next two rules have principal formulas marked by **1**.¹⁴

$1 \perp$ Elimination₀ For $\theta \in Fml$, if $C \Rightarrow \mathcal{D}_0:1 \perp$ then $C \Rightarrow \mathcal{D}:0\theta$ for \mathcal{D} as pictured.

$$\frac{\mathcal{D}_0}{\frac{1 \perp}{0\theta} 1 \perp E_0}$$

Also, $dpd(\mathcal{D}) = \{[0]^s \mid s \in dpd(\mathcal{D}_i)\}$.¹⁵

$1 \vee$ Elimination₀ If $C_2 \Rightarrow \mathcal{D}_2:1(\varphi_0 \vee \varphi_1)$, for both $i \in 2$ $C_i, v_i:1\varphi_i \Rightarrow \mathcal{D}_i:0\theta$, $v_0, v_1 \in Var$ are distinct, and $\{C_0, C_1, C_2\}$ is coherent, then $\bigcup_{i \in 3} C_i \Rightarrow \mathcal{D}:0\theta$ for \mathcal{D} as pictured.

$$\frac{\mathcal{D}_2 \quad \mathcal{D}_0 \quad \mathcal{D}_1}{\frac{1(\varphi_0 \vee \varphi_1) \quad 0\theta \quad 0\theta}{0\theta} 1 \vee E_0} \begin{matrix} [v_0:1\varphi_0] [v_1:1\varphi_1] \\ v_0, v_1 \end{matrix}$$

Let $dpd(\mathcal{D}) = \bigcup_{i \in 2} \{[i + 1]^s \mid s \in dpd(\mathcal{D}_i), \mathcal{D}_i(s) \neq v_i:1\varphi_i\} \cup \{[0]^s \mid s \in dpd(\mathcal{D}_2)\}$. Note that this rule, unlike $0 \vee E$, requires that its conclusion be marked by **0**. But see $1 \vee E_1$ in the next section.

The next rule differs from its **0**-cousin by requiring a minor premise.

$1 \supset$ Introduction If $C_0 \cup \{v:1\varphi\} \Rightarrow \mathcal{D}_0:1\psi$, $C_1 \Rightarrow \mathcal{D}_1:1\theta$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow \mathcal{D}:1(\varphi \supset \psi)$ for \mathcal{D} as pictured.

$$\frac{\mathcal{D}_0 \quad \mathcal{D}_1}{\frac{[v:1\varphi] \quad 1\psi}{1(\varphi \supset \psi)} 1 \supset I} 1\theta$$

Let $dpd(\mathcal{D}) = \{[0]^s \mid s \in dpd(\mathcal{D}_0), \mathcal{D}_0(s) \neq v:1\varphi\} \cup \{[1]^s \mid s \in dpd(\mathcal{D}_1)\}$.

The next rule is the only pure step-rule of IK.

¹⁴The reader may wonder why the conclusions for **1**-versions of $\perp E$ and $\vee E$ are marked only by **0**. The point is to minimize the class of instances of primitive rules; we lose nothing by restricting these rules to conclusions in $0Fml$.

¹⁵Along with $0 \perp E$, this rule makes $0 \perp$ and $1 \perp$ interdeducible. Were we to work with exploding IK-models, we would drop this rule.

Transfer₀ If $\{\} \Rightarrow \mathcal{D}_0:\mathbf{0}\varphi$ and $C \Rightarrow \mathcal{D}_1:\mathbf{1}\theta$, then $C \Rightarrow \mathcal{D}:\mathbf{1}\varphi$ for \mathcal{D} as pictured.

$$\frac{\begin{array}{cc} \mathcal{D}_0 & \mathcal{D}_1 \\ \mathbf{0}\varphi & \mathbf{1}\theta \end{array}}{\mathbf{1}\varphi} \text{Trn}_0$$

Let $dpd(\mathcal{D}) = \bigcup_{i \in 2} \{[i]\hat{s} \mid s \in dpd(\mathcal{D}_i)\}$.

The introduction rules for \square and \diamond will “freeze” a formula marked by $\mathbf{1}$ into a modal formula marked by $\mathbf{0}$; the elimination rules reverse this freezing.

\diamond Introduction If $C \Rightarrow \mathcal{D}_0:\mathbf{1}\varphi$, then $C \Rightarrow \mathcal{D}:\mathbf{0}\diamond\varphi$ for this \mathcal{D} .

$$\frac{\begin{array}{c} \mathcal{D}_0 \\ \mathbf{1}\varphi \end{array}}{\mathbf{0}\diamond\varphi} \diamond I$$

Let $dpd(\mathcal{D}) = \{[0]\hat{s} \mid s \in dpd(\mathcal{D}_0)\}$.

\diamond Elimination If $C_0, v:\mathbf{1}\varphi \Rightarrow \mathcal{D}_0:\mathbf{0}\psi$, \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\varphi$, $C_1 \Rightarrow \mathcal{D}_1:\mathbf{0}\diamond\varphi$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow \mathcal{D}:\mathbf{0}\psi$ for this \mathcal{D} .

$$\frac{\begin{array}{cc} [v:\mathbf{1}\varphi] & \\ \mathcal{D}_1 & \mathcal{D}_0 \\ \mathbf{0}\diamond\varphi & \mathbf{0}\psi \end{array}}{\mathbf{0}\psi} \diamond E^v$$

Let $dpd(\mathcal{D}) = \{[0]\hat{s} \mid s \in dpd(\mathcal{D}_1)\} \cup \{[1]\hat{s} \mid s \in dpd(\mathcal{D}_0), \mathcal{D}_0(s) \neq v:\mathbf{1}\varphi\}$.

\square Introduction If $C, v:\mathbf{1}\top \Rightarrow \mathcal{D}_0:\mathbf{1}\varphi$, and \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\top$, then $C \Rightarrow \mathcal{D}:\mathbf{0}\square\varphi$ for this \mathcal{D} .

$$\frac{\begin{array}{c} [v:\mathbf{1}\top] \\ \mathcal{D}_0 \\ \mathbf{1}\varphi \end{array}}{\mathbf{0}\square\varphi} \square I^v$$

Let $dpd(\mathcal{D}) = \{[0]\hat{s} \mid s \in dpd(\mathcal{D}_0), \mathcal{D}_0(s) \neq v:\mathbf{1}\top\}$.

\square Elimination If $C_0 \Rightarrow \mathcal{D}_0:\mathbf{0}\square\varphi$, $C_1 \Rightarrow \mathcal{D}_1:\mathbf{1}\theta$, and $\{C_0, C_1\}$ is coherent, then $C \Rightarrow \mathcal{D}:\mathbf{1}\varphi$ for this \mathcal{D} .

$$\frac{\begin{array}{cc} \mathcal{D}_0 & \mathcal{D}_1 \\ \mathbf{0}\square\varphi & \mathbf{1}\theta \end{array}}{\mathbf{1}\varphi} \square E$$

Let $dpd(\mathcal{D}) = \bigcup_{i \in 2} \{[i]\hat{s} \mid s \in dpd(\mathcal{D}_i)\}$.

Let \Rightarrow_{IK} be the type-assignment system obtained by stopping here. To finish defining \Rightarrow_{IK} we add one more rule.

\diamond Elimination⁺ If $C \Rightarrow \mathcal{D}_0:\mathbf{0}\diamond\top$ then $C \Rightarrow \mathcal{D}:\mathbf{1}\top$ for this \mathcal{D} .

$$\frac{\begin{array}{c} \mathcal{D}_0 \\ \mathbf{0}\diamond\top \end{array}}{\mathbf{1}\top} \diamond E^+$$

3.9 Remarks

A well-known point, but perhaps worth repeating: discharging is, in effect, variable-binding: the discharging superscripts bind all the occurrences of the associated variables at the leaves labeled by the discharged tagged formulas.

The barrier-conditions for $\diamond E$ and $\Box I$ correspond to the conditions on the eigenvariable for $\exists E$ and $\forall I$ in natural-deduction systems for first-order logic. The motivation for these conditions will be clarified by the soundness lemmas for these rules.

The minor premises for $\mathbf{1} \supset I$ and $\Box E$ play a role that resembles the role of the singular-existence premises for $\forall E$ in free first-order natural-deduction systems. This will be made clearer by the soundness lemmas for these rules.

$\diamond E^+$ differs from the other rules governing \diamond and \Box by virtue of having only one instance: from $\mathbf{0} \diamond \top$ to $\mathbf{1} \top$. I see no non-ad-hoc way to strengthen $\diamond E$ so as to make $\diamond E^+$ admissible.¹⁶

3.10 Remarks

When writing an earlier version of this paper, I viewed the occurrence of the marker in the conclusion of a deduction as a “moodal” operator on that deduction: we could rewrite $C \Rightarrow \mathcal{D}:m\varphi$ as $C \Rightarrow m(\mathcal{D}):\varphi$, indicating that relative to C $m(\mathcal{D})$ is a deduction of φ ; its mood is indicative if $m = \mathbf{0}$ and subjunctive if $m = \mathbf{1}$. I was led in this direction by thinking of mood in natural languages (and formal languages intended to reflect features of natural languages) as a matter of embeddable operators on formulas, along the lines formulated by Lloyd Humberstone in [4].¹⁷ (As a marker on formulas, m cannot be embedded; but as an operator on deductions, it can – if $dom(\mathcal{D}) \neq \{[\]\}$, \mathcal{D} contains step-markers above its root.) Work by Kai Wehmeier and Helge Rückert [9, 10] persuaded me that mood, as a feature of natural-languages, is best captured in a formal language by markers attaching to predicate-expressions, quantifier-expressions, and modal operators, rather than by operators on formulas. Thus the analogy between mood, properly understood, and step-markers as used in this paper is more misleading than helpful.¹⁸

¹⁶The following revision of Trn_0 would render $\diamond E^+$ admissible.

$$\frac{\begin{array}{cc} \mathcal{D}_0 & \mathcal{D}_1 \\ \mathbf{0}\varphi & \mathbf{0}\diamond\theta \end{array} Trn_0}{\mathbf{1}\varphi}$$

But it is preferable that Trn_0 be a pure one-step rule, and the special accommodation of the inference from $\mathbf{0} \diamond \top$ to $\mathbf{1} \top$ be undisguised.

¹⁷Although at that point I had not yet read [4].

¹⁸Although English speakers frequently use ‘suppose’ and ‘assume’ as synonyms, my own ideolect (perhaps corrupted by thinking about modal logic) prefers the indicative mood with ‘assume that’ and the subjunctive mood with ‘suppose that’. Perhaps I am not unique in this regard. If so, one might refer to assumption under the marker $\mathbf{1}$ as supposition, since model-theoretically it bears on an accessible world, and reserve ‘assume’ for assumption under the marker $\mathbf{0}$.

3.11 Definitions

Let $\Gamma \vdash_{IK} \chi$ [$\Gamma \vdash_{IK}^- \chi$] iff for some C and \mathcal{D} , $C \Rightarrow \mathcal{D}:\chi$ [$C \Rightarrow^- \mathcal{D}:\chi$] and $ran(C) \subseteq \Gamma$.

If Γ is finite, I will follow the standard convention of listing the members of Γ to the left of \vdash_{IK} without the curly brackets when writing $\Gamma \vdash_{IK} \chi$; ditto for \vdash_{IK}^- .

If $C \Rightarrow \mathcal{D}:\chi$ or $C \Rightarrow^- \mathcal{D}:\chi$, let $A(\mathcal{D}) = ran(C)$. $A(\mathcal{D})$ is the set of m-formulas assumed in \mathcal{D} .

3.12 Remarks

$\&$, \vee , \supset and \perp are what I call “level 0” logical constants, since their introduction and elimination rules involve no uses of any other logical constants.¹⁹ Were we to take \neg as a primitive logical constant, it would be of “level 1”, since its canonical elimination rule uses \perp .²⁰ Our introduction and elimination rules for \Box and \Diamond show them to also be of level 1, since they use \top .

Our elimination rules for \Box and \Diamond invert (in the sense of Prawitz) the corresponding introduction rules – i.e. they are “harmonious” (in Michael Dummett’s sense).

And they are sense-determining, in the following sense. Introduce a fresh monadic operator $\hat{\Box}$, and add introduction and elimination rules $\hat{\Box}I$ and $\hat{\Box}E$ by replacing \Box by $\hat{\Box}$ in the formulations of $\Box I$ and $\Box E$. In the resulting proof-theoretic system, \Box and $\hat{\Box}$ are interchangeable, preserving deductions. Similarly for \Diamond .

Our Natural Deduction system(s) can easily be transformed into Sequent Calculi, in which the right and left entry rules that correspond to our introduction and elimination rules for \Box and \Diamond are symmetric in the sense of Parisi.²¹

4 Some Deductions and Remarks

I will omit the subscript in \vdash_{IK} until further notice.

4.1 Lemmas

The following rules are admissible for $\Rightarrow = \Rightarrow_{IK}$ or \Rightarrow^-_{IK} .

Transfer_m If for $m \in \omega$, $m > 0$, $\{v_i:\mathbf{0}\varphi_i \mid i \in (m)\} \Rightarrow \mathcal{D}_0:\mathbf{0}\theta$, for each $i \in (m)$ $C_i \Rightarrow \mathcal{D}_i:\mathbf{1}\varphi_i$, and $\{C_i \mid i \in (m)\}$ is coherent, then there is a \mathcal{D} so that

¹⁹They are separated, in the sense of [11], p. 183.

²⁰It has this form.

$$\frac{\mathcal{D}_1 \quad \mathcal{D}_0}{\mathbf{0}\neg\varphi \quad \mathbf{0}\varphi} \perp$$

²¹See [5], p. 44.

$\bigcup_{i \in (m)} C_i \Rightarrow \mathcal{D}:\mathbf{1}\theta$. Here is the picture for $m = 1$.

$$\frac{\frac{\frac{[v:\mathbf{0}\varphi_1]}{\mathcal{D}_0} \quad \mathbf{0}\theta \quad \mathbf{0}\supset I}{\mathbf{0}(\varphi_1 \supset \theta)^v} \quad \mathcal{D}_1 \quad \mathbf{1}\varphi_1 \quad \text{Trn}_0 \quad \mathcal{D}_1}{\mathbf{1}(\varphi_1 \supset \theta)} \quad \mathbf{1}\varphi_1 \quad \mathbf{1}\supset E}{\mathbf{1}\theta}$$

For $m > 1$, iterate $\mathbf{0} \supset I$ m -times, and then $\mathbf{1} \supset E$ m -times. Let Transfer_m be this derived rule.

Transfer_1 lets us “complete” $\mathbf{1}\perp E_0$ thus.

1⊥ Elimination₁ For $\theta \in Fml$, if $C \Rightarrow \mathcal{D}_0:\mathbf{1}\perp$ then $C \Rightarrow \mathcal{D}:\mathbf{1}\theta$ for \mathcal{D} as pictured.

$$\frac{\frac{[v:\mathbf{0}\perp]}{\mathbf{0}\perp E} \quad \mathcal{D}_0 \quad \mathbf{0}\theta}{\mathbf{1}\perp} \quad \text{Trn}_1}{\mathbf{1}\theta^v}$$

Transfer_1 also gives us derived rules $\mathbf{1} \vee I$ and $\mathbf{1}\&E$ that, in an obvious sense, are $\mathbf{1}$ -versions of $\mathbf{0} \vee I$ and $\mathbf{0}\&E$. Here is one example; the remaining one is an exercise for the reader.

1∨ Introduction for $i \in 2$, if $C \Rightarrow \mathcal{D}_0:\mathbf{1}\varphi_i$ then $C \Rightarrow \mathcal{D}:\mathbf{1}(\varphi \supset \psi)$ for this \mathcal{D} .

$$\frac{\mathcal{D}_0 \quad \frac{[v:\mathbf{0}\varphi_i]}{\mathbf{0}(\varphi_0 \vee \varphi_1)} \quad \mathbf{0}\vee I}{\mathbf{1}\varphi_i \quad \mathbf{0}(\varphi_0 \vee \varphi_1)} \quad \text{Trn}_1}{\mathbf{1}(\varphi_0 \vee \varphi_1)^v}$$

Transfer_2 gives us a $\mathbf{1}$ -version of $\mathbf{0}\&I$ as a derived rule, as follows.

If for $i \in \{2\}$, $C_i \Rightarrow \mathcal{D}_i:\mathbf{1}\varphi_i$ and $\{C_0, C_1\}$ is coherent then $C \Rightarrow \mathcal{D}:\mathbf{1}(\varphi_0 \& \varphi_1)$ for this \mathcal{D} .

$$\frac{\mathcal{D}_0 \quad \mathcal{D}_1 \quad \frac{[v_0:\mathbf{0}\varphi_0] \quad [v_1:\mathbf{0}\varphi_1]}{\mathbf{0}(\varphi_0 \& \varphi_1)} \quad \mathbf{0}\&I}{\mathbf{1}\varphi_0 \quad \mathbf{1}\varphi_1 \quad \mathbf{0}(\varphi_0 \& \varphi_1)} \quad \text{Trn}_2}{\mathbf{1}(\varphi_0 \& \varphi_1)^{v_0, v_1}}$$

Remark We could have taken Transfer_2 as primitive, and then obtained $\mathbf{1} \supset E$ as a derived rule. But our approach is more economical.

1∨ Elimination₁ If $C_2 \Rightarrow \mathcal{D}_2:\mathbf{1}(\varphi_0 \vee \varphi_1)$, for both $i \in 2$ $C_i, v_i:\mathbf{1}\varphi_i \Rightarrow \mathcal{D}_i:\mathbf{1}\theta$, $v_0, v_1 \in Var$ are distinct, and $\{C_0, C_1, C_2\}$ is coherent, there is a deduction \mathcal{D} so that $\bigcup_{i \in 3} C_i \Rightarrow \mathcal{D}:\mathbf{1}\theta$.

We will construct such a \mathcal{D} as follows. Let σ_0 be $((\varphi_0 \supset \theta) \supset ((\varphi_1 \supset \theta) \supset \theta))$, and σ_1 be $((\varphi_0 \vee \varphi_1) \supset \sigma_0)$. σ_1 is an intuitionistic validity; so we may fix \mathcal{D}' to be the obvious deduction, constructed using $\mathbf{0} \vee E$ and $\mathbf{0} \supset I$, such that $\{\} \Rightarrow \mathcal{D}':\mathbf{0}\sigma_1$. Let \mathcal{D}'' be the following.

$$\frac{\mathcal{D}' \quad \mathcal{D}_2}{\mathbf{0}\sigma_1 \quad \mathbf{1}(\varphi_0 \vee \varphi_1)} \quad \text{Trn}_0}{\mathbf{1}\sigma_1}$$

For both $i \in 2$ let \mathcal{D}'_i be the following.

$$\frac{[v_i:\mathbf{1}\varphi_i] \quad \mathcal{D}_i \quad \mathcal{D}_2}{\frac{\mathbf{1}\theta \quad \mathbf{1}(\varphi_0 \vee \varphi_1)_{\mathbf{1}\supset I}}{\mathbf{1}(\varphi_i \supset \theta)}}$$

Let \mathcal{D} be the following.

$$\frac{\frac{\mathcal{D}'' \quad \mathcal{D}_2}{\mathbf{1}\sigma_1 \quad \mathbf{1}(\varphi_0 \vee \varphi_1)_{\mathbf{1}\supset E}} \quad \frac{\mathcal{D}'_0}{\mathbf{1}(\varphi_0 \supset \theta)_{\mathbf{1}\supset E}} \quad \mathcal{D}'_1}{\frac{\mathbf{1}((\varphi_1 \supset \theta) \supset \theta) \quad \mathbf{1}(\varphi_1 \supset \theta)_{\mathbf{1}\supset E}}{\mathbf{1}\theta}}$$

We may abbreviate \mathcal{D} thus.

$$\frac{\frac{\mathcal{D}_2 \quad \mathcal{D}_0 \quad \mathcal{D}_1}{\mathbf{1}(\varphi_0 \vee \varphi_1) \quad \mathbf{1}\theta \quad \mathbf{1}\theta}_{\mathbf{1}\theta \quad v_0, v_1} \quad [v_0:\mathbf{1}\varphi_0] \quad [v_1:\mathbf{1}\varphi_1]}{\mathbf{1}\theta \quad v_0, v_1}_{\mathbf{1}\vee E_1}$$

The reader might amuse him/herself by computing $dpd(\mathcal{D})$.

Necessitation If $\{\} \Rightarrow \mathcal{D}_0:\mathbf{0}\varphi$, then $\{\} \Rightarrow \mathcal{D}:\mathbf{0}\Box\varphi$ for the following \mathcal{D} .

$$\frac{[v:\mathbf{1}\top] \quad \mathcal{D}_0 \quad \mathbf{0}\varphi_{Trn_0}}{\frac{\mathbf{1}\varphi \quad \Box I}{\mathbf{0}\Box\varphi^v}}$$

The barrier for the indicated use of $\Box I$ is $\{\}$.

4.2 Lemmas

We will now consider some schematic deductions that, in effect, recover the axioms presented in [7].

(1) $\mathbf{0}\Diamond\perp \vdash \mathbf{0}\perp$, witnessed by the following.

$$\frac{\frac{\mathbf{0}\Diamond\perp \quad \frac{[v:\mathbf{1}\perp]_{\mathbf{1}\perp E_0}}{\mathbf{0}\perp}_{\Diamond E}}{\mathbf{0}\perp^v}}$$

The barrier for the indicated use of $\Diamond E$ is $\{\}$.

(2) $\mathbf{0}\Box(\varphi \supset \psi), \mathbf{0}\Box\varphi \vdash \mathbf{0}\Box\psi$, witnessed by the following.²²

$$\frac{\frac{\mu_0:\mathbf{0}\Box(\varphi \supset \psi)[v:\mathbf{1}\top]_{\Box E} \quad \mu_1:\mathbf{0}\Box\varphi [v:\mathbf{1}\top]_{\Box E}}{\mathbf{1}(\varphi \supset \psi) \quad \mathbf{1}\varphi_{\supset E}}}{\frac{\mathbf{1}\psi}{\mathbf{0}\Box\psi^v} \Box I}$$

The barrier for the indicated use of $\Box I$ is $\{\}$.

(3) $\mathbf{0}\Box(\varphi \supset \psi), \mathbf{0}\Diamond\varphi \vdash \mathbf{0}\Diamond\psi$, witnessed by the following.

$$\frac{\frac{\mu_0:\mathbf{0}\Box(\varphi \supset \psi) [v:\mathbf{1}\varphi]_{\Box E} \quad \mathbf{1}(\varphi \supset \psi) \quad [v:\mathbf{1}\varphi]_{\supset E}}{\mathbf{1}\psi}_{\Diamond I}}{\frac{\mu_1:\mathbf{0}\Diamond\varphi \quad \mathbf{0}\Diamond\psi}_{\Diamond E}}{\mathbf{0}\Diamond\psi^v}}$$

The barrier for the indicated use of $\Diamond E$ is $\{\}$.

(4) $\mathbf{0}(\Diamond\varphi \supset \Box\psi) \vdash \mathbf{0}\Box(\varphi \supset \psi)$, witnessed by the following

$$\frac{\frac{\frac{\mu:\mathbf{0}(\Diamond\varphi \supset \Box\psi) \quad \mathbf{0}\Diamond\varphi_{\supset E} \quad \frac{[v_0:\mathbf{1}\varphi]_{\Diamond I}}{\mathbf{0}\Box\psi} \quad [v_0:\mathbf{1}\varphi]_{\Box E}}{\mathbf{1}\psi} \quad [v_1:\mathbf{1}\top]_{\supset I}}{\mathbf{1}(\varphi \supset \psi)^{v_0}} \Box I}{\mathbf{0}\Box(\varphi \supset \psi)^{v_1}} \Box I$$

The discharging of $v:\mathbf{1}\top$ at $[0, 0]$ is vacuous. The barrier for the indicated use of $\Box I$ is $\{\}$.

(5) $\mathbf{0}\Diamond(\varphi \vee \psi) \vdash \mathbf{0}(\Diamond\varphi \vee \Diamond\psi)$, witnessed by the following.

$$\frac{\frac{\frac{[v_0:\mathbf{1}\varphi]_{\Diamond I} \quad [v_1:\mathbf{1}\psi]_{\Diamond I}}{\mathbf{0}\Diamond\varphi \quad \mathbf{0}\Diamond\psi} \quad \mathbf{0}\vee I}{[v_2:\mathbf{1}(\varphi \vee \psi)] \quad \mathbf{0}(\Diamond\varphi \vee \Diamond\psi)} \quad \mathbf{0}(\Diamond\varphi \vee \Diamond\psi)_{\mathbf{1}\vee E_0}}{\frac{\mu:\mathbf{0}\Diamond(\varphi \vee \psi) \quad \mathbf{0}(\Diamond\varphi \vee \Diamond\psi)^{v_0, v_1}}{\mathbf{0}(\Diamond\varphi \vee \Diamond\psi)^{v_2}} \Diamond E}$$

The barrier for the indicated use of $\Diamond E$ is $\{\}$.

4.3 Observation

For each axiom ψ of Plotkin and Sterling, $\vdash_{IK} \mathbf{0}\psi$. Proof: apply $\mathbf{0}\supset I$ as needed to (1)-(5) above.

4.4 Observations

A few other examples may be helpful.

²²This is, of course, the $\mathbf{0}$ -version of the K-rule.

(1) $\mathbf{1}\varphi \vdash_{IK} \mathbf{1}\top$, witnessed by the following.

$$\frac{[v:\mathbf{1}\varphi] \quad \overline{\mathbf{0}\top}}{\mathbf{1}\top} Trn_0$$

(2) $\mathbf{0}\Box\varphi, \mathbf{0}\Diamond\psi \vdash_{IK} \mathbf{0}\Diamond(\varphi\&\psi)$, witnessed by the following.

$$\frac{\frac{\frac{\mu_0:\mathbf{0}\Box\varphi \quad v:\mathbf{1}\psi}{\mathbf{1}\varphi} \Box E \quad [v:\mathbf{1}\psi]}{\mathbf{1}(\varphi\&\psi)} \mathbf{1}\&I}{\frac{\mu_1:\mathbf{0}\Diamond\psi \quad \mathbf{0}\Diamond(\varphi\&\psi)}{\mathbf{0}\Diamond(\varphi\&\psi)} \Diamond E} \Diamond I$$

The barrier for the indicated use of $\Diamond E$ is $\{\}$.

4.5 Exercises

The reader might like to construct deductions in IK to witness the following: $\vdash_{IK} \mathbf{0}\Box\top$; $\mathbf{0}\neg\Diamond\top \vdash_{IK} \mathbf{0}\Box\perp$; $\mathbf{0}\Box\perp, \mathbf{0}\Diamond\top \vdash_{IK} \mathbf{0}\perp$.

4.6 Observation

Consider a set Γ of formulas and a formula φ containing no occurrences of \Box or \Diamond , and an intuitionistic ND deduction \mathcal{D} of φ from Γ using the familiar “no step” rules. Form Γ' , φ' and \mathcal{D}' from Γ , φ and \mathcal{D} respectively by uniformly substituting formulas (which may contain occurrences of \Box or \Diamond) for members of \mathcal{S} . Form \mathcal{D}'' from \mathcal{D}' by prefixing the formula-label for each string in $dom(\mathcal{D}')$ by $\mathbf{0}$. Then \mathcal{D}'' witnesses that $\mathbf{0}\Gamma' \vdash_{IK} \mathbf{0}\varphi'$.

5 Substitution and Amputation in Deductions

5.1 Definitions

Consider $C, C', \mathcal{D}, \mathcal{D}', \chi, \chi'$ and ν so that $C, \nu:\chi' \Rightarrow \mathcal{D}:\chi, C' \Rightarrow \mathcal{D}':\chi'$, and $\{C, C'\}$ is coherent.

5.2 Definitions

Informally, $[\nu := \mathcal{D}']\mathcal{D}$ is the labeled tree formed by splicing a copy of \mathcal{D}' into \mathcal{D} at every leaf in $dpd(\mathcal{D})$ with label $\nu:\chi'$. As the notation indicates, $[\nu := \mathcal{D}']$ is a substitution function. Formally, we define $[\nu := \mathcal{D}']$ by induction on the stages of the inductive definition of \Rightarrow .

Base clause: if $\mathcal{D} = \{([\], \mu:\chi)\}$ then

$$[\nu := \mathcal{D}']\mathcal{D} = \begin{cases} \mathcal{D}' & \text{if } \nu:\chi' = \mu:\chi; \\ \mathcal{D} & \text{otherwise.} \end{cases}$$

For the induction step, the definition is straightforward. (Reminder: discharging, as indicated by variable-superscripts at root, in effect binds occurrences of the variable(s) at leaves in $d\text{pd}(\mathcal{D})$ with label(s) tagged by that variable, and this blocks substitution.)

5.3 Remarks

In some cases, $[\nu := \mathcal{D}']\mathcal{D}$ will be “bad” (i.e. not what we want), by virtue of failure of substitutability: a tag in a label of some undischarged leaf of \mathcal{D}' may get bound after substitution by some discharging step within \mathcal{D} .²³ But “bound” (i.e. discharging) tags in \mathcal{D} can be “relettered”, i.e. replaced by distinct “fresh” tags (variables not occurring in \mathcal{D}), to form a new but slightly different deduction. So we can define a revised substitution function $[\mathcal{D}'/\nu]$ as follows.

5.4 Definition

Fix an ω -ordering of Var . Given a deduction \mathcal{D} , let μ_1, \dots, μ_m be the variables (listed without repetitions) that tag the labels of members of $d\text{pd}(\mathcal{D}')$ and also that occur discharged (i.e. bound) in \mathcal{D} . Fix $\mu'_1, \dots, \mu'_m \in Var$ that don't occur in \mathcal{D} and are not in $\{\mu_{i \in (m)}\}$; do this in some unique way, e.g. μ'_1 is the first such variable in the ordering of Var , then μ'_2 is the next, etc. Form \mathcal{D}^* from \mathcal{D} by replacing occurrences of μ_i by μ'_i , for $i \in (m)$, say from left to right and high to low (this to make \mathcal{D}^* unique). Note: $C \Rightarrow \mathcal{D}^*:\chi$. We have made sure that applying $[\nu := \mathcal{D}']$ to \mathcal{D}^* involves no failure of substitutability. Let $[\mathcal{D}'/\nu]\mathcal{D} = [\nu := \mathcal{D}']\mathcal{D}^*$.

5.5 Observation

In the above situation, $C \cup C' \Rightarrow [\mathcal{D}'/\nu]\mathcal{D}:\chi$. Proof is by induction on the height of \mathcal{D} .

5.6 Definitions

Let \vdash be a consequence relation (on $MFml$) iff $\vdash \subseteq Power(MFml) \times MFml$ and \vdash satisfies these conditions: (i) (reiteration) for any $\chi \in MFml$ $\{\chi\} \vdash \chi$; (ii) (cut) for any $\Gamma_0, \Gamma_1 \subseteq MFml$ and $\chi \in MFml$, if $\Gamma_0 \vdash \gamma$ for every $\gamma \in \Gamma_1$ and $\Gamma_0 \cup \Gamma_1 \vdash \chi$ then $\Gamma_0 \vdash \chi$.

Consider a consequence relation \vdash . \vdash is monotonic iff for any $\Gamma_0, \Gamma_1 \subseteq MFml$ and $\chi \in MFml$, if $\Gamma_0 \vdash \chi$ and $\Gamma_0 \subseteq \Gamma_1$ then $\Gamma_1 \vdash \chi$. \vdash is finitary iff for any $\Gamma \subseteq MFml$ and $\chi \in MFml$, if $\Gamma \vdash \chi$ then for some finite $\Gamma_1 \subseteq \Gamma$, $\Gamma_1 \vdash \chi$.

For $\varphi \in Fml$, distinct $\alpha_1, \dots, \alpha_n \in S$, and $\psi_1, \dots, \psi_n \in Fml$, let $\left[\overrightarrow{\nu}; \overrightarrow{\psi} \right] \varphi$ be the result of simultaneously substituting ψ_i for all occurrences of α_i in φ , for

²³This is analogous to failure of substitutability in a first-order language when a variable occurring in a term that is substituted into a formula gets bound by a quantifier-prefix occurring in that formula. One difference: $[\nu := \mathcal{D}']\mathcal{D}$ could even fail to be a deduction relative to $C_0 \cup C_1$ (because of a mismatch of types, i.e. of formulas paired with bound variables).

$i \in (n)$. Similarly for $\chi \in Fml$ and $[\vec{v}; \vec{\psi}] \chi$. For $\Gamma \subseteq MFml$, $[\vec{v}; \vec{\psi}] \Gamma = \{[\vec{v}; \vec{\psi}] \gamma \mid \gamma \in \Gamma\}$. \vdash is structural iff for any $\Gamma \subseteq MFml$, $\chi \in MFml$, and \vec{v} and $\vec{\psi}$ as above, if $\Gamma \vdash \chi$ then $[\vec{v}; \vec{\psi}] \Gamma \vdash [\vec{v}; \vec{\psi}] \chi$.²⁴

5.7 Corollary

\vdash_{IK} is a finitary, monotonic structural consequence relation on $MFml$. Proof: the previous observation insures satisfaction of the cut condition.

5.8 Remark

The following questions might have occurred to an attentive reader. In presenting $\square I$, why not require that $A(\mathcal{D}_0) - \{v:\mathbf{1}\top\} \subseteq \mathbf{0}Fml$? And in presenting $\diamond E$, why not require that $ran(C_0 - \{v:\mathbf{1}\varphi\}) \subseteq \mathbf{0}Fml$? Both of these changes would have made our formulation of these rules simpler than imposing the barrier conditions. The answer: these changes would block 5.5, and thus block 5.7.

5.9 Definitions

Assume that $C, v_1:\chi_1, \dots, v_n:\chi_n \Rightarrow \mathcal{D}:\chi$, for each $i \in (n)$ $C'_i \Rightarrow \mathcal{D}'_i:\chi'_i$, and $\{C\} \cup \bigcup_{i \in (n)} C'_i$ is coherent. In this case, we define $[v_1, \dots, v_n := \mathcal{D}'_1, \dots, \mathcal{D}'_n]\mathcal{D}$, the result of simultaneously substituting each \mathcal{D}'_i for v_i , $i \in (n)$, into \mathcal{D} , by generalizing in the obvious way the case for $n = 1$. We define $[\mathcal{D}'_1, \dots, \mathcal{D}'_n/v_1, \dots, v_n]\mathcal{D}$ similarly.

5.10 Definitions

Consider C, \mathcal{D} and χ so that $C \Rightarrow \mathcal{D}:\chi$, and $s_1, \dots, s_m \in dom(\mathcal{D})$, listed without repetitions. Assume that for each $i \in (m)$ s_i is open in \mathcal{D} and $\{s_{i \in (m)}\}$ is an anti-chain in $dom(\mathcal{D})$. For $i \in (m)$, let χ_i be the m-formula label of s_i in \mathcal{D} . Consider distinct $v_1, \dots, v_m \in Var$, none occurring in \mathcal{D} . Let $\mathcal{D}^{\mathcal{S}}$, the result of surgery on \mathcal{D} at s_1, \dots, s_m using v_1, \dots, v_m , be as follows: $dom(\mathcal{D}^{\mathcal{S}}) = dom(\mathcal{D}) - \{s \mid \text{for some } i \in (m) s_i \preceq s\}$; $\mathcal{D}^{\mathcal{S}}(s) = \mathcal{D}(s)$ for each $s \in dom(\mathcal{D}^{\mathcal{S}})$ such that $s \notin \{s_{i \in (m)}\}$; for each $i \in (m)$ let $\mathcal{D}^{\mathcal{S}}(s_i) = \mu_i:\chi_i$.

For $i \in (m)$, let \mathcal{D}_i be the subdeduction of \mathcal{D} with root at s_i , i.e. $\mathcal{D}_i = \mathcal{D}_{s_i}$. In the above surgery on \mathcal{D} , each \mathcal{D}_i has been amputated at s_i , with v_i as a bandage on s_i .

5.11 Observation

For some $C_i \subseteq C$, $C_i \Rightarrow \mathcal{D}_i:\chi_i$. Note: this required that s_i be open in \mathcal{D} (i.e. no leaf of \mathcal{D} above s_i was discharged in \mathcal{D}).

²⁴See [11]

5.12 Observation

Let $C^{\$} = \{\mu:\xi \mid \text{for some } s \in dpd(\mathcal{D}), \text{ no } i \in (m) \text{ has } s_i \preceq s \text{ and } \mathcal{D}(s) = \mu:\xi\}$. Then $C^{\$} \cup \{v_i:\chi_i \mid i \in (m)\} \Rightarrow \mathcal{D}^{\$}:\chi$.

5.13 Observation

This surgery can be reversed:

$$\left[\vec{v} := \vec{\mathcal{D}} \right] \mathcal{D}^{\$} = \left[\vec{\mathcal{D}} / \vec{v} \right] \mathcal{D}^{\$} = \mathcal{D}.$$

5.14 Theorem

For $\Sigma \cup \{\theta\} \subseteq Fml$, $\mathbf{0}\Sigma \vdash_{IK} \mathbf{0}\theta$ iff $\mathbf{0}\Sigma \vdash_{IK}^- \mathbf{0}\theta$.

Proof. Right to left is trivial. Assume the left-side. Fix C and \mathcal{D} so that $ran(C) \subseteq \mathbf{0}\Sigma$ and $C \Rightarrow_{IK} \mathcal{D}:\mathbf{0}\theta$. If there are no uses of $\diamond E_+$ in \mathcal{D} , we have the right-side for free. Assume that $s_0 \in dom(\mathcal{D})$ was entered using $\diamond E_+$; so $\mathcal{D}(s_0) = \mathbf{1}\top$ and $\mathcal{D}(\langle 0 \rangle \wedge s_0) = \mathbf{0}\diamond\top$. Fix $v \in Var$ not occurring in \mathcal{D} . Let $\mathcal{D}^{\$}$ be the result of surgery on \mathcal{D} at s using v as the bandage. Let \mathcal{D}_{s_0} be the sub-deduction so amputated, and let $\mathcal{D}_{\langle 0 \rangle \wedge s_0}$ be its immediate proper subdeduction. So $s \in dom(\mathcal{D}_{s_0})$ iff $s \wedge s_0 \in dom(\mathcal{D})$, and $s \in dom(\mathcal{D}_{\langle 0 \rangle \wedge s_0})$ iff $s \wedge \langle 0 \rangle \wedge s_0 \in dom(\mathcal{D})$. Also, $\mathcal{D}_{\langle 0 \rangle \wedge s}(s) = \mathcal{D}(s \wedge \langle 0 \rangle \wedge s_0)$. So for some C_0 and $C^{\$}$, $C = C_0 \cup C^{\$}$, $C_0 \Rightarrow \mathcal{D}_{\langle 0 \rangle \wedge s_0}:\mathbf{0}\diamond\top$, $C^{\$}, v:\mathbf{1}\top \Rightarrow \mathcal{D}^{\$}:\mathbf{0}\theta$, and $ran(C^{\$}) \subseteq \mathbf{0}Fml$. Construct \mathcal{D}_1 using $\diamond E_0$ as pictured.

$$\frac{\begin{array}{c} [v:\mathbf{1}\top] \\ \mathcal{D}_{\langle 0 \rangle \wedge s_0} \quad \mathcal{D}^{\$} \\ \mathbf{0}\diamond\top \quad \mathbf{0}\theta \end{array}}{\mathbf{0}\theta \quad v} \diamond E_0$$

So $C \Rightarrow_{IK} \mathcal{D}':\mathbf{0}\theta$, and \mathcal{D}' contains one fewer use of $\diamond E_+$ than did \mathcal{D} . Iterate this procedure until all uses of $\diamond E_+$ have been eliminated.

6 Soundness of IK

For $\Gamma \subseteq MFml$ let $m^{-1}\Gamma = \{\varphi \mid m\varphi \in \Gamma\}$.

6.1 Soundness

Theorem for IK. If $\Gamma \vdash \chi$ then $\langle \Gamma, \chi \rangle$ is IK-valid.

Consider any IK-model \mathcal{M} with $F^{\mathcal{M}} = \langle W, R, \sqsubseteq \rangle$. It suffices to prove the following: for any C, \mathcal{D} and χ , if $C \Rightarrow \mathcal{D}:\chi$ then $\langle A(\mathcal{D}), \chi \rangle$ is \mathcal{M} -valid.

We do this by induction on the stages of the inductive definition of \Rightarrow , equivalently on the height of \mathcal{D} , call it $ht(\mathcal{D})$. Restate our goal thus: for every $n \in \omega$, every C, \mathcal{D} and χ , if $C \Rightarrow \mathcal{D}:\chi$ and $ht(\mathcal{D}) \leq n$, then for every $u \in W^{\mathcal{M}}$, (!) $\langle A(\mathcal{D}), \chi \rangle$ is \mathcal{M} -valid at u .

The base step is trivial. Given $n \in \omega$, assume the obvious Induction Hypothesis regarding the n th stage the inductive definition of \Rightarrow . Consider C, \mathcal{D}, χ and $u \in W$ so that $C \Rightarrow \mathcal{D}:\chi$ and $ht(\mathcal{D}) \leq n + 1$. We need a soundness lemma for each rule by which $[]$ might have been entered into \mathcal{D} .

In what follows, let (A) be this: u is a dead-end and $\mathcal{M}, u \Vdash A(\mathcal{D})$. And let (B) be this: given $v, \mathcal{M}, u, v \Vdash A(\mathcal{D})$.

Except for $\mathbf{0} \supset I$, the soundness lemmas for the introduction and elimination rules for the Boolean constants governed by $\mathbf{0}$ are relatively straightforward; so I will skip them.

Assume that $[]$ was entered into \mathcal{D} by $\mathbf{0} \supset I$, as pictured in 3.8. By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{0}\psi \rangle$ is \mathcal{M} -valid at u . Assume (A). So $A(\mathcal{D}) \subseteq \mathbf{0}Fml$ and $\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D})$. Consider a $u' \sqsupseteq u$ such that $\mathcal{M}, u' \models \varphi$. By the Persistence Lemma, $\mathcal{M}, u' \models \mathbf{0}^{-1}A(\mathcal{D})$; so $\mathcal{M}, u' \models \mathbf{0}^{-1}A(\mathcal{D}_0)$. So $\mathcal{M}, u' \Vdash A(\mathcal{D}_0)$. By (*) $\mathcal{M}, u' \Vdash \mathbf{0}\psi$; so $\mathcal{M}, u' \models \psi$. So $\mathcal{M}, u \models (\varphi \supset \psi)$. Assume (B). Consider a $u' \sqsupseteq u$ such that $\mathcal{M}, u' \models \varphi$. Again, by the Persistence Lemma, $\mathcal{M}, u' \models \mathbf{0}^{-1}A(\mathcal{D})$. But what about $\mathbf{1}^{-1}A(\mathcal{D})$? By the right-completeness of $F^{\mathcal{M}}$ we can fix a $v' \sqsupseteq v$ so that $u' Rv'$. Since $\mathcal{M}, v \models \mathbf{1}^{-1}A(\mathcal{D})$, by the Persistence Lemma $\mathcal{M}, v' \models \mathbf{1}^{-1}A(\mathcal{D})$; so $\mathcal{M}, u', v' \Vdash A(\mathcal{D})$. Since $\mathcal{M}, u', v' \Vdash \mathbf{0}\varphi$, $\mathcal{M}, u', v' \Vdash A(\mathcal{D}_0)$. By (*) $\mathcal{M}, u', v' \Vdash \mathbf{0}\psi$; so $\mathcal{M}, u' \models \psi$. So $\mathcal{M}, u \models (\varphi \supset \psi)$. (!) follows.

Assume that $[]$ was entered into \mathcal{D} by $\mathbf{1}\perp E_0$, as pictured in 3.8. By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{1}\perp \rangle$ is \mathcal{M} -valid at u . Assume (A). By (*) $\mathcal{M}, u \Vdash \mathbf{1}\perp$; so $\mathbf{1}\perp \in \mathbf{0}Fml$, a contradiction; so $\mathcal{M}, u \not\models A(\mathcal{D})$. (V1) follows vacuously. Assume (B). By (*) $\mathcal{M}, u, v \Vdash \mathbf{1}\perp$; so $\mathcal{M}, v \models \perp$, a contradiction; (V2) vacuously follows, yielding (!).

Assume that $[]$ was entered into \mathcal{D} by $\mathbf{1} \vee E_0$, as pictured in 3.8. By the IH, (*) $\langle A(\mathcal{D}_2), \mathbf{1}(\varphi_0 \vee \varphi_1) \rangle$ and (**) for both $i \in 2 \langle A(\mathcal{D}_i), \mathbf{0}\theta \rangle$ are \mathcal{M} -valid at u . Assume (A). Since $\mathcal{M}, u \Vdash A(\mathcal{D}_2)$, by (*) $\mathcal{M}, u \Vdash \mathbf{1}(\varphi_0 \vee \varphi_1)$; so $\mathbf{1}(\varphi_0 \vee \varphi_1) \in \mathbf{0}Fml$, a contradiction; so $\mathcal{M}, u \not\models A(\mathcal{D})$. (V1) follows vacuously. Assume (B). By (*), $\mathcal{M}, u, v \Vdash \mathbf{1}(\varphi_0 \vee \varphi_1)$; so $\mathcal{M}, v \models (\varphi_0 \vee \varphi_1)$. Fix $i \in 2$ so that $\mathcal{M}, v \models \varphi_i$. So $\mathcal{M}, u, v \Vdash \mathbf{1}\varphi_i$; so $\mathcal{M}, u, v \Vdash A(\mathcal{D}_i)$. By (**), $\mathcal{M}, u, v \Vdash \mathbf{0}\theta$, yielding (V2), and thus (!).

Assume that $[]$ was entered into \mathcal{D} by $\mathbf{1} \supset E$, as pictured in 3.8. By the IH (*) $\langle A(\mathcal{D}_0), \mathbf{1}\psi \rangle$ and (**) $\langle A(\mathcal{D}_1), \mathbf{1}(\psi \supset \varphi) \rangle$ are \mathcal{M} -valid at u . Assume (A). By (*) $\mathcal{M}, u \Vdash \mathbf{1}\psi$, a contradiction. (V1) follows vacuously. Assume (B). By (*), $\mathcal{M}, u, v \Vdash \mathbf{1}\psi$; by (ii) $\mathcal{M}, u, v \Vdash \mathbf{1}(\psi \supset \varphi)$; so $\mathcal{M}, v \models \{\psi, (\psi \supset \varphi)\}$; so $\mathcal{M}, v \models \varphi$; so $\mathcal{M}, u, v \Vdash \mathbf{1}\varphi$. (V2) follows, yielding (!).

Assume that $[]$ was entered into \mathcal{D} by $\mathbf{1} \supset I$, as pictured in 3.8. By the IH (*) $\langle A(\mathcal{D}_0), \mathbf{1}\psi \rangle$ and (**) $\langle A(\mathcal{D}_1), \mathbf{1}\theta \rangle$ are \mathcal{M} -valid at u . Assume (A). By (**), $\mathcal{M}, u \Vdash \mathbf{1}\theta$, a contradiction. (V1) follows vacuously.²⁵ Assume (B). Consider a v' so that $v \sqsubseteq v'$; assume that $\mathcal{M}, v' \models \varphi$. Since $\langle W, R, \sqsubseteq \rangle$ is left-complete, we may fix a u' so that $u \sqsubseteq u' Rv'$. Since $\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D}_0)$ and $\mathcal{M}, v \models \mathbf{1}^{-1}A(\mathcal{D}_0) - \{\varphi\}$, by the Persistence Lemma $\mathcal{M}, u' \models \mathbf{0}^{-1}A(\mathcal{D}_0)$ and $\mathcal{M}, v' \models \mathbf{1}^{-1}A(\mathcal{D}_0) - \{\varphi\}$. Thus $\mathcal{M}, u', v' \Vdash A(\mathcal{D}_0)$. By (*) $\mathcal{M}, u', v' \Vdash \mathbf{1}\psi$; so $\mathcal{M}, v' \models \psi$; so $\mathcal{M}, v \models (\varphi \supset \psi)$. So $\mathcal{M}, u, v \Vdash \mathbf{1}(\varphi \supset \psi)$. (V2) follows, yielding (!).

²⁵In effect, the minor premise $\mathbf{1}\theta$ says “No dead-end!”.

Assume that $[]$ was entered into \mathcal{D} by Trn_0 , as pictured in 3.8. Recall: $\{ \} \Rightarrow \mathcal{D}_0: \mathbf{0}\varphi$. By the IH (*) $\langle A(\mathcal{D}_1), \mathbf{1}\theta \rangle$ is \mathcal{M} -valid at u . Assume (A). By (*) $\mathcal{M}, u \Vdash \mathbf{1}\theta$, a contradiction. (V1) vacuously follows. Assume (B). By the IH, (**) $\{ \}, \mathbf{0}\varphi$ \mathcal{M} -valid at v . (Notice the switch from u to v .) If v is a dead-end, by (V1) for (**) $\mathcal{M}, v \Vdash \mathbf{0}\varphi$; so $\mathcal{M}, v \models \varphi$. Assume that v is not a dead-end; fix w so that vRw . By (V2) for (**) $\mathcal{M}, v, w \Vdash \mathbf{0}\varphi$; so $\mathcal{M}, v \models \varphi$. So $\mathcal{M}, u, v \Vdash \mathbf{1}\varphi$. (V2) follows, yielding (!).

Assume that $[]$ was entered into \mathcal{D} by $\diamond I$, as pictured in 3.8. So $A(\mathcal{D}) = A(\mathcal{D}_0)$. By the IH (*) $\langle A(\mathcal{D}_0), \mathbf{1}\varphi \rangle$ is \mathcal{M} -valid at u . Assume (A). By (*) $\mathcal{M}, u \Vdash \mathbf{1}\varphi$; so $\mathbf{1}\varphi \in \mathbf{0Fml}$, for a contradiction. (V1) follows vacuously. Assume (B). By (*) $\mathcal{M}, u, v \Vdash \mathbf{1}\varphi$; so $\mathcal{M}, v \models \varphi$; since uRv , $\mathcal{M}, u \models \diamond\varphi$; so $\mathcal{M}, u, v \Vdash \mathbf{0}\diamond\varphi$, proving (V2). (!) follows.

Assume that $[]$ was entered into \mathcal{D} by $\diamond E$, as pictured in 3.8; recall: \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\varphi$. By the IH, (*) $\langle A(\mathcal{D}_1), \mathbf{0}\diamond\varphi \rangle$ is \mathcal{M} -valid at u . Assume (A). By (*) $\mathcal{M}, u \Vdash \mathbf{0}\diamond\varphi$; so $\mathcal{M}, u \models \diamond\varphi$, a contradiction. (V1) vacuously follows. Assume (B). By (*) $\mathcal{M}, u, v \Vdash \mathbf{0}\diamond\varphi$; so $\mathcal{M}, u \models \diamond\varphi$. Fix w so that uRw and $\mathcal{M}, w \models \varphi$. So $\mathcal{M}, u, w \Vdash \mathbf{1}\varphi$. Fix $\{s_{i \in (m)}\}$ to be a barrier in \mathcal{D}_0 with exception for $v:\mathbf{1}\varphi$. Fix distinct $v_1, \dots, v_m \in Var$, none occurring in \mathcal{D} . Let $\mathcal{D}_0^S =$ the result of surgery on \mathcal{D}_0 at s_1, \dots, s_m using v_1, \dots, v_m . For each $i \in (m)$ fix θ_i so that $\mathbf{0}\theta_i$ is the m -formula label of s_i in \mathcal{D}_0 , and let \mathcal{D}'_i be the subdeduction of \mathcal{D}_0 with root at s_i . Consider any $i \in (m)$. Since s_i is open in \mathcal{D}_0 , $A(\mathcal{D}'_i) \subseteq A(\mathcal{D}_0)$; so $\mathcal{M}, u, v \Vdash A(\mathcal{D}'_i)$. Since $ht(\mathcal{D}'_i) \leq ht(\mathcal{D}_0)$, by the IH, $\langle A(\mathcal{D}'_i), \mathbf{0}\theta_i \rangle$ is \mathcal{M} -valid at u . So $\mathcal{M}, u, v \Vdash \mathbf{0}\theta_i$. By choice of $\{s_{i \in (m)}\}$ and \mathcal{D}_0^S ,

$$A(\mathcal{D}_0^S) \subseteq (A(\mathcal{D}_0) \cap \mathbf{0Fml}) \cup \{\mathbf{0}\theta_{i \in (m)}\} \cup \{\mathbf{1}\varphi\}.$$

So $\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D}_0^S)$. So $\mathcal{M}, u, w \Vdash A(\mathcal{D}_0^S)$. Since $ht(\mathcal{D}_0^S) \leq ht(\mathcal{D}_0) \leq n$, the IH insures that $\langle A(\mathcal{D}_0^S), \mathbf{0}\psi \rangle$ is \mathcal{M} -valid at u . So $\mathcal{M}, u, w \Vdash \mathbf{0}\psi$; so $\mathcal{M}, u \models \psi$; so $\mathcal{M}, u, v \Vdash \mathbf{0}\psi$. (V2) follows, yielding (!).

Assume that $[]$ was entered into \mathcal{D} by $\square I$, as pictured in 3.8; recall: \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\top$. By the IH (*) $\langle A(\mathcal{D}_0), \mathbf{1}\varphi \rangle$ is \mathcal{M} -valid at u . Assume (A). Given v , assume that uR^+v ;²⁶ fix a u' so that $u \sqsubseteq u'Rv$. Since $\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D})$, by the Persistence Lemma $\mathcal{M}, u' \models \mathbf{0}^{-1}A(\mathcal{D})$. So $\mathcal{M}, u' \Vdash A(\mathcal{D})$. Since $\mathcal{M}, v \models \top$, $\mathcal{M}, u', v \Vdash \mathbf{1}\top$; so $\mathcal{M}, u', v \Vdash A(\mathcal{D}_0)$. By the IH, $\langle A(\mathcal{D}_0), \mathbf{1}\varphi \rangle$ is \mathcal{M} -valid at u' . So $\mathcal{M}, u', v \Vdash \mathbf{1}\varphi$; so $\mathcal{M}, v \models \varphi$. Thus $\mathcal{M}, u \models \square\varphi$. (V1) follows. Assume (B). So $\mathcal{M}, u, v \Vdash A(\mathcal{D}_0)$. Fix $\{s_i\}_{i \in (m)}$ to be a barrier in \mathcal{D}_0 with exception for $v:\mathbf{1}\top$. Fix distinct $v_1, \dots, v_m \in Var$, none occurring in \mathcal{D} . Let $\mathcal{D}_0^S =$ the result of surgery on \mathcal{D}_0 at s_1, \dots, s_m using v_1, \dots, v_m . For each $i \in (m)$ fix θ_i such that $\mathbf{0}\theta_i$ is the m -formula label of s_i in \mathcal{D}_0 , and let \mathcal{D}'_i be the subdeduction of \mathcal{D}_0 with root at s_i . Consider any $i \in (m)$. Since s_i is open in \mathcal{D}_0 , $A(\mathcal{D}'_i) \subseteq A(\mathcal{D}_0)$; so $\mathcal{M}, u, v \Vdash A(\mathcal{D}'_i)$. Since $ht(\mathcal{D}'_i) \leq ht(\mathcal{D}_0)$, by the IH $\langle A(\mathcal{D}'_i), \mathbf{0}\theta_i \rangle$ is \mathcal{M} -valid at u . So $\mathcal{M}, u, v \Vdash \mathbf{0}\theta_i$. So

²⁶If there is no such v , vacuously $\mathcal{M}, u \models \square\varphi$.

$\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D}_0^S)$. Consider any v' so that uR^+v' ; ²⁷ fix a u' so that $u \sqsubseteq u'Rv'$. As in the previous case,

$$A(\mathcal{D}_0^S) \subseteq (A(\mathcal{D}_0) \cap \mathbf{0}Fml) \cup \{\mathbf{0}\theta_{i \in (m)}\} \cup \{\mathbf{1}\top\}.$$

By the Persistence Lemma $\mathcal{M}, u' \models \mathbf{0}^{-1}A(\mathcal{D}_0^S)$. Since $\mathcal{M}, v' \models \top$, $\mathcal{M}, u', v' \Vdash \mathbf{1}\top$. So $\mathcal{M}, u', v' \Vdash A(\mathcal{D}_0^S)$. Since $ht(\mathcal{D}_0^S) \leq ht(\mathcal{D}_0) \leq n$, by the IH $\langle A(\mathcal{D}_0^S), \mathbf{1}\varphi \rangle$ is \mathcal{M} -valid at u' . So $\mathcal{M}, u', v' \Vdash \mathbf{1}\varphi$; so $\mathcal{M}, v' \models \varphi$. Thus $\mathcal{M}, u \models \Box\varphi$; so $\mathcal{M}, u, v \Vdash \mathbf{0}\Box\varphi$. (V2) follows, yielding (!).

Assume that [] was entered into \mathcal{D} by $\Box E$, as pictured in 3.8; recall: $C_0 \Rightarrow \mathcal{D}_0:\mathbf{0}\Box\varphi$, $C_1 \Rightarrow \mathcal{D}_1:\mathbf{1}\theta$. By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{0}\Box\varphi \rangle$ and (**) $\langle A(\mathcal{D}_1), \mathbf{1}\theta \rangle$ are \mathcal{M} -valid at u . Assume (A). Since $A(\mathcal{D}_1) \subseteq A(\mathcal{D})$, by (**) $\mathcal{M}, u \Vdash \mathbf{1}\theta$, a contradiction that vacuously yields (V1). Assume (B). Since $A(\mathcal{D}_0) \subseteq A(\mathcal{D})$, by (*) $\mathcal{M}, u, v \Vdash \mathbf{0}\Box\varphi$; so $\mathcal{M}, u \models \Box\varphi$. Since $u \sqsubseteq u, R^+v$, and so $\mathcal{M}, v \models \varphi$. So $\mathcal{M}, u, v \Vdash \mathbf{1}\varphi$. (V2) follows, yielding (!).

Assume that [] was entered into \mathcal{D} by $\Diamond E^+$, as pictured in 3.8. By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{0}\Diamond\top \rangle$ is \mathcal{M} -valid at u . Assume (A). By (*) $\mathcal{M}, u \Vdash \mathbf{0}\Diamond\top$, for a contradiction; (V1) follows. Assume (B). Since $\mathcal{M}, v \models \top$, $\mathcal{M}, u, v \Vdash \mathbf{1}\top$. (V2) follows, yielding (!). Note: this is the only place in this proof in which the difference between \mathcal{M} -validity at u and \mathcal{M} -validity⁻ at u matters.²⁸

By induction, for any deduction \mathcal{D} with conclusion χ , $\langle A(\mathcal{D}), \chi \rangle$ is \mathcal{M} -valid at u . Thus for any deduction \mathcal{D} with conclusion χ , $\langle A(\mathcal{D}), \chi \rangle$ is \mathcal{M} -valid. Thus for any deduction \mathcal{D} with conclusion χ , $\langle A(\mathcal{D}), \chi \rangle$ is IK-valid.

6.2 Observation

\vdash_{IK}^- is sound with respect to IK-validity⁻. Proof is a slight variation of that for 6.1. (In verifying (V1⁻), don't assume that u is a dead-end. Of course we omit the last induction case, for $\Diamond E^+$.)

6.3 Observation

$\vdash_{\neq}^- \vdash$. Proof. The inclusion is trivial. Trivially $\mathbf{0}\Diamond\top \vdash \mathbf{1}\top$. Let $W = 1, \sqsubseteq = \{\langle 0, 0 \rangle\}$, and $0R0$; $F = \langle W, R, \sqsubseteq \rangle$ is an IK-frame. For any valuation function \mathcal{V} on $W \times S$, $\langle \mathbf{0}\Diamond\top, \mathbf{1}\top \rangle$ is not $\langle F, \mathcal{V} \rangle$ -valid⁻ at 0, this because $\langle F, \mathcal{V} \rangle, 0 \Vdash \mathbf{0}\Diamond\top$ but $\langle F, \mathcal{V} \rangle, 0 \not\Vdash \mathbf{1}\top$, violating (V1⁻). By 6.2, $\mathbf{0}\Diamond\top \not\vdash^- \mathbf{1}\top$.

6.4 Remark

One might think that by restricting our attention to \Diamond -free formulas, we could dispense with right-completeness in our definition of being an IK-frame; after all,

²⁷If there is no such v' , vacuously $\mathcal{M}, u \models \Box\varphi$.

²⁸We needed “ u is a dead-end” in (A) to make use of the IH.

the proof of the Persistence Lemma would then not need that IK-frames are right-complete. But that is not the case: we needed the right-completeness of IK-frames to handle $\mathbf{0} \supset I$.

Example. Let \sqsubseteq be reflexive on 3, $0 \sqsubseteq 2$, $0R1$, and let $F = \langle 3, \sqsubseteq, R \rangle$; for $\pi \in S$, let $\mathcal{V}(u, \pi) = 0$ for all $u \in 3$, and let $\mathcal{M} = \langle F, \mathcal{V} \rangle$. F is not an IK-frame, since it is not right-complete. Suppose that we were to consider it one, and so consider \mathcal{M} an IK-model. Check that $\mathbf{1} \neg \pi \vdash_{IK} \mathbf{0} \neg \Box \pi$. But $\langle \{\mathbf{1} \neg \pi\}, \mathbf{0} \neg \Box \pi \rangle$ would not be \mathcal{M} -valid at 0, since $\mathcal{M}, 1 \models \neg \pi$, but $\mathcal{M}, 2 \models \Box \pi$.

7 Canonical Models and the Completeness of IK

7.1 Definitions

Until further notice, let \vdash be any monotonic finitary consequence relation²⁹ on $MFml$ such that $\vdash_{IK} \sqsubseteq \vdash$.

Definitions For $\Sigma \subseteq Fml$, Σ is \vee -complete iff for any $\varphi_0, \varphi_1 \in Fml$, if $(\varphi_0 \vee \varphi_1) \in \Sigma$ then for some $i \in 2$ $\varphi_i \in \Sigma$.

Consider a $\Gamma \subseteq MFml$. Γ is \vee -complete iff for both markers \mathbf{m} , if $\mathbf{m}(\varphi_0 \vee \varphi_1) \in \Gamma$ then for some $i \in 2$ $\mathbf{m}\varphi_i \in \Gamma$.

For a marker \mathbf{m} let Γ be \mathbf{m} -closed under \vdash iff for every $\varphi \in Fml$ if $\Gamma \vdash \mathbf{m}\varphi$ then $\mathbf{m}\varphi \in \Gamma$. Γ is closed under \vdash iff Γ is \mathbf{m} -closed under \vdash for both markers \mathbf{m} . Γ is \vee -complete iff for both markers \mathbf{m} , if $\mathbf{m}(\varphi_0 \vee \varphi_1) \in \Gamma$ then for some $i \in 2$ $\mathbf{m}\varphi_i \in \Gamma$. Hereafter let ‘closed _{\vdash} ’ abbreviate ‘closed under \vdash ’.

7.2 Definitions

Let $W_{\vdash} = \{\Sigma \subseteq Fml \mid \Sigma \text{ is } \vee\text{-complete, } \mathbf{0}\Sigma \text{ is } \mathbf{0}\text{-closed under } \vdash, \text{ and } \perp \notin \Sigma\}$.

Consider $\Sigma_0, \Sigma_1 \subseteq Fml$. Let $\Sigma_0 R_{\vdash} \Sigma_1$ iff $\Sigma_0, \Sigma_1 \in W_{\vdash}$, $\Box^{-1}\Sigma_0 \subseteq \Sigma_1$ and $\Diamond\Sigma_1 \subseteq \Sigma_0$.

W_{\vdash} and R_{\vdash} are the canonical world-domain and accessibility relation under \vdash . For the rest of this section I will mostly omit the subscript for \vdash .

7.3 Observation

If $\Sigma \in W$ then $\top \in \Sigma$, and so $\Sigma \neq \{\}$. Proof. Assume that $\Sigma \in W$. Since $\{\} \vdash \mathbf{0}\top$, $\mathbf{0}\Sigma \vdash \mathbf{0}\top$; since $\mathbf{0}\Sigma$ is $\mathbf{0}IK$ -closed, $\mathbf{0}\top \in \mathbf{0}\Sigma$. So $\top \in \Sigma$.

7.4 Definition

For $\chi \in MFml$, Γ avoids χ under \vdash iff: Γ is closed under \vdash and \vee -complete, and $\chi \notin \Gamma$.

²⁹See section 5.6

7.5 0⊥-Avoidance Lemma for ⊢

For any $\Sigma_0, \Sigma_1 \subseteq Fml$, $\mathbf{0}\Sigma_0 \cup \mathbf{1}\Sigma_1$ avoids $\mathbf{0}\perp$ iff $\Sigma_0 \in W$ and either $\Sigma_1 = \{\}$ or $\Sigma_0 R \Sigma_1$.

Proof. Let $\Gamma = \mathbf{0}\Sigma_0 \cup \mathbf{1}\Sigma_1$. Assume the left-side. Since Γ is \vee -complete, Σ_0 is \vee -complete. Since Γ is $\mathbf{0}$ -closed $_{\perp}$, $\mathbf{0}\Sigma_0$ is. If $\perp \in \Sigma_0$ then $\mathbf{0}\perp \in \Gamma$, contrary to the left-side. Thus $\Sigma_0 \in W$. Assume that $\Sigma_1 \neq \{\}$; fix a $\psi \in \Sigma_1$. Since Γ is \vee -complete, Σ_1 is. Assume that $\mathbf{0}\Sigma_1 \vdash \mathbf{0}\varphi$. Fix $\varphi_1, \dots, \varphi_m \in \Sigma_1$ so that $\{\mathbf{0}\varphi_{i \in (m)}\} \vdash \mathbf{0}\varphi$. Assume that $m = 0$. Using Transfer_0 , $\mathbf{1}\psi \vdash \mathbf{1}\varphi$; since $\mathbf{1}\psi \in \Gamma$, $\mathbf{1}\varphi \in \Gamma$. Assume that $m > 0$. Using Transfer_m , $\{\mathbf{1}\varphi_{i \in (m)}\} \vdash \mathbf{1}\varphi$. Since $\{\mathbf{1}\varphi_{i \in (m)}\} \subseteq \Gamma$ and Γ is $\mathbf{1}$ -closed $_{\perp}$, $\mathbf{1}\varphi \in \Gamma$. Either way, $\varphi \in \Sigma_1$. Thus $\mathbf{0}\Sigma_1$ is $\mathbf{0}$ -closed $_{\perp}$. If $\perp \in \Sigma_1$ then $\mathbf{1}\perp \in \Gamma$; using $\mathbf{1}\perp E_0$, $\Gamma \vdash \mathbf{0}\perp$; since Γ is $\mathbf{0}$ -closed $_{\perp}$, $\mathbf{0}\perp \in \Gamma$, a contradiction. Thus $\Sigma_1 \in W$. Assume that $\theta \in \Sigma_1$. So $\mathbf{1}\theta \in \Gamma$; since $\mathbf{1}\theta \vdash \mathbf{0}\diamond\theta$ and Γ is $\mathbf{0}$ -closed $_{\perp}$, $\mathbf{0}\diamond\theta \in \Gamma$; so $\diamond\theta \in \Sigma_0$. Thus $\diamond\Sigma_1 \subseteq \Sigma_0$. Assume that $\varphi \in \square^{-1}\Sigma_0$; so $\mathbf{0}\square\varphi, \mathbf{1}\psi \in \Gamma$. Since $\{\mathbf{0}\square\varphi, \mathbf{1}\psi\} \vdash \mathbf{1}\varphi$ and Γ is $\mathbf{1}$ -closed $_{\perp}$, $\mathbf{1}\varphi \in \Gamma$; so $\varphi \in \Sigma_1$. Thus $\square^{-1}\Sigma_0 \subseteq \Sigma_1$. We have shown that $\Sigma_0 R \Sigma_1$.

Assume the right-side. If $\Sigma_1 = \{\}$, $\Gamma = \mathbf{0}\Sigma_0$; so Γ avoids $\mathbf{0}\perp$. Assume that $\Sigma_0 R \Sigma_1$. So $\Sigma_1 \in W$. Claim 1: Γ is $\mathbf{0}$ -closed $_{\perp}$. Assume that $\Gamma \vdash \mathbf{0}\theta$. Since \vdash is finitary we may fix $\psi_{i \in (m)} \in \Sigma_1$ so that $\mathbf{0}\Sigma_0 \cup \{\mathbf{1}\psi_{i \in (m)}\} \vdash \mathbf{0}\theta$. Since $\Sigma_1 \in W$, Σ_1 is closed under conjunction. Let ψ be $\bigwedge_{i \in (m)} \psi_i$. So $\mathbf{0}\Sigma_1 \vdash \mathbf{0}\psi$; since $\Sigma_1 \in W$, $\mathbf{0}\psi \in \mathbf{0}\Sigma_1$; so $\psi \in \Sigma_1$. Since $\Sigma_0 R \Sigma_1$, $\diamond\psi \in \Sigma_0$; so $\mathbf{0}\diamond\psi \in \mathbf{0}\Sigma_0$. Also, using $\mathbf{1}\&E$, $\mathbf{0}\Sigma_0 \cup \{\mathbf{1}\psi\} \vdash \mathbf{0}\theta$, witnessed by a deduction in which for some $v \in \text{Var } \{\}$ is a barrier with exception for $v:\mathbf{1}\psi$. Using $\diamond E$, $\mathbf{0}\Sigma_0 \cup \{\mathbf{0}\diamond\psi\} \vdash \mathbf{0}\theta$; so $\mathbf{0}\Sigma_0 \vdash \mathbf{0}\theta$. Since Σ_0 is $\mathbf{0}$ -closed $_{\perp}$, $\mathbf{0}\theta \in \mathbf{0}\Sigma_0$. Claim 1 follows. Claim 2: Γ is $\mathbf{1}$ -closed $_{\perp}$. Assume that $\Gamma \vdash \mathbf{1}\theta$. Fix $\psi \in \Sigma_1$ so that $\mathbf{0}\Sigma_0 \cup \{\mathbf{1}\psi\} \vdash \mathbf{1}\theta$. So $\mathbf{0}\Sigma_0 \cup \{\mathbf{1}\psi\} \vdash \mathbf{1}\theta$; using $\mathbf{1} \supset I$, $\mathbf{0}\Sigma_0 \cup \{\mathbf{1}\top\} \vdash \mathbf{1}(\psi \supset \theta)$, witnessed by a deduction in which for some $v \in \text{Var } \{\}$ is a barrier with exception for $v:\mathbf{1}\top$. Using $\square I$, $\mathbf{0}\Sigma_0 \vdash \mathbf{0}\square(\psi \supset \theta)$. Since Σ_0 is $\mathbf{0}$ -closed $_{\perp}$, $\mathbf{0}\square(\psi \supset \theta) \in \mathbf{0}\Sigma_0$; so $\square(\psi \supset \theta) \in \Sigma_0$. So $(\psi \supset \theta) \in \Sigma_1$; so $\mathbf{0}(\psi \supset \theta) \in \mathbf{0}\Sigma_1$. Note that $\mathbf{0}\Sigma_1 \vdash \mathbf{0}\psi$; so $\mathbf{0}\Sigma_1 \vdash \mathbf{0}\theta$. Since Σ_1 is $\mathbf{0}$ -closed $_{\perp}$, $\mathbf{0}\theta \in \mathbf{0}\Sigma_1$. So $\theta \in \Sigma_1$; so $\mathbf{1}\theta \in \Gamma$. Claim 2 follows. Claim 3: Γ is \vee -complete. Assume that $\mathbf{0}(\varphi_0 \vee \varphi_1) \in \Gamma$. So $(\varphi_0 \vee \varphi_1) \in \Sigma_0$; fixing an $i \in 2$ so that $\varphi_i \in \Sigma_0$, $\mathbf{0}\varphi_i \in \Gamma$. Similarly, assuming that $\mathbf{1}(\varphi_0 \vee \varphi_1) \in \Gamma$, we may fix $i \in 2$ so that $\varphi_i \in \Sigma_1$, and so $\mathbf{1}\varphi_i \in \Gamma$. Claim 3 follows. If $\mathbf{0}\perp \in \Gamma$, $\perp \in \Sigma_0$ for a contradiction; so $\mathbf{0}\perp \notin \Gamma$. Thus Γ avoids $\mathbf{0}\perp$.

7.6 Convention

For the rest of this section, let $\langle \zeta_i \rangle_{i \in \omega}$ list the members of Fml so that every member occurs infinitely often on the list; call the latter condition “infinite visitation”.

7.7 The Avoidance Theorem for ⊢

Consider any $\Gamma \subseteq MFml$ and $\chi \in MFml$ such that $\Gamma \not\vdash \chi$. There is a $\Sigma_0 \in W$ and a $\Sigma_1 \subseteq Fml$ such that $\Gamma \subseteq \mathbf{0}\Sigma_0 \cup \mathbf{1}\Sigma_1$, $\mathbf{0}\Sigma_0 \cup \mathbf{1}\Sigma_1$ avoids χ , and furthermore: (a) if $\Gamma \cup \{\mathbf{1}\top\} \vdash \chi$ then $\Sigma_1 = \{\}$; (b) otherwise $\Sigma_0 R \Sigma_1$.

Proof. Let

$$\Gamma' = \begin{cases} \Gamma & \text{if } \Gamma \cup \{\mathbf{1}\top\} \vdash \chi \\ \Gamma \cup \{\mathbf{1}\top\} & \text{otherwise.} \end{cases}$$

We will construct a $q \in \omega + 1$ and a “double sequence” $\langle \Phi_j, \Psi_j \rangle_{j \in q}$, and prove that $q = \omega$. For each $j \in q$ we will have $\Phi_j, \Psi_j \subseteq Fml$.

Let $0 \in q$ and $\Phi_0 = \Psi_0 = \{\}$. Given $j \in \omega$, assume that $j \in q$ and for some $n \in \omega$, $j \in [4n, 4n + 3]$. Fix that n ; so $4n \in q$. Let $4n + 1 \in q$ and $\Psi_{4n+1} = \Psi_{4n}$. If $\Gamma' \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \not\vdash \mathbf{0}\zeta_n$, let $4n + 2 \in q$, $\Phi_{4n+2} = \Phi_{4n+1} = \Phi_{4n}$ and $\Psi_{4n+2} = \Psi_{4n+1}$. Assume that $\Gamma' \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \vdash \mathbf{0}\zeta_n$. Let $\Phi_{4n+1} = \Phi_{4n} \cup \{\zeta_n\}$. If ζ_n is not a disjunction let $4n + 2 \in q$, $\Phi_{4n+2} = \Phi_{4n+1}$ and $\Psi_{4n+2} = \Psi_{4n+1}$. Assume that ζ_n is $(\varphi_0 \vee \varphi_1)$. If for some $i \in 2 \Gamma' \cup \mathbf{0}\Phi_{4n+1} \cup \mathbf{1}\Psi_{4n+1} \cup \{\mathbf{0}\varphi_i\} \not\vdash \chi$, fix such an i and let $4n + 2 \in q$, $\Phi_{4n+2} = \Phi_{4n+1} \cup \{\varphi_i\}$, and $\Psi_{4n+2} = \Psi_{4n+1}$. Otherwise (the bad case for $4n + 2$) let $q = 4n + 2$, and we are done. Now assume that $4n + 2 \in q$ (and so the bad case for $4n + 2$ did not obtain). Let $4n + 3 \in q$ and $\Phi_{4n+3} = \Phi_{4n+2}$. If $\Gamma' \cup \mathbf{0}\Phi_{4n+2} \cup \mathbf{1}\Psi_{4n+2} \not\vdash \mathbf{1}\zeta_n$, let $\Phi_{4n+3} = \Phi_{4n+2} = \Phi_{4n+2}$. Assume that $\Gamma' \cup \mathbf{0}\Phi_{4n+2} \cup \mathbf{1}\Psi_{4n+2} \vdash \mathbf{1}\zeta_n$. Let $\Psi_{4n+3} = \Psi_{4n+2} \cup \{\zeta_n\}$. If ζ_n is not a disjunction let $4n + 4 \in q$, $\Psi_{4n+4} = \Psi_{4n+3}$ and $\Phi_{4n+4} = \Phi_{4n+3}$. Assume that ζ_n is $(\varphi_0 \vee \varphi_1)$. If for some $i \in 2 \Gamma' \cup \mathbf{0}\Phi_{4n+2} \cup \mathbf{1}\Psi_{4n+2} \cup \{\mathbf{1}\varphi_i\} \not\vdash \chi$, fix such an i ; let $4n + 4 \in q$, $\Psi_{4n+4} = \Psi_{4n+3} \cup \{\varphi_i\}$ and $\Phi_{4n+4} = \Phi_{4n+3}$. Otherwise (the bad case for $4n + 4$) let $q = 4n + 4$, and we are done.

Claim 1: for every $j \in \omega$, (i) $j \in q$, (ii) $\Gamma' \cup \mathbf{0}\Phi_j \cup \mathbf{1}\Psi_j \not\vdash \chi$, (iii) if $j > 0$ and j is even then the bad case for j does not obtain. Proof by induction on j .

The base-step. 0 satisfies (i) by stipulation, (ii) by assumption, and (iii) vacuously.

The induction step. Given j , assume the obvious IH. Fix $n \in \omega$ so that $4n \leq j \leq 4n + 3$. Case 1: $j = 4n$. So $4n + 1 \in q$, and $j + 1$ satisfies (i). If $\Gamma' \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \not\vdash \mathbf{0}\zeta_n$, the IH implies that $j + 1$ satisfies (ii)-(iii), and so $j + 2$ satisfies (i)-(iii). Assume that $\Gamma' \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \vdash \mathbf{0}\zeta_n$. If $\Gamma' \cup \mathbf{0}\Phi_{4n+1} \cup \mathbf{1}\Psi_{4n+1} \vdash \chi$, cutting $\mathbf{0}\zeta_n$ shows that $\Gamma' \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \vdash \chi$, contrary to the IH. So $4n + 1$ satisfies (ii), and vacuously satisfies (iii). Case 2: $j = 4n + 1$. If ζ_n is not a disjunction, clearly $4n + 2$ satisfies (i)-(iii). Assume that ζ_n is $(\varphi_0 \vee \varphi_1)$. Assume that for both $i \in 2 \Gamma' \cup \mathbf{0}\Phi_{4n+1} \cup \mathbf{1}\Psi_{4n+1} \cup \{\mathbf{0}\varphi_i\} \vdash \chi$. Since

$$\Gamma' \cup \mathbf{0}\Phi_{4n+1} \cup \mathbf{1}\Psi_{4n+1} \cup \{\mathbf{0}(\varphi_0 \vee \varphi_1)\} = \Gamma' \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \cup \{\mathbf{0}\zeta_n\},$$

using $\mathbf{0} \vee E$ gives a witness that $\Gamma' \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \cup \{\mathbf{0}\zeta\} \vdash \chi$, contrary to our IH. So $4n + 2$ satisfies (i)-(iii). Similar arguments for $j = 4n + 3$ and $j = 4n + 4$ show that such $j + 1$ satisfy (i)-(iii). Claim 1 follows. Thus $q = \omega$.

Let $\Sigma_0 = \bigcup_{j \in \omega} \Phi_j$ and $\Sigma_1 = \bigcup_{j \in \omega} \Psi_j$. Let $\Gamma^* = \mathbf{0}\Sigma_0 \cup \mathbf{1}\Sigma_1$. Clearly $\Gamma' \subseteq \Gamma^*$. Assume that $\chi \in \Gamma^*$. By the infinite visitation condition there is an n so that either ζ_n is $\mathbf{0}^{-1}\chi$ and $\chi \in \mathbf{0}\Phi_{4n+1}$, violating Claim 1.(ii) for $4n + 1$, or ζ_n is $\mathbf{1}^{-1}\chi$ and $\chi \in \mathbf{1}\Psi_{4n+3}$, violating Claim 1.(ii) for $4n + 3$. So $\chi \notin \Gamma^*$.

Claim 2: Γ^* is closed $_{\vdash}$. Given $\theta \in Fml$, assume that $\Gamma^* \vdash \mathbf{0}\theta$. By the infinite visitation condition we may fix an $n \in \omega$ so that θ is ζ_n and $\Gamma' \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \vdash \mathbf{0}\theta$. So $\theta \in \Phi_{4n+1}$; so $\theta \in \Sigma_0$; so $\mathbf{0}\theta \in \Gamma^*$. Thus $\mathbf{0}\Gamma^*$ is $\mathbf{0}$ -closed $_{\vdash}$. A similar argument shows that $\mathbf{1}\Gamma^*$ is $\mathbf{1}$ -closed $_{\vdash}$. Claim 2 follows.

Claim 3: Γ^* is \vee -complete. Consider any $\varphi_0, \varphi_1 \in Fml$. Assume that $\mathbf{0}(\varphi_0 \vee \varphi_1) \in \Gamma^*$. By the infinite visitation condition we may fix an $n \in \omega$ so that $(\varphi_0 \vee \varphi_1)$ is ζ_n

and $\Gamma' \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \vdash \mathbf{0}(\varphi_0 \vee \varphi_1)$. So for some $i \in 2$ $\varphi_i \in \Phi_{4n+2}$; so $\varphi_i \in \Sigma_0$; so $\mathbf{0}\varphi_i \in \Gamma^*$. If $\mathbf{1}(\varphi_0 \vee \varphi_1) \in \Gamma^*$, a similar argument shows that for some $i \in 2$ $\mathbf{1}\varphi_i \in \Gamma^*$. Claim 3 follows. So Γ^* avoids χ .

Claim 4: Γ^* avoids $\mathbf{0}\perp$. If $\mathbf{0}\perp \in \Gamma^*$ then $\Gamma^* \vdash \chi$ using $\mathbf{0}\perp E$, and then $\chi \in \Gamma^*$, since Γ^* is closed $_{\vdash}$, for a contradiction.

By the $\mathbf{0}\perp$ -Avoidance Lemma, (*) either $\Sigma_1 = \{\}$ or $\Sigma_0 R \Sigma_1$. Assume that $\Gamma \cup \{\mathbf{1}\top\} \vdash \chi$; so $\mathbf{1}\top \notin \Gamma^*$; if $\Sigma_1 \neq \{\}$ then $\Sigma_1 \in W$, and so $\top \in \Sigma_1$, a contradiction. So $\Sigma_1 = \{\}$. Assume that $\Gamma \cup \{\mathbf{1}\top\} \not\vdash \chi$; because we started with Γ' in the above construction, $\mathbf{1}\top \in \Gamma^*$. So $\top \in \Sigma_1$; so $\Sigma_1 \neq \{\}$; so $\Sigma_0 R \Sigma_1$.

7.8 Lindenbaum’s Lemma for \vdash

If $\Phi \subseteq Fml$, $\psi \in Fml$, and $\mathbf{0}\Phi \not\vdash \mathbf{0}\psi$, then for some $\Sigma \in W$, $\psi \notin \Sigma$ and $\Phi \subseteq \Sigma$.

Proof. Assume the if-clause. Applying the Avoidance Theorem, with $\Gamma = \mathbf{0}\Phi$ and taking χ to be $\mathbf{0}\psi$, we may fix Σ_0 and Σ_1 so that $\mathbf{0}\Phi \subseteq \mathbf{0}\Sigma_0 \cup \mathbf{1}\Sigma_1$, $\Sigma_0 \in W$, and $\mathbf{0}\psi \notin \mathbf{0}\Sigma_0$, which yields $\psi \notin \Sigma_0$.

7.9 Definition

Let $\sqsubseteq_{\vdash} = \subseteq | W_{\vdash}$. Let $F_{\vdash} = \langle W_{\vdash}, R_{\vdash}, \sqsubseteq_{\vdash} \rangle$; F_{\vdash} is the canonical frame for \vdash . In this section I will usually omit the subscript.

7.10 Remark

The proofs of the next two lemmas are “one-sided” cousins of the proof of the Avoidance Theorem.

7.11 The Unbox Lemma for \vdash

If $\Sigma \subseteq Fml$, $\Psi \in W$, $\mathbf{0}\Sigma$ is $\mathbf{0}$ -closed $_{\vdash}$, and $\square^{-1}\Sigma \subseteq \Psi$, then for some Φ , $\Sigma \subseteq \Phi$ and $\Phi R \Psi$.

Proof. Assume the if-clause. Let $\Gamma = \mathbf{0}\Sigma \cup \mathbf{1}\Psi$. We will define a $q \in \omega + 1$ and a sequence $\langle \Phi_j \rangle_{j \in q}$ as follows. Let $0 \in q$ and $\Phi_0 = \{\}$. Assume that $2n \in q$. Let $2n + 1 \in q$. If $\Gamma \cup \mathbf{0}\Phi_{2n} \not\vdash \mathbf{0}\zeta_n$ let $\Phi_{2n+2} = \Phi_{2n+1} = \Phi_{2n}$. Assume that $\Gamma \cup \mathbf{0}\Phi_{2n} \vdash \mathbf{0}\zeta_n$. Let $\Phi_{2n+1} = \Phi_{2n} \cup \{\zeta_n\}$, and if ζ_n is not a disjunction let $\Phi_{2n+2} = \Phi_{2n+1}$. Assume that ζ_n is $(\varphi_0 \vee \varphi_1)$. If for some $i \in 2$,

$$(*_{2n+2}) \text{ for every } \sigma \in Fml, \text{ if } \Gamma \cup \mathbf{0}\Phi_{2n+1} \cup \{\mathbf{0}\varphi_i\} \vdash \mathbf{0}\square\sigma \text{ then } \sigma \in \Psi,$$

fix such an i and let $\Phi_{2n+2} = \Phi_{2n+1} \cup \{\varphi_i\}$. If neither $i \in 2$ satisfies condition $(*_{2n+2})$ (the bad case for $2n+2$), let $q = 2n + 2$.

Claim 1: for every $j \in \omega$, (i) $j \in q$; (ii) $\Gamma \cup \mathbf{0}\Phi_j$ is $\mathbf{1}$ -closed $_{\vdash}$; (iii) for every $\sigma \in Fml$, if $\Gamma \cup \mathbf{0}\Phi_j \vdash \mathbf{0}\square\sigma$ then $\sigma \in \Psi$; (iv) if $j > 0$ and j is even, the bad case for j does not obtain. Proof is by induction.

Base step. 0 satisfies (i) by stipulation. Given $\delta \in Fml$, assume that $\Gamma \vdash \mathbf{1}\delta$. Since \vdash is finitary and Ψ is closed under conjunction, we may fix $\psi \in \Psi$ so that $\mathbf{0}\Sigma \cup \{\mathbf{1}\psi\} \vdash \mathbf{1}\delta$. So $\mathbf{0}\Sigma \cup \{\mathbf{1}\psi\} \vdash \mathbf{1}\delta$. Using $\mathbf{1} \supset I$, $\mathbf{0}\Sigma \cup \{\mathbf{1}\top\} \vdash \mathbf{1}(\psi \supset \delta)$, witnessed

by a deduction \mathcal{D} in which for some $v \in Var \{ \}$ is a barrier with exception for $v:1\top$. Using $\Box I$, $\mathbf{0}\Sigma \vdash \mathbf{0}\Box(\psi \supset \delta)$. So $\Box(\psi \supset \delta) \in \Sigma$. Since $\Box^{-1}\Sigma \subseteq \Psi$, $(\psi \supset \delta) \in \Psi$; so $\mathbf{0}(\psi \supset \delta) \in \mathbf{0}\Psi$. Since $\Psi \in W$, Ψ is closed under conjunction; so $\psi \in \Psi$; so $\mathbf{0}\psi \in \mathbf{0}\Psi$. Since $\Psi \in W$, it is $\mathbf{0}$ -closed $_{\vdash}$; so $\delta \in \Psi$; so $\mathbf{1}\delta \in \Gamma$. So 0 satisfies (ii). Given $\sigma \in Fml$, assume that $\Gamma \vdash \mathbf{0}\Box\sigma$. Now fix $\psi \in \Psi$ so that $\mathbf{0}\Sigma \cup \{\mathbf{1}\psi\} \vdash \mathbf{0}\Box\sigma$. As above, $\mathbf{0}\Sigma \cup \{\mathbf{1}\psi\} \vdash \mathbf{0}\Box\sigma$, which is witnessed by a deduction in which for some $v \in Var \{ \}$ is a barrier with exception for $v:1\psi$. Using $\Diamond E$, $\mathbf{0}\Sigma \cup \{\mathbf{0}\Diamond\psi\} \vdash \mathbf{0}\Box\sigma$. So $\mathbf{0}\Sigma \vdash \mathbf{0}(\Diamond\psi \supset \Box\sigma)$. By 4.2.(4), $\mathbf{0}\Sigma \vdash \mathbf{0}\Box(\psi \supset \sigma)$. Since Σ is $\mathbf{0}$ -closed $_{\vdash}$, $\Box(\psi \supset \sigma) \in \Sigma$; so $(\psi \supset \sigma) \in \Psi$. Since $\psi \in \Psi$, $\mathbf{0}\Psi \vdash \mathbf{0}\sigma$; so $\sigma \in \Psi$. So 0 satisfies (iii). 0 vacuously satisfies (iv).

Induction step. Given j , assume the obvious IH. Fix n so that $2n \leq j \leq 2n + 1$. Assume that $j = 2n$. So $2n + 1 < q$. If $\Gamma \cup \mathbf{0}\Phi_{2n} \not\vdash \zeta_n$, the IH insures that $2n + 1$ satisfies (ii)-(iii), it vacuously satisfies (iv), and $2n + 2$ trivially satisfies (i)-(iv). Assume that $\Gamma \cup \mathbf{0}\Phi_{2n} \vdash \zeta_n$. Given $\delta \in Fml$, assume that $\Gamma \cup \mathbf{0}\Phi_{2n+1} \vdash \mathbf{1}\delta$. Cutting $\mathbf{0}\zeta_n$, $\Gamma \cup \mathbf{0}\Phi_{2n} \vdash \mathbf{1}\delta$. By the IH $2n$ satisfies (ii); so $\delta \in \Psi$; so $2n + 1$ satisfies (ii). Similarly, for a given $\sigma \in Fml$ if $\Gamma \cup \mathbf{0}\Phi_{2n+1} \vdash \mathbf{0}\Box\sigma$, cutting $\mathbf{0}\zeta_n$ yields $\Gamma \cup \mathbf{0}\Phi_{2n} \vdash \mathbf{0}\Box\sigma$; so by the IH $\sigma \in \Psi$. So $2n + 2$ satisfies (iii). $2n + 1$ vacuously satisfies (iv). If ζ_n is not a disjunction, $2n + 2$ trivially satisfies (i)-(iv). Assume that ζ_n is $(\varphi_0 \vee \varphi_1)$. Subclaim: for some $i \in 2$, i satisfies $(^*_{2n+2})$. Assume not; for both $i \in 2$ fix $\sigma_i \in Fml$ so that $\Gamma \cup \mathbf{0}\Phi_{2n+1} \cup \{\mathbf{0}\varphi_i\} \vdash \mathbf{0}\Box\sigma_i$ but $\sigma_i \notin \Psi$. Let σ be $(\sigma_0 \vee \sigma_1)$. Since $\Psi \neq \{ \}$, $\mathbf{1}\top \in \Gamma$; so for each $i \in 2$ one use of $\Box E$ gives a witness that $\Gamma \cup \mathbf{0}\Phi_{2n+1} \cup \{\mathbf{0}\varphi_i\} \vdash \mathbf{1}\sigma_i$; using $\mathbf{1}\vee I$, $\Gamma \cup \mathbf{0}\Phi_{2n+1} \cup \{\mathbf{0}\varphi_i\} \vdash \mathbf{1}\sigma$. Using $\mathbf{1}\vee E$, $\Gamma \cup \mathbf{0}\Phi_{2n+1} \vdash \mathbf{1}\sigma$. Cutting $\mathbf{0}\zeta_n$, $\Gamma \cup \mathbf{0}\Phi_{2n} \vdash \mathbf{1}\sigma$. By the IH, $2n$ satisfies (ii); so $\mathbf{1}\sigma \in \Gamma \cup \mathbf{0}\Phi_{2n}$; so $(\sigma_0 \vee \sigma_1) \in \Psi$. Since $\Psi \in W$, for some $i \in 2$ $\sigma_i \in \Psi$, a contradiction. The subclaim follows. So $2n + 2 \in q$ and $2n + 2$ satisfies (iv). Fix an $i \in 2$ for which $\Phi_{2n+2} = \Phi_{2n+1} \cup \{\varphi_i\}$. Given $\delta \in Fml$, assume that $\Gamma \cup \mathbf{0}\Phi_{2n+2} \vdash \mathbf{1}\delta$. Fix $\theta \in \Psi$ so that $\mathbf{0}\Sigma_0 \cup \{\mathbf{1}\theta\} \cup \mathbf{0}\Phi_{2n+2} \vdash \mathbf{1}\delta$. So $\mathbf{0}\Sigma \cup \{\mathbf{1}\theta\} \cup \mathbf{0}\Phi_{2n+2} \vdash \mathbf{1}\delta$. Using $\mathbf{1}\supset I$, $\mathbf{0}\Sigma_0 \cup \mathbf{0}\Phi_{2n+2} \cup \{\mathbf{1}\top\} \vdash \mathbf{1}(\theta \supset \delta)$, and for some v this is witnessed by a deduction in which $\{ \}$ is a barrier with exception for $v:1\top$. Using $\Box I$, $\mathbf{0}\Sigma \cup \mathbf{0}\Phi_{2n+2} \vdash \mathbf{0}\Box(\theta \supset \delta)$. Since i satisfies $(^*_{2n+2})$, $(\theta \supset \delta) \in \Psi$. Since $\mathbf{0}\Psi \vdash \mathbf{0}\theta$, $\mathbf{0}\Psi \vdash \mathbf{0}\delta$. Since Ψ is $\mathbf{0}$ -closed, $\delta \in \Psi$. So $\mathbf{1}\delta \in \Gamma \cup \mathbf{0}\Phi_{2n+2}$. Thus $2n + 2$ satisfies (ii). Since i satisfies $(^*_{2n+2})$, $2n + 2$ satisfies (iii).

Claim 1 follows by induction. Thus $q = \omega$. Let $\Phi = \bigcup \Phi_{j \in \omega}$. Clearly $\Sigma \subseteq \Phi$.

Claim 2: $\mathbf{0}\Phi \cup \mathbf{1}\Psi$ avoids $\mathbf{0}\perp$. Assume that $\mathbf{0}\Phi \cup \mathbf{1}\Psi \vdash \mathbf{0}\perp$. By the infinite visitation condition we may fix $n \in \omega$ so that $\Gamma \cup \mathbf{0}\Phi_{2n} \vdash \mathbf{0}\theta$ and ζ_n is θ ; so $\zeta_n \in \Phi_{2n+1}$; so $\theta \in \Phi$. So $\mathbf{0}\Phi \cup \mathbf{1}\Psi$ is $\mathbf{0}$ -closed $_{\vdash}$. Assume that $\mathbf{0}\Phi \cup \mathbf{1}\Psi \vdash \mathbf{1}\theta$. Since \vdash is finitary we may fix $j \in \omega$ and $\psi \in \Psi$ so that $\mathbf{0}\Sigma \cup \{\mathbf{1}\psi\} \cup \mathbf{0}\Phi_j \vdash \mathbf{1}\theta$. Using $\mathbf{1}\&E$, $\mathbf{0}\Sigma_0 \cup \{\mathbf{1}\psi\} \cup \mathbf{0}\Phi_j \vdash \mathbf{1}\theta$. So $\mathbf{0}\Sigma_0 \cup \{\mathbf{1}\top\} \cup \mathbf{0}\Phi_j \vdash \mathbf{1}(\psi \supset \theta)$, witnessed by a deduction in which, for some $v \in Var$, $\{ \}$ is a barrier except for $v:1\top$. Using $\Box I$, $\mathbf{0}\Sigma \cup \mathbf{0}\Phi_j \vdash \mathbf{0}\Box(\psi \supset \theta)$. By Claim 1(iii), $(\psi \supset \theta) \in \Psi$. Since $\Psi \in W$, $\psi \in \Psi$, and so $\theta \in \Psi$. Thus $\mathbf{0}\Phi \cup \mathbf{1}\Psi$ is $\mathbf{1}$ -closed $_{\vdash}$, and so is closed $_{\vdash}$. Assume that $\mathbf{0}(\varphi_0 \vee \varphi_1) \in \mathbf{0}\Phi \cup \mathbf{1}\Psi$; so $(\varphi_0 \vee \varphi_1) \in \Phi$. Fix $n \in \omega$ so that ζ_n is $(\varphi_0 \vee \varphi_1)$ and $(\varphi_0 \vee \varphi_1) \in \Phi_{2n}$. So $\Gamma \cup \mathbf{0}\Phi_{2n} \vdash \zeta_n$; for some $i \in 2$ $\varphi_i \in \Phi_{2n+2}$, and so $\varphi_i \in \Phi$. Assume that $\mathbf{1}(\varphi_0 \vee \varphi_1) \in \mathbf{0}\Phi \cup \mathbf{1}\Psi$; so $(\varphi_0 \vee \varphi_1) \in \Psi$. Since $\Psi \in W$, it is \vee -complete; fix $i \in 2$ so that $\varphi_i \in \Psi$; so $\mathbf{1}\varphi_i \in \mathbf{0}\Phi \cup \mathbf{1}\Psi$. Thus $\mathbf{0}\Phi \cup \mathbf{1}\Psi$ is \vee -complete. Assume

that $\mathbf{0}\perp \in \mathbf{0}\Phi \cup \mathbf{1}\Psi$. Using $\mathbf{0}\perp E$, $\mathbf{0}\Phi \cup \mathbf{1}\Psi \vdash \mathbf{1}\perp$; since $\mathbf{0}\Phi \cup \mathbf{1}\Psi$ is $\mathbf{1}$ -closed $_{\vdash}$, $\perp \in \Psi$, contrary to $\Psi \in W$. So $\mathbf{0}\perp \notin \mathbf{0}\Phi \cup \mathbf{1}\Psi$. Claim 2 follows.

Since $\Psi \in W$, $\Psi \neq \{\}$. So by the $\mathbf{0}\perp$ -Avoidance Lemma, $\Phi R\Psi$.

7.12 The Left Lemma for \vdash

If $\Sigma_0 R\Sigma_1 \sqsubseteq \Sigma'_1$ then for some $\Sigma'_0 \in W$ $\Sigma_0 \sqsubseteq \Sigma'_0 R\Sigma'_1$.

Proof. Assume the if-clause. So $\square^{-1}\Sigma_0 \subseteq \Sigma'_1$. Take Σ_0 and Σ'_1 to be Σ and Ψ in the above formulation of the Unbox Lemma. So that lemma yields a Σ'_0 (called Φ in the formulation of the lemma) that is as desired.

7.13 The Diamond Lemma for \vdash

If $\Phi \in W$, $\Sigma \subseteq Fml$, Σ is closed under conjunction, and $\diamond\Sigma \subseteq \Phi$, then for some Ψ , $\Phi R\Psi$ and $\Sigma \subseteq \Psi$.

Proof. Assume the if-clause. Let $\Gamma = \mathbf{0}\Phi \cup \mathbf{1}\Sigma$. We will define a $q \in \omega + 1$, and a sequence $\langle \Psi_j \rangle_{j \in q}$ as follows. Let $0 \in q$ and $\Psi_0 = \{\}$. Assume that $2n \in q$. Let $2n + 1 \in q$. If $\Gamma \cup \mathbf{1}\Psi_{2n} \not\vdash \mathbf{1}\zeta_n$ let $2n + 2 \in q$ and $\Psi_{2n+2} = \Psi_{2n+1} = \Psi_{2n}$. Assume that $\Gamma \cup \mathbf{1}\Psi_{2n} \vdash \mathbf{1}\zeta_n$. Let $\Psi_{2n+1} = \Psi_{2n} \cup \{\zeta_n\}$, and if ζ_n is not a disjunction let $2n + 2 \in q$ and $\Psi_{2n+2} = \Psi_{2n+1}$. Assume that ζ_n is $(\varphi_0 \vee \varphi_1)$. If for some $i \in 2$,

$$(*_{2n+2}) \text{ for every } \sigma \in \Sigma, \diamond(\varphi_i \& \sigma \& \wedge \Psi_{2n+1}) \in \Phi,$$

fix such an i and let $\Psi_{2n+2} = \Psi_{2n+1} \cup \{\varphi_i\}$. If neither $i \in 2$ satisfies $(*_{2n+2})$, (the bad case for $2n + 2$), let $q = 2n + 2$.

Claim 1: for every $j \in \omega$, (i) $j \in q$, (ii) $\Gamma \cup \mathbf{1}\Psi_j$ is $\mathbf{0IK}$ -closed, (iii) for every $\sigma \in \Sigma$, $\diamond(\sigma \& \wedge \Psi_j) \in \Phi$, and (iv) if $j > 0$ and j is even, the bad case for j does not obtain. Proof is by induction.

Base Step. 0 satisfies (i) by stipulation. Given $\delta \in Fml$, assume that $\Gamma \vdash \mathbf{0}\delta$. Since \vdash is finitary and Σ is closed under conjunction, we may fix $\rho \in \Sigma$ so that $\mathbf{0}\Phi \cup \{\mathbf{1}\rho\} \vdash \mathbf{0}\delta$. (If $l = 0$, ρ is $\wedge\{\}$ which is \top .) So $\mathbf{0}\Phi \cup \{\mathbf{1}\rho\} \vdash \mathbf{0}\delta$; this is witnessed by a deduction in which for some $v \in Var\{\}$ is a barrier with exception for $v:\mathbf{1}\rho$. By one use of $\diamond E$, $\mathbf{0}\Phi \cup \{\mathbf{0}\diamond\rho\} \vdash \mathbf{0}\delta$. Since $\diamond\Sigma \subseteq \Phi$, $\mathbf{0}\Phi \vdash \mathbf{0}\delta$. Since Φ is $\mathbf{0IK}$ -closed, $\mathbf{0}\delta \in \Gamma$. So 0 satisfies (ii). Since $\diamond\Sigma \subseteq \Phi$, 0 satisfies (iii). 0 vacuously satisfies (iv).

Induction step. Given j , assume the obvious IH. Fix n so that $2n \leq j \leq 2n + 1$. Assume that $j = 2n$. So $2n + 1 \in q$. If $\Gamma \cup \mathbf{1}\Psi_{2n} \not\vdash \mathbf{1}\zeta_n$, the IH insures that $2n + 1$ satisfies (ii)-(iii), it vacuously satisfies (iv), and $2n + 2$ trivially satisfies (i)-(iv). Assume that $\Gamma \cup \mathbf{1}\Psi_{2n} \vdash \mathbf{1}\zeta_n$. Given $\delta \in Fml$, assume that $\Gamma \cup \mathbf{1}\Psi_{2n+1} \vdash \mathbf{0}\delta$. Cutting $\mathbf{1}\zeta_n$, $\Gamma \cup \mathbf{1}\Psi_{2n} \vdash \mathbf{0}\delta$; by the IH $2n$ satisfies (ii); so $\mathbf{0}\delta \in \Gamma \cup \mathbf{1}\Psi_{2n}$; so $\mathbf{0}\delta \in \Gamma \cup \mathbf{1}\Psi_{2n+1}$. So $2n + 1$ satisfies (ii). Fix $\rho \in \Sigma$ so that $\mathbf{0}\Phi \cup \{\mathbf{1}\rho\} \cup \mathbf{1}\Psi_{2n} \vdash \mathbf{1}\zeta_n$. Let ψ be $\wedge\Psi_{2n}$. Using admissible rules,

$$\mathbf{0}\Phi \cup \{\mathbf{1}\top\} \vdash \mathbf{1}((\rho \& \psi) \supset (\rho \& \zeta_n \& \psi)).$$

Using $\square I$, with empty barrier, $\mathbf{0}\Phi \vdash \mathbf{0}\square((\rho \& \psi) \supset (\rho \& \zeta_n \& \psi))$. Using 4.2.(3),

$$\mathbf{0}\Phi \cup \{\mathbf{0}\diamond(\rho \& \psi)\} \vdash \mathbf{0}\diamond(\rho \& \zeta_n \& \psi).$$

By the IH, $2n$ satisfies (iii); so $\diamond(\rho \& \psi) \in \Phi$; so $\mathbf{0}\Phi \vdash \mathbf{0}\diamond(\rho \& \zeta_n \& \psi)$; so $\diamond(\rho \& \zeta_n \& \psi) \in \Phi$. Since $\bigwedge \Psi_{2n+1}$ is $(\zeta_n \& \psi)$, $2n + 1$ satisfies (iii). If ζ_n is not a disjunction, $2n + 2$ trivially satisfies (i)-(iv). Assume that ζ_n is $(\varphi_0 \vee \varphi_1)$. Subclaim: for some $i \in 2$, i satisfies $(^*_{2n+2})$. Assume otherwise. Let ψ' be $(\zeta_n \& \psi)$. For both $i \in 2$ fix $\sigma_i \in \Sigma$ so that $\diamond(\varphi_i \& \sigma_i \& \psi') \notin \Phi$. Let σ be $(\sigma_0 \& \sigma_1)$; so $\sigma \in \Sigma$. We have shown that $\diamond(\sigma \& \psi') \in \Phi$. Using Necessitation and 4.2.(3),

$$\mathbf{0}\diamond(\sigma \& \psi') \vdash \mathbf{0}\diamond((\varphi_0 \& \sigma \& \psi') \vee (\varphi_1 \& \sigma \& \psi')).$$

Using 4.2.(5),

$$\mathbf{0}\diamond((\varphi_0 \& \sigma \& \psi') \vee (\varphi_1 \& \sigma \& \psi')) \vdash \mathbf{0}(\diamond(\varphi_0 \& \sigma \& \psi') \vee \diamond(\varphi_1 \& \sigma \& \psi')).$$

Since Φ is $\mathbf{0}$ -closed, $\diamond(\varphi_0 \& \sigma \& \psi') \vee \diamond(\varphi_1 \& \sigma \& \psi') \in \Phi$; since Φ is \vee -complete, for some $i \in 2$, $\diamond(\varphi_i \& \sigma \& \psi') \in \Phi$. Fix such an i . Check that $\mathbf{0}\diamond(\varphi_i \& \sigma \& \psi') \vdash \mathbf{0}\diamond(\varphi_i \& \sigma_i \& \psi')$. Since Φ is $\mathbf{0}$ -closed, $\diamond(\varphi_i \& \sigma_i \& \psi') \in \Phi$, a contradiction. The subclaim follows. Fix the $i \in 2$ for which $\Psi_{2n+2} = \Psi_{2n+1} \cup \{\varphi_i\}$. Given $\theta \in Fml$, assume that $\Gamma \cup \mathbf{1}\Psi_{2n+1} \cup \{\mathbf{1}\varphi_i\} \vdash \mathbf{0}\theta$. So $\Gamma \cup \{\mathbf{1}(\varphi_i \& \psi')\} \vdash \mathbf{0}\theta$. Fix $\rho_1, \dots, \rho_m \in \Sigma$ so that

$$\mathbf{0}\Phi \cup \{\mathbf{1}\rho_{i \in (m)}\} \cup \{\mathbf{1}(\varphi_i \& \psi')\} \vdash \mathbf{0}\theta.$$

Let ρ be $\bigwedge_{i \in (m)} \rho_i$. Using $\mathbf{1}\&E$ and $\square E$, $\mathbf{0}\Phi \cup \{\mathbf{1}(\varphi_i \& \rho \& \psi')\} \vdash \mathbf{0}\theta$, witnessed by a deduction in which for some $v \in Var\{\}$ is a barrier with exception for $v:\mathbf{1}(\varphi_i \& \rho \& \psi')$. Using $\diamond E$, $\Gamma \cup \{\mathbf{0}\diamond(\varphi_i \& \rho \& \psi')\} \vdash \mathbf{0}\theta$. Since i satisfies $(^*_{2n+2})$, $\diamond(\varphi_i \& \rho \& \psi') \in \Phi$; so $\Gamma \vdash \mathbf{0}\theta$; since 0 satisfies (iii), $\mathbf{0}\theta \in \Gamma \cup \mathbf{1}\Psi_{2n+2}$. So $2n + 2$ satisfies (ii). Since i satisfies $(^*_{2n+2})$, $2n + 2$ satisfies (iii) and (iv).

Claim 1 follows by induction. Thus $q = \omega$. Let $\Psi = \bigcup \Psi_{j \in \omega}$. Clearly $\Gamma \subseteq \mathbf{0}\Phi \cup \mathbf{1}\Psi$; so $\Sigma \subseteq \Psi$.

Claim 2: $\mathbf{0}\Phi \cup \mathbf{1}\Psi$ avoids $\mathbf{0}\perp$. Given $\delta \in Fml$, assume that $\mathbf{0}\Phi \cup \mathbf{1}\Psi \vdash \mathbf{0}\delta$. Since \vdash is finitary, we may fix $j \in \omega$ so that $\Gamma \cup \mathbf{1}\Psi_j \vdash \mathbf{0}\delta$. By Claim 1.(ii), $\mathbf{0}\delta \in \Gamma \cup \mathbf{1}\Psi_j$. Thus $\mathbf{0}\Phi \cup \mathbf{1}\Psi$ is $\mathbf{0}$ -closed. Assume that $\mathbf{0}\Phi \cup \mathbf{1}\Psi \vdash \mathbf{1}\delta$. By the infinite visitation condition we may fix n so that ζ_n is δ and $\Gamma \cup \mathbf{1}\Psi_{2n} \vdash \mathbf{1}\delta$. So $\delta \in \Psi_{2n+1}$; so $\mathbf{1}\delta \in \mathbf{0}\Phi \cup \mathbf{1}\Psi$. So $\mathbf{0}\Phi \cup \mathbf{1}\Psi$ is $\mathbf{1}$ -closed. If $\mathbf{0}\perp \in \mathbf{0}\Phi \cup \mathbf{1}\Psi$, $\perp \in \Phi$, contrary to $\Phi \in W$. Claim 2 follows.

Since $\bigwedge\{\}$ is \top and Σ is closed under conjunction, $\top \in \Sigma$; so $\diamond\top \in \Psi$; so $\Psi \neq \{\}$. By the $\mathbf{0}\perp$ -Avoidance Lemma, $\Phi R\Psi$.

7.14 The Right Lemma for \vdash

If $\Sigma_0 \sqsubseteq \Sigma'_0$ and $\Sigma_0 R \Sigma_1$ then for some $\Sigma'_1 \in W$ $\Sigma'_0 R \Sigma'_1$ and $\Sigma_1 \sqsubseteq \Sigma'_1$.

Proof. Assume the if-clause. Since $\Sigma_1 \in W$, $\mathbf{0}\Sigma_1$ is $\mathbf{0}$ -closed \vdash , and so Σ_1 is closed under conjunction. Also $\diamond\Sigma_1 \subseteq \Sigma'_0$. Take Σ'_0 and Σ_1 to be Φ and Σ in the statement of the Diamond Lemma. So the Diamond lemma yields a Σ'_1 (called Ψ in the statement of that lemma) which is as required.

7.15 Corollary

F_{\vdash} is a IK-frame.

Proof: the Left Lemma yields left completeness, and the Right Lemma yields right completeness of F_{\perp} .

7.16 Definitions

For $\Sigma \in W_{\perp}$ and $\gamma \in S$, let

$$\mathcal{V}_{\perp}(\Sigma, \gamma) = \begin{cases} 1 & \text{if } \gamma \in \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{M}_{\perp} = \langle F_{\perp}, \mathcal{V}_{\perp} \rangle$ is the canonical model for \vdash . By 7.16, it is an IK-model. Where confusion is unlikely, I will omit the subscript.

Let the canonical IK-model based on S be $\mathcal{M}_{\vdash_{IK}}$.

7.17 The Canonical Model Theorem for \vdash

For any formula φ based on S , for any $\Sigma \in W$ (*) $\mathcal{M}_{\perp}, \Sigma \models \varphi$ iff $\varphi \in \Sigma$.

Proof by induction on the construction of formulas. The base step is trivial. So are some of the cases under the induction step. We'll consider the less-trivial cases under the induction step.

Assume that φ is $(\varphi_0 \supset \varphi_1)$. Consider any $\Phi \in W$. Assume that $\mathcal{M}, \Phi \models \varphi$. Let $\Psi = \Phi \cup \{\varphi_0\}$. Claim: $\mathbf{0}\Psi \vdash \mathbf{0}\varphi_1$. Assume not. By Lindenbaum's Lemma, we may fix a $\Phi' \in W$ so that $\Psi \subseteq \Phi'$ and $\varphi_1 \notin \Phi'$. Since $\varphi_0 \in \Phi'$, by the IH $\mathcal{M}, \Phi' \models \varphi_0$. Since $\Phi \subseteq \Phi'$, by the Persistence Lemma $\mathcal{M}, \Phi' \models \varphi_1$. By the IH, $\varphi_1 \in \Phi'$, a contradiction. The claim follows. Since $\mathbf{0}\Phi \cup \{\mathbf{0}\varphi_0\} \vdash \mathbf{0}\varphi_1$, $\mathbf{0}\Phi \vdash \mathbf{0}\varphi$; since Φ is $\mathbf{0}$ -closed, $\mathbf{0}\varphi \in \mathbf{0}\Phi$; so $\varphi \in \Phi$. Now assume that $\varphi \in \Phi$. Consider any Φ' so that $\Phi \sqsubseteq \Phi'$. Clearly $\varphi \in \Phi'$. Assume that $\mathcal{M}, \Phi' \models \varphi_0$. By the IH, $\varphi_0 \in \Phi'$. Since $\{\mathbf{0}\varphi_0, \mathbf{0}\varphi\} \vdash \mathbf{0}\varphi_1$ and $\mathbf{0}\Phi'$ is $\mathbf{0}$ -closed $_{\perp}$, $\mathbf{0}\varphi_1 \in \mathbf{0}\Phi'$; so $\varphi_1 \in \Phi'$. By the IH, $\mathcal{M}, \Phi' \models \varphi_1$. Thus (*).

Assume that φ is $\diamond\varphi_0$. Consider any $\Phi \in W$. Assume that $\mathcal{M}, \Phi \models \varphi$. Fix a $\Phi_1 \in W$ so that $\Phi R \Phi_1$ and $\mathcal{M}, \Phi_1 \models \varphi_0$. By the IH, $\varphi_0 \in \Phi_1$. So $\mathbf{1}\varphi_0 \in \mathbf{0}\Phi \cup \mathbf{1}\Phi_1$. Using $\diamond I$, $\mathbf{0}\Phi \cup \mathbf{1}\Phi_1 \vdash \mathbf{0}\diamond\varphi_0$. By the $\mathbf{0}\perp$ -Avoidance Lemma, $\mathbf{0}\Phi \cup \mathbf{1}\Phi_1$ is $\mathbf{0}$ -closed $_{\perp}$; so $\mathbf{0}\varphi \in \mathbf{0}\Phi \cup \mathbf{1}\Phi_1$; so $\varphi \in \Phi$. Now assume that $\varphi \in \Phi$. Let $\Sigma =$ the closure of $\{\varphi_0\}$ under conjunction. For every $n \in \omega$, let φ'_n be the n -fold conjunction of φ_0 ; check that $\diamond\varphi'_n \in \Phi$. By the Diamond Lemma we may fix a Φ_1 such that $\Sigma \subseteq \Phi_1$ and $\Phi R \Phi_1$. Since $\varphi_0 \in \Phi_1$, by the IH, $\mathcal{M}, \Phi_1 \models \varphi_0$. So $\mathcal{M}, \Phi \models \varphi$. Thus (*).

Assume that φ is $\Box\varphi_0$. Consider any $\Phi \in W$. Assume that $\mathcal{M}, \Phi \models \varphi$. Case 1: $\mathbf{0}\Phi \cup \{\mathbf{1}\top\} \vdash \mathbf{1}\varphi_0$. Using $\Box I$, with empty barrier, $\mathbf{0}\Phi \vdash \mathbf{0}\Box\varphi_0$. So $\varphi \in \Phi$. Case 2: otherwise. Claim: $\mathbf{0}\Phi \vdash \mathbf{1}\varphi_0$. Assume not. By the Avoidance Theorem there is a $\Phi_0 \in W$ and a $\Phi_1 \subseteq Fml$ so that $\Phi \sqsubseteq \Phi_0$ and $\mathbf{0}\Phi_0 \cup \mathbf{1}\Phi_1$ avoids $\mathbf{1}\varphi_0$; by case assumption and the furthermore-clause of that theorem, $\Phi_0 R \Phi_1$. If $\varphi_0 \in \Phi_1$ then $\mathbf{1}\varphi_0 \in \mathbf{0}\Phi_0 \cup \mathbf{1}\Phi_1$, contrary to avoidance; so $\varphi_0 \notin \Phi_1$. By the IH, $\mathcal{M}, \Phi_1 \not\models \varphi_0$. Since $\Phi R^+ \Phi_1$, this contradicts $\mathcal{M}, \Phi \models \varphi$. The claim follows. Using $\Box I$, $\mathbf{0}\Phi \vdash \mathbf{0}\Box\varphi_0$. Since $\Phi \in W$, $\mathbf{0}\Phi$ is $\mathbf{0}$ -closed $_{\perp}$; so $\varphi \in \Phi$. Now assume that $\varphi \in \Phi$. Consider any Φ_0 and Φ_1 so that $\Phi \sqsubseteq \Phi_0 R \Phi_1$. Using $\Box E$, $\{\mathbf{0}\varphi, \mathbf{1}\top\} \vdash \mathbf{1}\varphi_0$. Since $\top \in \Phi_1$,

$\mathbf{0}\Phi_0 \cup \mathbf{1}\Phi_1 \vdash \mathbf{1}\varphi_0$. By the $\mathbf{0}\perp$ -Avoidance Lemma, $\mathbf{0}\Phi_0 \cup \mathbf{1}\Phi_1$ is $\mathbf{0}$ -closed \vdash ; so $\varphi_0 \in \Phi_1$. By the IH, $\mathcal{M}, \Phi_1 \models \varphi_0$. Thus $\mathcal{M}, \Phi \models \varphi$. Thus (*).

The theorem follows by induction.

7.18 Completeness Theorem for \vdash_{IK} (with respect to IK-models)³⁰

For every $\Gamma \subseteq MFml$ and $\chi \in MFml$, if $\langle \Gamma, \chi \rangle$ is IK-valid then $\Gamma \vdash_{IK} \chi$.

Proof. Let \mathcal{M} = the canonical IK-model. Given Γ and χ , assume the if-clause. Assume that $\Gamma \not\vdash_{IK} \chi$. Applying the Avoidance Theorem to \vdash_{IK} , fix $\Sigma_0 \in W$ and $\Sigma_1 \subseteq Fml$ so that $\mathbf{0}\Sigma_0 \cup \mathbf{1}\Sigma_1$ avoids χ and $\Gamma \subseteq \mathbf{0}\Sigma_0 \cup \mathbf{1}\Sigma_1$. By the Canonical Model Lemma $\mathcal{M}, \Sigma_0 \models \Sigma_0$. Case 1: $\Sigma_1 = \{\}$. So $\Gamma \subseteq \mathbf{0}Fml$. If $\diamond\top \in \Sigma_0$, $\top \in \Sigma_1$ (using $\diamond E^+$) for a contradiction; so Σ_0 is a dead-end in the canonical frame for \vdash_{IK} . Thus $\mathcal{M}, \Sigma_0 \Vdash \mathbf{0}\Sigma_0$; so $\mathcal{M}, \Sigma_0 \Vdash \Gamma$; so $\chi \in \mathbf{0}Fml$ and $\mathcal{M}, \Sigma_0 \Vdash \chi$. Let χ be $\mathbf{0}\theta$. So $\mathcal{M}, \Sigma_0 \models \theta$. By the Canonical Model Lemma, $\theta \in \Sigma_0$; so $\chi \in \mathbf{0}\Sigma_0$, a contradiction. Case 2: otherwise. By the Avoidance Theorem (the furthermore-clause), $\Sigma_0 R \Sigma_1$. Since $\Sigma_1 \in W$, the Canonical Model Lemma entails that $\mathcal{M}, \Sigma_1 \models \Sigma_1$. Thus $\mathcal{M}, \Sigma_0, \Sigma_1 \Vdash \mathbf{0}\Sigma_0 \cup \mathbf{1}\Sigma_1$; so $\mathcal{M}, \Sigma_0, \Sigma_1 \Vdash \Gamma$; so $\mathcal{M}, \Sigma_0, \Sigma_1 \Vdash \chi$. Let χ be $m\theta$. Assume that m is $\mathbf{0}$. So $\mathcal{M}, \Sigma_0 \models \theta$; by the Canonical Model Lemma $\theta \in \Sigma_0$; so $\chi \in \mathbf{0}\Sigma_0$, a contradiction. Assume that m is $\mathbf{1}$. So $\mathcal{M}, \Sigma_1 \models \theta$; by the Canonical Model Lemma $\theta \in \Sigma_1$; so $\chi \in \mathbf{1}\Sigma_1$, a contradiction. Since both cases yield contradictions, $\Gamma \vdash_{IK} \chi$.

7.19 Completeness Theorem for \vdash_{IK}^- (with respect to IK-models)

For any $\Gamma \subseteq MFml$ and $\chi \in MFml$, if $\langle \Gamma, \chi \rangle$ is IK-valid $^-$ then $\Gamma \vdash_{IK}^- \chi$.

Proof. Given Γ and χ as described, assume the if-clause. Case 1: $\Gamma \cup \{\chi\} \subseteq \mathbf{0}Fml$. So $\langle \Gamma, \chi \rangle$ is IK-valid. By 7.18 $\Gamma \vdash_{IK} \chi$. By 5.14, $\Gamma \vdash_{IK}^- \chi$. Case 2: $\Gamma \subseteq \mathbf{0}Fml$ and $\chi \in \mathbf{1}Fml$. By 2.7.2.(4) $\langle \Gamma, \mathbf{0}\perp \rangle$ is IK-valid $^-$, and so is also IK-valid. By 7.18 $\Gamma \vdash_{IK} \mathbf{0}\perp$; by 5.14 $\Gamma \vdash_{IK}^- \mathbf{0}\perp$. With one use of $\mathbf{0}\perp E$ we can witness $\Gamma \vdash_{IK}^- \chi$. Case 3: $\Gamma \not\subseteq \mathbf{0}Fml$. Assume that $\Gamma \not\vdash_{IK}^- \chi$. Claim: $\Gamma \not\vdash_{IK} \chi$. Assume that $\Gamma \vdash_{IK} \chi$. Replace any use of $\diamond E^+$ in a witnessing deduction by assumption of $\mathbf{1}\top$ to obtain a witness that $\Gamma \cup \{\mathbf{1}\top\} \vdash_{IK}^- \chi$. By the case assumption, $\Gamma \vdash_{IK}^- \mathbf{1}\top$. So $\Gamma \vdash_{IK}^- \chi$, a contradiction that yields the claim. By 7.18, $\langle \Gamma, \chi \rangle$ is not IK-valid. By 2.7.1.(1,2), $\langle \Gamma, \chi \rangle$ is not IK-valid $^-$, a contradiction. So $\Gamma \vdash_{IK}^- \chi$.

8 Classical K

We can form a one-step version of Classical K by supplementing IK with any of the usual classicalizing rules or axioms, for example, rule $\mathbf{0}$ -Excluded Middle.

³⁰This is inference- (sometimes called strong-) completeness, as opposed to formula- (sometimes called weak) completeness.

8.1 Definition

Define $\Rightarrow_{CK} [\Rightarrow_{CK}^-]$ by adding the following rule to those defining $\Rightarrow_{IK} [\Rightarrow_{IK}^-]$.

0-Excluded Middle If $C_0, v_0:\mathbf{0}\varphi \Rightarrow_{CK} \mathcal{D}_0:\chi$, $C_1, v_1:\mathbf{0}\neg\varphi \Rightarrow_{CK} \mathcal{D}_1:\chi$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow_{CK} \mathcal{D}:\chi$, for \mathcal{D} as pictured.

$$\frac{\begin{array}{c} [v_0:\mathbf{0}\varphi] \quad [v_1:\mathbf{0}\neg\varphi] \\ \mathcal{D}_0 \quad \mathcal{D}_1 \\ \chi \quad \chi \end{array}}{\chi \quad \mathbf{0EM}^{v_0, v_1}}$$

Let $dpd(\mathcal{D}) = (dpd(\mathcal{D}_0) - \{s \mid \mathcal{D}_0(s) = v_0:\mathbf{0}\varphi\}) \cup (dpd(\mathcal{D}_1) - \{s \mid \mathcal{D}_1(s) = v_1:\mathbf{0}\neg\varphi\})$.

Define $\vdash_{CK} [\vdash_{CK}^-]$ from $\Rightarrow_{CK} [\Rightarrow_{CK}^-]$ in the obvious way.

8.2 Observation

Consider a set Γ of formulas and a formula φ containing no occurrences of \square or \diamond , and an classical ND deduction \mathcal{D} of φ from Γ using the familiar “no step” introduction and elimination rules, and perhaps the rule of excluded middle. Form Γ' , φ' and \mathcal{D}' from Γ , φ and \mathcal{D} as in 4.6. Form \mathcal{D}'' from \mathcal{D}' by prefixing the formula-label for each string in $dom(\mathcal{D}')$ by $\mathbf{0}$. Then \mathcal{D}'' witnesses that $\mathbf{0}\Gamma' \vdash_{CK} \mathbf{0}\varphi'$.

Thus $\vdash_{CK} \mathbf{0}Fml = \vdash_{CK}^- \mathbf{0}Fml$ = the result of prefixing ‘ $\mathbf{0}$ ’ to all formulas in the familiar “no step” consequence relation \vdash_K .

8.3 Lemma

Next, a derived rule.

1-Excluded Middle If $C_0, v_0:\mathbf{1}\varphi \Rightarrow_{CK} \mathcal{D}_0:\mathbf{1}\theta$, $C_1, v_1:\mathbf{1}\neg\varphi \Rightarrow_{CK} \mathcal{D}_1:\mathbf{1}\theta$, $\{C_0, C_1\}$ is coherent and $\mu \notin dom(C_0 \cup C_1)$, then for some CK-deduction \mathcal{D} and $\mu \in Var$, $C_0 \cup C_1 \cup \{\mu:\mathbf{1}\top\} \Rightarrow_{CK} \mathcal{D}:\mathbf{1}\theta$.

We construct such a \mathcal{D} as follows. Let σ_0 be $((\neg\varphi \supset \theta) \supset \theta)$, and let σ_1 be $((\varphi \supset \theta) \supset \sigma_0)$. So σ_1 is a classical tautology. By 8.2 we may fix a CK-deduction \mathcal{D}' such that $\{\} \Rightarrow_{CK} \mathcal{D}':\mathbf{0}\sigma_1$. Fix $\mu \in Var - dom(C_0 \cup C_1)$. Let \mathcal{D}'' be the following.

$$\frac{\begin{array}{c} \mathcal{D}' \\ \mathbf{0}\sigma_1 \quad \mu:\mathbf{1}\top \end{array}}{\mathbf{1}\sigma_1} \quad Trn_0$$

Let \mathcal{D}'_0 and \mathcal{D}'_1 be as follows, respectively.

$$\frac{\begin{array}{c} [v_0:\mathbf{1}\varphi] \\ \mathcal{D}_0 \\ \mathbf{1}\theta \quad \mu:\mathbf{1}\top \end{array}}{\mathbf{1}(\varphi \supset \theta)^{v_0}} \mathbf{1}\supset I \qquad \frac{\begin{array}{c} [v_1:\mathbf{1}\neg\varphi] \\ \mathcal{D}_1 \\ \mathbf{1}\theta \quad \mu:\mathbf{1}\top \end{array}}{\mathbf{1}(\neg\varphi \supset \theta)^{v_1}} \mathbf{1}\supset I$$

Let \mathcal{D} be the following.

$$\frac{\begin{array}{c} \mu: \mathbf{1}\top \\ \mathcal{D}' \quad \mathcal{D}_0 \\ \mathbf{1}\sigma_1 \quad \mathbf{1}(\varphi \supset \theta)_{\mathbf{1}\supset E} \end{array}}{\mathbf{1}\sigma_0} \quad \frac{\mathcal{D}_1}{\mathbf{1}(\neg\varphi \supset \theta)_{\mathbf{1}\supset E}}}{\mathbf{1}\theta}$$

8.4 Observations

Next, two important schematic deductions.

(1) $\mathbf{0}\neg\Diamond\neg\varphi \vdash_{CK} \mathbf{0}\Box\varphi$, as witnessed by the following.

$$\frac{\begin{array}{c} [v_1: \mathbf{1}\neg\varphi]_{\Diamond I} \\ v_3: \mathbf{0}\neg\Diamond\neg\varphi \quad \mathbf{0}\Diamond\neg\varphi_{\mathbf{0}\supset I} \\ \mathbf{0}\perp_{\mathbf{0}\perp E} \end{array}}{[v_0: \mathbf{1}\varphi] \quad \mathbf{1}\varphi \quad [v_2: \mathbf{1}\top]}_{\mathbf{1}EM} \frac{\mathbf{1}\varphi^{v_0, v_1}}{\mathbf{0}\Box\varphi^{v_2}}$$

(2) $\mathbf{0}\neg\Box\varphi \vdash_{CK} \mathbf{0}\Diamond\neg\varphi$, as witnessed by the following, taking \mathcal{D}' to witness the previous observation.

$$\frac{\begin{array}{c} [v_3: \mathbf{0}\neg\Diamond\neg\varphi] \\ \mathcal{D}' \\ \mu: \mathbf{0}\neg\Box\varphi \quad \mathbf{0}\Box\varphi_{\mathbf{0}\supset I} \\ \mathbf{0}\perp_{\mathbf{0}\perp E} \end{array}}{[v_4: \mathbf{0}\Diamond\neg\varphi] \quad \mathbf{0}\Diamond\neg\varphi_{\mathbf{0}EM}}{\mathbf{0}\Diamond\neg\varphi}$$

Since $\mathbf{0}\Diamond\neg\varphi \vdash_{IK} \mathbf{0}\neg\Box\varphi$ and $\vdash_{IK} \subseteq \vdash_{CK}$, $\neg\Box$ and $\Diamond\neg$ are equivalent under \vdash_{CK} . Since we also have $\mathbf{0}\neg\neg\varphi \vdash_{CK} \mathbf{0}\varphi$ and $\mathbf{0}\varphi \vdash_{CK} \mathbf{0}\neg\neg\varphi$, we have the familiar classical interdefinability of \Box and \Diamond .³¹

8.5 Definition

A CK-frame $\langle W, R, \sqsubseteq \rangle$ is an IK-frame such that $\sqsubseteq = id|W$. A CK-model is an IK-model with a CK-frame. An inference (Γ, χ) is CK-valid iff it is \mathcal{M} -valid for every CK-model \mathcal{M} .

8.6 Observation

For any CK-model \mathcal{M} , $u \in W^{\mathcal{M}}$ and $\varphi_{i \in 2} \in Fml$, $\mathcal{M}, u \models (\varphi_0 \supset \varphi_1)$ iff either $\mathcal{M}, u \not\models \varphi_0$ or $\mathcal{M}, u \models \varphi_1$.

³¹ So taking \Box to be defined as $\neg\Diamond\neg$ we could drop $\Box I$ and $\Box E$ in the definition of \Rightarrow_{CK} without changing \vdash_{CK} ; and taking \Diamond to be defined we could drop $\Diamond I$, $\Diamond E_0$ and $\Diamond E_1$ with similar lack of effect.

8.7 Theorem

\vdash_{CK} is sound with respect to \mathcal{M} -validity for \mathcal{M} a CK-model, i.e. for any $\Gamma \subseteq MFml$ and $\chi \in MFml$, if $\Gamma \vdash_{CK} \chi$ then $\langle \Gamma, \chi \rangle$ is CK-valid. Furthermore, \vdash_{CK}^- is sound with respect to \mathcal{M} -validity $^-$.

Proof: a straightforward induction on the stages of the definition of \Rightarrow_{CK} (i.e. on the height of classical deductions), using 8.6.

8.8 Observation

$\vdash_{CK}^- \subsetneq \vdash_{CK}$. The proof of 6.3 transfers.

8.9 Definitions

Consider a $\Gamma \subseteq MFml$. Γ is maximally CK-consistent iff $\Gamma \not\vdash_{CK} \mathbf{0} \perp$, and for any Γ' such that $\Gamma \subseteq \Gamma' \subseteq MFml$, if $\Gamma' \not\vdash_{CK} \mathbf{0} \perp$ then $\Gamma = \Gamma'$.

8.10 Lemma

For every Σ , $\Sigma \in W_{\vdash_{CK}}$ iff $\Sigma \in W_{\vdash_{CK}^-}$ and Σ is maximally CK-consistent.

Proof. Given $\Sigma \in W_{\vdash_{CK}}$, this the key point: if Σ is $\mathbf{0}$ -closed $_{\vdash_{CK}}$ then for any $\varphi \in Fml$, $(\varphi \vee \neg\varphi) \in \Sigma$, and thus (since Σ is \vee -complete) either $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$. With that, the lemma follows by well-known arguments.

8.11 Observation

$F_{\vdash_{CK}}$ is a CK-frame. Proof: use the previous lemma. So $\mathcal{M}_{\vdash_{CK}}$ is a CK-model; call it the canonical CK-model based on vocabulary set S .

The obvious Completeness Theorem for CK follow by slight variations of the arguments used for 7.18.

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