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Simplified Tableaux for STIT Imagination Logic

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Abstract
We show how to correct the analytic tableaux system from the paper Olkhovikov and Wansing (*Journal of Philosophical Logic*, *47*(2), 259–279, 2018).

Keywords Logic of imagination · STIT logic · Tableaux · Canonical models

In [1] a tableau calculus for STIT imagination logic is presented. If formulas $I_a A$ and $\neg I_a B$, saying that agent a imagines that A and that a does not imagine that B, are true at a moment/history-pair (m, h) from a model $\mathcal{M}$, then there must be a moment/history pair that witnesses the non-equivalence of A and B in $\mathcal{M}$. As stated in [1], the tableau rule for pairs $I_a A$ and $\neg I_a B$ locates such witnessing pairs in the future of the moment m, but for the soundness and completeness proofs this causes problems, which were overlooked in [1]. In the present paper we simplify the calculus and present a detailed soundness and completeness proof that includes proofs of some assumptions tacitly made in [1]. The soundness proof refers to the canonical model construction from the completeness proof for the axiom system given in [1]. In order to keep this paper reasonably self-contained, we here include not only the syntax and semantics of STIT imagination logic but also the definition of the canonical model and two truth lemmas from [1].

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1 STIT Imagination Logic

The language of STIT imagination logic consists of a countably infinite set Var of propositional variables, with typical element denoted p. Moreover, we assume a given finite non-empty set Ag of agents, and use indices a, a_1, a_2, etc., to stand for pairwise distinct agents. The set $Form$ of all formulas (or sentences) generated from Var and Ag has elements denoted A, B, etc. The syntax of STIT imagination logic is then defined in BNF as follows:

Propositional variables: $p \in Var$
Agents: $a \in Ag$
Formulas: $A \in Form$
$A := p \mid \neg A \mid A \wedge A \mid SA \mid [c]_a A \mid I_a A$

Here $\neg$ and $\wedge$ stand for classical negation and conjunction, the other Boolean connectives are defined as usual. Moreover, SA is to be read as "A is settled true", $[c]_a A$ as "agent a c-stit realizes A", and $I_a A$ as "agent a imagines that A". We will also use PA as a shorthand for $\neg S \neg A$.

Formulas are evaluated in imagination models at moment/history-pairs. An imagination model is a tuple $\mathcal{M} = \langle Tree, \leq, Ag, Choice, \{N_a \mid a \in Ag\}, V\rangle$, where:

- $Tree$ is a non-empty set of moments, and $\leq$ is a partial order on $Tree$ satisfying the conditions $\forall m_1, m_2 \exists m(m \leq m_1 \wedge m \leq m_2)$ (historical connection), and $\forall m_1, m_2, m((m_1 \leq m \wedge m_2 \leq m) \rightarrow (m_1 \leq m_2 \vee m_2 \leq m_1))$ (no backward branching).
- The set $Hist(\mathcal{M})$ of all histories of $\mathcal{M}$ is then just the set of all maximal $\leq$-chains in $Tree$. A history h is said to pass through a moment m iff $m \in h$. The set of all histories passing through $m \in Tree$ is denoted by H_m.
- Ag is the assumed finite set of all agents acting in $Tree$ and is assumed to be disjoint from all the other items in $\mathcal{M}$.
- $Choice$ is a function defined on the set $Tree \times Ag$, such that for an arbitrary $(m, a) \in Tree \times Ag$, the value of this function, that is to say $Choice(m, a)$ (more commonly denoted $Choice^m_a$) is a partition of H_m. If $h \in H_m$, then $Choice^m_a(h)$ denotes the element of $Choice^m_a$, to which h belongs. In the special case when we have $Choice^m_a = \{H_m\}$, it is said that the agent a has a *vacuous* choice at the moment m. $Choice$ is assumed to satisfy the following two restrictions:
  - "No choice between undivided histories": for arbitrary $m \in Tree$, $a \in Ag$, $e \in Choice^m_a$, and $h, h' \in H_m$: $(h \in e \wedge \exists m'(m < m' \wedge m' \in h \cap h')) \rightarrow h' \in e$.
  - "Independence of agents". If f is a function defined on Ag such that $\forall a \in Ag\ (f(a) \in Choice^m_a)$, then $\bigcap_{a \in Ag} f(a) \neq \varnothing$.
- The set of moment/history-pairs of $\mathcal{M}$ is defined as $MH(\mathcal{M}) := \{(m, h) \mid m \in Tree, h \in H_m\}$.
- For every $a \in Ag$, we have $N_a\colon MH(\mathcal{M}) \longrightarrow 2^{(2^{MH(\mathcal{M})})}$.
- V is an evaluation function for atomic sentences, i.e., $V\colon Var \longrightarrow 2^{MH(\mathcal{M})}$.

Definition 1 The satisfaction relation between formulas and moment/history-pairs in an imagination model $\mathcal{M}$ is then defined inductively as follows:

$$\mathcal{M}, (m, h) \models p \Leftrightarrow (m, h) \in V(p), \text{ for atomic } p;$$
$$\mathcal{M}, (m, h) \models (A \wedge B) \Leftrightarrow \mathcal{M}, (m, h) \models A \text{ and } \mathcal{M}, (m, h) \models B;$$
$$\mathcal{M}, (m, h) \models \neg A \Leftrightarrow \mathcal{M}, (m, h) \not\models A;$$
$$\mathcal{M}, (m, h) \models SA \Leftrightarrow \forall h' \in H_m(\mathcal{M}, (m, h') \models A);$$
$$\mathcal{M}, (m, h) \models [c]_a A \Leftrightarrow \forall h' \in Choice_a^m(h)(\mathcal{M}, (m, h') \models A);$$
$$\mathcal{M}, (m, h) \models I_a A \Leftrightarrow (i)\ \forall h' \in Choice_a^m(h)\ (\|A\|_{\mathcal{M}} \in N_a((m, h')))$$
$$\text{and } (ii)\ \exists h'' \in H_m\ (\|A\|_{\mathcal{M}} \notin N_a((m, h''))),$$

where $\|A\|_{\mathcal{M}} := \{(m, h) \in MH(\mathcal{M}) \mid \mathcal{M}, (m, h) \models A\}$.

In what follows, we use "positive condition" and "negative condition" to refer to the first and second conjunct in the semantic clause for $I_a A$, respectively.

The axiom system L for STIT imagination logic is given as follows:

(A0) Propositional tautologies.
(A1) S is an $S5$ modality.
(A2) For every $a \in Ag$, $[c]_a$ is an $S5$ modality.
(A3) $SA \to [c]_a A$ for every $a \in Ag$.
(A4) $(P[c]_{a_1} A_1 \wedge \ldots \wedge P[c]_{a_n} A_n) \to P([c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n)$, provided that all the
$a_1, \ldots, a_n$ are pairwise different.
(A5) $I_a A \to ([c]_a I_a A \wedge \neg S I_a A)$ for every $a \in Ag$.

Rules of inference:

(R1) Modus ponens.
(R2) From A infer SA.
(R3) From $A \leftrightarrow B$ infer $I_a A \leftrightarrow I_a B$ for every $a \in Ag$.

Remark 1 Observe that (A5) can be equivalently replaced in this system by the two axiomatic schemes (A5.1) $I_a A \to [c]_a I_a A$ and (A5.2) $\neg S I_a A$ for every $a \in Ag$.

In [1], the technique of canonical models is used to prove that L is strongly complete with respect to the class of all imagination models. It is shown that if Θ is an L-consistent set of sentences, then there exists an imagination model that at some moment/history pair satisfies every formula from Θ.

In order to define the canonical L-model, let W be the set of all L-maxiconsistent sets of sentences and denote the members of W as w, w', w_1 etc. Set wRw' iff $\{A \mid SA \in w\} \subseteq w'$, and set $w \simeq_a w'$ iff $\{A \mid [c]_a A \in w\} \subseteq w'$. Then (A1) and (A2) ensure that all these relations are equivalence relations; moreover, (A3) ensures that $\simeq_a \subseteq R$ for every $a \in Ag$.

We denote equivalence classes of W with respect to R by X, X', X_1, etc. The set of all such equivalence classes is denoted by Ξ. When restricted to an arbitrary $X \in \Xi$, the relation R turns into the universal relation, but relations of the form $\simeq_a$ can remain non-trivial equivalence relations breaking X up into several equivalence classes. We denote the family of equivalence classes corresponding to $\simeq_a \restriction X$ by $E(X, a)$. In the canonical model, a special role is played by the following set: $\Sigma = \{\neg p \mid p \in Var\} \cup \{SA \leftrightarrow A \mid \text{ for arbitrary } A\} \cup \{[c]_a A \leftrightarrow A \mid \text{ for arbitrary } A\}$.

The following facts about Σ were noted and discussed in [1]:

(F1) There exists exactly one element in W, which extends Σ. We denote this element by $\mathbf{w}$.

(F2) It follows from the definitions of Σ and R that the R-equivalence set containing $\mathbf{w}$, contains $\mathbf{w}$ only.

The canonical model is based on a tree of depth 2. First, we choose an element $0 \notin \Xi \cup W$ and define our set of moments: $Tree = \{0\} \cup \Xi \cup W$. Then, for arbitrary $x, y \in Tree$ we define $x \leq y$ iff $x = y$, or $y \in x$ or $x = 0$. This allows for a simple description of the set of histories. Every history has the form $h_w = \langle 0, X, w \rangle$, where $X \in \Xi$ and $w \in X$. Thus, the set of histories is in one-to-one correspondence with W.

Thirdly, we define the choice function. It assigns a vacuous choice to every agent at every moment m, if $m \notin \Xi$. That is to say, the only choice of every agent at every such moment will be just the set of all histories passing through this moment. Otherwise, i.e., for the case when $m = X \in \Xi$, we define the choice function as follows: $Choice^X_a = \{H \mid \exists e \in E(X, a)(H = \{h_w \mid w \in e\})\}$.

Next, we define the imagination neighbourhoods: $N_a((m, h)) = \varnothing$ for every $a \in Ag$ and every $m \notin \Xi$. For the case when $m = X \in \Xi$, we need one further auxiliary notion. For every sentence A we set $Ext(A)$ (read: extension of A) to be $\{(X, h_w) \mid A \in w \in X\}$ if $A \notin \mathbf{w}$; otherwise we set $Ext(A) = \{(X, h_w) \mid A \in w \in X\} \cup \{(m, h_w) \mid m \notin \Xi \text{ and } m \in h_w\}$. Having defined the extensions, we set $N_a((X, h_w)) = \{Ext(A) \mid I_a A \in w\}$ for arbitrary $w \in X \in \Xi$. Finally, we define the evaluation function for variables in the following way: $V(p) = \{(X, h_w) \mid p \in w \in X \in \Xi\}$.

It is shown in [1] that the so defined structure, call it $\mathcal{M}$, is indeed an imagination model. Moreover, the following truth lemmas are shown.

Lemma 1 (**Truth Lemma 1**) *Let $m \notin \Xi$ and $m \in h$. Then, for any sentence A, the following holds: $\mathcal{M}, (m, h) \models A \Leftrightarrow A \in \mathbf{w}$.*

Lemma 2 (**Truth Lemma 2**) *Let $X \in \Xi$ and $w \in X$. Then, for any sentence A, the following holds: $\mathcal{M}, (X, h_w) \models A \Leftrightarrow A \in w$.*

2 A Tableau Calculus

One variant of a correct system of analytic tableaux rules for stit imagination logic can be given by the sets of structural rules and decomposition rules as presented below in Tables 1 and 2, respectively. Direct comparison between these tables and Tables 1 and 2 of [1] shows that in the new rules the expressions of the form $m \in h$ and $m \prec m_1$ are deleted altogether, sometimes to be replaced with expressions of the form $h \lhd^m_a h$. We also introduce a new structural rule, REF0, and the structural rule

Table 1 New structural tableau rules

REF0	REF	SYM	TRAN	IND
$A, (m, h)$	$h_i \lhd_a^m h_k$	$h_i \lhd_a^m h_k$	$h_i \lhd_a^m h_k$	$h_{l_1} \lhd_{a_1}^m h_{l_1}$
			$h_k \lhd_a^m h_l$	$\ldots\ h_{l_k} \lhd_{a_k}^m h_{l_k}$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$h \lhd_a^m h$	$h_i \lhd_{a_1}^m h_i$	$h_k \lhd_a^m h_i$	$h_i \lhd_a^m h_l$	$h_{l_1} \lhd_{a_1}^m h_n\ \ldots h_{l_k} \lhd_{a_k}^m h_n$
$a \in Ag$	$a_1 \in Ag$			

where h_n is new, $k > 1$, and $a_1, \ldots, a_k$ are pairwise distinct

REF is modified so as to allow for switching between agent indices. Finally, the rule for $I_a A$ and $\neg I_a B$ is replaced by a similar, but simpler, rule for $\|A\| \in N_a((m, h_i))$ and $\|B\| \notin N_a((m, h_i))$.

Table 2 New decomposition rules for STIT imagination logic

$$\begin{array}{c} \neg\neg A, (m, h) \\ \downarrow \\ A, (m, h) \end{array} \qquad \begin{array}{c} (A \wedge B), (m, h) \\ \downarrow \\ A, (m, h), \quad B, (m, h) \end{array} \qquad \begin{array}{c} \neg(A \wedge B), (m, h) \\ \swarrow \qquad \searrow \\ \neg A, (m, h) \qquad \neg B, (m, h) \end{array}$$

$$\begin{array}{c} SA, (m, h_i), \\ h_k \lhd_a^m h_k \\ \downarrow \\ A, (m, h_k) \end{array} \qquad \begin{array}{c} \neg SA, (m, h_i) \\ \downarrow \\ \neg A, (m, h_k) \\ \text{where } h_k \text{ is new} \end{array}$$

$$\begin{array}{c} [c]_a A, (m, h_i), \\ h_i \lhd_a^m h_k \\ \downarrow \\ A, (m, h_k) \end{array} \qquad \begin{array}{c} \neg[c]_a A, (m, h_i) \\ \downarrow \\ \neg A, (m, h_k), \\ h_i \lhd_a^m h_k \\ \text{where } h_k \text{ is new} \end{array}$$

$$\begin{array}{c} I_a A, (m, h_i), \\ h_i \lhd_a^m h_k \\ \downarrow \\ \|A\| \in N_a((m, h_k)), \\ h_l \lhd_a^m h_l, \\ \|A\| \notin N_a((m, h_l)) \ \text{ where } h_l \text{ is new} \end{array} \qquad \begin{array}{c} \neg I_a A, (m, h_i), h_l \lhd_a^m h_l \\ \swarrow \qquad \searrow \\ \|A\| \in N_a((m, h_l)) \qquad \|A\| \notin N_a((m, h_k)), \\ \qquad\qquad h_i \lhd_a^m h_k \\ \qquad\qquad \text{where } h_k \text{ is new} \end{array}$$

$$\begin{array}{c} \|A\| \in N_a((m, h_i)), \\ \|B\| \notin N_a((m, h_i)) \\ \swarrow \qquad \searrow \\ A, (m_k, h_{k_1}), \ \neg B, (m_k, h_{k_1}) \qquad \neg A, (m_l, h_{l_1}), \ B, (m_l, h_{l_1}) \\ \text{where } m_k, h_{k_1} \text{ are new} \qquad \text{where } m_l, h_{l_1} \text{ are new} \end{array}$$

The tableau rules are utilized to process semantic information about imagination models, and we will use (i) expressions $h_i \lhd_a^m h_l$ to indicate that the histories h_i and h_l are both in H_m and are choice-equivalent for agent a at moment m, and (ii) statements $\|A\| \in N_a((m,h))$ ($\|A\| \notin N_a((m,h))$) to express that the truth set of A belongs (does not belong) to $N_a((m,h))$.

The resulting system of analytic tableaux is designed to work with finite sets of signed formulas of the form $\Gamma_{(m,h)}$, where Γ is a set of imagination stit formulas, m a moment name, and h a history name. We define that $\Gamma_{(m,h)} = \{A, (m,h) \mid A \in \Gamma\}$, and say that $\Gamma_{(m,h)}$ is *signed with* the moment-history name pair (m,h).

A tableau is a tree which has a finite set of signed formulas as its root; its nodes are finite sets of certain expressions. To finite sets of signed formulas, decomposition rules and structural tableau rules can be applied to complete the tableau. A tableau is said to be *complete* iff each of its branches is complete. A branch is complete if there is no possibility to apply one more rule to expand this branch. A tableau branch is said to be *closed* iff there are expressions of the form $A, (m,h)$ and $\neg A, (m,h)$ on the branch.[1] A closed branch is considered complete. A tableau is called *closed* iff all of its branches are closed, and it is called *open* if it is not closed.

The indices $i, k, l, \ldots$ used in the tableau rules are natural numbers, and a *new* index is the smallest natural number not already used in the tableau. In models constructed from open tableau branches, we shall interpret an agent index a by a itself. Note that it may happen that a rule is applied to an expression from a tableau node more than once if the rule requires additional input and some suitable additional input is introduced at later nodes. If, for instance, the decomposition rule for formulas SA is applied to the expressions $SA, (m,h_i)$, $h_k \lhd_a^m h_k$, and later on the branch a new expression $h_l \lhd_a^m h_l$ is introduced, then the rule has to be applied also to $SA, (m,h_i)$, $h_l \lhd_a^m h_l$.

Definition 2 Let $\Delta \cup \{A\}$ be a finite set of formulas.[2] $\Delta \vdash A$ ("A is derivable from Δ") iff there exists a closed and complete tableau for $\Delta_{(m,h_0)} \cup \{\neg A, (m,h_0)\}$.

Since we have simplified the calculus, we here reconsider the examples of tableaux from [1] (Tables 3 and 4). In these tables, we assume that $Ag = \{a\}$.

In what follows we let $\mathcal{M} = \langle Tree, \leq, Ag, Choice, \{\overline{N}_a \mid a \in Ag\}, V\rangle$ stand for the canonical stit imagination model as defined in Section 1.

Definition 3 Let b be a tableau branch. The pair of functions $(\mathbb{M}, \mathbb{H})$ such that $\mathbb{M} : \{m_k \mid m_k \text{ occurs on } b\} \longrightarrow Tree$ and $\mathbb{H} : \{h_k \mid h_k \text{ occurs on } b\} \longrightarrow Hist(\mathcal{M})$ is said to be *faithful to b* iff the following conditions hold:

1. Whenever $A, (m_k, h_l)$ occurs on b, there exist $w \in X \in \Xi$ such that $A \in w$, $\mathbb{M}(m_k) = X$ and $\mathbb{H}(h_l) = h_w = (0, X, w)$ (so that by Truth Lemma 2, we will also have $\mathcal{M}, (\mathbb{M}(m_k), \mathbb{H}(h_l)) \models A$).

[1] This definition of closed branch is simpler than the one used in [1] (and also more standard).

[2] The difference from the derivability definition given in [1] is that in Definition 2 we demand the finiteness of the set of premises.

Table 3 Examples of open tableaux

$\varnothing \not\vdash \neg I_a(p \to p), (m, h_0)$:	$\varnothing \not\vdash \neg I_a I_a p, (m, h_0)$:
$\neg\neg I_a(p \to p), (m, h_0)$	$\neg\neg I_a I_a p, (m, h_0)$
$\downarrow$	$\downarrow$
$I_a(p \to p), (m, h_0)$	$I_a I_a p, (m, h_0)$
$\downarrow$	$\downarrow$
$h_0 \lhd_a^m h_0$	$h_0 \lhd_a^m h_0$
$\downarrow$	$\downarrow$
$\|p \to p\| \in N_a((m, h_0))$,	$\|I_a p\| \in N_a((m, h_0))$,
$h_1 \lhd_a^m h_1$,	$h_1 \lhd_a^m h_1$,
$\|p \to p\| \notin N_a((m, h_1))$	$\|I_a p\| \notin N_a((m, h_1))$

2. Whenever $h_i \lhd_a^m h_k$ occurs on b, then $\mathbb{H}(h_k) \in Choice_a^{\mathbb{M}(m)}(\mathbb{H}(h_i))$, and, moreover, there exist $w, u \in X \in \Xi$ such that $\mathbb{M}(m) = X$, $\mathbb{H}(h_i) = h_w = (0, X, w)$, and $\mathbb{H}(h_k) = h_u = (0, X, u)$.
3. Whenever $\|A\| \in N_a((m, h))$ occurs on b, then $\|A\|_{\mathcal{M}} \in \overline{N}_a((\mathbb{M}(m), \mathbb{H}(h)))$.
4. Whenever $\|A\| \notin N_a((m, h))$ occurs on b, then $\|A\|_{\mathcal{M}} \notin \overline{N}_a((\mathbb{M}(m), \mathbb{H}(h)))$.

Lemma 3 *Let b be a tableau branch. If $(\mathbb{M}, \mathbb{H})$ is faithful to b and a tableau rule is applied to b, then the application produces at least one extension b' of b, such that some (not necessarily proper) extension $(\mathbb{M}', \mathbb{H}')$ of $(\mathbb{M}, \mathbb{H})$ is faithful to b'.*

Proof Assume the hypothesis of the lemma. We have to consider every tableau rule. If extended branches are obtained by applying one of the rules for $\neg\neg A$, $(A \wedge B)$ or $\neg(A \wedge B)$, then obviously $(\mathbb{M}, \mathbb{H})$ is faithful to at least one extension b' of b.

Suppose the rule for formulas SA is applied to $SA, (m, h_i)$. Then branch b' extends branch b by $A, (m, h_k)$ for an arbitrary $h_k \lhd_a^m h_k$ on b such that $a \in Ag$. Since $(\mathbb{M}, \mathbb{H})$ is faithful to b, we have, by Definition 3.1, that, for some $w \in X \in \Xi$ such that $SA \in w$, $\mathbb{M}(m) = X$ and $\mathbb{H}(h_i) = h_w = (0, X, w)$, whence, by Truth Lemma 2, also $\mathcal{M}, (\mathbb{M}(m), \mathbb{H}(h_i)) \models SA$. By the definition of satisfaction, it holds that $\mathcal{M}, (\mathbb{M}(m), \bar{h}) \models A$ for all $\bar{h} \in H_{\mathbb{M}(m)} = H_X = \{h_u \mid u \in X\}$. By Definition 3.2, $\mathbb{H}(h_k)$ must be in $Choice_a^{\mathbb{M}(m)}(\mathbb{H}(h_k)) \subseteq H_{\mathbb{M}(m)}$, therefore, we must have $\mathbb{H}(h_k) = h_u$ for some $u \in X$. On the other hand, we must have $\mathcal{M}, (\mathbb{M}(m), h_u) \models A$ so that, by Truth Lemma 2, $A \in u$. Summing this up, we see that $(\mathbb{M}, \mathbb{H})$ is faithful to b'.

Suppose now that the rule for $\neg SA$ is applied to $\neg SA, (m, h_i)$, so that b is extended by $\neg A, (m, h_k)$, for a new history name h_k. Since $(\mathbb{M}, \mathbb{H})$ is faithful to b, we have, by Definition 3.1, that, for some $w \in X \in \Xi$ such that $\neg SA \in w$, $\mathbb{M}(m) = X$ and $\mathbb{H}(h_i) = h_w = (0, X, w)$, whence, by Truth Lemma 2, also $\mathcal{M}, (\mathbb{M}(m), \mathbb{H}(h_i)) \models \neg SA$. Thus, there is an $\bar{h} \in H_{\mathbb{M}(m)} = H_X = \{h_u \mid u \in X\}$ such that $\mathcal{M}, (\mathbb{M}(m), \bar{h}) \models \neg A$. But then, take an $u \in X$ such that $\bar{h} = h_u$. By Truth Lemma 2, we must have $\neg A \in u$, therefore, $(\mathbb{M}, \mathbb{H} \cup \{(h_k, h_u)\})$ is seen to be faithful to b'.

Table 4 Examples of closed tableaux (some Boolean steps skipped)

$$\varnothing \vdash I_a\neg(p \wedge q) \rightarrow I_a(\neg p \vee \neg q):$$

$$\neg(I_a\neg(p \wedge q) \rightarrow I_a(\neg p \vee \neg q)), (m, h_0)$$
$$\downarrow$$
$$I_a\neg(p \wedge q), (m, h_0),$$
$$\neg I_a(\neg p \vee \neg q), (m, h_0)$$
$$\downarrow$$
$$h_0 \lhd_a^m h_0$$
$$\downarrow$$
$$\|\neg(p \wedge q)\| \in N_a((m, h_0)),$$
$$\|\neg(p \wedge q)\| \notin N_a((m, h_1)),$$
$$h_1 \lhd_a^m h_1$$
$$\swarrow \qquad \searrow$$
$$\|\neg p \vee \neg q\| \in N_a((m, h_1)) \qquad \|\neg p \vee \neg q\| \notin N_a((m, h_2)),\ h_0 \lhd_a^m h_2$$
$$\swarrow \qquad \searrow \qquad\qquad \downarrow$$
$$\neg\neg(p \wedge q), (m_1, h_3), \quad \neg(p \wedge q), (m_2, h_4), \quad \|\neg(p \wedge q)\| \in N_a((m, h_2)),$$
$$(\neg p \vee \neg q), (m_1, h_3) \quad \neg(\neg p \vee \neg q), (m_2, h_4) \quad h_5 \lhd_a^m h_5, \|\neg(p \wedge q)\| \notin N_a((m, h_5))$$
$$\swarrow \qquad \downarrow$$
$$\neg(p \wedge q), (m_3, h_6), \qquad \neg\neg(p \wedge q), (m_4, h_7),$$
$$\neg(\neg p \vee \neg q), (m_3, h_6) \qquad (\neg p \vee \neg q), (m_4, h_7)$$

$$\varnothing \vdash I_a p \rightarrow ([c]_a I_a p \wedge \neg S I_a p):$$

$$\neg(I_a p \rightarrow ([c]_a I_a p \wedge \neg S I_a p)), (m, h_0)$$
$$\downarrow$$
$$I_a p, (m, h_0),$$
$$\neg([c]_a I_a p \wedge \neg S I_a p), (m, h_0)$$
$$\swarrow \qquad \searrow$$
$$\neg[c]_a I_a p, (m, h_0) \qquad \neg\neg S I_a p, (m, h_0)$$
$$\downarrow \qquad \downarrow$$
$$\neg I_a p, (m, h_1), \qquad S I_a p, (m, h_0)$$
$$h_0 \lhd_a^m h_1 \qquad \downarrow$$
$$\downarrow \qquad h_0 \lhd_a^m h_0$$
$$h_1 \lhd_a^m h_1 \qquad \downarrow$$
$$\downarrow \qquad \|p\| \in N_a((m, h_0)),$$
$$\|p\| \in N_a((m, h_1)), \qquad h_4 \lhd_a^m h_4,$$
$$h_2 \lhd_a^m h_2, \qquad \|p\| \notin N_a((m, h_4))$$
$$\|p\| \notin N_a((m, h_2)) \qquad \downarrow$$
$$\swarrow \qquad \searrow \qquad I_a p, (m, h_4)$$
$$\|p\| \in N_a((m, h_2)) \qquad \|p\| \notin N_a((m, h_3)), \qquad \downarrow$$
$$\swarrow \quad \downarrow \qquad h_1 \lhd_a^m h_3 \qquad \|p\| \in N_a((m, h_4))$$
$$p, (m_2, h_5), \quad p, (m_3, h_6), \qquad \downarrow \qquad \downarrow \quad \searrow$$
$$\neg p, (m_2, h_5) \quad \neg p, (m_3, h_6) \qquad h_0 \lhd_a^m h_3 \qquad p, (m_4, h_7), \quad p, (m_5, h_8),$$
$$\downarrow \qquad \neg p, (m_4, h_7) \quad \neg p, (m_5, h_8)$$
$$\|p\| \in N_a((m, h_3))$$
$$\swarrow \qquad \searrow$$
$$p, (m_6, h_9), \qquad p, (m_7, h_{10}),$$
$$\neg p, (m_6, h_9) \qquad \neg p, (m_7, h_{10})$$

Next, suppose the rule for $[c_a]A$ is applied to $[c_a]A, (m, h_i)$. Then we obtain b' as an extension of b by $A, (m, h_k)$ for an arbitrary $h_i \lhd_a^m h_k$ on b. Since $(\mathbb{M}, \mathbb{H})$ is faithful to b, we have, by Definition 3.1, that, for some $w \in X \in \Xi$ such that $[c_a]A \in w$, $\mathbb{M}(m) = X$ and $\mathbb{H}(h_i) = h_w = (0, X, w)$, whence, by Truth Lemma 2, also $\mathcal{M}, (\mathbb{M}(m), \mathbb{H}(h_i)) \models [c_a]A$. Then, by the definition of satisfaction, it holds that $\mathcal{M}, (\mathbb{M}(m), \bar{h}) \models A$ for all $\bar{h} \in Choice_a^{\mathbb{M}(m)}(\mathbb{H}(h_i)) \subseteq H_{\mathbb{M}(m)} = H_X$. By Definition 3.2, $\mathbb{H}(h_k) \in Choice_a^{\mathbb{M}(m)}(\mathbb{H}(h_i)) \subseteq H_{\mathbb{M}(m)} = H_X$. Therefore, for some $u \in X$, we must have both $\mathbb{H}(h_k) = h_u$ and $\mathcal{M}, (\mathbb{M}(m), \mathbb{H}(h_k)) \models A$, whence, by Truth Lemma 2, we get $A \in u$. Therefore, $(\mathbb{M}, \mathbb{H})$ is also faithful to b'.

Assume that the rule for $\neg[c_a]A$ is applied to $\neg[c_a]A, (m, h_i)$, so that b is extended by $h_i \lhd_a^m h_k$ and $\neg A, (m, h_k)$, for a new history name h_k. Since $(\mathbb{M}, \mathbb{H})$ is faithful to b, we have, by Definition 3.1, that, for some $w \in X \in \Xi$ such that $\neg[c_a]A \in w$, $\mathbb{M}(m) = X$ and $\mathbb{H}(h_i) = h_w = (0, X, w)$, whence, by Truth Lemma 2, also $\mathcal{M}, (\mathbb{M}(m), \mathbb{H}(h_i)) \models \neg[c_a]A$. Therefore, there is an $\bar{h} \in Choice_a^{\mathbb{M}(m)}(\mathbb{H}(h_i)) \subseteq H_{\mathbb{M}(m)} = H_X = \{h_u \mid u \in X\}$ such that $\mathcal{M}, (\mathbb{M}(m), \bar{h}) \models \neg A$. But then, take an $u \in X$ such that $\bar{h} = h_u$. By Truth Lemma 2, we must have $\neg A \in u$. Therefore, $(\mathbb{M}, \mathbb{H} \cup \{(h_k, h_u)\})$ is seen to be faithful to b'.

Suppose that the rule for $I_a A$ is applied to $I_a A, (m, h_i)$. Then the branch b' is obtained as an extension of b by $\|A\| \in N_a((m, h_k))$ for an arbitrary $h_i \lhd_a^m h_k$ occurring on b and by $h_l \lhd_a^m h_l$ and $\|A\| \notin N_a((m, h_l))$ for a new history name h_l. Since $(\mathbb{M}, \mathbb{H})$ is faithful to b, we have, by Definition 3.1, that, for some $w \in X \in \Xi$ such that $I_a A \in w$, $\mathbb{M}(m) = X$ and $\mathbb{H}(h_i) = h_w = (0, X, w)$, whence, by Truth Lemma 2, $\mathcal{M}, (\mathbb{M}(m), \mathbb{H}(h_i)) \models I_a A$. This further means that:

$$\forall \bar{h} \in Choice_a^{\mathbb{M}(m)}(\mathbb{H}(h_i))(\|A\|_{\mathcal{M}} \in \overline{N}_a((\mathbb{M}(m), \bar{h}))) \tag{1}$$

and

$$\exists \bar{h}' \in H_{\mathbb{M}(m)}(\|A\|_{\mathcal{M}} \notin \overline{N}_a((\mathbb{M}(m), \bar{h}'))) \tag{2}$$

We first consider Eq. 2 and note that $H_{\mathbb{M}(m)} = H_X = \{h_u \mid u \in X\}$. Therefore, we can choose an $u \in X$ such that $\|A\|_{\mathcal{M}} \notin \overline{N}_a((\mathbb{M}(m), h_u))$. Of course, we will also have $h_u \in Choice_a^{\mathbb{M}(m)}(h_u)$, since $Choice_a^{\mathbb{M}(m)}$ is a partition of $H_{\mathbb{M}(m)}$. Turning to Eq. 1, by Definition 3.2, we know that $\mathbb{H}(h_k) \in Choice_a^{\mathbb{M}(m)}(\mathbb{H}(h_i))$. Therefore, we must have that $\|A\|_{\mathcal{M}} \in \overline{N}_a((\mathbb{M}(m), \mathbb{H}(h_k)))$. Summing this up, we see that $(\mathbb{M}, \mathbb{H} \cup \{(h_l, h_u)\})$ must be faithful to b'.

Next, assume that the rule for $\neg I_a A$ is applied to $\neg I_a A, (m, h_i)$. Then there are two extended branches. The branch b' extends b by $\|A\| \in N_a((m, h_l))$ for an arbitrary h_l with $h_l \lhd_a^m h_l$ on b. The branch b'' extends branch b by $\|A\| \notin N_a((m, h_k))$ and $h_i \lhd_a^m h_k$ for some new history name h_k. Since $(\mathbb{M}, \mathbb{H})$ is faithful to b, we have, by Definition 3.1, that, for some $w \in X \in \Xi$ such that $\neg I_a A \in w$, $\mathbb{M}(m) = X$ and $\mathbb{H}(h_i) = h_w = (0, X, w)$, whence, by Truth Lemma 2, $\mathcal{M}, (\mathbb{M}(m), \mathbb{H}(h_i)) \models \neg I_a A$. The latter means that one of the following two alternatives hold: either (a) $\forall \bar{h} \in H_{\mathbb{M}(m)}$ $(\|A\|_{\mathcal{M}} \in \overline{N}_a((\mathbb{M}(m), \bar{h}))$, or (b) $\exists \bar{h}' \in Choice_a^{\mathbb{M}(m)}(\mathbb{H}(h_i))$ with $\|A\|_{\mathcal{M}} \notin \overline{N}_a((\mathbb{M}(m), \bar{h}'))$.

In case (a), by Definition 3.2, we know that $\mathbb{H}(h_l) \in Choice_a^{\mathbb{M}(m)}(\mathbb{H}(h_l)) \subseteq H_{\mathbb{M}(m)}$ so that $\|A\|_{\mathcal{M}} \in \overline{N}_a((\mathbb{M}(m), \mathbb{H}(h_l)))$. Therefore, $(\mathbb{M}, \mathbb{H})$ is faithful to b'.

In case (b), given that $Choice_a^{\mathbb{M}(m)}(\mathbb{H}(h_i)) \subseteq H_{\mathbb{M}(m)} = H_X = \{h_u \mid u \in X\}$, we can choose a $v \in X$ such that both $h_v \in Choice_a^{\mathbb{M}(m)}(\mathbb{H}(h_i))$ and $\|A\|_{\mathcal{M}} \notin \overline{N}_a((\mathbb{M}(m), h_v))$. But then $(\mathbb{M}, \mathbb{H} \cup \{(h_k, h_v)\})$ must be faithful to b''.

Suppose that the rule for $\|A\| \in N_a((m, h_i))$ and $\|B\| \notin N_a((m, h_i))$ is applied to the respective configurations on our branch. Again there are two extended branches. The branch b' extends branch b by $A, (m_k, h_{k_1})$ and $\neg B, (m_k, h_{k_1})$ for a new moment name m_k and a new history name h_{k_1}. The branch b'' extends b by $\neg A, (m_l, h_{l_1})$ and $B, (m_l, h_{l_1})$ for a new moment name m_l and a new history name h_{l_1}. Since $(\mathbb{M}, \mathbb{H})$ is faithful to b, we have, by Definition 3.3–4, that both $\|A\|_{\mathcal{M}} \in \overline{N}_a((\mathbb{M}(m), \mathbb{H}(h_i)))$ and $\|B\|_{\mathcal{M}} \notin \overline{N}_a((\mathbb{M}(m), \mathbb{H}(h_i)))$. Therefore, $\|A\|_{\mathcal{M}} \neq \|B\|_{\mathcal{M}}$, whence it is easily seen that A cannot be logically equivalent to B. Therefore, either (a) $\{A, \neg B\}$ is L-consistent or (b) $\{\neg A, B\}$ is L-consistent. Case (a): we extend $\{A, \neg B\}$ to an L-maxiconsistent set u and we let Y be the element of Ξ containing u. It follows from Truth Lemma 2 that $(\mathbb{M} \cup \{(m_k, Y)\}, \mathbb{H} \cup \{(h_{k_1}, h_u)\})$ must be faithful to b'. Case (b): analogous to the previous case. Finally, we consider the structural tableau rules. If $(\mathbb{M}, \mathbb{H})$ is faithful to b, and one of the rules REF0, REF, SYM, or TRAN is applied to obtain a branch b', then $(\mathbb{M}, \mathbb{H})$ is also faithful to b', and Definition 3.2 is satisfied in virtue of Definition 3.1 and by the fact that for every agent $a \in Ag$ and every $m \in Tree$, the relation $\{(h, h') \mid h' \in Choice_a^m(h)\}$ is an equivalence relation on H_m.

For IND, assume that $(\mathbb{M}, \mathbb{H})$ is faithful to b. Then, by Definition 3.2, for some $u_1, \ldots, u_k \in X \in \Xi$ we have $\mathbb{M}(m) = X$ and $\mathbb{H}(h_{l_1}) = h_{u_1}, \ldots, \mathbb{H}(h_{l_k}) = h_{u_k}$. By the independence of agents condition, the set $\bigcap_{1 \leq j \leq k} Choice_{a_j}^{\mathbb{M}(m)}(\mathbb{H}(h_{l_j})) \subseteq H_{\mathbb{M}(m)} = H_X = \{h_u \mid u \in X\}$ must be non-empty. Therefore, choose an arbitrary $\bar{h}$ in this set. For some $u \in X$ we must have $\bar{h} = h_u$ so that Definition 3.2 and $(\mathbb{M}, \mathbb{H} \cup \{(h_n, h_u)\})$ is faithful to b'. □

Theorem 1 (**Soundness**) *Assume that $\Gamma_{(m,h)}$ is a finite non-empty set of signed imagination stit formulas such that Γ is satisfiable in an imagination stit model. Then $\Gamma_{(m,h)}$ does not have a closed tableau.*

Proof Assume the conditions of the theorem. Then Γ is L-consistent, and hence can be extended to an L-maxiconsistent set u. We let X be the element of Ξ containing u. Then the pair of functions $\{(m, X), (h, h_u)\}$ is clearly faithful to the only branch of the tableau $\{\Gamma_{(m,h)}\}$. Assume that we can unfold this tableau to a closed one. Then it follows, by an obvious induction on the maximal length of branches in this tableau, that at least for one branch b in this closed tableau there is an extension $(\mathbb{M}, \mathbb{H})$ of $\{(m, X), (h, h_u)\}$ which is faithful to b.

But, given that b is closed, there must be an imagination stit formula C and a moment-history name pair (m', h') such that both $C, (m', h')$ and $\neg C, (m', h')$ occur on b. It follows then, by Definition 3.1, that both $\mathcal{M}, (\mathbb{M}(m'), \mathbb{H}(h')) \models C$ and $\mathcal{M}, (\mathbb{M}(m'), \mathbb{H}(h')) \models \neg C$, which is a contradiction. Hence, there is no complete closed tableau for $\Gamma_{(m,h)}$ □

Corollary 1 *Let $\Delta \cup \{A\}$ be a finite set of imagination stit formulas. If $\Delta \cup \{\neg A\}$ is satisfiable in an imagination stit model, then $\Delta \not\vdash A$.*

Proof Immediately by Theorem 1 and Definition 2 □

We now turn to proving completeness.

Lemma 4 *Let b be a tableau branch. Then all of the following conditions hold for every $a, a' \in Ag$, all moment names m, m', all history names h_i, h_j, h_k, h_l, and all imagination stit formulas A, B:*

1. *If $A, (m, h_i)$ and $B, (m', h_i)$ occur on b, then $m = m'$.*
2. *If $A, (m, h_i)$ and $h_k \lhd_a^{m'} h_l$ occur on b, then either $m = m'$ or $h_i \notin \{h_k, h_l\}$.*
3. *If $h_i \lhd_a^m h_j$ and $h_k \lhd_{a'}^{m'} h_l$ occur on b, then either $m = m'$ or $\{h_i, h_j\} \cap \{h_k, h_l\} = \varnothing$.*

Proof By induction on the construction of b. If b consists of a set of formulas signed by a single moment-history pair (m, h), then the satisfaction of the Lemma is immediate. Assume that b' is obtained from a branch b satisfying the Lemma by applying one of the rules. We will refer to the fact that the i-th part of the Lemma is satisfied by b' (resp. b) as $i_{b'}$ (resp. i_b). We have to consider the following cases:

Case 1. Suppose that one of the rules for $\neg\neg A$, $(A \wedge B)$ or $\neg(A \wedge B)$ is applied to b. Then no new configurations of the forms (m, h_i) and $h_i \lhd_a^m h_j$ are created, therefore b' still satisfies the Lemma.
Case 2. Assume the rule for SA is applied to $SA, (m, h_i)$. Then we obtain b' as an extension of b by $A, (m, h_k)$ for an arbitrary $h_k \lhd_a^m h_k$ on b, where $a \in Ag$. We show the three parts of the Lemma for b' as follows:

$1_{b'}$: Given that $h_k \lhd_a^m h_k$ occurs on b, it follows from 2_b that for every $m' \neq m$ and every formula B, $B, (m', h_k)$ does not occur on b'. Since the only possibly new element of b is $A, (m, h_k)$, we can infer $1_{b'}$.
$2_{b'}$: Given that $h_k \lhd_a^m h_k$ occurs on b', it follows from 3_b that for every $m' \neq m$, every $a' \in Ag$, and every pair of history names h_l, h_{l_1} such that $h_k \in \{h_l, h_{l_1}\}$, the element $h_l \lhd_{a'}^{m'} h_{l_1}$ does not occur on b. Since the only possibly new element of b is $A, (m, h_k)$, we can infer $2_{b'}$.
$3_{b'}$: Given that b' does not contain any new elements of the form $h_l \lhd_{a'}^{m'} h_{l_1}$, $3_{b'}$ follows from 3_b.

Case 3. Assume the rule for $\neg SA$ is applied to $\neg SA, (m, h_i)$. Then b is extended by $\neg A, (m, h_k)$, for a new history name h_k so that no violations of the Lemma are created.
Case 4. Next, suppose the rule for $[c_a]A$ is applied to $[c_a]A, (m, h_i)$. Then we obtain b' as an extension of b by $A, (m, h_k)$ for an arbitrary $h_i \lhd_a^m h_k$ on b. We show the three parts of the Lemma for b' as follows:

$1_{b'}$: Given that $h_i \lhd_a^m h_k$ occurs on b, it follows from 2_b that for every $m' \neq m$ and every formula B, $B, (m', h_k)$ does not occur on b. Since the only possibly new element of b' is $A, (m, h_k)$, we can infer $1_{b'}$.
$2_{b'}$: Given that $h_i \lhd_a^m h_k$ occurs on b, it follows from 3_b that for every $m' \neq m$, every $a' \in Ag$, and every pair of history names h_l, h_{l_1} such that $h_k \in \{h_l, h_{l_1}\}$,

the element $h_l \lhd_{a'}^{m'} h_{l_1}$ does not occur on b. Since the only possibly new element of b' is $A, (m, h_k)$, we can infer $2_{b'}$.
$3_{b'}$: Given that the rule for $[c_a]A$ does not generate any new elements of the form $h_l \lhd_{a'}^{m'} h_{l_1}$, $3_{b'}$ follows from 3_b.

Case 5. Assume that the rule for $\neg[c_a]A$ is applied to $\neg[c_a]A, (m, h_i)$, so that b is extended by $h_i \lhd_a^m h_k$ and $\neg A, (m, h_k)$, for a new history name h_k. It is clear that the addition of $\neg A, (m, h_k)$ cannot lead to any violations of the Lemma, since h_k is new. We have to consider then the consequences of adding $h_i \lhd_a^m h_k$:

$1_{b'}$: It is clear that the addition of $h_i \lhd_a^m h_k$ cannot lead to violations of $1_{b'}$.
$2_{b'}$: Given that $\neg[c_a]A, (m, h_i)$ occurs on b, it follows from 1_b that for every $m' \neq m$ and every formula B, $B, (m', h_i)$ does not occur on b. Moreover, h_k is new so that we cannot have any occurrences of the form $B, (m', h_k)$ on b either. Therefore, $2_{b'}$ holds.
$3_{b'}$: Given that $\neg[c_a]A, (m, h_i)$ occurs on b, it follows from 2_b that for every $m' \neq m$, every $a' \in Ag$, and every pair of history names h_l, h_{l_1} such that $h_i \in \{h_l, h_{l_1}\}$, the element $h_l \lhd_{a'}^{m'} h_{l_1}$ does not occur on b. Moreover, h_k is new and cannot coincide with one of h_l, h_{l_1}. Therefore, $3_{b'}$ holds.

Case 6. Suppose that the rule for $I_a A$ is applied to $I_a A, (m, h_i)$. Then the branch b' is obtained as an extension of b by $\|A\| \in N_a((m, h_k))$ for an arbitrary $h_i \lhd_a^m h_k$ occurring on b and by $h_l \lhd_a^m h_l$ and $\|A\| \notin N_a((m, h_l))$ for a new history name h_l. The only element that can lead to violations of the Lemma is $h_l \lhd_a^m h_l$, but, given that h_l is new, such violations are clearly excluded.
Case 7. Next, assume that the rule for $\neg I_a A$ is applied to $\neg I_a A, (m, h_i)$. Then there are two further options.
Case 7a.The branch b' extends b by $\|A\| \in N_a((m, h_l))$ for an arbitrary h_l with $h_l \lhd_a^m h_l$ on b. Addition of this element can lead to no further violations of the Lemma.
Case 7b. The branch b' extends branch b by $\|A\| \notin N_a((m, h_k))$ and $h_i \lhd_a^m h_k$ for some new history name h_k. The only significant addition in this case is $h_i \lhd_a^m h_k$. We show the three parts of the Lemma for b' as follows:

$1_{b'}$: An immediate consequence of 1_b.
$2_{b'}$: Given that $\neg I_a A, (m, h_i)$ occurs on b, it follows from 1_b that for every $m' \neq m$ and every formula B, $B, (m', h_i)$ does not occur on b. Moreover, h_k is new so that we cannot have any occurrences of the form $B, (m', h_k)$ on b either. Therefore, $2_{b'}$ holds.
$3_{b'}$: Given that $\neg I_a A, (m, h_i)$ occurs on b, it follows from 2_b that for every $m' \neq m$, every $a' \in Ag$, and every pair of history names h_l, h_{l_1} such that $h_i \in \{h_l, h_{l_1}\}$, the element $h_l \lhd_{a'}^{m'} h_{l_1}$ does not occur on b. Moreover, h_k is new and cannot coincide with one of h_l, h_{l_1}. Therefore, $3_{b'}$ holds.

Case 8. Suppose that the rule for $\|A\| \in N_a((m, h_i))$ and $\|B\| \notin N_a((m, h_i))$ is applied to the respective configurations on our branch. Again there are two further options.

Case 8a. The branch b' extends branch b by $A, (m_k, h_{k_1})$ and $\neg B, (m_k, h_{k_1})$ for a new moment name m_k and a new history name h_{k_1}. Since all the additions only involve new moment and history names, the satisfaction of the Lemma by b' is an immediate consequence of its satisfaction by b.
Case 8b. The branch b' extends b by $A, (m_l, h_{l_1})$ and $\neg B, (m_l, h_{l_1})$ for a new moment name m_l and a new history name h_{l_1}. This case is symmetric to Case 8a.
Case 9. Suppose that the rule REF0 is applied to $A, (m, h)$. Then the branch b' is obtained as an extension of b by $h \lhd_a^m h$ for an arbitrary $a \in Ag$. We show the three parts of the Lemma for b' as follows:

$1_{b'}$: An immediate consequence of 1_b.
$2_{b'}$: Given that $A, (m, h)$ occurs on b, it follows from 1_b that for every $m' \neq m$ and every formula B, $B, (m', h)$ does not occur on b. Therefore, $2_{b'}$ holds.
$3_{b'}$: Given that $A, (m, h)$ occurs on b, it follows from 2_b that for every $m' \neq m$, every $a' \in Ag$, and every pair of history names h_l, h_{l_1} such that $h \in \{h_l, h_{l_1}\}$, the element $h_l \lhd_{a'}^{m'} h_{l_1}$ does not occur on b. Therefore, $3_{b'}$ holds.

Case 10. Assume that one of the rules REF, SYM is applied to $h_i \lhd_a^m h_k$. Then the branch b' is obtained as an extension of b by either $h_i \lhd_a^m h_i$ or $h_k \lhd_a^m h_i$. Observe that, by the inclusions $\{h_i\} \subseteq \{h_k, h_i\} \subseteq \{h_i, h_k\}$, the satisfaction of the Lemma by b' is an immediate consequence of its satisfaction by b.
Case 11. Assume that the rule TRAN is applied to $h_i \lhd_a^m h_k$ and $h_k \lhd_a^m h_l$. Then the branch b' is obtained as an extension of b by $h_i \lhd_a^m h_l$. We show the three parts of the Lemma for b' as follows:

$1_{b'}$: An immediate consequence of 1_b.
$2_{b'}$: Given that $h_i \lhd_a^m h_k$ occurs on b, it follows from 2_b that for every $m' \neq m$ and every formula B, $B, (m', h_i)$ does not occur on b. Given that $h_k \lhd_a^m h_l$ occurs on b, it follows from 2_b that for every $m' \neq m$ and every formula B, $B, (m', h_l)$ does not occur on b. Therefore, $2_{b'}$ holds.
$3_{b'}$: Given that $h_i \lhd_a^m h_k$ occurs on b, it follows from 3_b that for every $m' \neq m$, every $a' \in Ag$, and every pair of history names h_{k_1}, h_{l_1} such that $h_i \in \{h_{k_1}, h_{l_1}\}$, the element $h_{k_1} \lhd_{a'}^{m'} h_{l_1}$ does not occur on b. Given that $h_k \lhd_a^m h_l$ occurs on b, it follows from 3_b that for every $m' \neq m$, every $a' \in Ag$, and every pair of history names h_{k_1}, h_{l_1} such that $h_k \in \{h_{k_1}, h_{l_1}\}$, the element $h_{k_1} \lhd_{a'}^{m'} h_{l_1}$ does not occur on b. Therefore, $3_{b'}$ holds.

Case 12. Assume that the rule IND is applied to $h_{l_1} \lhd_{a_1}^m h_{l_1}, \ldots, h_{l_k} \lhd_{a_k}^m h_{l_k}$. Then the branch b' is obtained as an extension of b by $h_{l_1} \lhd_{a_1}^m h_n, \ldots, h_{l_k} \lhd_{a_k}^m h_n$ for a new history name h_n. We show the three parts of the Lemma for b' as follows:

$1_{b'}$: An immediate consequence of 1_b.
$2_{b'}$: Let $1 \leq j \leq k$. Then, given that $h_{l_j} \lhd_{a_j}^m h_{l_j}$ occurs on b, it follows from 2_b that for every $m' \neq m$ and every formula B, $B, (m', h_{l_j})$ does not occur on b. Moreover, since h_n is new, it follows that for every $m' \neq m$ and every formula B, $B, (m', h_n)$ does not occur on b. Therefore, $2_{b'}$ holds.
$3_{b'}$: Let $1 \leq j \leq k$. Then, given that $h_{l_j} \lhd_{a_j}^m h_{l_j}$ occurs on b, it follows from 3_b that for every $m' \neq m$, every $a' \in Ag$, and every pair of history names h_{k_1},

h_{k_2} such that $h_{l_j} \in \{h_{k_1}, h_{k_2}\}$, the element $h_{k_1} \lhd^{m'}_{a'} h_{k_2}$ does not occur on b. Moreover, since h_n is new, it follows that for every $m' \neq m$, every $a' \in Ag$, and every pair of history names h_{k_1}, h_{k_2} such that $h_n \in \{h_{k_1}, h_{k_2}\}$, the element $h_{k_1} \lhd^{m'}_{a'} h_{k_2}$ does not occur on b. Therefore, $3_{b'}$ holds. □

Lemma 5 *Let b be an open branch of a complete tableau. Then the following statements hold for every imagination stit formula A, every moment name m, all history names h_i, h_k, and every $a, a_1 \in Ag$:*

1. *If $A, (m, h_i)$ occurs on b, then $h_i \lhd^m_a h_i$ occurs on b.*
2. *If $h_i \lhd^m_a h_k$ occurs on b, then both $h_i \lhd^m_{a_1} h_i$ and $h_k \lhd^m_{a_1} h_k$ occur on b.*
3. *If one of $\|A\| \in N_a((m, h_k))$, $\|A\| \notin N_a((m, h_k))$ occurs on b, then $h_k \lhd^m_{a_1} h_k$ occurs on b.*

Proof (Part 1). By completeness of b and rule REF0.

(Part 2). By completeness of b and rules REF, SYM.

(Part 3). Since the occurrences of the forms $\|A\| \in N_a((m, h_k))$, $\|A\| \notin N_a((m, h_k))$ are never present in the sets of signed formulas, they can only appear as a product of some rule application. We have to consider the following cases:

Case 1. The occurrence is of the form $\|A\| \in N_a((m, h_k))$. Two further options are possible:
Case 1a. The occurrence was generated by an application of the rule for $I_a A$ to $I_a A, (m, h_i)$ and $h_i \lhd^m_a h_k$. Then $h_i \lhd^m_a h_k$ must also occur on b and we are done by Part 2.
Case 1b. The occurrence was generated by an application of the rule for $\neg I_a A$ to $\neg I_a A, (m, h_i)$ and $h_k \lhd^m_a h_k$. Then $h_k \lhd^m_a h_k$ must also occur on b and we are done by Part 2.
Case 2. The occurrence is of the form $\|A\| \notin N_a((m, h_k))$. Again, two further options are possible:
Case 2a. The occurrence of $\|A\| \notin N_a((m, h_k))$ was generated by an application of the rule for $I_a A$ to $I_a A, (m, h_i)$ and $h_i \lhd^m_a h_l$. Then this same application also generated an occurrence of $h_k \lhd^m_a h_k$, and we are done by Part 2.
Case 2b. The occurrence of $\|A\| \notin N_a((m, h_k))$ was generated by an application of the rule for $\neg I_a A$ to $\neg I_a A, (m, h_i)$ and $h_l \lhd^m_a h_l$. Then this same application also generated an occurrence of $h_i \lhd^m_a h_k$, and we are done by Part 2. □

Corollary 2 *Let b be an open branch of a complete tableau. If a moment name m occurs on b then some history name h is such that for an arbitrary $a \in Ag$ $h \lhd^m_a h$ occurs on b.*

Proof If m occurs on b then the following cases are possible: (a) for some history name h and some formula A, $A, (m, h)$ occurs on b; (b) for some history names h, h_i and some $a_1 \in Ag$, the element $h \lhd^m_{a_1} h_i$ occurs on b; (c) for some history name

h, some $a_1 \in Ag$ and some formula A, one of the elements $\|A\| \in N_{a_1}((m, h))$, $\|A\| \notin N_{a_1}((m, h))$ occurs on b.

These cases are disposed of by Parts 1, 2, and 3 of Lemma 5, respectively. □

Definition 4 Let b be an open branch of a complete tableau. Then the structure $Temp_b = (Tree_b, \leq_b)$ induced by b is defined as follows:

1. $Tree_b := \{\dagger\} \cup \{\, \mu_m, \mu_h \mid (\exists a \in Ag)(h \lhd_a^m h$ occurs on $b)\,\}$.
2. $\leq_b$: = the reflexive closure of $\{\, (\dagger, \mu_m), (\dagger, \mu_h), (\mu_m, \mu_h) \mid (\exists a \in Ag)(h \lhd_a^m h$ occurs on $b)\,\}$.

Lemma 6 *Let b be an open branch of a complete tableau. Then:*

1. *In the structure $Temp_b = (Tree_b, \leq_b)$ induced by b, $\leq_b$ is a partial order on $Tree_b$ satisfying the absence of backward branching and historical connection.*
2. *The set of histories induced by $Temp_b$ is exactly the set:*

$$\{\, \chi_{(m,h)} \mid (\exists a \in Ag)(h \lhd_a^m h \textit{ occurs on } b)\,\},$$

where $\chi_{(m,h)} = (\dagger, \mu_m, \mu_h)$, and this set is non-empty.

Proof (Part 1). Reflexivity and transitivity we have by definition of $\leq_b$. Antisymmetry follows from the fact that the sets of moment and history names are disjoint and that † is different from any moment indexed by either a moment name or a history name.

As for the absence of backward branching, assume that $m_1, m_2, m_3 \in Tree_b$ are pairwise different and that we have both $m_1 <_b m_3$ and $m_2 <_b m_3$. Then m_3 cannot be †, since † has no $<_b$-predecessors. Also, m_3 cannot be μ_m for some moment name m since in this case † will be the only $<_b$-predecessor of m_3, hence we must have $m_1 = \dagger = m_2$, contrary to our assumption. Therefore, m_3 must be of the form μ_h for some history name h. Then μ_h is not $\leq_b$-comparable to any μ_{h_1} for $h_1 \neq h$. The following cases are then possible:

Case 1. $\{m_1, m_2\} = \{\dagger, \mu_m\}$ for some moment name. Then we have $\dagger <_b \mu_m$ and the absence of backward branching is satisfied.

Case 2. $m_1 = \mu_m$, $m_2 = \mu_{m_0}$ for some moment names $m \neq m_0$. Then, by definition of $\leq_b$, there must be $a_1, a_2 \in Ag$ such that both $h \lhd_{a_1}^m h$ and $h \lhd_{a_2}^{m_0} h$ occur on b, which is in contradiction with Lemma 4.3.

Finally, the historical connection is satisfied since † is the $\leq_b$-least element of $Tree_b$.

(Part 2). Whenever $h \lhd_a^m h$ occurs on b for some $a \in Ag$, $\chi_{(m,h)}$ is clearly a $\leq_b$-chain. Furthermore, this chain cannot be extended by any moment μ_{h_1} such that $h_1 \neq h$, since μ_{h_1} is $\leq_b$-incomparable to μ_h, and it cannot be extended by any moment μ_{m_1} such that $m_1 \neq m$, since μ_{m_1} is $\leq_b$-incomparable to μ_m. Therefore, any such $\chi_{(m,h)}$ is also a maximal $\leq_b$-chain and hence a history. Also, we must have at least one chain of the form $\chi_{(m,h)} \in Hist(Temp_b)$, since b contains at least the initial set of formulas signed by a pair of moment-history names and hence, by Lemma 5.1, must contain at least one element of the form $h \lhd_a^m h$.

Now let χ be a maximal $\leq_b$-chain in $Tree_b$. Then clearly $\dagger \in \chi$ since $\dagger$ is the $\leq_b$-least element of $Tree_b$. But we cannot have $\chi = \{\dagger\}$, since there is at least one $\chi_{(m,h)} \in Hist(Temp_b)$, and we clearly have $\chi = \{\dagger\} \subset \chi_{(m,h)}$ which contradicts the maximality of χ. Therefore, one of the following two cases obtains:

Case 1. For some moment name m, $\mu_m \in \chi$. Then all the other moments of the form μ_{m_1} must be outside χ since all these moments are $\leq_b$-incomparable to μ_m. Since $\mu_m \in Tree_b$, then, by definition of $Tree_b$, there must be a history name h and an $a \in Ag$ such that $h \lhd_a^m h$ occurs on b. But then also $\mu_h \in Tree_b$ and hence $\chi_{(m,h)} \in Hist(Temp_b)$. Therefore, we cannot have $\chi = \{\dagger, \mu_m\} \subset \chi_{(m,h)}$ since this is in contradiction with the maximality of χ. Thus, there must be a history name h_1 such that $\mu_{h_1} \in \chi$, and we can only have one moment indexed with a history name in χ since all such moments are pairwise $\leq_b$-incomparable. Also, since χ is a $\leq_b$-chain, we must have either $\mu_m \leq_b \mu_{h_1}$ or $\mu_{h_1} \leq_b \mu_m$, but the definition of $\leq_b$ is incompatible with the latter option. Therefore, we must have $\mu_m \leq_b \mu_{h_1}$, whence, by definition of $\leq_b$, $h_1 \lhd_a^m h_1$ must occur on b for some $a \in Ag$. The latter means that $\chi = \chi_{(m,h_1)}$.

Case 2. For some history name h, $\mu_h \in \chi$. This case is similar to Case 1. Again, all the other moments of the form μ_{h_1} must be outside χ since all these moments are $\leq_b$-incomparable to μ_h. Since $\mu_h \in Tree_b$, then, by definition of $Tree_b$, there must be a moment name m and an $a \in Ag$ such that $h \lhd_a^m h$ occurs on b. But then also $\mu_m \in Tree_b$ and hence $\chi_{(m,h)} \in Hist(Temp_b)$. Therefore, we cannot have $\chi = \{\dagger, \mu_h\} \subset \chi_{(m,h)}$ since this is in contradiction with the maximality of χ. Moreover, Lemma 4.3 implies that for no $m_1 \neq m$ and no $a_1 \in Ag$ with $a_1 \neq a$, $h \lhd_{a_1}^{m_1} h$ can occur on b. Therefore, $\chi_{(m,h)}$ is the only maximal $\leq_b$-chain extending $\{\dagger, \mu_h\}$, which means that $\chi = \chi_{(m,h)}$.

Part 2 of the Lemma is thereby proven. □

Corollary 3 *Let b be an open branch of a complete tableau and $Temp_b = (Tree_b, \leq_b)$ be the structure induced by b. Let m be a moment name, h_i, h_k history names, $a \in Ag$, and let A be an imagination stit formula. Then the following statements are true:*

1. *If m occurs on b, then $\mu_m \in Tree_b$ and $H_{\mu_m} = \{\chi_{(m,h)} \mid (\exists a \in Ag)(h \lhd_a^m h$ occurs on $b)\} \neq \varnothing$.*
2. *If $h_i \lhd_a^m h_k$ occurs on b, then $(\mu_m, \chi_{(m,h_i)}), (\mu_m, \chi_{(m,h_k)}) \in MH(Temp_b)$.*
3. *If one of $A, (m, h_i)$, $\|A\| \in N_a((m, h_i))$, $\|A\| \notin N_a((m, h_i))$ occurs on b, then $(\mu_m, \chi_{(m,h_i)}) \in MH(Temp_b)$.*

Proof (Part 2). By Lemma 5.2 and Lemma 6.2.

(Part 3). By Lemma 5.1, Lemma 5.3, Lemma 6.2.

(Part 1).Note that m can only occur on b in one of the contexts of the form $h_i \lhd_a^m h_k$, $A, (m, h_i)$, $\|A\| \in N_a((m, h_i))$ or $\|A\| \notin N_a((m, h_i))$. It follows from Part 2 and Part 3 of the corollary that in all of these cases $\mu_m \in Tree_b$. Therefore, for some history name h and some $a \in Ag$, we must have $h \lhd_a^m h$ on b. The latter means that $\chi_{(m,h)} \in Hist(Temp_b)$ and, of course, $\mu_m \in \chi_{(m,h)}$. Therefore, the set $\{\chi_{(m,h)} \mid (\exists a \in Ag)(h \lhd_a^m h \text{ occurs on } b)\}$ must be non-empty. The fact that this set is exactly H_{μ_m} follows from Lemma 6.2. □

Definition 5 Let b be an open branch of a complete tableau. Then the structure $\mathcal{M}_b = (Tree_b, \leq_b, Choice_b, \{Nb_a \mid a \in Ag\}, V_b)$ induced by b is defined as follows:

1. $Tree_b$ and $\leq_b$ are defined according to Definition 4.
2. $(Choice_b)_a^{\mu_m} = \{h_{\lhd_a^m} \mid h \lhd_a^m h \text{ occurs on } b\}$.
3. $(Choice_b)_a^{\alpha} = \{H_\alpha\}$, if $\alpha \neq \mu_m$ for any moment name m.
4. $Nb_a((\mu_m, \chi_{(m,h)})) = \{\|A\|_{\mathcal{M}_b} \mid \|A\| \in N_a((m,h)) \text{ occurs on } b\}$
5. $Nb_a((\alpha, \chi_{(m,h)})) = \varnothing$, if $\alpha \neq \mu_m$ for any moment name m.
6. $V_b(p) = \{(\mu_m, \chi_{(m,h)}) \mid p, (m,h) \text{ occurs on } b\}$.

In the above definition we assume that $h_{\lhd_a^m} = \{\chi_{(m,h_i)} \mid h \lhd_a^m h_i \text{ occurs on } b\}$.

Remark 2 Contrary to first appearance, the neighbourhood functions Nb_a are well-defined. We can define the depth of I-nesting of a formula A, $dI(A)$. If A contains no imagination operator I_a, then $dI(A) = 0$. If A has the form $\neg B$, SB, or $[c_a]B$, then $dI(A) = dI(B)$. If A is a conjunction $(B \wedge C)$, then $dI(A)$ is $max(dI(B), dI(C))$. If A has the shape $I_a B$, then $dI(A) = dI(B) + 1$. We can show that $Nb_a((m,h))$ is well-defined by a double induction first on the depth of I-nesting and then on the construction of A. If $dI(A) = 0$, then the truth set $\|A\|_{\mathcal{M}_b}$ is well-defined because it is defined independently of neighbourhood functions, and thus $Nb_a((m,h))$ is well-defined. Suppose that $dI(A) = n+1$, and $Nb_a((m,h))$ is well-defined for formulas B with $dI(B) \leq n$, i.e. $\|B\|_{\mathcal{M}_b}$ is well-defined. Then (i) A has the shape $I_a B$ or (ii) A has the form $(B \wedge C)$ with $dI(B) \leq n+1$ and $dI(C) \leq n+1$. In case (i), we may use the induction hypothesis to conclude that $\|A\|_{\mathcal{M}_b}$ is well-defined. In case (ii), we may note that $\|(B \wedge C)\|_{\mathcal{M}_b} = \|B\|_{\mathcal{M}_b} \cap \|C\|_{\mathcal{M}_b}$. By induction on the construction of A, $\|B\|_{\mathcal{M}_b}$ and $\|C\|_{\mathcal{M}_b}$ are well-defined, and thus their intersection is well-defined. But then $Nb_a((m,h))$ is well-defined. Hence, $Nb_a((m,h))$ is well-defined for every formula A.

Remark 3 It is clear that we have $Hist(\mathcal{M}_b) = Hist(Temp_b)$ and $MH(\mathcal{M}_b) = MH(Temp_b)$ for every complete and open branch b. Therefore, in what follows, we will use Lemma 6 and Corollary 3 also in application to the structures of the form $\mathcal{M}_b$.

Lemma 7 *Let b be an open branch of a complete tableau. Then the structure $\mathcal{M}_b = (Tree_b, \leq_b, Choice_b, \{Nb_a \mid a \in Ag\}, V_b)$ induced by b, as given in Definition 5, is an imagination stit model.*

Proof It follows from Corollary 3 that the clauses of Definition 5 are indeed meaningful. Moreover, it follows from Lemma 6.1 that the constraints on imagination stit models which are only relevant to $Tree_b$ and $\leq_b$ are satisfied by $\mathcal{M}_b$. As for the no choice between undivided histories constraint, it follows from Lemma 6.2 that $\mathcal{M}_b$ only has undivided histories at †. In this latter moment every agent has the unique (and vacuous) choice so that the constraint is satisfied trivially. We check the other constraints.

Claim 1. $(Choice_b)^{\alpha}_a$ induces a partition on the set H_α for $\alpha \in Tree_b$ and $a \in Ag$.

This is obvious if $\alpha \neq \mu_m$ for any moment name m. On the other hand, if $\alpha = \mu_m$ for some moment name m, then consider the set $h_{\lhd^m_a}$ for some $h \lhd^m_a h$ occurring on b. The latter occurrence means, by definition of $h_{\lhd^m_a}$, that we have $\chi_{(m,h)} \in h_{\lhd^m_a}$. Therefore, all the elements of $(Choice_b)^{\mu_m}_a$ are non-empty. Next, if $\chi \in H_{\mu_m}$, then, by Corollary 3.1, $\chi = \chi_{(m,h_i)}$ for some history name h_i such that, for some $a_1 \in Ag$, $h_i \lhd^m_{a_1} h_i$ occurs on b. But then, by Lemma 5.2, also $h_i \lhd^m_a h_i$ must occur on b, whence, further, $\chi = \chi_{(m,h_i)} \in h_{i\lhd^m_a}$. Since χ was chosen in H_{μ_m} arbitrarily, it follows that the union of the sets of the form $h_{\lhd^m_a}$ makes up H_{μ_m}.

Now, assume that $h_{l\lhd^m_a}$ and $h_{n\lhd^m_a}$ are two different elements of $(Choice_b)^{\mu_m}_a$. We will show that $h_{l\lhd^m_a} \cap h_{n\lhd^m_a} = \varnothing$. Indeed, assume, for *reductio*, that $\chi \in h_{l\lhd^m_a} \cap h_{n\lhd^m_a}$. Then there must be a history name h_k such that $\chi = \chi_{(m,h_k)}$ and both $h_l \lhd^m_a h_k$ and $h_n \lhd^m_a h_k$ occur on b. But then by the rules SYM and TRAN and completeness of b, both $h_l \lhd^m_a h_n$ and $h_n \lhd^m_a h_l$ must occur on b as well. Now choose an arbitrary $\chi' \in h_{l\lhd^m_a}$. We must have $\chi' = \chi_{(m,h_i)}$ for some history name such that $h_l \lhd^m_a h_i$ occurs on b. But then, by the rule TRAN, the occurrence of $h_n \lhd^m_a h_l$ on b, and the completeness of b, $h_n \lhd^m_a h_i$ occurs on b as well so that $\chi' = \chi_{(m,h_i)} \in h_{n\lhd^m_a}$. Since $\chi' \in h_{l\lhd^m_a}$ was chosen arbitrarily, this shows that $h_{l\lhd^m_a} \subseteq h_{n\lhd^m_a}$. In a symmetric fashion, one can also show that $h_{n\lhd^m_a} \subseteq h_{l\lhd^m_a}$, whence $h_{l\lhd^m_a} = h_{n\lhd^m_a}$ contrary to our assumption. Therefore, any two different elements of $(Choice_b)^{\mu_m}_a$ are disjoint. Claim 1 is thus proven.

Claim 2. $\mathcal{M}_b$ satisfies independence of agents.

Let $f : Ag \longrightarrow 2^{2^{Hist(\mathcal{M}_b)}}$ be such that for a given $\mu \in Tree_b$ and every $a \in Ag$ it is true that $f(a) \in (Choice_b)^{\mu}_a$. Then, if $\mu \neq \mu_m$ for any moment name m, we must have $\bigcap_{a \in Ag} f(a) = H_\mu \neq \varnothing$.

On the other hand, if for some moment name m we have $\mu = \mu_m$, we may assume wlog that $Ag = \{a_1, \ldots, a_k\}$. Then for some history names $h_{l_1}, \ldots, h_{l_k}$ such that for some $a_{l_1}, \ldots, a_{l_k}$ the elements $h_{l_1} \lhd^m_{a_{l_1}} h_{l_1}, \ldots, h_{l_k} \lhd^m_{a_{l_k}} h_{l_k}$ occur on b, we must have:

$$f = \{(a_1, h_{l_1 \lhd^m_{a_1}}), \ldots, (a_k, h_{l_k \lhd^m_{a_k}})\}. \tag{3}$$

But then, by Lemma 5.2, the elements $h_{l_1} \lhd^m_{a_1} h_{l_1}, \ldots, h_{l_k} \lhd^m_{a_k} h_{l_k}$ must also occur on b. Therefore, by IND and the completeness of b there must be a history name h_n such that all of $h_{l_1} \lhd^m_{a_1} h_n, \ldots, h_{l_k} \lhd^m_{a_k} h_n$ must also occur on b. By Corollary 3.2, the latter means that $\chi_{(m,h_n)} \in H_{\mu_m}$. By Eq. 3 and Definition 5, we also know that $\chi_{(m,h_n)} \in \bigcap_{a \in Ag} f(a)$, so that the independence of agents is satisfied. □

Lemma 8 (**Truth Lemma 3**) *Let b be an open branch of a complete tableau. Consider the structure $\mathcal{M}_b = (Tree_b, \leq_b, Choice_b, \{Nb_a \mid a \in Ag\}, V_b)$ induced by b. Let A, (m, h) occur on b. Then $\mathcal{M}_b, (\mu_m, \chi_{(m,h)}) \models A$.*

Proof The proof proceeds by induction on the construction of A.

Basis. Case 1. Assume that $A = p \in Var$. Then, if $p, (m, h)$ occurs on b, we have, by Corollary 3.3, that $(\mu_m, \chi_{(m,h)}) \in MH(\mathcal{M}_b)$, and, by Definition 5, that $(\mu_m, \chi_{(m,h)}) \in V_b(p)$. Therefore, $\mathcal{M}_b, (\mu_m, \chi_{(m,h)}) \models p = A$.
Case 2. Assume that $A = \neg p$ for some $p \in Var$. Then, if $\neg p, (m, h)$ occurs on b, we have, by Corollary 3.3, that $(\mu_m, \chi_{(m,h)}) \in MH(\mathcal{M}_b)$. Furthermore, since b is open, $p, (m, h)$ cannot occur on b, hence, by Definition 5, we get that $(\mu_m, \chi_{(m,h)}) \notin V_b(p)$. Therefore, $\mathcal{M}_b, (\mu_m, \chi_{(m,h)}) \models \neg p = A$.
Induction step. The cases when A is of the form $\neg\neg B$, $B \wedge C$, or $\neg(B \wedge C)$ are trivially solved by a reference to the respective rule plus induction hypothesis, so we omit them. We consider the modal cases in some detail:

Case 1. A is of the form SB. If $SB, (m, h)$ occurs on b, then, by Corollary 3.3, $(\mu_m, \chi_{(m,h)}) \in MH(\mathcal{M}_b)$ so that $\mu_m \in Tree_b$. Moreover, by Corollary 3.1, $H_{\mu_m} = \{\chi_{(m,h)} \mid (\exists a \in Ag)(h \lhd^m_a h \text{ occurs on } b)\} \neq \varnothing$. Now assume that $\chi' \in H_{\mu_m}$. Then $\chi' = \chi_{(m,h_i)}$ for some history name h_i such that, for some $a \in Ag$, $h_i \lhd^m_a h_i$ occurs on b. But then, by the rule for SB and the completeness of b, also $B, (m, h_i)$ must occur on b. Therefore, by induction hypothesis, we must have $\mathcal{M}_b, (\mu_m, \chi_{(m,h_i)}) \models B$. Since $\chi' = \chi_{(m,h_i)}$ was chosen in H_{μ_m} arbitrarily, this means that $\mathcal{M}_b, (\mu_m, \chi_{(m,h)}) \models SB = A$.
Case 2. A is of the form $\neg SB$. If $\neg SB, (m, h)$ occurs on b, then, by Corollary 3.3, $(\mu_m, \chi_{(m,h)}) \in MH(\mathcal{M}_b)$ so that $\mu_m \in Tree_b$. By the rule for $\neg SB$ and the completeness of b, also $\neg B, (m, h_i)$ must occur on b for some history name h_i. By Corollary 3.3, $\chi_{(m,h_i)} \in H_{\mu_m}$, and, by induction hypothesis, $\mathcal{M}_b, (\mu_m, \chi_{(m,h_i)}) \models \neg B$, which means that $\mathcal{M}_b, (\mu_m, \chi_{(m,h)}) \models \neg SB = A$.
Case 3. A is of the form $[c_a]B$ for some $a \in Ag$. If $[c_a]B, (m, h)$ occurs on b, then, by Corollary 3.3, $(\mu_m, \chi_{(m,h)}) \in MH(\mathcal{M}_b)$ so that $\mu_m \in Tree_b$. Therefore, for some $a_1 \in Ag$, $h \lhd^m_{a_1} h$ occurs on b. By Lemma 5.2, also $h \lhd^m_a h$ occurs on b. Therefore, by Definition 5, we must have $\chi_{(m,h)} \in h_{\lhd^m_a}$ so that $(Choice_b)^{\mu_m}_a(\chi_{(m,h)}) = h_{\lhd^m_a} = \{\chi_{(m,h_i)} \mid h \lhd^m_a h_i \text{ occurs on } b\}$. Now, let $\chi_{(m,h_i)} \in (Choice_b)^{\mu_m}_a(\chi_{(m,h)})$ be arbitrary. Then $h \lhd^m_a h_i$ occurs on b and hence, by the rule for $[c_a]B$ and the completeness of b, also $B, (m, h_i)$ occurs on b. Therefore, by induction hypothesis, we must have $\mathcal{M}_b, (\mu_m, \chi_{(m,h_i)}) \models B$. Since $\chi_{(m,h_i)}$ was chosen in $(Choice_b)^{\mu_m}_a(\chi_{(m,h)})$ arbitrarily, this means that $\mathcal{M}_b, (\mu_m, \chi_{(m,h)}) \models [c_a]B = A$.
Case 4. A is of the form $\neg[c_a]B$ for some $a \in Ag$. If $\neg[c_a]B, (m, h)$ occurs on b, then, by Corollary 3.3, $(\mu_m, \chi_{(m,h)}) \in MH(\mathcal{M}_b)$ so that $\mu_m \in Tree_b$. By the rule for $\neg[c_a]B$ and the completeness of b, also $\neg B, (m, h_i)$ and $h \lhd^m_a h_i$ must occur on b for some history name h_i. But then, $\chi_{(m,h_i)} \in h_{\lhd^m_a} = (Choice_b)^{\mu_m}_a(\chi_{(m,h)})$, and, by induction hypothesis, $\mathcal{M}_b, (\mu_m, \chi_{(m,h_i)}) \models \neg B$, which means that $\mathcal{M}_b, (\mu_m, \chi_{(m,h)}) \models \neg[c_a]B = A$.

Case 5. A is of the form $I_a B$ for some $a \in Ag$. If $I_a B, (m, h)$ occurs on b, then, by Corollary 3.3, $(\mu_m, \chi_{(m,h)}) \in MH(\mathcal{M}_b)$ so that $\mu_m \in Tree_b$. Therefore, by Corollary 3.1, for some $a_1 \in Ag$, $h \lhd^m_{a_1} h$ occurs on b. By Lemma 5.2, also $h \lhd^m_a h$ occurs on b. Therefore, by Definition 5, we must have $\chi_{(m,h)} \in h_{\lhd^m_a}$ so that $(Choice_b)^{\mu_m}_a(\chi_{(m,h)}) = h_{\lhd^m_a} = \{\chi_{(m,h_i)} \mid h \lhd^m_a h_i \text{ occurs on } b\}$, and, by Corollary 3.1, $H_{\mu_m} = \{\chi_{(m,h)} \mid (\exists a \in Ag)(h \lhd^m_a h \text{ occurs on } b)\} \neq \varnothing$. Therefore, in the first place, if $\chi_{(m,h_i)} \in (Choice_b)^{\mu_m}_a(\chi_{(m,h)})$ is arbitrary, then $h \lhd^m_a h_i$ occurs on b and hence, by the rule for $I_a B$ and the completeness of b, also $\|B\| \in N_a((m, h_i))$ occurs on b. By definition of Nb_a, this implies that $\|B\|_{\mathcal{M}_b} \in Nb_a((\mu_m, \chi_{(m,h_i)}))$. Since $\chi_{(m,h_i)}$ was chosen in $(Choice_b)^{\mu_m}_a(\chi_{(m,h)})$ arbitrarily, this means that the positive condition for the truth of $I_a B$ at $(\mu_m, \chi_{(m,h)})$ is satisfied. Moreover, in the second place, since b is complete, and we have shown above that $h \lhd^m_a h$ occurs on b, the rule for $I_a B$ gets applied on b at least once, which means that, for some history name h_l, both $h_l \lhd^m_a h_l$ and $\|B\| \notin N_a((m, h_l))$ occur on b. Therefore, we must have that $\chi_{(m,h_l)} \in H_{\mu_m}$. We show that in this case also $\|B\|_{\mathcal{M}_b} \notin Nb_a((\mu_m, \chi_{(m,h_l)}))$. Indeed, if $\|B\|_{\mathcal{M}_b} \in Nb_a((\mu_m, \chi_{(m,h_l)}))$, then, by definition of Nb_a, there must be an imagination stit formula C such that both $\|C\|_{\mathcal{M}_b} = \|B\|_{\mathcal{M}_b}$ and $\|C\| \in N_a((m, h_l))$ occurs on b. But then, by its completeness, b must contain at least one application of the rule for $\|C\| \in N_a((m, h_l))$ and $\|B\| \notin N_a((m, h_l))$. Therefore, one of the two following options will hold: (a) b contains $C, (m_{k_1}, h_{k_2})$ and $\neg B, (m_{k_1}, h_{k_2})$ for some m_{k_1} and h_{k_2} or (b) b contains $\neg C, (m_{l_1}, h_{l_2})$ and $B, (m_{l_1}, h_{l_2})$ for some m_{l_1} and h_{l_2}. In case (a), it follows from the induction hypothesis that both $\mathcal{M}_b, (\mu_{m_{k_1}} \chi_{(m_{k_1}, h_{k_2})}) \models C$ and $\mathcal{M}_b, (\mu_{m_{k_1}} \chi_{(m_{k_1}, h_{k_2})}) \models \neg B$, which is in contradiction with our assumption that $\|C\|_{\mathcal{M}_b} = \|B\|_{\mathcal{M}_b}$; in case (b), the contradiction to the assumption that $\|C\|_{\mathcal{M}_b} = \|B\|_{\mathcal{M}_b}$ can be derived in a symmetric way. This latter contradiction shows that the configuration $\|C\| \in N_a((m, h_l))$ can occur on b for no imagination stit formula C such that $\|C\|_{\mathcal{M}_b} = \|B\|_{\mathcal{M}_b}$. Therefore, we must have $\|B\|_{\mathcal{M}_b} \notin Nb_a((\mu_m, \chi_{(m,h_l)}))$ so that the negative condition for the truth of $I_a B$ at $(\mu_m, \chi_{(m,h)})$ is also satisfied.

Summing this up, we must have $\mathcal{M}_b, (\mu_m, \chi_{(m,h)}) \models I_a B = A$.

Case 6. A is of the form $\neg I_a B$ for some $a \in Ag$. If $\neg I_a B, (m, h)$ occurs on b, then, by Corollary 3.3, $(\mu_m, \chi_{(m,h)}) \in MH(\mathcal{M}_b)$ so that $\mu_m \in Tree_b$. Therefore, for some $a_1 \in Ag$, $h \lhd^m_{a_1} h$ occurs on b. By Lemma 5.2, also $h \lhd^m_a h$ occurs on b. Therefore, by Definition 5, we must have $\chi_{(m,h)} \in h_{\lhd^m_a}$ so that $(Choice_b)^{\mu_m}_a(\chi_{(m,h)}) = h_{\lhd^m_a} = \{\chi_{(m,h_i)} \mid h \lhd^m_a h_i \text{ occurs on } b\}$, and, by Corollary 3.1, $H_{\mu_m} = \{\chi_{(m,h)} \mid (\exists a \in Ag)(h \lhd^m_a h \text{ occurs on } b)\} \neq \varnothing$. There are two further subcases, according to the branching of the decomposition rule for negated imagination ascriptions.

Case 6.1. $\|B\| \in N_a((m, h_l))$ occurs on b for every history name h_l with $h_l \lhd^m_a h_l$ on b. Now, assume that $\chi_{(m,h_i)} \in H_{\mu_m}$ be arbitrary. Then there exists an $a_1 \in Ag$ such that $h_i \lhd^m_{a_1} h_i$ occurs on b. By Lemma 5.2 and the completeness of b, this also means that $h_i \lhd^m_a h_i$ occurs on b, whence, by our assumption, also $\|B\| \in N_a((m, h_i))$ must occur on b. By Definition 5, the latter means that $\|B\|_{\mathcal{M}_b} \in Nb_a((\mu_m, \chi_{(m,h_i)}))$. Since $\chi_{(m,h_i)} \in H_{\mu_m}$ was chosen arbitrarily, this means that the negative condition for $I_a B$ is violated at $(\mu_m, \chi_{(m,h)})$, so that we must have $\mathcal{M}_b, (\mu_m, \chi_{(m,h)}) \models \neg I_a B = A$.

Case 6.2. For some history name h_k it is true that both $h \lhd_a^m h_k$ and $\|B\| \notin N_a((m, h_k))$ occur on b. Then $\chi_{(m,h_k)} \in (Choice_b)_a^{\mu_m}(\chi_{(m,h)})$. We show that in this case also $\|B\|_{\mathcal{M}_b} \notin Nb_a((\mu_m, \chi_{(m,h_k)}))$. Indeed, if $\|B\|_{\mathcal{M}_b} \in Nb_a((\mu_m, \chi_{(m,h_k)}))$, then, by definition of Nb_a, there must be an imagination stit formula C such that both $\|C\|_{\mathcal{M}_b} = \|B\|_{\mathcal{M}_b}$ and $\|C\| \in N_a((m, h_k))$ occurs on b. But then, by its completeness, b must contain at least one application of the rule for $\|C\| \in N_a((m, h_k))$ and $\|B\| \notin N_a((m, h_k))$. Therefore, one of the two following options will hold: (a) b contains $C, (m_{k_1}, h_{k_2})$ and $\neg B, (m_{k_1}, h_{k_2})$ for some m_{k_1} and h_{k_2} or (b) b contains $\neg C, (m_{l_1}, h_{l_2})$ and $B, (m_{l_1}, h_{l_2})$ for some m_{l_1} and h_{l_2}. In case (a), it follows from the induction hypothesis that both $\mathcal{M}_b, (\mu_{m_{k_1}} \chi_{(m_{k_1},h_{k_2})}) \models C$ and $\mathcal{M}_b, (\mu_{m_{k_1}} \chi_{(m_{k_1},h_{k_2})}) \models \neg B$, which is in contradiction with our assumption that $\|C\|_{\mathcal{M}_b} = \|B\|_{\mathcal{M}_b}$; in case (b), the contradiction to the assumption that $\|C\|_{\mathcal{M}_b} = \|B\|_{\mathcal{M}_b}$ can be derived in a symmetric way. This latter contradiction shows that the configuration $\|C\| \in N_a((m, h_k))$ can occur on b for no imagination stit formula C such that $\|C\|_{\mathcal{M}_b} = \|B\|_{\mathcal{M}_b}$. Therefore, we must have $\|B\|_{\mathcal{M}_b} \notin Nb_a((\mu_m, \chi_{(m,h_k)}))$, and, summing this up with the earlier established fact that $\chi_{(m,h_k)} \in (Choice_b)_a^{\mu_m}(\chi_{(m,h)})$, we see that the positive condition for $I_a B$ is violated at $(\mu_m, \chi_{(m,h)})$. Therefore, we, again, must have $\mathcal{M}_b, (\mu_m, \chi_{(m,h)}) \models \neg I_a B = A$. □

Theorem 2 (**Completeness**) *Let $\Delta \cup \{A\}$ be a finite set of imagination stit formulas. If $\Delta \not\vdash A$, then $\Delta \cup \{\neg A\}$ is satisfiable in an imagination stit model.*

Proof Suppose that $\Delta \not\vdash A$. Then there is no complete and closed tableau for $\Delta_{(m,h_0)} \cup \{\neg A, (m, h_0)\}$. Let b be an open branch of a complete tableau for this set and let $\mathcal{M}_b$ be the model induced by b. By the previous lemma, it follows that $\mathcal{M}_b, (\mu_m, \chi_{(m,h_0)}) \models B$ for every $B \in \Delta$ and $\mathcal{M}_b, (\mu_m, \chi_{(m,h_0)}) \models \neg A$, thus $\mathcal{M}_b, (\mu_m, \chi_{(m,h_0)}) \not\models A$. □

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References

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