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An Axiomatic System and a Tableau Calculus for STIT Imagination Logic

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Abstract We formulate a Hilbert-style axiomatic system and a tableau calculus for the STIT-based logic of imagination recently proposed in Wansing (2015). Completeness of the axiom system is shown by the method of canonical models; completeness of the tableau system is also shown by using standard methods.

1 Introduction

There exists an important difference between imagination and the propositional attitudes of belief and knowledge. Whilst it is widely assumed that, as a matter of psychological fact, it is impossible for agents to form beliefs at will, it seems to be clear that imagination may operate in voluntary mode. However, until recently the agentiveness of imagination has not been accounted for in the sparse literature on the logic of imagination ascriptions. Both I. Niiniluoto [10] and A. Costa Leite [8] treat expressions of the form 'agent j imagines that' as normal modal operators.

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Also, there is disagreement concerning the extent to which imagination is regulated by logical principles. Modelling imagination as a normal modality comes with the commitment to a number of properties that in analogy to the well-known problem of logical omniscience in epistemic logic may be said to give rise to a problem of logical omni-imagination. Imagination then is closed under valid implication and imagined implication. Moreover, all valid propositions of the underlying non-modal logic are imagined. This may be seen as a rather strong idealization and as being descriptively inadequate. According to some authors, such as G. Priest [11], imagination even fails to be governed by *any* logical principles.

A position intermediate between these two extremes is adopted in [18], where the semantics of a propositional logic of imagination ascriptions for single agents has been motivated and introduced. The main feature of this approach is that 'to imagine' is seen as an action verb. It is assumed that subjects often and typically have direct control over their imaginations. This idea suggests to model imagination ascriptions by using the modal logic of agency, more concretely the *seeing-to-it-that* (STIT) theory developed by N.D. Belnap, M. Perloff, M. Xu, F. von Kutschera, and J. Horty, cf. [2, 6] and the references given there. In [18] it is argued that imagination is closed under valid equivalence, and the suggested STIT imagination logic is compared with Niiniluoto's [10] semantics of imagination ascriptions. The semantics of the agentive imagination operator merges the semantics of the *dstit*-operator from STIT theory and the interpretation of the necessity operator in the smallest classical modal logic **E**, see [4]. As a result, imagination is closed under valid equivalence. In this framework, propositional content is represented by sets of moment/history-pairs. The point is not that propositions are necessarily to be identified with sets of moment/history-pairs, but that for imagination we have closure under some criterion of propositional identity. This conception agrees with understanding imagination in terms of mental pictures that are assumed to comprise propositional content. If an agent a imagines that B, and B is logically equivalent with C, then the mental picture that a creates in her or his act of imagining that B does not differ from the picture created by imagining that C, cf. [18].[1]

In the present paper, we provide two different kinds of proof systems for the semantically introduced logic of imagination: a Hilbert-style axiomatization and a tableau calculus. Both systems offer algorithmic insight into the STIT imagination logic. We shall not repeat the motivating considerations from [18] but otherwise keep the paper self-contained. Whereas in [18] the focus is on introducing the semantics of the modal imagination operators, in the present paper we include other modal operators as well.

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[1]Imagination differs from knowledge not just in being agentive. Knowledge is factive, imagination clearly is not. Whilst justification is generally seen as a necessary condition of possessing knowledge, this is clearly not the case for imagination. Epistemic extensions of STIT logic have been presented, for example, in [3, 16], for a survey see [14].

2 Semantics

We first present the language of propositional STIT imagination logic and assume a countably infinite set Var of propositional variables, the connectives $\neg$ (negation), $\wedge$ (conjunction),[2] and the following set of modalities:

1. SA understood as 'A is settled true'; the dual modality is PA understood as 'A is possible'.

2. $[c]_a A$ understood as 'agent a $cstit$-realizes A'; another action modality, namely, $[d]_a A$ to be read as 'agent a $dstit$-realizes A', is in this setting a defined one with the following definition: $[c]_a A \wedge \neg SA$.

3. $I_a A$ understood as 'agent a imagines that A'.

All the agent indices are assumed to stand for pairwise different agents. For these modalities we assume the following 'stit-plus-neighbourhood' semantics defined in [18].

An imagination model is a tuple $\mathcal{M} = \langle Tree, \leq, Ag, Choice, \{N_a \mid a \in Ag\}, V \rangle$, where:

-   $Tree$ is a non-empty set of moments, and $\leq$ is a partial order on $Tree$ such that

$$\forall m_1, m_2 \exists m (m \leq m_1 \wedge m \leq m_2),$$

and

$$\forall m_1, m_2, m((m_1 \leq m \wedge m_2 \leq m) \rightarrow (m_1 \leq m_2 \vee m_2 \leq m_1)).$$

-   The set $History$ of all histories of $\mathcal{M}$ is then just the set of all maximal $\leq$-chains in $Tree$. A history h is said to pass through a moment m iff $m \in h$. The set of all histories passing through $m \in Tree$ is denoted by H_m.

-   Ag is a finite set of all agents acting in $Tree$ and is assumed to be disjoint from all the other items in $\mathcal{M}$. The set Ag is fixed for the language of STIT imagination logic and does not vary from model to model.

-   $Choice$ is a function defined on the set $Tree \times Ag$, such that for an arbitrary $(m, a) \in Tree \times Ag$, the value of this function, that is to say $Choice(m, a)$ (more commonly denoted $Choice_a^m$) is a partition of H_m. If $h \in H_m$, then $Choice_a^m(h)$ denotes the element of $Choice_a^m$ to which h belongs. In the special case when we have $Choice_a^m = \{H_m\}$, it is said that the agent a has a *vacuous* choice at the moment m. In our models, $Choice$ is assumed to satisfy the following two restrictions:

    -   "No choice between undivided histories": for arbitrary $m \in Tree, a \in Ag, e \in Choice_a^m$, and $h, h' \in H_m$:

    $$(h \in e \wedge \exists m'(m < m' \wedge m' \in h \cap h')) \rightarrow h' \in e.$$

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[2]The other Boolean connectives are defined as usual.

- "Independence of agents". If f is a function defined on Ag such that $\forall a \in Ag\ (f(a) \in Choice_a^m)$, then $\bigcap_{a \in Ag} f(a) \neq \varnothing$.
- The set of moment/history-pairs in $\mathcal{M}$, that is to say, the set

$$MH(\mathcal{M}) = \{(m, h) \mid m \in Tree, h \in H_m\}$$

  is then to be used as a set of points, where formulas are evaluated.
- For every $a \in Ag$, we have $N_a \colon MH(\mathcal{M}) \to 2^{(2^{MH(\mathcal{M})})}$. N_a is thus a neighbourhood function, defining, for every moment history pair (m, h) the set of propositions imagined by the agent a at the moment m in history h.
- V is an evaluation function for atomic sentences, i.e., $V \colon Var \to 2^{MH(\mathcal{M})}$.

Definition 1 The relation of satisfaction of sentences in the above defined language at moment/history-pairs in an imagination model $\mathcal{M}$ is then defined inductively as follows:

$\mathcal{M}, (m, h) \vDash p \Leftrightarrow (m, h) \in V(p)$, for atomic p;

$\mathcal{M}, (m, h) \vDash (A \wedge B) \Leftrightarrow \mathcal{M}, (m, h) \vDash A$ and $\mathcal{M}, (m, h) \vDash B$;

$\mathcal{M}, (m, h) \vDash \neg A \Leftrightarrow \mathcal{M}, (m, h) \nvDash A$;

$\mathcal{M}, (m, h) \vDash SA \Leftrightarrow \forall h' \in H_m (\mathcal{M}, (m, h') \vDash A)$;

$\mathcal{M}, (m, h) \vDash [c]_a A \Leftrightarrow \forall h' \in Choice_a^m(h)(\mathcal{M}, (m, h') \vDash A)$;

$\mathcal{M}, (m, h) \vDash I_a A \Leftrightarrow \forall h' \in Choice_a^m(h)$

$\qquad (\{(m', h''') \in MH(\mathcal{M}) \mid \mathcal{M}, (m', h''') \vDash A\} \in N_a((m, h')))$ and

$\qquad \exists h'' \in H_m(\{(m', h''') \in MH(\mathcal{M}) \mid \mathcal{M}, (m', h''') \vDash A\} \notin N_a((m, h''')))$.

If by $\|A\|_{\mathcal{M}}$ we denote the set $\{(m, h) \in MH(\mathcal{M}) \mid \mathcal{M}, (m, h) \vDash A\}$, i.e., the truth set of A in model $\mathcal{M}$, then the above satisfaction clause for formulas $I_a A$ can more compactly be rewritten as:

$$\mathcal{M}, (m, h) \vDash I_a A \Leftrightarrow$$

$\forall h' \in Choice_a^m(h)\ (\|A\|_{\mathcal{M}} \in N_a((m, h')))$ and $\exists h'' \in H_m\ (\|A\|_{\mathcal{M}} \notin N_a((m, h'')))$.

A picture may help to familiarize oneself with the semantics, see Fig. 1.

Definition 2 A formula is said to be valid in an imagination model $\mathcal{M}$ iff A is satisfied by every moment/history-pair in $\mathcal{M}$, and A is said to be valid (simpliciter) iff A is valid in every imagination model.

The language of STIT imagination logic, as presented here, does not contain temporal operators. Therefore, it may seem natural to use the alternative, atemporal Kripke STIT semantics developed in [1] for single agents (and in [9] for collective agents) and to extend it by neighbourhood functions. This is a legitimate move, but we refrain from making it because we view the branching-time structures augmented by agent choice functions as the intended semantics of STIT theory.

Note that in [18] it has been pointed out that $I_a A$ can be defined as $[d]_a \square_a A$ (notation adjusted), where $\square_a A$ is true in a model $\mathcal{M}$ at moment/history-pair (m, h)
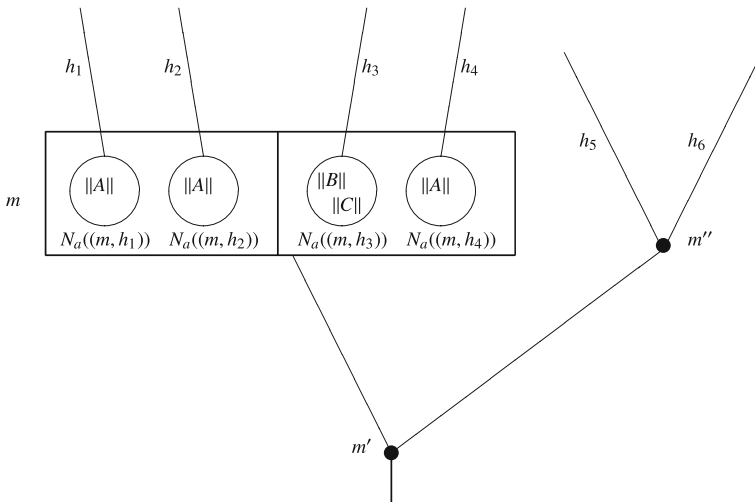
Fig. 1 Two choice-cells of the single agent a at moment m; $\{h_1, h_2, h_3, h_4\} = H_m$; $(m, h_1) \vDash I_a A$ and $(m, h_2) \vDash I_a A$; $\|A\| \notin N_a((m, h_3))$; $(m, h_4) \nvDash I_a A$

iff $\|A\|_{\mathcal{M}} \in N_a(m, h)$. The latter can be understood as saying that the proposition expressed by A belongs to the set of propositions contained in s's mental image at (m, h) in model $\mathcal{M}$. There are, however, reasons to take the operators I_a as *primitive* connectives. T. Williamson [13] considers imagination as working in voluntary and *involuntary* mode. Treating I_a as primitive guards against misinterpreting $[d]_a \square_a A$ as saying that agent a sees to it that a imagines in involuntary mode that A, which could be made true by performing some indirect actions, such as taking drugs, so as to make sure that $\square_a A$ is the case. If imagining is seen as a homogeneous activity, it seems natural to consider the non-normal modal operator $\square_a$ as incorporated in the primitive agentive operator I_a.[3]

3 Axiomatization

For the semantically presented logic we propose the following axiomatization:

(A0)   Propositional tautologies.
(A1) S is an $S5$ modality.
(A2)   For every $a \in Ag$, $[c]_a$ is an $S5$ modality.
(A3) $SA \rightarrow [c]_a A$ for every $a \in Ag$.
(A4) $(P[c]_{a_1} A_1 \wedge \ldots \wedge P[c]_{a_n} A_n) \rightarrow P([c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n)$, provided that all the $a_1 \ldots a_n$ are pairwise different.
(A5) $I_a A \rightarrow ([c]_a I_a A \wedge \neg S I_a A)$ for every $a \in Ag$.

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[3]In [16] (notation adjusted), $[d]_a B_a A$ is suggested as a formalization of "agent a forms the implicit belief that A", where B_a is a KD45 modal operator. In this case, it is desired not to exclude the option of understanding $[d]_a B_a A$ as a statement of *indirect* belief formation.

Rules are as follows:

(R1)   Modus ponens.
(R2)   From A infer SA.
(R3)   From $A \leftrightarrow B$ infer $I_a A \leftrightarrow I_a B$ for every $a \in Ag$.

Note Observe that (A4) actually is a family of schemas parametrized by n. The above axiomatization is just the axiomatization of *dstit* logic proposed by Ming Xu plus the axiomatization of the logic of I_a as a minimal neighbourhood modal system for the smallest classical (or congruential) modal logic **E** plus the special axiom (A5) stating the agentive character of the imagination operator. Note also that the converse of (A5) easily follows from (A2), so that we actually have a biconditional here.

The above axiomatization can easily be show to be sound with respect to the class of all imagination models. Our aim now is to get a strong completeness theorem for this system L with respect to the above semantics, in the following form: if Θ is an L-consistent set of sentences, then Θ has a model.

In what follows we will always use 'consistency' to mean 'L-consistency' and we let $\vdash$ stand for the relation of L-derivability.

In order to get the main theorem, we use the technique of canonical models, which is an adaptation of the corresponding techniques for the two respective parts of our system as mentioned in the Note above. In particular, we draw on [2, ch. 17] in many matters relevant to the pure STIT part of the following construction.

More precisely, we let W to be the set of all L-maxiconsistent sets of sentences and we denote the members of W as w, w', w_1 etc. We set wRw' iff $\{A \mid SA \in w\} \subseteq w'$, and we set $w \simeq_a w'$ iff $\{A \mid [c]_a A \in w\} \subseteq w'$. By standard modal logic, (A1) and (A2) ensure that all these relations are relations of equivalence; moreover, (A3) ensures that $\simeq_a \subseteq R$ for every $a \in Ag$.

Indeed, let $w \simeq_a w'$ and let $SA \in w$. By (A3) and maxiconsistency of w, we get $[c]_a A \in w$, whence by $w \simeq_a w'$ we get that $A \in w'$. Since A was arbitrary, this means that wRw'.

In the sequel, we will be denoting equivalence classes of W with respect to R by X, X', X_1, etc. The set of all such equivalence classes will be denoted by Ξ. When restricted to an arbitrary $X \in \Xi$, the relation R turns into the universal relation, but relations of the form $\simeq_a$ can remain non-trivial equivalences breaking X up into several equivalence classes. We will denote the family of equivalence classes corresponding to $\simeq_a \restriction X$ by $E(X, a)$.

For the construction of the canonical model below, we will need to pick among the elements of W a maxiconsistent set which is the only element of the respective R- and $\sim_a$-equivalence classes. In order to do this, we consider maxiconsistent sets extending the following set of formulas:

$$\Sigma = \{\neg p \mid p \in Var\} \cup \{SA \leftrightarrow A \mid \text{ for arbitrary } A\} \cup \{[c]_a A \leftrightarrow A \mid \text{ for arbitrary } A\}.$$

The following facts are worth noting:

(F1)   There exists exactly one element in W, which extends Σ. We will denote this element by **w**. Indeed, one easily sees that Σ pre-determines every non-modal formula by fixing the literals. The modalities S and $[c]_a$ are then just vacuous

in virtue of the definition of Σ. Finally, every maxiconsistent set extending Σ will have to contain $\neg I_a A$ for every formula A and every $a \in Ag$. For suppose otherwise. Then for some $w \in W$ such that $\Sigma \subseteq w$, for some formula A and for some $a \in Ag$ we will have $I_a A \in w$. Then, by (A5) and maxiconsistency of w we will get $\neg S I_a A \in w$. Therefore, by definition of Σ and maxiconsistency of w, we will get $\neg I_a A \in w$, which contradicts the assumption that $w \in W$. Therefore, the statements beginning with I_a-modalities are also fixed for every $w \in W$, for which $\Sigma \subseteq w$. It is also easy to see that such a maxiconsistent w extending Σ must exist, since Σ itself is obviously consistent[4]

(F2)   It follows from the definitions of Σ and R that the R-equivalence set containing $\mathbf{w}$ contains $\mathbf{w}$ only. We will denote this equivalence set by $\mathbf{X}$.

We now proceed to the definition of our canonical model, which will be based on a tree of depth 2. First, we choose[5] an element $0 \notin \Xi \cup W$ and define our set of moments:

$$Tree = \{0\} \cup \Xi \cup W.$$

We then set the following partial order on $Tree$. For arbitrary $x, y \in Tree$ we have $x \leq y$ iff $x = y$, or $y \in x$ or $x = 0$. This allows for a simple description of the set of histories in our frame. Every history turns out to have the form $h_w = \langle 0, X, w \rangle$, where $X \in \Xi$ and $w \in X$. Thus, our set of histories is in one-to-one correspondence with W.

Thirdly, we define the choice function. It assigns a vacuous choice to every agent at every moment m, if $m \notin \Xi$. That is to say, the only choice of every agent at every such moment will be just the set of all histories passing through this moment. Otherwise, i.e. for the case when $m = X \in \Xi$, we define the choice function as follows:

$$Choice_a^X = \{H \mid \exists e \in E(X, a)(H = \{h_w \mid w \in e\})\}.$$

Next, we need to define the imagination neighbourhoods. We do this in the following way. $N_a((m, h)) = \varnothing$ for every $a \in Ag$ and every $m \notin \Xi$. For the case when $m = X \in \Xi$, we need one further auxiliary notion. For every sentence A we set $Ext(A)$ (read: 'the extension of A') to be $\{(X, h_w) \mid A \in w \in X\}$ if $A \notin \mathbf{w}$; otherwise we set

$$Ext(A) = \{(X, h_w) \mid A \in w \in X\} \cup \{(m, h_w) \mid m \notin \Xi \text{ and } m \in h_w\}.$$

Having defined the extensions, we set

$$N_a((X, h_w)) = \{Ext(A) \mid I_a A \in w\}$$

for arbitrary $w \in X \in \Xi$.

Finally, we define the evaluation function for variables in the following way:

$$V(p) = \{(X, h_w) \mid p \in w \in X \in \Xi\}.$$

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[4] Σ is satisfiable and thus consistent. Indeed, consider a model consisting of a single moment, where every agent has a vacuous choice, every set of imagination neighbourhoods is empty and every variable valuation is empty as well.

[5] We also assume, in view of the definition of $\leq$ below, that 0 is not an element of any element of $\Xi \cup W$.

We need to show that the canonical model $\mathcal{M}$ defined above is a model of our logic. The semantic restrictions are mostly seen to hold immediately; in particular, the no-choice-between-undivided-histories restriction holds because we only have undivided histories at the moment 0, where only vacuous choices are allowed. The only exception is the independence-of-agents restriction, which we treat below.

Lemma 1 (**On Independence**) *Let $m \in Tree$ and let f be a function on Ag such that $\forall a \in Ag \ (f(a) \in Choice_a^m)$. Then $\bigcap_{a \in Ag} f(a) \neq \varnothing$.*

Proof If $m \notin \Xi$, then the statement of the lemma is obvious, since every agent will have a vacuous choice. We treat the case when $m = X \in \Xi$. Consider a function f as described in the lemma. For every $f(a)$ we fix $e_{f(a)} \in E(X, a)$ such that $f(a) = \{h_w \mid w \in e_{f(a)}\}$ and we fix, further, an arbitrary $w_{f(a)} \in e_{f(a)}$. Since $e_{f(a)}$ is an $\simeq_a$-equivalence class, there is a set $\Gamma_{f(a)}$ of sentences of the form $[c]_a A$ shared by all the members of $e_{f(a)}$ and only those members. Also, since X is an R-equivalence class, there is a set Δ of sentences of the form SA shared by all (and only) members of X. Consider, then, the following set of sentences:

$$\Lambda = (\bigcup_{a \in Ag} \Gamma_{f(a)}) \cup \Delta.$$

We claim that Λ is consistent. Assume otherwise. In this case Λ contains a finite inconsistent subset. Given that S and $[c]_a$ are $S5$-modalities, we can assume that this inconsistent subset has the following elements:

$$SB, [c]_{a_1} A_1, \ldots, [c]_{a_n} A_n,$$

where all the $a_1 \ldots a_n$ are pairwise different (and moreover, $Ag = \{a_1 \ldots a_n\}$). We know, further, that for all $1 \leq i \leq n$ we have $SB, [c]_{a_i} A_i \in w_{f(a_i)}$. So, choose an arbitrary $w \in X$. For every $1 \leq i \leq n$ we have $w_{f(a_i)} R w$, therefore, we must also have $P[c]_{a_i} A_i \in w$ for every $1 \leq i \leq n$. Indeed, if it were otherwise, we would have $S\neg[c]_{a_i} A_i \in w$ since w is maxiconsistent. But then, given that $w R w_{f(a_i)}$, we would have $\neg[c]_{a_i} A_i \in w_{f(a_i)}$, a contradiction.

Thus, we have in fact that

$$P[c]_{a_1} A_1 \wedge \ldots \wedge P[c]_{a_n} A_n \in w,$$

therefore, by (A4), we also have

$$P([c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n) \in w.$$

This, in turn, means that the set

$$\{A \mid SA \in \Delta\} \cup \{[c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n\}$$

is consistent: otherwise, we would have that

$$\{A \mid SA \in \Delta\} \vdash \neg([c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n),$$

and, by standard modal $S5$-reasoning, that

$$\Delta \vdash S\neg([c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n),$$

which, given that $w \in X$ and hence $\Delta \subseteq w$, would mean inconsistency of w, a contradiction.

Therefore, we may choose an arbitrary maxiconsistent w' extending $\{A \mid SA \in \Delta\} \cup \{[c]_{a_1} A_1 \wedge \ldots \wedge [c]_{a_n} A_n\}$, and by the fact that this set contains $\{A \mid SA \in \Delta\}$ we know that wRw' and thus $w' \in X$ and further $SB \in w'$. This means that our finite subset in fact has a model and is not inconsistent. Therefore, since the finite set was arbitrary, Λ is consistent as well. Consider, then, an arbitrary maxiconsistent w'' extending Λ. Since $\Delta \subseteq w''$, we have $w'' \in X$, and since $\Gamma_{f(a)} \subseteq w''$ for arbitrary $a \in Ag$, we have $w'' \simeq_a w_{f(a)}$ for every such a. This means, in turn, that $w'' \in e_{f(a)}$ for every $a \in Ag$, and so $h_{w''} \in \bigcap_{a \in Ag} f(a) \neq \varnothing$. $\qquad \square$

By now, the only ingredient to be added is the Truth Lemma; we divide it into two parts as follows.

Lemma 2 (**Truth Lemma 1**) *Let $m \notin \Xi$ and $m \in h$. Then, for any sentence A, the following holds:*

$$\mathcal{M}, (m, h) \vDash A \Leftrightarrow A \in \mathbf{w}.$$

Proof We use induction on the construction of A. If $A = p \in Var$, then $A \notin \mathbf{w}$, and also $(m, h) \notin V(A)$, since $m \notin \Xi$. Therefore, $\mathcal{M}, (m, h) \nvDash A$.

The Boolean cases are then trivial.

If $A = SB$, then $\mathcal{M}, (m, h) \vDash A$ iff $\mathcal{M}, (m, h') \vDash B$ for every h' such that $m \in h'$ iff $A \in \mathbf{w}$ by induction hypothesis (since we have proved IH for arbitrary h going through m).

If $A = [c]_a B$, then $\mathcal{M}, (m, h) \vDash A$ iff $\mathcal{M}, (m, h') \vDash B$ for every h' such that $m \in h'$ and $h' \in Choice_a^m(h)$ iff $A \in \mathbf{w}$ by induction hypothesis (cf. the commentary on the previous case).

If $A = I_a B$, then $A \notin \mathbf{w}$ by (F1). We also have $\mathcal{M}, (m, h) \nvDash A$, since, given that $m \notin \Xi$, all the choices at m are vacuous. $\qquad \square$

Lemma 3 (**Truth Lemma 2**) *Let $X \in \Xi$ and $w \in X$. Then, for any sentence A, the following holds:*

$$\mathcal{M}, (X, h_w) \vDash A \Leftrightarrow A \in w.$$

Proof Again, we use induction on the construction of A. The atomic case we have by definition of V, and the Boolean cases are obvious. We consider the modal cases.

Let $A = SB$, and assume that $SB \in w$. Then take any $h_{w'}$ passing through X. In the context of $\mathcal{M}$ this means that $w' \in X$, which in turn means that wRw'. Therefore, we have $B \in w'$ and, by induction hypothesis, $\mathcal{M}, (X, h_{w'}) \vDash B$. Since $h_{w'}$ was arbitrary, this means that $\mathcal{M}, (X, h_w) \vDash SB$.

On the other hand, assume that $SB \notin w$. This means that the set

$$\alpha = \{C \mid SC \in w\} \cup \{\neg B\}$$

is consistent. Indeed, otherwise we would have

$$\{C \mid SC \in w\} \vdash B,$$

and further, by standard $S5$ reasoning

$$\{SC \mid SC \in w\} \vdash SB,$$

and so, given, maxiconsistency of w, we would have $SB \in w$, contrary to our assumption. Therefore, consider an arbitrary $w' \in W$ extending α. By definition, $w' \in X$, therefore $h_{w'}$ goes through X and we have, by induction hypothesis, that $\mathcal{M}, (X, h_{w'}) \nvDash B$.

Let $A = [c]_a B$, and let $[c]_a B \in w$. Then take any $h_{w'}$ such that $h_{w'} \in Choice_a^X(h_w)$. In the context of $\mathcal{M}$ this means that $w \simeq_a w'$. Therefore, we have $B \in w'$ and, by induction hypothesis, $\mathcal{M}, (X, h_{w'}) \vDash B$. Since $h_{w'}$ was arbitrary, this means that $\mathcal{M}, (X, h_w) \vDash [c]_a B$.

On the other hand, assume that $[c]_a B \notin w$. This means that the set

$$\beta = \{C \mid [c]_a C \in w\} \cup \{\neg B\}$$

is consistent. Indeed, otherwise we would have

$$\{C \mid [c]_a C \in w\} \vdash B,$$

and further, by standard $S5$ reasoning

$$\{[c]_a C \mid [c]_a C \in w\} \vdash [c]_a B,$$

and so, given, maxiconsistency of w, we would have $[c]_a B \in w$, contrary to our assumption. Therefore, consider an arbitrary $w' \in W$ extending β. By definition, $w' \simeq_a w$, and also $w' \in X$ given that $\simeq_a \subseteq R$. Therefore $h_{w'}$ goes through X and moreover $h_{w'} \in Choice_a^X(h_w)$. By induction hypothesis, we have that $\mathcal{M}, (X, h_{w'}) \nvDash B$, and so, putting all together, that $\mathcal{M}, (X, h_w) \nvDash [c]_a B$.

Let $A = I_a B$. First of all, note that by induction hypothesis and Lemma 2 we have the following identity:

$$Ext(B) = \{(m, h) \mid \mathcal{M}, (m, h) \vDash B\}. \tag{1}$$

Now, assume that $I_a B \in w$. Then, by (A5), we also have $[c]_a I_a B \in w$ and $\neg S I_a B \in w$. Take any $h_{w'}$ such that $h_{w'} \in Choice_a^X(h_w)$. In the context of $\mathcal{M}$ this means that $w \simeq_a w'$. Therefore, we have $I_a B \in w'$. By definition of N_a, this means that $Ext(B) \in N_a((X, h_{w'}))$. On the other hand, the fact that $\neg S I_a B \in w$ means that the set

$$\gamma = \{C \mid SC \in w\} \cup \{\neg I_a B\}$$

is consistent. Indeed, otherwise we would have

$$\{C \mid SC \in w\} \vdash I_a B,$$

and further, by standard $S5$ reasoning

$$\{SC \mid SC \in w\} \vdash S I_a B,$$

and so, given maxiconsistency of w, we would have $S I_a B \in w$, contrary to our assumption. Therefore, consider an arbitrary $w'' \in W$ extending γ. By definition,

$w'' \in X$ so that $h_{w''}$ goes through X, and we have $Ext(B) \notin N_a((X, h_{w''}))$ by definition of N_a.

Putting all this together, we get that, by Eq. 1, $\{(m, h) \mid \mathcal{M}, (m, h) \vDash B\} \in N_a((X.h_{w'}))$ for every $h_{w'} \in Choice_a^X(h_w)$ and $\{(m, h) \mid \mathcal{M}, (m, h) \vDash B\} \notin N_a((X, h_{w''}))$ for some $h_{w'}$ going through X. That is to say, we get that $\mathcal{M}, (X, h_w) \vDash I_a B$.

On the other hand, if $I_a B \notin w$, then, of course, $Ext(B) \notin N_a(X, h_w)$, and given the fact that $h_w \in Choice_a^X(h_w)$ and the identity (1), we get that $\mathcal{M}, (X, h_w) \nvDash I_a B$ immediately. $\qquad \square$

Now we are ready for the main result of this section.

Theorem 1 (**Completeness**) *Let Θ be a consistent set of sentences. Then Θ has a model.*

Proof Consider the model $\mathcal{M}$ defined above. By Lemma 1 and the reasoning given immediately before it, $\mathcal{M}$ is an imagination model. Take any maxiconsistent set w extending Θ and its corresponding R-equivalence class X. Then, by Lemma 3, we have $\mathcal{M}, (X, h_w) \vDash \Theta$. $\qquad \square$

We also get compactness of the STIT imagination logic axiomatized by L as a standard consequence of strong completeness.

4 Tableaux

We now devise a tableau proof system to implement the systematic search for countermodels to inferences. The tableau calculus we shall present is based on the tableau calculus for *dstit* logic in [17] (see also [15]) and the labelled tableau calculus for **E** in [7].

The tableau rules are utilized to process semantic information about imagination models, and we will use (i) expressions $h_i \lhd_a^m h_l$ to indicate that the histories h_i and h_l are choice-equivalent for agent a at moment m, (ii) statements $m_k \in h_k$ to express that moment m_k belongs to history h_k, (iii) expressions $m \prec m_k$ to state that moment m is earlier than moment m_k, and (iv) statements $\|A\| \in N_a((m, h))$ ($\|A\| \notin N_a((m, h))$) to express that the truth set of A belongs (does not belong) to $N_a((m, h))$. Moreover, it must be ensured that a model induced by a complete tableau satisfies the independence of agents condition and that $\lhd_a^m$ indeed designates an equivalence relation. To guarantee the latter properties, the structural tableau rules of Table 1 are assumed.

We annotate formulas by names of moment/history-pairs. If Δ is a set of formulas, then $\Delta^0 := \{A, (m, h_0) \mid A \in \Delta\}$. A tableau is a rooted tree; its nodes are sets of certain expressions. In this section we shall use '$\vdash$' not to denote L-derivability, but to form derivability statements (i.e., sequents) in the tableau calculus we are about to define. If $\Delta \vdash A$ is a sequent, then the root of the tableau for $\Delta \vdash A$ is $\Delta^0 \cup \{m \in h_0, m \prec m_0, m_0 \in h_0\} \cup \{\neg A, (m, h_0)\}$. To expressions from this

Table 1 Structural tableau rules

REF	SYM	TRAN	IND
$m \in h_i$	$h_i \lhd_a^m h_k$	$h_i \lhd_a^m h_k$	$h_{l_1} \lhd_m^{a_1} h_{l_1}$
$\downarrow$	$\downarrow$	$h_k \lhd_a^m h_l$	$\ldots h_{l_k} \lhd_{a_k}^m h_{l_k}$
$h_i \lhd_a^m h_i$	$h_k \lhd_a^m h_i$	$\downarrow$	$\downarrow$
		$h_i \lhd_a^m h_l$	$m \prec m_n$
			$m \in h_n,\ m_n \in h_n,$
			$h_{l_1} \lhd_m^{a_1} h_n \ \ldots h_{l_k} \lhd_m^{a_k} h_n$
			where n is a new natural number not already occurring on the tableau

root, decomposition rules and structural tableau rules can be applied to complete the tableau. A tableau is said to be *complete* iff each of its branches is complete. A branch is complete if there is no possibility to apply one more rule to expand this branch. A tableau branch is said to be *closed* iff there are expressions of the form $A, (m, h)$ and $\neg A, (m, h)$ on the branch or expressions of the form $\|A\| \in N_a((m, h))$ and $\|A\| \notin N_a((m, h))$. A closed branch is considered complete. A tableau is called *closed* iff all of its branches are closed, and it is called *open* if it is not closed.

The indices $i, k, l, \ldots$ used in the tableau rules are natural numbers, and a *new* index is the smallest natural number not already used in the tableau. In models constructed from open tableau branches, we shall interpret an agent index a by a itself. Note that it may happen that a rule is applied to an expression from a tableau node more than once if the rule requires additional input and suitable additional input is introduced at later nodes. If, for instance, the decomposition rule for formulas SA is applied to the expressions $SA, (m, h_i),\ m \in h_k$, and later on the branch a new expression $m \in h_l$ is introduced, then the rule has to be applied also to $SA, (m, h_i),\ m \in h_l$.

The tableau calculus for STIT imagination logic consists of the tableau rules presented in the Tables 1 and 2. Syntactic consequence is then defined as follows.

Definition 3 Let $\Delta \cup \{A\}$ be a set of formulas. $\Delta \vdash A$ ('A is derivable from Δ') iff there exists a closed and complete tableau for $\Delta^0 \cup \{m \in h_0, m \prec m_0, m_0 \in h_0\} \cup \{\neg A, (m, h_0)\}$.

If there is a complete and open branch on a tableau with root $\Delta^0 \cup \{m \in h_0, m \prec m_0, m_0 \in h_0\} \cup \{\neg A, (m, h_0)\}$, then there exists a countermodel to the sequent $\Delta \vdash A$. The construction of a countermodel is guided by the open branch, but there are limits to directly reading off a countermodel. An open branch may contain an expression $\|A\| \in N_a((m, h))$ and thereby provide the information that in the countermodel the truth set of A is a neighbourhood of agent a at moment/history-pair (m, h), but that does not fully specify the countermodel.

Examples of tableaux The examples in Table 3 show open and complete tableaux from which a countermodel can be read off to some extent. Then in Table 4 we present two examples of closed tableaux. In the first example from Table 4 the

Table 2 Decomposition rules for STIT imagination logic

$$\neg\neg A, (m, h) \qquad (A \wedge B), (m, h) \qquad \neg(A \wedge B), (m, h)$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \swarrow \qquad \searrow$$
$$A, (m, h) \qquad A, (m, h), \ B, (m, h) \qquad \neg A, (m, h) \qquad \neg B, (m, h)$$

$$S A, (m, h_i), \qquad\qquad \neg S A, (m, h_i)$$
$$m \in h_k \qquad\qquad\qquad \downarrow$$
$$\downarrow \qquad\qquad\qquad \neg A, (m, h_k),$$
$$A, (m, h_k) \qquad\qquad m \in h_k, \ m_k \in h_k,$$
$$m \prec m_k, \ \text{where } k \text{ is new}$$

$$[c]_a A, (m, h_i), \qquad\qquad \neg[c]_a A, (m, h_i)$$
$$h_i \vartriangleleft_a^m h_k \qquad\qquad\qquad \downarrow$$
$$\downarrow \qquad\qquad\qquad \neg A, (m, h_k)$$
$$A, (m, h_k) \qquad\qquad m \in h_k, \ m_k \in h_k,$$
$$h_i \vartriangleleft_a^m h_k,$$
$$m \prec m_k, \ \text{where } k \text{ is new}$$

$$I_a A, (m, h_i), \qquad\qquad\qquad\qquad \neg I_a A, (m, h_i)$$
$$h_i \vartriangleleft_a^m h_k \qquad\qquad\qquad\qquad\qquad \swarrow \qquad\qquad \searrow$$
$$\downarrow \qquad\qquad\qquad \|A\| \in N_a((m, h_l)), \qquad \|A\| \notin N_a((m, h_k)),$$
$$\|A\| \in N_a((m, h_k)), \ m \prec m_l, \qquad \text{for every } h_l \text{ with} \qquad m \in h_k, \ m_k \in h_k,$$
$$m \in h_l, \ m_l \in h_l, \qquad m \in h_l \text{ on the branch} \qquad m \prec m_k, \ h_i \vartriangleleft_a^m h_k,$$
$$\|A\| \notin N_a((m, h_l)), \ \text{where } l \text{ is new} \qquad\qquad\qquad \text{where } k \text{ is new}$$

$$I_a A, (m, h_i),$$
$$\neg I_a B, (m, h_i)$$
$$\swarrow \qquad\qquad\qquad \searrow$$
$$m \prec m_k, \ m_k \in h_k, \ m \in h_k \qquad m \prec m_l, \ m_l \in h_l, \ m \in h_l$$
$$A, (m_k, h_k), \ \neg B, (m_k, h_k), \qquad \neg A, (m_l, h_l), \ B, (m_l, h_l),$$
$$\text{where } k \text{ is new} \qquad\qquad \text{where } l \text{ is new}$$

branching two-premises tableau rule for the imagination operator is applied. In the second example from Table 4 we prove an instance of axiom (A5) and apply the other decomposition rules for the imagination operator.

Note that the open branches in the first two examples do not provide full information concerning the construction of a countermodel. The countermodel consists of three distinct moments of time m, m_0, and m_1. The moments m_0 and m_1 are later than moment m and are assumed to be incomparable with respect to the temporal order. There are two histories; history h_0 passes through m and m_0, and history h_1 passes through m and m_1. In the first example the induced model is such that the truth set of $p \to p$, i.e., the set of all moment/history-pairs, belongs to $N_a((m, h_0))$ but not to $N_a((m, h_1))$. The remaining features of the countermodel are left unspecified. In both examples the open tableau does not give any information concerning the evaluation of p and thus the evaluation of p is arbitrary. In the second example the induced model is such that the truth set of $I_a p$ belongs to $N_a((m, h_0))$ but not to $N_a((m, h_1))$. $\|I_a p\|$ = $\{(m^*, h^*) \mid \forall h' \in Choice_a^{m^*}(h^*)(\|p\| \in N_a((m^*, h'))) \text{ and } \exists h'' \in H_{m^*}(\|p\| \notin N_a((m^*, h'')))\}$. Since no information about $\|p\|$ is provided on the open branch,

Table 3 Examples of open tableaux

$\varnothing \nvdash \neg I_a(p \to p), (m, h_0):$	$\varnothing \nvdash \neg I_a I_a p, (m, h_0):$
$\neg\neg I_a(p \to p), (m, h_0)$	$\neg\neg I_a I_a p, (m, h_0)$
$m \in h_0, m \prec m_0, m_0 \in h_0$	$m \in h_0, m \prec m_0, m_0 \in h_0$
$\downarrow$	$\downarrow$
$I_a(p \to p), (m, h_0)$	$I_a I_a p, (m, h_0)$
$\downarrow$	$\downarrow$
$h_0 \lhd_a^m h_0$	$h_0 \lhd_a^m h_0$
$\downarrow$	$\downarrow$
$\|p \to p\| \in N_a((m, h_0))$	$\|I_a p\| \in N_a((m, h_0))$
$m \prec m_1, m \in h_1, m_1 \in h_1$	$m \prec m_1, m \in h_1, m_1 \in h_1$
$\|p \to p\| \notin N_a((m, h_1))$	$\|I_a p\| \notin N_a((m, h_1))$
$\downarrow$	$\downarrow$
$h_1 \lhd_a^m h_1$	$h_1 \lhd_a^m h_1$

we may set $\|I_a p\| = \varnothing$. Thus, in the induced countermodel $\varnothing \in N_a((m, h_0))$ and $\varnothing \notin N_a((m, h_1))$.

Note that in the second example from Table 4 the final node of the leftmost branch is obtained by applying the rule for negated imagination operators to $\neg I_a p, (m, h_1)$ and that the final nodes of the other two branches are obtained by applying the rule for non-negated imagination operators to $I_a p, (m, h_0)$. Note also that the rule TRAN gives rise to infinite tableaux, cf. Example 3.4.7 in [12].
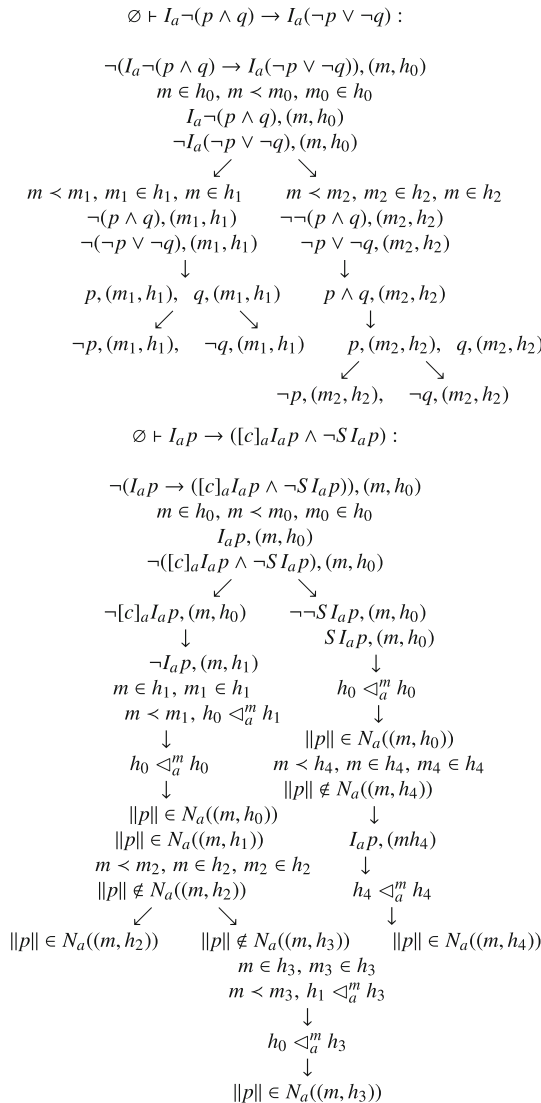
Definition 4 Let $\mathcal{M} = \langle Tree, \leq, Ag, Choice, \{\overline{N}_a \mid a \in Ag\}, V \rangle$ be an imagination model and let b be a tableau branch. The model $\mathcal{M}$ is said to be faithful to b iff there exists a function, $f : S \to MH(\mathcal{M})$, where $S = \{(m_k, h_i) \mid m_k \in h_i \text{ occurs on } b\} \subseteq M \times H$ with $M = \{m_k \mid m_k \text{ occurs on } b\}$, $H = \{h_k \mid h_k \text{ occurs on } b\}$, such that the following conditions hold:

1. For every expression $A, (m, h)$ on b, it holds that $\mathcal{M}, f((m, h)) \vDash A$.
2. $(\forall h, h' \in H) (\forall m \in M) \texttt{left}(f((m, h))) = \texttt{left}(f((m, h')))$ and $(\forall m, m' \in M) (\forall h \in H) \texttt{right}(f((m, h))) = \texttt{right}(f((m', h)))$, where $\texttt{left}$ and $\texttt{right}$ are the left and right projection functions. Thus, it is possible to define two auxiliary functions related to f, $\pi_1 : M \to Tree$ and $\pi_2 : H \to History$ by requiring that for $m \in M$, $\pi_1(m) = \overline{m}$, if $f((m, \ldots)) = (\overline{m}, \ldots)$, and for $h \in H$, $\pi_2(h) = \overline{h}$, if $f((\ldots, h)) = (\ldots, \overline{h})$.
3. If $m_i \in h_k$ occurs on b, then $\pi_2(h_k) \in H_{\pi_1(m_i)}$.
4. If $h_i \lhd_a^m h_k$ occurs on b, then $\pi_2(h_k) \in Choice_a^{\pi_1(m)}(\pi_2(h_i))$.
5. If $\|A\| \in N_a((m, h))$ occurs on b, then $\|A\|_{\mathcal{M}} \in \overline{N}_a(f((m, h)))$.

The function f is said to show that $\mathcal{M}$ is faithful to branch b.

Lemma 4 *Let $\mathcal{M} = \langle Tree, \leq, Ag, Choice, \{\overline{N}_a \mid a \in Ag\}, V \rangle$ be an imagination model, and let b be a tableau branch. If $\mathcal{M}$ is faithful to b and a tableau rule is*

Table 4 Examples of closed tableaux

$$\varnothing \vdash I_a\neg(p \wedge q) \rightarrow I_a(\neg p \vee \neg q) :$$

$$\neg(I_a\neg(p \wedge q) \rightarrow I_a(\neg p \vee \neg q)), (m, h_0)$$
$$m \in h_0,\ m \prec m_0,\ m_0 \in h_0$$
$$I_a\neg(p \wedge q), (m, h_0)$$
$$\neg I_a(\neg p \vee \neg q), (m, h_0)$$

$$m \prec m_1,\ m_1 \in h_1,\ m \in h_1 \qquad m \prec m_2,\ m_2 \in h_2,\ m \in h_2$$
$$\neg(p \wedge q), (m_1, h_1) \qquad\quad \neg\neg(p \wedge q), (m_2, h_2)$$
$$\neg(\neg p \vee \neg q), (m_1, h_1) \qquad\quad \neg p \vee \neg q, (m_2, h_2)$$

$$p, (m_1, h_1),\quad q, (m_1, h_1) \qquad p \wedge q, (m_2, h_2)$$

$$\neg p, (m_1, h_1),\quad \neg q, (m_1, h_1) \qquad p, (m_2, h_2),\quad q, (m_2, h_2)$$

$$\neg p, (m_2, h_2),\quad \neg q, (m_2, h_2)$$

$$\varnothing \vdash I_a p \rightarrow ([c]_a I_a p \wedge \neg S I_a p) :$$

$$\neg(I_a p \rightarrow ([c]_a I_a p \wedge \neg S I_a p)), (m, h_0)$$
$$m \in h_0,\ m \prec m_0,\ m_0 \in h_0$$
$$I_a p, (m, h_0)$$
$$\neg([c]_a I_a p \wedge \neg S I_a p), (m, h_0)$$

$$\neg[c]_a I_a p, (m, h_0) \qquad\qquad \neg\neg S I_a p, (m, h_0)$$
$$\downarrow \qquad\qquad\qquad\qquad S I_a p, (m, h_0)$$
$$\neg I_a p, (m, h_1)$$
$$m \in h_1,\ m_1 \in h_1 \qquad\qquad h_0 \vartriangleleft_a^m h_0$$
$$m \prec m_1,\ h_0 \vartriangleleft_a^m h_1$$
$$\downarrow \qquad\qquad\qquad \|p\| \in N_a((m, h_0))$$
$$h_0 \vartriangleleft_a^m h_0 \qquad\qquad m \prec h_4,\ m \in h_4,\ m_4 \in h_4$$
$$\downarrow \qquad\qquad\qquad \|p\| \notin N_a((m, h_4))$$
$$\|p\| \in N_a((m, h_0)) \qquad\qquad\qquad \downarrow$$
$$\|p\| \in N_a((m, h_1)) \qquad\qquad\qquad I_a p, (m h_4)$$
$$m \prec m_2,\ m \in h_2,\ m_2 \in h_2 \qquad\qquad \downarrow$$
$$\|p\| \notin N_a((m, h_2)) \qquad\qquad\qquad h_4 \vartriangleleft_a^m h_4$$

$$\|p\| \in N_a((m, h_2)) \qquad \|p\| \notin N_a((m, h_3)) \qquad \|p\| \in N_a((m, h_4))$$
$$m \in h_3,\ m_3 \in h_3$$
$$m \prec m_3,\ h_1 \vartriangleleft_a^m h_3$$
$$\downarrow$$
$$h_0 \vartriangleleft_a^m h_3$$
$$\downarrow$$
$$\|p\| \in N_a((m, h_3))$$

applied to b, then the application produces at least one extension b' of b, such that $\mathcal{M}$ is faithful to b'.

Proof Assume that f is a function that shows $\mathcal{M}$ to be faithful to b. We have to consider every tableau rule. If extended branches are obtained by applying one of the rules for $\neg\neg A$, $(A \wedge B)$ or $\neg(A \wedge B)$, then obviously f shows $\mathcal{M}$ to be faithful to at least one extension b' of b.

Suppose the rule for formulas SA is applied to $SA, (m, h_i)$. Then branch b' extends branch b by $A, (m, h_k)$ for all $m \in h_k$ on b. Since f is faithful to b, we have $\mathcal{M}, f((m, h_i)) \vDash SA$. By the definition of satisfaction, it holds that $\mathcal{M}, (\pi_1(m), \bar{h}) \vDash A$ for all $\bar{h} \in H_{\pi_1(m)}$. Since $\pi_2(h_k) \in H_{\pi_1(m)}$, we have $\mathcal{M}, f((m, h_k)) \vDash A$ for all h_k with $m \in h_k$ on b and f shows $\mathcal{M}$ to be faithful to b'.

Suppose now that the rule for $\neg SA$ is applied to $\neg SA, (m, h_i)$, so that b is extended by $m \prec m_k, m \in h_k, m_k \in h_k$, and $\neg A, (m, h_k)$, for a new index k. Since f shows $\mathcal{M}$ to be faithful to b, we have $\mathcal{M}, f((m, h_i)) \vDash \neg SA$. Thus, there is $\bar{h} \in H_{\pi_1(m)}$ with $\mathcal{M}, (\pi_1(m), \bar{h}) \vDash \neg A$. We define f' to be the same function as f, except that $f'((m, h_k)) = (\pi_1(m), \bar{h})$ and $f'((m_k, h_k)) = (\overline{m}, \bar{h})$ for some $\overline{m} \in \bar{h}$ with $\pi_1(m) \leq \overline{m}$. Clearly, the function f' satisfies Condition 2 of Definition 4, and the functions π_1, π_2 of f' are appropriately expanded, i.e., $\pi_2(h_k) = \bar{h}$ and $\pi_1(m_k) = \overline{m}$. Then $\mathcal{M}, f'((m, h_k)) \vDash \neg A, \pi_2(h_k) \in H_{\pi_1(m)}$, and $\pi_2(h_k) \in H_{\pi_1(m_k)}$. The function f' shows $\mathcal{M}$ to be faithful to the extended branch b'.

Next, suppose the rule for $[c_a]A$ is applied to $[c_a]A, (m, h_i)$. Then we obtain b' as an extension of b by $A, (m, h_k)$ for all $h_i \lhd_a^m h_k$ on b. Since f is faithful to b, we have $\mathcal{M}, f((m, h_i)) \vDash [c_a]A$. Then, by the definition of satisfaction, it holds that $\mathcal{M}, (\pi_1(m), \bar{h}) \vDash A$ for all $\bar{h} \in Choice_a^{\pi_1(m)}(\pi_2(h_i))$. Since $\pi_2(h_k) \in Choice_a^{\pi_1(m)}(\pi_2(h_i))$, we have $\mathcal{M}, f((m, h_k)) \vDash A$ for all h_k with $h_i \lhd_a^m h_k$ on b and f shows $\mathcal{M}$ to be faithful to b'.

Assume that the rule for $\neg[c_a]A$ is applied to $\neg[c_a]A, (m, h_i)$, so that b is extended by $m \prec m_k, m \in h_k, m_k \in h_k, h_i \lhd_a^m h_k$, and $\neg A, (m, h_k)$, for a new index k. Since f shows $\mathcal{M}$ to be faithful to b, we have $\mathcal{M}, f((m, h_i)) \vDash \neg[c_a]A$. Thus, there is $\bar{h} \in Choice_a^{\pi_1(m)}(\pi_2(h_i))$ with $\mathcal{M}, (\pi_1(m), \bar{h}) \vDash \neg A$. We define f' to be the same function as f, except that $f'((m, h_k)) = (\pi_1(m), \bar{h})$ and $f'((m_k, h_k)) = (\overline{m}, \bar{h})$ for some $\overline{m} \in \bar{h}$ with $\pi_1(m) \leq \overline{m}$. The auxiliary functions π_1, π_2 of f' are appropriately expanded. Then $\mathcal{M}, f'((m, h_k)) \vDash \neg A, \pi_2(h_k) \in H_{\pi_1(m_k)}$, and $\pi_2(h_k) \in H_{\pi_1(m)}$ (since $\pi_2(h_k) = \bar{h} \in Choice_a^{\pi_1(m)}(\pi_2(h_i)) \subseteq H_{\pi_1(m)}$). The function f' shows $\mathcal{M}$ to be faithful to the extended branch b'.

Suppose that the rule for $I_a A$ is applied to $I_a A, (m, h_i)$. Then the branch b' is obtained as an extension of b by $\|A\| \in N_a((m, h_k))$ for all $h_i \lhd_a^m h_k$ occurring on b and by $m \prec m_l, m \in h_l, m_l \in h_l$, and $\|A\| \notin N_a((m, h_l))$ for a new index l. Since f is faithful to b, we have $\mathcal{M}, f((m, h_i)) \vDash I_a A$. This means that $\forall \bar{h} \in Choice_a^{\pi_1(m)}(\pi_2(h_i)) (\|A\|_\mathcal{M} \in \overline{N}_a((\pi_1(m), \bar{h})))$ and $\exists \bar{h}' \in H_{\pi_1(m)}$ $(\|A\|_\mathcal{M} \notin \overline{N}_a((\pi_1(m), \bar{h}')))$. Since $\pi_2(h_k) \in Choice_a^{\pi_1(m)}(\pi_2(h_i))$, we have $\|A\| \in \overline{N}_a((\pi_1(m), \pi_2(h_k)))$ for all $h_i \lhd_a^m h_k$ on the branch. We define f' to be exactly as f, except that $f'((m, h_l)) = (\pi_1(m), \bar{h}')$ and $f'((m_l, h_l)) = (\overline{m}, \bar{h}')$ for some $\overline{m} \in \bar{h}'$ with $\tau_1(m) \leq \overline{m}$. The auxiliary functions π_1, π_2 of f' are appropriately expanded. Then $\pi_2(h_l) \in H_{\pi_1(m)}$ (since $\pi_2(h_l) = \bar{h}'$ and $\bar{h}' \in H_{\pi_1(m)}$) and $\pi_2(h_l) \in H_{\pi_1(m_l)}$ (since $\pi_1(h_l) = \overline{m}$ and $\overline{m} \in \bar{h}'$). The function f' shows $\mathcal{M}$ to be faithful to b'.

Next, assume that the rule for $\neg I_a A$ is applied to $\neg I_a A, (m, h_i)$. Then there are two extended branches. The branch b' extends b by $\|A\| \in N_a((m, h_l))$ for every h_l with $m \in h_l$ on b. The branch b'' extends branch b by $\|A\| \notin N_a((m, h_k)), m \in h_k$, $m_k \in h_k, m \prec m_k, h_i \lhd_a^m h_k$ for some new index k. Since f is faithful to b, we have $\mathcal{M}, f((m, h_i)) \vDash \neg I_a A$. Thus either (a) $\forall \bar{h} \in H_{\pi_1(m)} (\|A\|_\mathcal{M} \in \overline{N}_a((\pi_1(m), \bar{h}))$,

or (b) $\exists \bar{h}' \in Choice_a^{\pi_1(m)}(\pi_2(h_i))$ with $\|A\|_{\mathcal{M}} \notin \overline{N}_a((\pi_1(m), \bar{h}'))$. The function f shows $\mathcal{M}$ to be faithful to branch b'. In case (b), we define f' to be exactly as f, except that $f'((m, h_k)) = (\pi_1(m), \bar{h})$ and $f'((m_k, h_k)) = (\overline{m}, \bar{h})$ for some $\overline{m} \in \bar{h}$ with $\tau_1(m) \leq \overline{m}$. The auxiliary functions π_1, π_2 of f' are appropriately expanded. Then $\pi_2(h_k) \in H_{\pi_1(m)}$ (since $\pi_2(h_k) = \bar{h}$ and $\bar{h} \in H_{\pi_1(m)}$) and $\pi_2(h_k) \in H_{\pi_1(m_k)}$ (since $\pi_1(h_k) = \overline{m}$ and $\overline{m} \in \bar{h}$). The function f' shows $\mathcal{M}$ to be faithful to b''.

Suppose that the rule for $I_a A$ and $\neg I_a B$ is applied to $I_a A, (m, h_i)$ and $\neg I_a B, (m, h_i)$. Again there are two extended branches. The branch b' extends branch b by $m \prec m_k, m_k \in h_k, m \in h_k, A, (m_k, h_k), \neg B, (m_k, h_k)$ for a new index k. The branch b'' extends b by $m \prec m_l, m_l \in h_l, m \in h_l, \neg A, (m_l, h_l), B, (m_l, h_l)$ for a new index l. Since f is faithful to b, we have $\mathcal{M}, f((m, h_i)) \vDash I_a A$ and $\mathcal{M}, f((m, h_i)) \vDash \neg I_a B$. It is easily seen that therefore $|A\|_{\mathcal{M}} \neq |B\|_{\mathcal{M}}$. Hence, either (a) there is a moment/history-pair $(\overline{m}, \bar{h}) \in MH(\mathcal{M})$ with $\mathcal{M}, (\overline{m}, \bar{h}) \vDash A$ and $\mathcal{M}, (\overline{m}, \bar{h}) \vDash \neg B$ or (b) there is a moment/history-pair $(\overline{m}, \bar{h}) \in MH(\mathcal{M})$ with $\mathcal{M}, (\overline{m}, \bar{h}) \vDash \neg A$ and $\mathcal{M}, (\overline{m}, \bar{h}) \vDash B$. Case (a): We define f' to be the same function as f, except that $f'((m, h_k)) = (\pi_1(m), \bar{h})$ and $f'((m_k, h_k)) = (\overline{m}, \bar{h})$ for some $\overline{m} \in \bar{h}$ with $\pi_1(m) \leq \overline{m}$. The auxiliary functions π_1, π_2 of f' are appropriately expanded. Then $\mathcal{M}, f'((m, h_k)) \vDash A, \mathcal{M}, f'((m_k, h_k)) \vDash \neg B$, and $\pi_2(h_k) \in H_{\pi_1(m)}$. The function f' shows $\mathcal{M}$ to be faithful to the extended branch b'. Case (b): analogous to the previous case.

Finally, we consider the structural tableau rules. If f shows $\mathcal{M}$ to be faithful to b, and one of the rules REF, SYM, or TRAN is applied to obtain a branch b', then f shows $\mathcal{M}$ to be faithful to b' and Condition 4 from Definition 4 is satisfied, because for every agent $a \in Ag$ and every $m \in Tree$, the relation $\{(h, h') \mid h' \in Choice_a^m(h)\}$ is an equivalence relation.

For IND, we have $\pi_2(h_n) \in Choice_{a_1}^{\pi_1(m)}(\pi_2(h_{l_1})), \ldots, \pi_2(h_n) \in Choice_{a_k}^{\pi_1(m)}(\pi_2(h_{l_k}))$, because $\mathcal{M}$ satisfies the independence of agents condition. Moreover, since $\pi_2(h_n) \in Choice_{a_1}^{\pi_1(m)}(\pi_2(h_{l_1})) \subseteq H_{\pi_1(m)}$, it follows that $\pi_2(h_n) \in H_{\pi_1(m)}$. Thus, the function f shows $\mathcal{M}$ to be faithful to a branch b, if the IND-rule is applied to b. $\square$

Theorem 2 (**Soundness**) *If $\Delta \nvdash A$, then $\Delta \nvdash A$.*

Proof If $\Delta \nvdash A$, then there is a model $\mathcal{M}$ and a moment/history-pair $(m', h') \in MH(\mathcal{M})$, such that for all $B \in \Delta, \mathcal{M}, (m', h') \vDash B$ and $\mathcal{M}, (m', h') \nvDash A$. We consider a complete tableau for $\Delta^0 \cup \{\neg A, (m, h_0)\} \cup \{m \in h_0, m \prec m_0, m_0 \in h_0\}$, such that every branch starts with the single-node branch b consisting of $\Delta^0 \cup \{\neg A, (m, h_0)\} \cup \{m \in h_0, m \prec m_0, m_0 \in h_0\}$. Let $f((m, h_0)) = (m', h')$. Then f shows $\mathcal{M}$ to be faithful to b. (Without loss of generality, we may assume that in $\mathcal{M}$'s set of moments there is a moment m'' later than m', so that we can put $f(m_0, h_0) = (m'', \pi_2(h_0))$. If necessary, we can add such a moment and assume that agents have vacuous choices at $(m'', \pi_2(h_0))$ and that $(m'', \pi_2(h_0)) \notin V(p)$ for every atom p.) By the previous lemma, after applying a tableau rule to branch b, the model $\mathcal{M}$ is faithful to at least one branch which is an extension of b. Therefore, in the

tableau there is a complete branch b' to which $\mathcal{M}$ is faithful. If every branch of such a tableau is closed, then (i) there are formulas $C, \neg C$, and a pair (m'', h'') such that $C, (m'', h'')$ and $\neg C, (m'', h'')$ occur on b', or (ii) there is a formula D, an agent index a, and a pair (m''', h''') such that $\|D\| \in N_a((m''', h'''))$ and $\|D\| \notin N_a((m''', h'''))$ occur on b'. Since $\mathcal{M}$ is faithful to b', there is a function f' such that we have the contradiction $\mathcal{M}, f'((m'', h'')) \vDash C$ and $\mathcal{M}, f'((m'', h'')) \vDash \neg C$ or the contradiction $\|D\|_{\mathcal{M}} \in \overline{N}_a((m''', h'''))$ and $\|D\|_{\mathcal{M}} \notin \overline{N}_a((m''', h'''))$. Hence, there is no complete closed tableau for $\Delta^0 \cup \{\neg A, (m, h_0)\} \cup \{m \in h_0, m \prec m_0, m_0 \in h_0\}$ and thus $\Delta \nvdash A$. $\square$

In order to show completeness, we define for a given open branch of a complete tableau a model. Then we show that the defined model satisfies the formulas occurring on the open branch.

Definition 5 Let b be an open branch of a complete tableau. Then the model $\mathcal{M}_b = (Tree, \leq, Ag, Choice, \{\overline{N}_a \mid a \in Ag\}, V)$ induced by b is defined as follows:

1. $Tree := \{m \mid (m, h) \text{ occurs on } b\}$.
2. $\leq \; := $ the reflexive, transitive closure of $\{(m_i, m_j) \mid m_i \prec m_j \text{ occurs on } b, m_i, m_j \in Tree\}$.
3. $Ag := \{a \mid a \text{ is an agent index occurring on } b\}$.
4. $Choice_a^m(h) := \{h_l \mid h \lhd_a^m h_l \text{ occurs on } b\}$
   for all $a \in Ag, m \in Tree, m \in h$ occurring on b.
5. $\overline{N}_a((m, h)) := \{\|A\|_{\mathcal{M}_b} \mid \|A\| \in N_a((m, h)) \text{ occurs on } b\}$.
6. $V(p) := \{(m, h) \mid p, (m, h) \text{ occurs on } b\}$;
   $V(p) = \varnothing$ for every other atomic formula p.

Since b is a complete and *open* branch, $(m, h) \notin V(p)$ if $\neg p, (m, h)$ occurs on b. Because of the reflexive and transitive closure and since every moment m_k introduced by a tableau rule is a $\prec$-successor of the root moment m from the first node in a tableau, the ordered set $(Tree, \leq)$ is a tree and thus a branching time structure. Given this branching time structure, we have induced sets of histories and moment/history-pairs. Since b is an open branch of a *complete* tableau, by the structural rules REF, SYM, and TRAN, $\lhd_a^m$ is an equivalence relation defined on H_m. Therefore, $Choice_a^m(h)$ is the corresponding equivalence class of h. By rule IND, the independence of agents condition is satisfied, cf. [17].

Note Contrary to first appearance, the neighbourhood functions $\overline{N}_a$ are well-defined. We can define the depth of I-nesting of a formula A, $dI(A)$. If A contains no imagination operator I_a, then $dI(A) = 0$. If A has the form $\neg B$ or $[c_a]B$, then $dI(A) = dI(B)$. If A is a conjunction $(B \wedge C)$, then $dI(A)$ is $max(dI(B), dI(C))$. If A has the shape $I_a B$, then $dI(A) = dI(B) + 1$. We can show that $\overline{N}_a((m, h))$ is well-defined by a double induction first on the depth of I-nesting and then on the construction of A. If $dI(A) = 0$, then the truth set $\|A\|_{\mathcal{M}_b}$ is well-defined because it

is defined independently of neighbourhood functions, and thus $\overline{N}_a((m, h))$ is well-defined. Suppose that $dI(A) = n + 1$, and $\overline{N}_a((m, h))$ is well-defined for formulas B with $dI(B) \leq n$, i.e., $\|B\|_{\mathcal{M}_b}$ is well-defined. Then (i) A has the shape $I_a B$ or (ii) A has the form $(B \wedge C)$ with $dI(B) \leq n + 1$ and $dI(C) \leq n + 1$. In case (i), we may use the induction hypothesis to conclude that $\|A\|_{\mathcal{M}_b}$ is well-defined. In case (ii), we may note that $\|(B \wedge C)\|_{\mathcal{M}_b} = \|B\|_{\mathcal{M}_b} \cap \|C\|_{\mathcal{M}_b}$. By induction on the construction of A, $\|B\|_{\mathcal{M}_b}$ and $\|C\|_{\mathcal{M}_b}$ are well-defined, and thus their intersection is well-defined. But then $\overline{N}_a((m, h))$ is well-defined. Hence, $\overline{N}_a((m, h))$ is well-defined for every formula A.

Lemma 5 *If b is an open branch of a complete tableau and $\mathcal{M}_b = (Tree, \leq, Ag, Choice, \{\overline{N}_a \mid a \in Ag\}, V)$ is the model induced by b, then (if $A, (m, h)$ occurs on b, then $\mathcal{M}_b, (m, h) \vDash A$).*

Proof The proof is by induction on the number of connectives in A. Suppose that A is atomic and hence contains no connectives and $A, (m, h)$ occurs on b. By definition of the valuation function V, $(m, h) \in V(A)$, so that $\mathcal{M}_b, (m, h) \vDash A$.

If A is a negated atom $\neg p$ and the expression $\neg p, (h, m)$ occurs on b, then, as noted above, $\mathcal{M}_b, (m, h) \not\vDash p$, and thus $\mathcal{M}_b, (m, h) \vDash A$.

If A has the shape $\neg\neg B$, $B \wedge C$, or $\neg(B \wedge C)$ we may use the induction hypothesis and the fact that the tableau is complete.

Suppose that A has the form SB. If $A, (m, h)$ occurs on b, then for every h_k with $m \in h_k$ on b we have $B, (m, h_k)$ on b (since the tableau is complete) and, by the induction hypothesis, $\mathcal{M}_b, (m, h_k) \vDash B$. By the definition of H_m, it follows that $\mathcal{M}_b, (m, h) \vDash SB$.

Let A have the form $\neg SB$. If $A, (m, h)$ occurs on b, then, by completeness of the tableau, there is a pair (m, h_k) on b with $\neg B, (m, h_k)$ on b. By the induction hypothesis and definition of M_b, $\mathcal{M}_b, (m, h_k) \vDash \neg B$. Therefore, $\mathcal{M}_b, (m, h) \vDash \neg SB$.

If A has the form $[c]_a B$ and $A, (m, h)$ occurs on b, then $B, (m, h_k)$ occurs on b for every h_k with $h \lhd_a^m h_k$ on b. By the induction hypothesis and the definition of $\mathcal{M}_b$ it follows that $\mathcal{M}_b, (m, h) \vDash [c]_a B$.

If A has the shape $\neg[c]_a B$ and $A, (m, h)$ occurs on b, then there exists an h_k with $h \lhd_a^m h_k$ and $\neg B, (m, h_k)$ occurring on b. By definition of $\mathcal{M}_b$, it follows that $h_k \in Choice_a^m(h)$ and, by the induction hypothesis, it follows that $\mathcal{M}_b, (m, h_k) \vDash \neg B$. Therefore $\mathcal{M}_b, (m, h) \vDash \neg[c]_a B$.

If A has the form $I_a B$ and $A, (m, h)$ occurs on b, then $\|B\| \in N_a((m, h))$ occurs on b for every h_k with $h \lhd_a^m h_k$ on b. Moreover, there exists an h_l with $h \lhd_a^m h_l$ and $\|B\| \notin N_a((m, h_l))$ occurring on b. By definition of $\mathcal{M}_b$, $\|B\|_{\mathcal{M}_b} \in \overline{N}_a((m, h))$ for every h_k with $h_k \in Choice_a^m(h)$ and $\|B\|_{\mathcal{M}_b} \notin \overline{N}_a((m, h_l))$ with $h_l \in Choice_a^m(h)$. Hence $\mathcal{M}_b, (m, h) \vDash I_a B$.

Suppose that A has the form $\neg I_a B$ and that $A, (m, h)$ occurs on b. Then there are two cases, according to the branching of the decomposition rule for negated imagination ascriptions. In the first case $\|B\| \in N_a((m, h_l))$ occurs on b for every h_l with $m \in h_l$ on b. By definition of $\mathcal{M}_b$, $\|B\|_{\mathcal{M}_b} \in \overline{N}_a((m, h_l))$ for every h_l

$\in H_m$. But then $\mathcal{M}_b, (m, h) \nvDash I_a B$, i.e., $\mathcal{M}_b, (m, h) \vDash A$. In the second case there exists an h_k with $h \vartriangleleft_a^m h_k$ and $\|B\| \notin N_a((m, h_k))$ occurring on b. By the definition of $\mathcal{M}_b$, $h_k \in Choice_a^m(h)$. Since b is open, $\|B\|_{\mathcal{M}_b} \notin \overline{N}_a((m, h_k))$. But then $\mathcal{M}_b, (m, h) \nvDash I_a B$, i.e., $\mathcal{M}_b, (m, h) \vDash A$. $\square$

Theorem 3 (**Completeness**) *If $\Delta \nvdash A$, then $\Delta \nvDash A$.*

Proof Suppose that $\Delta \nvdash A$. Then there is no complete and closed tableau for $\Delta^0 \cup \{\neg A, (m, h_0)\} \cup \{m \in h_0, m \prec m_0, m_0 \in h_0\}$. Let b be an open branch of a complete tableau for this set and let $\mathcal{M}_b$ be the model induced by b. By the previous lemma, it follows that $\mathcal{M}_b, (m, h_0) \vDash A$ for every formula $A \in \Delta$ and $\mathcal{M}_b, (m, h_0) \vDash \neg A$, thus $\mathcal{M}_b, (m, h_0) \nvDash A$. Therefore, $\Delta \nvDash A$. $\square$

As a corollary to tableau soundness and completeness we may note that if one complete tableau for $\Delta^0 \cup \{\neg A, (m, h_0)\} \cup \{m \in h_0, m \prec m_0, m_0 \in h_0\}$ corresponding to the sequent $\Delta \vdash A$ is open, then every complete tableau is open, and if one complete tableau for $\Delta^0 \cup \{\neg A, (m, h_0)\} \cup \{m \in h_0, m \prec m_0, m_0 \in h_0\}$ is closed, then so is every tableau corresponding to $\Delta \vdash A$.

5 Conclusion

The present paper addresses and solves one of the open problems listed in [18], namely to give proof-theoretic characterizations of the semantically presented STIT imagination logic. The other items are still on the agenda: addressing the programme of reducing non-propositional imagination ascriptions to propositional ones as suggested by Niiniluoto [10], the development of a first-order extension of STIT imagination logic with identity, the addition of conception operators as in [8], and the development of the idea of "strategic imagination". In [5] is has been shown that *dstit*-logic for groups of agents is undecidable. We conjecture that single-agent STIT imagination logic is decidable.

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