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Cut Elimination for GLS Using the Terminability of its Regress Process

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Abstract The system GLS, which is a modal sequent calculus system for the provability logic GL, was introduced by G. Sambin and S. Valentini in *Journal of Philosophical Logic, 11*(3), 311–342, (1982), and in *12*(4), 471–476, (1983), the second author presented a syntactic cut-elimination proof for GLS. In this paper, we will use regress trees (which are related to search trees) in order to present a simpler and more intuitive syntactic cut derivability proof for GLS_1, which is a (more connectively and inferentially economical) variant of GLS without the cut rule.

Keywords Modal logic · GL · Gentzen-style logic · Cut-elimination · Regress trees

1 Introduction

The modal logic GL (for Gödel and Löb) is the logic obtained when we attach to classical propositional logic the two modal axioms

A1. $\Box(A \to B) \to \Box A \to \Box B$
A2. $\Box(\Box A \to A) \to \Box A$

and also the inference rule
(Nec) If A then $\Box A$.

Historically, GL was created as an attempt to treat the provability predicate in Peano arithmetic, $Pr(\ulcorner\cdot\urcorner)$, as a modal operator; and thus the theorems of GL were designed to be provable in Peano arithmetic under any interpretation, $*$, where an

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interpretation of a propositional formula, A, is a sentence in the language of arithmetic, A^*, such that (atomic sentence)* is any sentence (except for $\bot^*$ which is $0 = 1$), $*$ commutes with boolean connectives and $(\Box A)^* = Pr(\ulcorner A^* \urcorner)$.

Thus, for example, the second axiom, A2, was added due to the fact [4] that $Pr(\ulcorner Pr(\ulcorner A \urcorner) \to A \urcorner) \to Pr(\ulcorner A \urcorner)$ is always provable in Peano arithmetic for every sentence A in the language of arithmetic (where $\ulcorner A \urcorner$ is the Gödel number of A). Similarly for (Nec) and A1.

The importance of GL lies in the fact that if A is not provable in GL then there exists an interpretation, $*$, such that A^* is not provable in Peano arithmetic [7] (thus making GL a *provability logic*).

In [5], G. Sambin and S. Valentini presented GLS, which (they showed) is a Gentzen-style sequent calculus system corresponding to GL. In the same paper, they proved that, given Γ, Δ, sets of formulas, it is possible to decide whether $\Gamma \vdash \Delta$ is derivable or not in GLS' (which is basically equivalent to GLS *without the cut rule*) by showing that $\Gamma \vdash \Delta$ must have a finite *search tree* [3].

In this paper, instead of search trees, we will use similar constructs called *regress trees*.[1] A regress tree is, essentially, an attempt to construct a (legal) proof tree for a given sequent; thus, when we want to invert the GLR inference rule then, unlike in search trees, we examine each one of the possible premisses *separately*—that is, we examine each one in a *separate* regress tree.

Do note that, for various reasons, the nodes of a regress tree are not sequents but are expressions of the form $\Gamma \mathrel{\varepsilon\!\!-} \Delta$, and are called *regressants*.[2]

In this paper, we will use regress trees and a regressant-related induction parameter (namely, the height of the regressant's highest regress tree) in order to present a (complete) syntactic cut-derivability proof for GLS_1. This proof, we believe, is simpler and more intuitive than the one presented in [9] or the ones presented in [1] and in [6].[3]

Finally, let us stress that the method of search trees (which can very easily be adapted to regress trees) has been used to obtain decidability and completeness results in numerous logic systems (for example, in [3] and [8]); and indeed, it was used in [5] to show that GL is decidable and complete with respect to finite, transitive and irreflexive Kripke frames. Thus, unlike the methods utilized in the proofs mentioned above, this method is more than just an ad-hoc measure used to obtain a simple cut-derivability result. Moreover, as far as we know, the utilization of regress trees in order to obtain a syntactic cut-derivability proof is a novel approach, which is likely to be helpful in other logics as well.

[1] Stemming from one of the meanings of the word "regress," which is "The reasoning involved when one assumes the conclusion is true and reasons backward to the evidence."

[2] One reason we use this notation is to avoid confusion with the expression $\Gamma \vdash \Delta$ which sometimes is a short for "the sequent $\Gamma \vdash \Delta$ is provable," which might not be true since Γ, Δ can be *any* sets of formulas whatsoever. This notation also serves to indicate that the context is of regress trees and not of the (corresponding) Gentzen proof system.

[3] See also [2] for a resolution of certain issues concerning the proof in [9].

2 The Systems $\mathbf{GLS_1}$ and RGL

First, a few conventions:

1. The formulas in our language (well formed modal formulas) are constructed using $\bot$, boolean variables, $\rightarrow$ and $\Box$.
2. Upper case Greek letters such as Γ, Δ, Θ will represent sets of formulas.
3. A formula is called *prime* if it is atomic or boxed.
4. Γ is *atomic* (*prime*) if all the formulas in Γ are atomic (prime).

2.1 The System $\mathbf{GLS_1}$

(1) Initial sequents: $\Gamma, \bot \vdash \Delta \quad A, \Gamma \vdash \Delta, A \quad A$ is prime

(2) $\rightarrow$ -left : $\dfrac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}$

(3) $\rightarrow$ -right : $\dfrac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B}$

(4) $\bot$ -right : $\dfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A \rightarrow \bot}$

(5) -left : $\dfrac{\Gamma \vdash \Delta, A}{\Gamma, A \rightarrow \bot \vdash \Delta}$

(6) GLR : $\dfrac{\Gamma, \Box\Gamma, \Box A \vdash A}{\Phi, \Box\Gamma \vdash \Box A, \Psi}$ (where Φ, Ψ are any sets of formulas)

Note:

1. It is straightforward to prove that rules (2)-(5) are invertible; i.e., if the conclusion (denominator) is provable in $\mathrm{GLS_1}$ then so is the premiss (numerator).
2. The formulas Φ, Ψ in the GLR rule are called, respectively, the weakening and strengthening formulas of the rule.
3. We can easily prove that weakening and strengthening are derived rules.
4. We do not include a cut rule in our system.
5. Proving cut-derivability for $\mathrm{GLS_1}$ is equivalent to proving cut-elimination for GLS.

2.2 The System RGL

Definition 1 1. For any Γ, Δ, we call the expression of the form $\Gamma \mathrel{\varepsilon\!\!-} \Delta$ a *regressant*.
2. $\Gamma \mathrel{\varepsilon\!\!-} \Delta$ is a *prime* regressant if one of the following applies:

   a. $\bot \in \Gamma$.
   b. There exists a *prime formula*, A, such that $A \in \Gamma \cap \Delta$.
   c. Γ is prime and Δ is atomic, $\bot \notin \Gamma$ and $\Gamma \cap \Delta = \emptyset$.

Note that if $\Gamma \mathrel{\varepsilon\!\!-} \Delta$ is a prime regressant, then $\Gamma \vdash \Delta$ is either an initial sequent (cases a. and b.) or it is obviously not derivable in $\mathrm{GLS_1}$ since it is not an initial sequent and cannot be the conclusion of any of the inference rules of $\mathrm{GLS_1}$ (case c.).

The Rules of Regress:

(1) Primality: $\Gamma \rightarrowtail \Delta$ is a prime regressant

(2) $\rightarrow$ -left : $\dfrac{\Gamma \rightarrowtail \Delta, A \qquad B, \Gamma \rightarrowtail \Delta}{\Gamma, A \rightarrow B \rightarrowtail \Delta}$

(3) $\rightarrow$ -right : $\dfrac{\Gamma, A \rightarrowtail \Delta, B}{\Gamma \rightarrowtail \Delta, A \rightarrow B}$

(4) $\perp$ -right : $\dfrac{\Gamma, A \rightarrowtail \Delta}{\Gamma \rightarrowtail \Delta, A \rightarrow \perp}$

(5) $\perp$ -left : $\dfrac{\Gamma \rightarrowtail \Delta, A}{\Gamma, A \rightarrow \perp \rightarrowtail \Delta}$

(6) GLR: $\dfrac{\Gamma, \Box\Gamma, \Box A_i \rightarrowtail A_i}{\Phi, \Box\Gamma \rightarrowtail \Box A_1, \ldots, \Box A_n, \Psi}$ (for all $i \in \{1, \ldots, n\}$ and such that Φ, Ψ are atomic but the denominator is *not* a prime regressant). We call $\Box A_i$ the p.f. (*principle formula*) of the rule.

Note that these rules are intended to be applied "upwards" from the denominator to the numerator; thus, for example, we regress $\Gamma \rightarrowtail \Delta, A \rightarrow B$ using rule (3) in order to obtain $\Gamma, A \rightarrowtail \Delta, B.$.

Definition 2 A *regress tree*, T, for a regressant $\Gamma \rightarrowtail \Delta$ is a (graph theoretical) directed rooted tree such that:

a). $\Gamma \rightarrowtail \Delta$ is the *root* of T.
b). If R, a non-prime regressant, is a node in T then it has either exactly one child, R_1, such that $\dfrac{R_1}{R}$ is a regress rule, or it has exactly two children, R_1, R_2, such that $\dfrac{R_1 \ R_2}{R}$ is a regress rule (namely rule (2)).
c). T has no nodes or edges other than those that are required by the previous items.

Thus, for every Γ, Δ we start from the root $\Gamma \rightarrowtail \Delta$ and try to regress it (upwards) until we reach prime regressants, at which point we stop. We call this process the *regress process* for GLS_1.[4]

This leads us to the following definition:

Definition 3 For every regressant R, we denote by $M_h(R)$ as the height of R's highest regress tree.

The proof in [5] that every *search tree* in GLS' is finite can easily be modified to obtain a similar result for search trees in GLS_1 (defined analogously). But,

[4]For example, if the root is $\Box A \rightarrowtail \Box B_1, \Box B_2$ we can use the GLR regress rule to regress it to $A, \Box A, \Box B_1 \rightarrowtail B_1$ or to regress it to $A, \Box A, \Box B_2 \rightarrowtail B_2$ (and therefore $\Box A \rightarrowtail \Box B_1, \Box B_2$ has *at least* two associated regress trees).

since $M_h(R)$ is obviously equal to the height of R's search tree in GLS_1, we can immediately deduce that for every regressant R, *$M_h(R)$ is finite*.

The following corollary is readily seen as true:

Corollary 1 *If R, R_1 are two regressants such that R_1 is a node in one of R's regress trees, but not the root, then $M_h(R_1) < M_h(R)$.*

3 Cut Derivability for GLS_1

Note that in this section, we will, at times, forgo writing "$\Gamma \vdash \Delta$ is derivable in GLS_1," and simply write $\Gamma \vdash \Delta$. It will be clear from the context when we use this abbreviation.

Proposition 1 *The following two are equivalent for every formula A and any $\Gamma, \Delta, \Theta, \Omega, \Phi, \Psi$:*

1. *If $\Gamma \vdash \Delta, A$ and $A, \Theta \vdash \Omega$ then $\Gamma, \Theta \vdash \Delta, \Omega$ (cut derivability).*
2. *If $A \to A, \Phi \vdash \Psi$ then $\Phi \vdash \Psi$.*

Proof The proof is straightforward and therefore is omitted. □

Theorem 1 (**Cut-derivability for GLS_1**) *For any regressant $\Gamma \Leftarrow \Delta$ and for any formula A, if $A \to A, \Gamma \vdash \Delta$ then $\Gamma \vdash \Delta$.*

Proof By primary induction (P.I.) on the complexity of A and secondary induction (S.I.) on $M_h(\Gamma \Leftarrow \Delta)$.

Case 1. A is atomic.

I. $\Gamma \Leftarrow \Delta$ is prime. The only non-immediate case is when Γ is prime and Δ is atomic. By invertibility, we have that $\Gamma \vdash \Delta, A$ and it must be an initial sequent. Now, if $\Gamma \nvdash \Delta$ then we must have that $A \in \Gamma$, but since $A, \Gamma \vdash \Delta$ we have that $\Gamma \vdash \Delta$ — contradiction. So $\Gamma \vdash \Delta$.

II. $\Gamma \Leftarrow \Delta$ is not prime and is the denominator of one of regress rules (2)–(5). For example, let $\Delta = \Delta', B \to C$; now, by invertibility, $A \to A, \Gamma \vdash \Delta', B \to C$ implies that $A \to A, B, \Gamma \vdash \Delta', C$, and since $\dfrac{B, \Gamma \Leftarrow \Delta', C}{\Gamma \Leftarrow \Delta', B \to C}$ is an application of regress rule (3) we can use the S.I.H. to deduce $B, \Gamma \vdash \Delta', C$. We can now use the ($\to$-right) inference rule to get $\Gamma \vdash \Delta$. The other cases are similar.

III. The only regress rule applicable is GLR. Then the (derivable) sequent $\Gamma \vdash \Delta, A$ must be an initial sequent or the conclusion of a GLR inference rule. In the first case it must be because $A \in \Gamma$, in the second case A must be a strengthening formula. Both cases imply that $\Gamma \vdash \Delta$.

Case 2. $A = B \to C$. If $(B \to C) \to (B \to C), \Gamma \vdash \Delta$ then, by invertibility, we can also derive $B \to C, \Gamma \vdash \Delta$ and $\Gamma \vdash \Delta, B \to C$; and, again by invertibility, we can derive $S_1 = \Gamma \vdash \Delta, B$ and $S_2 = C, \Gamma \vdash \Delta$ and also $S_3 = B, \Gamma \vdash \Delta, C$. Now, we can derive $S_4 = B \to B, \Gamma \vdash \Delta, C$ from S_1 and S_3 using the ($\to$-left) rule;[5] similarly, we can derive $C \to C, B \to B, \Gamma \vdash \Delta$ from S_4 and S_2. We can now apply the P.I.H. twice to get $\Gamma \vdash \Delta$.

Case 3. $A = \Box B$.

I. $\Gamma \Leftarrow \Delta$ is prime. Again, the only non-immediate case is when Γ is prime and Δ is *atomic*. Now, $\Box B, \Gamma \vdash \Delta$ is derivable and must be an initial sequent, which means that $\Gamma \vdash \Delta$ is also an initial sequent.

II. $\Gamma \Leftarrow \Delta$ is not prime and is the denominator of one of regress rules (2)–(5). Done Similarly to the case where A is atomic.

III. The only regress rule applicable is GLR. First, we know that Γ, Δ are prime, Δ contains a boxed formula, $\bot \notin \Gamma$ and $\Gamma \cap \Delta = \emptyset$. We also know that we can prove $S = \Box B, \Gamma \vdash \Delta$ and $S' = \Gamma \vdash \Delta, \Box B$. We can assume that $\Box B \notin \Gamma \cup \Delta$ (otherwise it is immediate). Thus, the only applicable rule in obtaining both S and S' is GLR.
Now, if $\Box B$ was a weakening or strengthening formula in any one of them we are done. Thus, we may assume we have two proofs ending with: (where $\Gamma = \Phi, \Box\Gamma'$; $\Delta = \Box D, \Box\Delta', \Psi$ and Φ, Ψ are atomic)

$$\underbrace{\frac{\Gamma', \Box\Gamma', \Box B \vdash B}{\Phi, \Box\Gamma' \vdash \Box D, \Box\Delta', \Psi, \Box B}}_{S'} \qquad \underbrace{\frac{B, \Box B, \Gamma', \Box\Gamma', \Box D \vdash D}{\Box B, \Phi, \Box\Gamma' \vdash \Box D, \Box\Delta', \Psi}}_{S}$$

Which means we also have proofs ending with:

$$\begin{array}{c} S_1 \\ S_2 \end{array} \frac{\Gamma', \Box\Gamma', \Box B \vdash B}{\Box\Gamma' \vdash \Box B} \qquad \frac{B, \Box B, \Gamma', \Box\Gamma', \Box D \vdash D}{\Box B, \Box\Gamma' \vdash \Box D} \begin{array}{c} S_3 \\ S_4 \end{array}$$

Now, we can obtain $S_5 = \Box B \to \Box B, \Gamma', \Box\Gamma' \vdash B$ from S_2 and S_1; we can obtain $S_6 = B, \Box B \to \Box B, \Gamma', \Box\Gamma', \Box D \vdash D$ from S_2 and S_3, and, finally, we can obtain $S_7 = B \to B, \Box B \to \Box B, \Gamma', \Box\Gamma', \Box D \vdash D$ from S_5 and S_6. We can now apply the P.I.H. to obtain $S_8 = \Box B \to \Box B, \Gamma', \Box\Gamma', \Box D \vdash D$ from S_7.
Note that

$$\frac{\Gamma', \Box\Gamma', \Box D \Leftarrow D}{\Phi, \Box\Gamma' \Leftarrow \Box D, \Box\Delta', \Psi}$$

is an application of the regress rule GLR, thus we can apply the S.I.H. to obtain $\Gamma', \Box\Gamma', \Box D \vdash D$ from S_8. Now apply GLR to obtain $\Gamma \vdash \Delta$. $\Box$

[5]Technically, S_1 and S_3 are not in the right form to allow a legal application of the ($\to$-left) rule; however, they can easily be weakened and strengthened to the right form, thus allowing us to obtain S_4. This remark will apply to all the following instances involving a similar “illegal” use of the ($\to$-left) rule.

We can now deduce the following:

Corollary 2
The system GLS allows cut-elimination.

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