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The Intensional Many - Conservativity Reclaimed

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Abstract Following on Westerståhl's argument that many is not Conservative [9], I propose an intensional account of Conservativity as well as intensional versions of EXT and Isomorphism closure. I show that an intensional reading of many can easily possess all three of these, and provide a formal statement and proof that they are indeed proper intensionalizations. It is then discussed to what extent these intensionalized properties apply to various existing readings of many.

Keywords Generalized Quantifiers · Many · Intensionality

1 Introduction

In the theory of Generalized Quantifiers, much weight is given to the property of Conservativity, which for a binary quantifier Q can be paraphrased as

$$QAB \text{ if and only if } QA(A \text{ and } B)$$

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Conservativity is often suggested as a linguistic universal (eg [1, 5]), as it seems almost trivially true for virtually every natural language determiner. For instance, all of the following seem obvious enough:

No man is perfect.	⇔	No man is a perfect man.
Seven women are running.	⇔	Seven women are women who are running.
All good philosophers are wise.	⇔	All good philosophers are good philosophers who are wise.
Many men smoke.	⇔	Many men are men who smoke.

The last one, however, is actually problematic.

1.1 The Problem

Westerståhl [9] coined the following classic example to demonstrate the problem. In a certain class at a certain college 10 out of the 30 students got the highest grade on a certain exam, which is unusually many. Those same 10 students are the only ones in the class who are right-handed, which is unusually few. Let A be the set of students in the class, B_1 the set of students at the college who got the highest grade in their class, and B_2 the set of right-handed students at the college.

Thus, the assumptions from the example are expressed roughly as follows:

$$\mathbf{many}(A, B_1), \textit{not}\ \mathbf{many}(A, B_2)$$

If Conservativity were true of many, from this we could then conclude.

$$\mathbf{many}(A, A \cap B_1), \textit{not}\ \mathbf{many}(A, A \cap B_2)$$

But of course $A \cap B_1$ and $A \cap B_2$ are in fact the same set. Hence "many" can not be Conservative, or at least not without giving it two different interpretations to arbitrarily fix the problem.

1.2 Issues

It is hard to argue with the formal part of this argument, but it does leave something to be desired. For as soon as we translate the result back into natural language, serious problems with our intuitions arise. If we give up on Conservativity for this case and reject the conclusion that $\mathbf{many}(A, B_1)$ and *not* $\mathbf{many}(A, A \cap B_2)$, then we have to in turn accept the opposite of at least one of these. Hence we would be forced to accept one of the following natural language sentences:

- *Not many students in the class are students in the class who got the highest grade on the exam.*
  (While we at the same time accept many students in the class did get the highest grade.),

or

- *Many students in the class are right-handed students in the class.*
  (While we at the same time accept not many students in the class are right-handed.)

Neither of these is a particularly attractive statement to endorse, and then there is the question of which of the two we should pick. Westerståhl offers no answer to this question, and it is hard to see how anyone could; they seem equally counterintuitive and "resolve" the inconsistency equally well. So how do we get out of this problem?

I would say that rather than a straightforward case against Conservativity for many, what the example really provides is a complication arising from a different problem.

We saw before that $A \cap B_1$ and $A \cap B_2$ were the same set. Let us call this set C. Do we now have any intuitions about the sentence "Many students in the class are C"? Of course not. There is no obviously correct way to parse C as something we would have intuitions about. To have an idea about whether 10 students in a class being C is many or not, we need to know not the set itself but the *property* it is representing -and hence presenting the same set as an instantiation of different properties leads to different intuitions. This, then, is our problem.

The theory of Generalized Quantifiers as formalized by Barwise & Cooper [1] and van Benthem [2] is inherently extensional: while it involves possible universes and how quantifiers deal with them, it does not allow properties to be identified as more than subsets of a specific universe. We can use it to talk about "right-handed students at the college, in this particular world/situation", but not of right-handedness as a property in its own right identified independent of any one universe. We are limited to identifying properties by their local extensions, whereas many requires an intensional approach.

This, of course, is not a particularly new thought. The fact that many is intensional has been generally agreed upon after being pointed out by Keenan and Stavi [5]. What *is* interesting here is that we shall see that when it is treated in this way, Conservativity is reclaimed.

In the next section we will construct an intensional framework for generalized quantifiers and create an intensional version of Conservativity. We will then show that this move resolves the issues created by the example, and further support this position by providing a specific reading of many which works well for it and is (Intensionally) Conservative.

The point of doing this is not to suggest that this is the single best reading of many, or even that it is the single best framework in which to consider such readings. Rather, the point is to demonstrate that when cast into a proper intensional form, Conservativity can be reclaimed as an important standard by which to judge quantifiers, even previously problematic ones like many.

In Section 3, we take a look at some other partly intensional readings of many that have been proposed and see to what extent they can meet this standard.

2 An Intensional Framework

2.1 Framework

Definition 1 Where L is a set of predicates closed under Boolean combination, a *structure S for L* is a triple $< W, D, [\![\cdot]\!] >$ where W is a non-empty set of *possible worlds*,[1] D assigns to each world $m \in W$ a non-empty set *D(m)* referred to as the *domain* of that world, and $[\![\cdot]\!]$ assigns to each predicate $A \in L$ its *intension* $[\![A]\!]$, which in turn for each world m determines the *extension* $[\![A]\!]^m$ of A. We demand that $[\![A]\!]^m \subseteq D(m)$ and that intensions satisfy the following rules:

$$[\![A \wedge B]\!]^m = [\![A]\!]^m \cap [\![B]\!]^m$$
$$[\![A \vee B]\!]^m = [\![A]\!]^m \cup [\![B]\!]^m$$
$$[\![\neg A]\!]^m = D(m) - [\![A]\!]^m$$

More generally, a *property on S* is a function which assigns to each $m \in W$ a subset of $D(m)$.

Thus we may identify each $m \in W$ with the first-order model $< D(m), [\![\cdot]\!]^m >$ (where the derived interpretation function $[\![\cdot]\!]^m$ simply assigns to each predicate its extension in m, as previously defined). From now on we will refer to possible worlds as models. Also, we will use capital letter from the beginning of the alphabet (*A, B, C*) for predicates and boldface capitals from the end of the alphabet (**X**, **Y**, **Z**) for properties and write quantifiers in boldface.

We now get to the essential non-cosmetic change, which is that quantifiers are applied to properties rather than extensions.

Definition 2 An *intensional quantifier* **Q** is a function whose input consists of two properties on the same S and a model in W and whose output is the evaluation true or false.

We will write $\mathbf{Q}_m\mathbf{XY}$ to denote that this evaluation is true -and hence $\mathbf{Q}_m[\![A]\!][\![B]\!]$ when the properties in question are the intensions of the predicates A and B.[2]

[1] The set of worlds W serves as a basis from which to derive intensional standards that are not (heavily) dependent on the interpretations in any one world. The idea here is *not* that W would include every logical possibility, but rather that it is made up of worlds which are much like the actual world except (possibly) for the issues at hand, for which they will by and large correspond to our expectations and the things we consider normal and plausible

[2] In more traditional intensional semantics, the thing we call a *structure* above is referred to as a *model*, and $\mathbf{Q}_m\mathbf{XY}$ would be expressed as $S \models QXY[m]$.

2.2 Intensional Conservativity

For the sake of generality, we define Intensional Conservativity in terms of arbitrary properties, rather than only those which are the intensions of predicates. To do this, we first need to define a property-conjunction operation, which obviously is just to say that $\mathbf{X} \wedge \mathbf{Y}$ is the unique property satisfying

$$\forall m : (\mathbf{X} \wedge \mathbf{Y})^m = (\mathbf{X}^m) \cap (\mathbf{Y}^m)$$

It is now a straightforward task to rephrase the definition of Conservativity into Intensional Conservativity, which we define as follows:

$$\text{For all } S,\ \text{for all properties } \mathbf{X}, \mathbf{Y} \text{on} S, \text{ for all } m \in W,$$

$$\mathbf{Q}_m\mathbf{XY} \Leftrightarrow \mathbf{Q}_m\mathbf{X}(\mathbf{X} \wedge \mathbf{Y})$$

To see that Conservativity is now possible, let us take another look at the earlier example. Let the predicate C stand for students at the particular college in question, and A for students at the particular class. Let R stand for right-handedness and H for getting the highest grade in class.

Now the complex predicates $B_1 = C \wedge H,\ B_2 = C \wedge R$ are appropriate to express the assumptions of the example, which amount to

$$\mathbf{many}_m [\![A]\!][\![B_1]\!],\ \textit{not}\ \mathbf{many}_m [\![A]\!][\![B_2]\!]$$

The question is: can **many** be interpreted in a way that satisfies the above while also being intensionally conservative?

It can. From these assumptions, Intensional Conservativity merely lets us conclude that

$$\mathbf{many}_m [\![A]\!][\![A \wedge B_1]\!],\ \neg\mathbf{many}_m [\![A]\!][\![A \wedge B_2]\!]$$

Since $[\![A \wedge B_1]\!]$ and $[\![A \wedge B_2]\!]$ are not the same properties, this does not lead to a contradiction.

2.2.1 A Sample Reading

While technically the above is enough to conclude the argument, it will carry more weight when we have an actual single interpretation $\mathbf{Q}$ that is a reasonable reading of many and satisfies these conditions.

For this we use just one further simplifying assumption, that W is finite.[3]

[3]This assumption may sometimes be undesirable, but keep in mind that this reading is merely an illustrative example. We shall see in Section 2.6 that there is a broad general form such that any reading of that form will possess Conservativity and other key properties. Thus, for certain infinite W the average could be generalized using series summation $\left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{|\mathbf{Y}^{w_i} \cap \mathbf{X}^{w_i}|}{|\mathbf{X}^{w_i}|}\right)$ or integration $\left(\int_W h(w) \frac{|\mathbf{Y}^{w} \cap \mathbf{X}^{w}|}{|\mathbf{X}^{w}|} \mathrm{d}w,\ where\ \int_W h(w)\mathrm{d}w = 1\right)$, or be replaced by an intensional standard based on a probability function on W, a subset of particularly 'normal' or normative worlds, or some other notion (see also Section 3). For any of these, the desirable properties remain attainable.

Given this, consider the following definition, which says roughly that many students have property $\mathbf{Y}$ if the relative number of students who have that property is larger than the average of that same number taken over all models:[4]

$$\mathbf{Q}_m\mathbf{XY} \Leftrightarrow \left(\frac{|\mathbf{Y}^m \cap \mathbf{X}^m|}{|\mathbf{X}^m|} > \frac{1}{|W|} \sum_{n \in W} \frac{|\mathbf{Y}^n \cap \mathbf{X}^n|}{|\mathbf{X}^n|} \right)$$

The example establishes that $|[\![A]\!]^m| = 30$, while $|[\![B_1]\!]^m \cap [\![A]\!]^m| = |[\![B_2]\!]^m \cap [\![A]\!]^m| = 10$. Since it is rare for as many as a third of students to get a top grade, we may expect $\frac{|[\![B_1]\!]^n \cap [\![A]\!]^n|}{|[\![A]\!]^n|}$ to be lower on average, and thus we obtain $\mathbf{Q}_m[\![A]\!][\![B_1]\!]$. At the same time, since right-handedness is commonplace, we may expect $\frac{|[\![B_2]\!]^n \cap [\![A]\!]^n|}{|[\![A]\!]^n|}$ to average significantly higher than one-third, so that we do *not* get $\mathbf{Q}_m[\![A]\!][\![B_2]\!]$.

This takes care of the basic setup. We should now see if we get $\mathbf{Q}_m[\![A]\!][\![A \wedge B_1]\!]$, $\neg\mathbf{Q}_m[\![A]\!][\![A \wedge B_2]\!]$. And indeed we do. To see that the definition satisfies Intensional Conservativity - and therefore gives those results - it is enough to note that (for all $\mathbf{X}, \mathbf{Y}$)

$$|\mathbf{Y}^m \cap \mathbf{X}^m| = |(\mathbf{X}^m \cap \mathbf{Y}^m) \cap \mathbf{X}^m| = |(\mathbf{X} \wedge \mathbf{Y})^m \cap \mathbf{X}^m|.$$

This of course is but a single possible interpretation of a single possible reading of many, but it seems likely that a variety of other options will work equally well, and we will see later that this is indeed the case. Thus, when intensionality is properly accounted for, Conservativity does not need to be given up as a universal property of natural language determiners, not even for many.

2.3 On Scandinavians and the Reverse Reading

Taking an intensional approach to *many* not only helps to reclaim Conservativity, it also resolves a different issue: that of the so-called Reverse Reading whereby a quantifier will sometimes seem to take its arguments in the opposite order from what the sentence structure would suggest.

A famous example of this is found in [9]. Consider the following sentences:

(1) Many winners of the Nobel Prize in Literature are Scandinavian.
(2) Many Scandinavians have won the Nobel Prize in Literature.
(3) Many Scandinavians are Nobel Prize winners in Literature.

As of the year 1984, 14 out of a total of 81 winners of the Nobel Prize in Literature are Scandinavians. This would seem surprisingly many, and it is generally agreed that the sentence (1) is true here. Furthermore, it is generally felt that from an intuitive point at least, sentence (2) should be true.

Sentence (3) would seem to be a slightly different way of phrasing sentence (2). However, Westerståhl argues that (3) is clearly false, on the basis that 14 is a very small number compared to the number of Scandinavians. He goes on to suggest that while (3) certainly corresponds to a possible reading of (2), the preferred reading of

[4] To get around division by zero, we may harmlessly use $\frac{0}{0} = 1$.

(2) is expressed by (1). Thus, the logical form of (2) would have the arguments of the quantifier reversed relative to what the surface form would suggest.

Contrary to this view, I maintain that (2) and (3) should be rendered the same way and can be found to be true without resorting to a reversed reading equivalent to (1). To see how this may be done, let us again take the example reading of *many* we used earlier:

$$\mathbf{Q}_m\mathbf{XY} \Leftrightarrow \left(\frac{|\mathbf{Y}^m \cap \mathbf{X}^m|}{|\mathbf{X}^m|} > \frac{1}{|W|}\sum_{n\in W}\frac{|\mathbf{Y}^n \cap \mathbf{X}^n|}{|\mathbf{X}^n|}\right)$$

Using **S** for "Scandinavian" and **N** for "Nobel Prize in Literature winner", sentence (3) would be true iff the following holds.

$$\left(\frac{|\mathbf{N}^m \cap \mathbf{S}^m|}{|\mathbf{S}^m|} > \frac{1}{|W|}\sum_{n\in W}\frac{|\mathbf{N}^n \cap \mathbf{S}^n|}{|\mathbf{S}^n|}\right)$$

On the left-hand side, we have the relative number of Nobel Prize in Literature winners among Scandinavians in this world. This of course is a tiny number. So why is it wrong to say that this reading is clearly false?

The trick is that the important comparison here is not between Prize winners and Scandinavians, nor even between Scandinavian Prize winners and other Scandinavians. Rather, the comparison that matters is between the this possible world and others.

As is conventional, let us assume for the sake of argument that the actual world is fairly normal in the sense that other worlds by and large have a similar amount of Scandinavians as the actual world. Thus, the division by $|\mathbf{S}^m|$ for the actual world is by and large comparable to the division by $|\mathbf{S}^n|$ in others. This suggests the comparison will be true so long as $|\mathbf{N}^m \cap \mathbf{S}^m|$ is substantially larger than the average $\frac{1}{|W|}\sum_{n\in W}|\mathbf{N}^n \cap \mathbf{S}^n|$ across all worlds. But the reason we take (1) to be true in the example is exactly that among the possible worlds we consider there are generally substantially less Scandinavian Nobel Prize winners than in the real world. Thus, so long as W is chosen in a way appropriate to the example this reading will predict that (3) is true.

2.4 Other key properties

Conservativity is not the only property taken to apply to virtually all natural language determiners. Two important others are Extension (which I will mostly refer to by the abbreviation EXT)[5] and Isomorphism closure. Let us see how well many does on intensionalized versions of those.

2.4.1 Intensional EXT

We start with Extension. Extension roughly states that when a domain M is extended to M', the interpretation relative to that domain remains the same.

[5]Given how much here revolves around intensions and extensions, to do otherwise could invite confusion.

For traditional binary quantifiers, this is defined as follows paraphrasing ([10, p. 281]):

$$\textit{If } A, B \subseteq M \subseteq M'$$
$$\textit{then } Q_M AB \Leftrightarrow Q_{M'} AB$$

The point of EXT is domain restriction; it serves to make everything in $M - (A \cup B)$ irrelevant to the interpretation of $Q_M AB$.

Under the circumstances one might well think that the highly context- dependent many stands a poor chance of satisfying any version of EXT. Yet it is quite possible.

In fact, we shall intensionalize a more broadly defined property EXT*, defined as:

$$\textit{If } A, B \subseteq M, A, B \subseteq M'$$
$$\textit{then } Q_M AB \Leftrightarrow Q_{M'} AB$$

(It's worth pointing out that in the traditional approach the difference is largely irrelevant, as regular EXT gives $Q_M AB = Q_{A\cup B} AB = Q_{M'} AB$. However, EXT* is more convenient to work with when intensionalizing.) We define our Intensional version of EXT as follows:

$$\text{If } \mathbf{X}^m = \mathbf{X}^{m'}, \mathbf{Y}^m = \mathbf{Y}^{m'}, \text{ then } \mathbf{Q}_m\mathbf{XY} \Leftrightarrow \mathbf{Q}_{m'}\mathbf{XY}$$

This amounts to saying that $\mathbf{Q}_m\mathbf{XY}$ depends on m only insofar as it depends on the interpretations of $\mathbf{X}$ and $\mathbf{Y}$ in m: where those stay the same, so does the evaluation.

This sounds like a tall order, but it is satisfied by the interpretation from our earlier example. To see this, it suffices to note that

$$\frac{|\mathbf{Y}^m \cap \mathbf{X}^m|}{|\mathbf{X}^m|} = \frac{|\mathbf{Y}^{m'} \cap \mathbf{X}^{m'}|}{|\mathbf{X}^{m'}|}.$$

There are some important caveats to this result. First of all, Intensional EXT does *not* mean the quantifier only "has access to" the interpretations in the local universe. It still has access to the properties themselves. What it does mean is that insofar as the quantifier has access to more than the local interpretations of **X** and **Y**, it only has such access in a model-independent way.

For example, in the reading for "many" we used in Section 2.2.1, the quantifier used this access to **X** and **Y** to generate the comparison standard $\frac{1}{|W|}\sum_{n\in W}\frac{|\mathbf{Y}^n\cap\mathbf{X}^n|}{|\mathbf{X}^n|}$. Such behavior is not undesirable, and arguably is part of the point of using an intensionalized definition.

Second, even this intensional version might not be possible or desirable for every reading we want to model. Those who compare things against alternatives (e.g., [3, 8]) risk running foul of it. More on this in Section 3.

2.4.2 *Intensional Isomorphism Closure*

Next, we consider Isomorphism closure, sometimes abbreviated ISOM. In the traditional version, this can be rendered as follows ([10, p. 281]):

$$\textit{If } f \textit{ is a bijection from} M \textit{ to } M',$$
$$\textit{then } Q_M AB \Leftrightarrow Q_{M'} f[A] f[B]$$

The point of Isomorphism closure is to ensure that quantifiers cannot distinguish between individual elements in a universe, or even across universes.

Since models in our formalism come with interpretation functions, the Intensional version is slightly more complicated:

> *If there is a bijection* $f : D(m) \rightarrow D(m')$
> *with* $f[\mathbf{X}^m] = \mathbf{X}^{m'}$, $f[\mathbf{Y}^m] = \mathbf{Y}^{m'}$
> *then* $\mathbf{Q}_m\mathbf{XY} \Leftrightarrow \mathbf{Q}_{m'}\mathbf{XY}$

We demand not only a bijection f from $D(m)$ to $D(m')$, but also that the interpretations of **X** and **Y** in the two models are related through this same bijection. This is similar to demanding that f is an isomorphism, except that the demand is more of a local one for each pair.[6] (Also, note that since f still works on the level of domains rather than involving properties, the conclusion is phrased a bit differently.)

It is straightforward enough to see that our earlier interpretation of many satisfies this property as well. The key part of that interpretation was the following comparison.

$$\frac{|\mathbf{Y}^m \cap \mathbf{X}^m|}{|\mathbf{X}^m|} > \frac{1}{|W|}\sum_{n \in W}\frac{|\mathbf{Y}^n \cap \mathbf{X}^n|}{|\mathbf{X}^n|}$$

Let us focus on the left side first. Since $f[\mathbf{X}^m] = \mathbf{X}^{m'}$ and f is a bijection, it follows that $|\mathbf{X}^m| = |\mathbf{X}^{m'}|$.

To see that $|\mathbf{Y}^m \cap \mathbf{X}^m| = |\mathbf{Y}^{m'} \cap \mathbf{X}^{m'}|$, note that since f is a bijection, the following holds:

$$\begin{aligned} f[\mathbf{Y}^m \cap \mathbf{X}^m] &= f[\mathbf{Y}^m] \cap f[\mathbf{X}^m] \\ &= \mathbf{Y}^{m'} \cap \mathbf{X}^{m'} \end{aligned}$$

Therefore as before $|\mathbf{Y}^m \cap \mathbf{X}^m| = |\mathbf{Y}^{m'} \cap \mathbf{X}^{m'}|$. Thus the left side of the equation is the same for m and m'. This is trivially true for the right side, and therefore $\mathbf{Q}_m\mathbf{XY} \Leftrightarrow \mathbf{Q}_{m'}\mathbf{XY}$.

2.5 Relation with Extensional Properties

One may wonder whether we are justified in believing that the 'lifted' properties we have come up with in this section represent the most appropriate way of intensionalizing. But they are far from arbitrary. In all three cases they can be naturally related to their original counterpart through a straightforward lifting function.

Definition 3 For a non-intensional quantifier Q, define its *intensional lift* Q^* *as* follows*:*

$$Q^*_m\mathbf{XY} \Leftrightarrow Q_{D(m)}\mathbf{X}^m\mathbf{Y}^m$$

[6] As a first thought it might look desirable to go much further and that f be an actual isomorphism; i.e. that $f[\mathbf{X}^m] = \mathbf{X}^{m'}$ holds for all properties. However, one can always define, say, a property **X** for which $\mathbf{X}^m$ and $\mathbf{X}^{m'}$ do not even have the same number of elements. Thus, making such a broad demand would guarantee that no such f exists for any structure, rendering the whole thing worthless. Therefore we are forced to work only with those properties which work well with f (for at least one f).

This lifting function leads to the following correspondence theorem.

Theorem 1 *Where Q is a non-intensional quantifier and Q^* is its lift:*

- Q^* *satisfies Intensional Conservativity if and only if Q is Conservative*
- Q^* *satisfies Intensional EXT if and only if Q satisfies EXT*, where EXT* is like EXT but applies for any* M, M' *such that* $A, B \subseteq M$, $A, B \subseteq M'$
- Q^* *satisfies Intensional Isomorphism closure if and only if Q satisfies Isomorphism closure*

Thus, all three of them are natural and true broadenings of their original counterparts. For proof of the above, see Appendix A.1.

The lifting function suggests another matter of some interest: under which conditions can an intensional quantifier (or at least a quantifier expressed in terms of this framework) be interpreted as the lift of a traditional extensional one? This question is answered in Appendix A.2.

(Of course, appropriate readings of 'many' cannot be interpreted as such lifts.)

2.6 General form

The intensionalized properties described above obviously apply to far more than the simple example reading of many. We will generalize that reading greatly to obtain a general form of intensional quantifier they also apply to. Besides being interesting in its own right, this will be useful when looking at other approaches in the next section.

Our sample reading was as follows:

$$\mathbf{Q}_m\mathbf{XY} \Leftrightarrow \left(\frac{|\mathbf{Y}^m \cap \mathbf{X}^m|}{|\mathbf{X}^m|} > \frac{1}{|W|}\sum_{n\in W}\frac{|\mathbf{Y}^n \cap \mathbf{X}^n|}{|\mathbf{X}^n|}\right)$$

Here the fraction of **X**'s in a particular model that are also **Y** had to be larger than the average of that same fraction over all models. To generalize this, we replace "fraction of **X**'s in a particular model that are also **Y**" with an arbitrary function a (an Actual value of something) depending only on $|\mathbf{X}^m|$ and $|(\mathbf{X} \wedge \mathbf{Y})^m|$, "larger than" with an arbitrary relation $\succ$, and "the average of …" with an arbitrary function st (an intensionally determined standard value) depending only on **X**, $\mathbf{X} \wedge \mathbf{Y}$ and W. Formally, then, we get the following.

Definition 4 A quantifier Q has the *general form* if the following is true

$$\mathbf{Q}_m\mathbf{XY} \Leftrightarrow a(|\mathbf{X}^m|, |(\mathbf{X} \wedge \mathbf{Y})^m|) \succ st(\mathbf{X}, \mathbf{X} \wedge \mathbf{Y}, W)$$

with a, st, $\succ$ as above.

It is perhaps not immediately obvious that All and Some have the general form, but this can be shown to be true if the right choices are made. These and other examples are listed below.

Quantifier	$a(\|\mathbf{X}^m\|, \|(\mathbf{X} \wedge \mathbf{Y})^m\|)$	$\succ$	$st(\mathbf{X}, \mathbf{X} \wedge \mathbf{Y}, W)$
All	$\frac{\|(\mathbf{X} \wedge \mathbf{Y})^m\|}{\|\mathbf{X}^m\|}$	$\geq$	1
Some	$\|(\mathbf{X} \wedge \mathbf{Y})^m\|$	$\geq$	1
At least n	$\|(\mathbf{X} \wedge \mathbf{Y})^m\|$	$\geq$	n
Exactly n	$\|(\mathbf{X} \wedge \mathbf{Y})^m\|$	$=$	n
At most n	$\|(\mathbf{X} \wedge \mathbf{Y})^m\|$	$\leq$	n
Most	$\frac{\|(\mathbf{X} \wedge \mathbf{Y})^m\|}{\|\mathbf{X}^m\|}$	$>$	$\frac{1}{2}$
More than x % of	$\frac{\|(\mathbf{X} \wedge \mathbf{Y})^m\|}{\|\mathbf{X}^m\|}$	$>$	$\frac{x}{100}$

Theorem 2 *Every quantifier that has the general form (and indeed, every quantifier whose evaluation depends only on* $\mathbf{X}$, $\mathbf{X} \wedge \mathbf{Y}$, $|\mathbf{X}^m|$, $|(\mathbf{X} \wedge \mathbf{Y})^m|$ *and* W*) satisfies Intensional Conservativity, Intensional EXT and Intensional Isomorphism closure.*

Proof A straightforward substitution will show that this is true for Intensional Conservativity. Details left to the reader.

Intensional EXT says that $\mathbf{Q}_m\mathbf{XY} \Leftrightarrow \mathbf{Q}_{m'}\mathbf{XY}$ whenever $\mathbf{X}^m = \mathbf{X}^{m'}$, $\mathbf{Y}^m = \mathbf{Y}^{m'}$. Now if $\mathbf{X}^m = \mathbf{X}^{m'}$, $\mathbf{Y}^m = \mathbf{Y}^{m'}$ then trivially $|\mathbf{X}^m| = |\mathbf{X}^{m'}|$, $|(\mathbf{X} \wedge \mathbf{Y})^m| = |(\mathbf{X} \wedge \mathbf{Y})^{m'}|$. Since the only way in which a quantifier $\mathbf{Q}$ of the general form depends on the specific models m, m' is through its dependence on those cardinalities, $\mathbf{Q}_m\mathbf{XY} \Leftrightarrow \mathbf{Q}_{m'}\mathbf{XY}$ follows.

For Intensional Isomorphism Closure, suppose $h : D(m) \to D(m')$ is a bijection and $\mathbf{X}^{m'} = h[\mathbf{X}^m]$, $\mathbf{Y}^{m'} = h[\mathbf{Y}^m]$. Then clearly $|\mathbf{X}^m| = |\mathbf{X}^{m'}|$. Similarly,

$$\begin{aligned} |(\mathbf{X} \wedge \mathbf{Y})^m| &= |\mathbf{X}^m \cap \mathbf{Y}^m| \\ &= |h[\mathbf{X}^m \cap \mathbf{Y}^m]| \\ &= |h[\mathbf{X}^m] \cap h[\mathbf{Y}^m]| \\ &= |\mathbf{X}^{m'} \cap \mathbf{Y}^{m'}| \\ &= |(\mathbf{X} \wedge \mathbf{Y})^{m'}| \end{aligned}$$

Thus, the arguments of a are invariant under replacing m by m' under these circumstances, which leads to $\mathbf{Q}_m\mathbf{XY} \Leftrightarrow \mathbf{Q}_{m'}\mathbf{XY}$. □

3 Other readings of Many

As mentioned in the introduction, I am not the first to notice that any proper treatment of many should have at least an intensional component to it. Thus, through the years a number of readings that have such a component have been proposed. However, it has not yet been looked into how these readings fare with regards to Conservativity. In this section we will investigate some of them to find out just that.

To avoid confusion, we will rephrase these treatments in terms of the framework and notational conventions we have been using so far.

3.1 Fernando and Kamp

Fernando and Kamp's account [4] states that "... the arguments of many ... cannot be interpreted simply by their extensions" and uses a probability-based method for the intensional component. The idea is that a given number of **X**'s that are **Y** qualifies as many if one would have expected there to be less. The quantifier is given by

$$\mathbf{Many}_m(\mathbf{X}, \mathbf{Y}) \Leftrightarrow \bigvee_{n \geq 1} (|(\mathbf{X} \wedge \mathbf{Y})^m| \geq n) \wedge n\text{−is} - \text{many}(\mathbf{X}, \mathbf{Y})$$

The probability-driven component n−is − many($\mathbf{X}, \mathbf{Y}$) comes in a simple version and a more complex one. The simple version asserts that the probability of there being less than n **X**'s that are **Y** is sufficiently high. It is of the form $P(\{m' : |(\mathbf{X} \wedge \mathbf{Y})^{m'}| < n\}) > c$, for a world-independent probability function P and constant c.

While it would be fairly easy to express this in our general form (left to the reader), it unfortunately is also symmetrical. Thus we are more interested in the more advanced reading.

In the advanced version, we do not merely use the probability of there being less than n such objects, but conditionalize this probability against that of having exactly as many **X**'s are there happen to be. This gives us the following n−is − many($\mathbf{X}, \mathbf{Y}$):

$$P\left(\left\{m' : |(\mathbf{X} \wedge \mathbf{Y})^{m'}| < n\right\} \mid \left\{m' : |\mathbf{X}^{m'}| = |\mathbf{X}^m|\right\}\right) > c$$

Because of this actual world-dependent component, this reading does not have the general form. However, since it depends only on $\mathbf{X}, \mathbf{X} \wedge \mathbf{Y}, |\mathbf{X}^m|, |(\mathbf{X} \wedge \mathbf{Y})^m|$ and the independent c and P it still satisfies Intensional Conservativity, EXT and Isomorphism closure.

3.2 Cohen

The Relative Proportional reading introduced by Cohen [3] is based on the notion of *alternatives*. The alternatives of a property are other properties which it is appropriate to compare it to.

For instance, when considering the sentence "Many Scandinavians won a Nobel Prize in Literature" (see also Section 2.3), the alternatives to Scandinavian would be various (non-Scandinavian) nationalities. This sentence would be considered true under this reading if the proportion of Scandinavians who have won a Nobel Prize is (significantly) larger than the average proportion of people who have done so from other backgrounds.[7]

Formally, we take $\mathbf{many}_m(\mathbf{X}, \mathbf{Y})$ to be true iff the following holds:

$$\frac{|\mathbf{X}^m \cap \mathbf{Y}^m|}{|\mathbf{X}^m \cap \bigcup\{\mathbf{Z}^m|\mathbf{Z} \in A\}|} > \frac{|\bigcup\{\mathbf{Z}^m|\mathbf{Z} \in A\} \cap \mathbf{Y}^m|}{|\bigcup\{\mathbf{Z}^m|\mathbf{Z} \in A\}|}$$

Here A is a set of pairs of alternatives for $\mathbf{X}$ and Y, given by

$$A = \{\mathbf{X}' \wedge \mathbf{Y}'|\mathbf{X}' \in \text{ALT}(\mathbf{X}), \mathbf{Y}' \in \text{ALT}(\mathbf{Y})\}$$

where ALT($\mathbf{X}$) gives a set of properties considered to be alternatives to $\mathbf{X}$, including $\mathbf{X}$ itself. It is important to keep in mind that such alternatives are necessarily disjoint everywhere.

The above looks a bit complex because it accounts for the possibility that the alternatives are not exhaustive (that is, that there exist objects that don't fall under any alternative) either for $\mathbf{X}$ or for $\mathbf{Y}$. If they are exhaustive for both it simplifies considerably, leaving

$$\frac{|\mathbf{X}^m \cap \mathbf{Y}^m|}{|\mathbf{X}m|} > \frac{|\mathbf{Y}^m|}{|D(m)|}$$

It is not hard to see that this reading is Symmetric. Cohen admits this much, but does not consider it a significant problem. He also notes in his abstract that this reading is not Conservative (in the regular sense), which similarly he does not necessarily consider to be an important issue. It is not a big surprise then that Intensional Conservativity does not necessarily hold either.

To test this, let A remain as before and let A' be the version of A obtained when $\mathbf{X}$ is replaced by $\mathbf{X} \wedge \mathbf{Y}$. This raises the question what kind of alternatives are in ALT($\mathbf{X} \wedge \mathbf{Y}$). A straightforward choice for this would be to let ALT($\mathbf{X} \wedge \mathbf{Y}$) $= A$.[8] Hence we get

$$A' = \{\mathbf{X}' \wedge \mathbf{Z}'|\mathbf{X}' \in \text{ALT}(Y), \mathbf{Z}' \in A\}.$$

But because of the nature of $\mathbf{Z}'$, it always either implies or contradicts $\mathbf{X}'$. Therefore what we in fact end up with is $A' = A$. We now obtain

$$\begin{aligned}
\mathbf{many}(\mathbf{X}, \mathbf{X} \wedge \mathbf{Y}) \Leftrightarrow\ & \frac{|\mathbf{X}^m \cap (\mathbf{X} \wedge \mathbf{Y})^m|}{|\mathbf{X}^m \cap \bigcup\{\mathbf{Z}^m|\mathbf{Z} \in A\}|} > \frac{|\bigcup\{\mathbf{Z}^m|\mathbf{Z} \in A\} \cap (\mathbf{X} \wedge \mathbf{Y})^m|}{|\bigcup\{\mathbf{Z}^m|\mathbf{Z} \in A\}|} \\
\Leftrightarrow\ & \frac{|(\mathbf{X} \wedge \mathbf{Y})^m|}{|\mathbf{X}^m \cap \bigcup\{\mathbf{Z}^m|\mathbf{Z} \in A\}|} > \frac{|(\mathbf{X} \wedge \mathbf{Y})^m|}{|\bigcup\{\mathbf{Z}^m|\mathbf{Z} \in A\}|} \\
\Leftrightarrow\ & |\mathbf{X}^m \cap \bigcup\{\mathbf{Z}^m|\mathbf{Z} \in A\}| < |\bigcup\{\mathbf{Z}^m|\mathbf{Z} \in A\}|
\end{aligned}$$

The latter is a tautology, so Intensional Conservativity does not hold.

[7] Though note that strictly speaking, 'other' here would include Scandinavian itself.

[8] Admittedly this decision is a crucial step, and making a different choice here might potentially lead to a different outcome. Still, the choice seems appropriate enough and no alternative that actually gives a different outcome comes to mind.

With the reading depending so much on the extensions of alternatives, we shouldn't expect Intensional EXT to hold either, and it doesn't. Pick $\mathbf{X}$, $\mathbf{Y}$, m, m' such that $\mathbf{many}_m(\mathbf{X}, \mathbf{Y})$ is true, $\mathbf{X}^{m'} = \mathbf{X}^m$, $\mathbf{Y}^{m'} = \mathbf{Y}^m$ and every alternative to $\mathbf{X}$ or $\mathbf{Y}$ (except $\mathbf{X}$ and $\mathbf{Y}$ themselves) has empty extension. Then $\mathbf{many}_{m'}(\mathbf{X}, \mathbf{Y})$ reduces to

$$\frac{|\mathbf{X}^m \cap \mathbf{Y}|}{|\mathbf{X}^m \cap \mathbf{Y}|} > \frac{|\mathbf{X}^m \cap \mathbf{Y}|}{|\mathbf{X}^m \cap \mathbf{Y}|},$$

a contradiction.

It is worth pointing out that an important motivation behind the Relative Proportional reading was to provide an alternative to what Cohen calls the Reverse Interpretation view. Thus, as we have seen in Section 2.3, the good news is that even if the Relative Proportional reading is not as successful as one may hope, the intensional approach has allowed us to provide an alternative of our own which does satisfy Intensional Conservativity (as well as Intensional EXT and Intensional Isomorphism closure).

3.3 Tanaka

Similar to Cohen, Tanaka's account [8] is based on sets of alternatives, based on taxonomic knowledge. It distinguishes between taking alternatives to the subject or the predicate, and between comparing alternatives of the same level (the Sister-alt reading) or a higher level (the Mother-alt reading).

For instance, in the sentence "Many Scandinavians have won the Nobel Prize in Literature", which Tanaka also discusses, the "sisters" of Scandinavian would be various other nationalities, whereas the "Mother" property would include people of any nationality.

This leads to four possible readings[9] of "Many X's are Y", which can be paraphrased as follows:

	$S - ALT$	$M - ALT$
Subject	$\frac{\lvert\mathbf{Y}^m \cap \mathbf{X}^m\rvert}{\lvert\mathbf{Y}^m\rvert} > \frac{\lvert\mathbf{Y}^m \cap \text{sister}(\mathbf{X})^m\rvert}{\lvert\mathbf{Y}^m\rvert}$	$\frac{\lvert\mathbf{Y}^m \cap \mathbf{X}^m\rvert}{\lvert\mathbf{Y}^m\rvert} > \frac{\lvert\mathbf{X}^m\rvert}{\lvert(\text{mother}(\mathbf{Y}))^m\rvert}$
Predicate	$\frac{\lvert\mathbf{X}^m \cap \mathbf{Y}^m\rvert}{\lvert\mathbf{X}^m\rvert} > \frac{\lvert\mathbf{X}^m \cap \text{sister}(\mathbf{Y})^m\rvert}{\lvert\mathbf{X}^m\rvert}$	$\frac{\lvert\mathbf{X}^m \cap \mathbf{Y}^m\rvert}{\lvert\mathbf{X}^m\rvert} > \frac{\lvert\mathbf{Y}^m\rvert}{\lvert(\text{mother}(\mathbf{X}))^m\rvert}$

In the relative M-ALT Subject reading, the relative amount of **Y**s that are **X** is compared to the proportion of **X**'s among the 'mother' of **Y**. In the earlier example, this would mean comparing the proportion of Scandinavians who have won the Nobel Prize in Literature to the proportion of Scandinavians among all humans.

The M-ALT readings are not (Intensionally) Conservative: it is easy enough to see that both of them turn into a tautology if $\mathbf{Y}$ is replaced by $\mathbf{X} \wedge \mathbf{Y}$.

In the relative S-ALT Subject reading, the relative amount of **Y**s that are **X** is compared to the same value for some sister of **X**. It is admittedly not entirely clear to me if this means comparing to a single sister picked arbitrarily, comparing to some

[9] In addition to two absolute readings which we are not interested in here.

constructed 'arbitrary' sister, taking an average among all sisters or something else. Still, it seems unlikely that Intensional Conservativity can be attained.

Since sisters are disjoint, we get $|\text{sister}(\mathbf{X})^m \cap (\mathbf{X} \wedge \mathbf{Y})^m| = |\emptyset| = 0$, and similarly $|\mathbf{X}^m \cap \text{sister}(\mathbf{X} \wedge \mathbf{Y})^m| = |\emptyset| = 0$. A more charitable interpretation based on some constructed 'arbitrary' sister which may overlap the original sister would not help here either: only the part that does overlap the original would be left, so both readings would still produce a tautology.

Another possible interpretation could be to take an average over all sisters, writing the Subject-focused reading as

$$\frac{|\mathbf{Y}^m \cap \mathbf{X}^m|}{|\mathbf{Y}^m|} > \sum_{\mathbf{Z} \in \text{sisters}(\mathbf{X})} \frac{|\mathbf{Y}^m \cap \mathbf{Z}^m|}{|\mathbf{Y}^m|}$$

But even if we do this, replacing **Y** with $\mathbf{X} \wedge \mathbf{Y}$ will make the reading either trivially false (if **X** itself is counted among the sisters) or trivially true (if it is not).

To make matters particularly odd, Tanaka makes it a point to propose a revised notion of Conservativity, wherein focal mapping determines which element is conservative. This could mean that for some or all of the readings above, he would have us replace not **Y** but **X** by $\mathbf{X} \wedge \mathbf{Y}$ to test for Conservativity. But the fact of the matter is that this changes nothing. Replacing **X** by $\mathbf{X} \wedge \mathbf{Y}$ above turns all four readings into tautologies in essentially the same ways. As it stands I fail to see how his readings could satisfy the notion he introduces.

As for Intensional EXT, it fails for much the same reason it fails for Cohen's reading. The proof for this is left as an exercise for the reader.

3.4 Lappin

Lappin provides the only thoroughly intensional treatment I am aware of [6], and it might hold up well. It works by constructing a set S of normative possible situations, then comparing the amount of **X**'s that are **Y** in the actual situation sa with the amounts in the normative ones. [10] Thus it is broadly defined as follows:

$$\mathbf{many}_{sa}(\mathbf{X}, \mathbf{Y}) \Leftrightarrow S \neq \emptyset, \text{ and for every } sn \in S, \; |\mathbf{X}^{sa} \cap \mathbf{Y}^{sa}| \geq |\mathbf{X}^{sn} \cap \mathbf{Y}^{sn}|$$

This account looks good and simple, but is held back by a highly underdefined S. One of the choices for S Lappin discusses is based on historical averages; another aims to be similar to the Fernando & Kamp approach. Some of his less useful suggestions involve using the following, where C is "a comparison set determined in sa":

$$S = \{sn | sn = sa \,\&\, |\mathbf{X}^{sa} \cap \mathbf{Y}^{sa}| \geq |\mathbf{X}^{sn} \cap C|\}$$
$$S = \{sn | sn = sa \,\&\, |\mathbf{X}^{sa} \cap \mathbf{Y}^{sa}| \geq |\mathbf{Y}^{sn} \cap C|\}$$
$$S = \{sn | sn = sa \,\&\, |\mathbf{X}^{sa} \cap \mathbf{Y}^{sa}| \geq |C|\}$$

[10] Lappin uses "situation" where we would use "world" or "model".

The first conjunct in each of these ensures that only sa is considered for S. Since C is also determined using only sa, the readings generated by these choices for S have $\mathbf{Q}_{sa}\mathbf{XY}$ depend only on $\mathbf{X}^{sa}$, $\mathbf{Y}^{sa}$ and sa. Thus, these readings are in fact non-intensional ones, and therefore will not be able to overcome Westerståhl's problematic example as discussed in the introduction. Any way to get around this would involve taking the intensions of **X** and **Y** into account when choosing C.

The examples above show that some extra conditions on S are needed to separate useful readings from less useful ones. To find them, we look to our general form.

In line with this, we can make things easier for ourselves by rephrasing the broad definition of many as

$$\mathbf{many}_{sa}(\mathbf{X}, \mathbf{Y}) \Leftrightarrow \left(S \neq \emptyset \;\&\; |\mathbf{X}^{sa} \cap \mathbf{Y}^{sa}| \geq \max_{sn \in S} |\mathbf{X}^{sn} \cap \mathbf{Y}^{sn}| \right)$$

This comes close to matching our general form, provided the right-hand side does not require too much. Specifically, we get the restriction that one must be able to determine S using only $\mathbf{X}$, $\mathbf{X} \wedge \mathbf{Y}$ and W.

3.5 Solt

Like Lappin, Solt provides a broad account [7] which can cover a lot of possible readings of many by varying a somewhat underspecified parameter. In this case, the readings are built based on a 'neutral range' N_S of amounts that are not considered either many or few. As Solt puts it, "the full range of readings available to *many* and *few* can be derived via manipulation of two elements: the structure of the scale (whether or not an upper bound is assumed) and the choice of the neutral range on that scale" [7, p. 177].

The structure of the scale corresponds to the difference between cardinal and proportional readings. In both cases, the general reading ultimately amounts to

$$|(\mathbf{X} \wedge \mathbf{Y})^m| \geq \sup N_S$$

When it comes to determining N_S, Solt finds that there is sometimes merit to involving possible worlds as Fernando & Kamp and Lappin do, but argues that this is often inappropriate. Instead, she favors constructing N_S as a range around an (implicit) comparison point p_c. A general recipe to determine p_c (in the absence of cues like 'compared to' and 'for a') is not provided.

Still, the general reading above easily fits our general form from Section 2.6, allowing us to say that when a possible world-based approach is taken, Intensional Conservativity can be guaranteed simply by demanding N_S depend only on $\mathbf{X}$, $\mathbf{X} \wedge \mathbf{Y}$ and W.

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Appendix: Reductions

A.1 Lifting Theorem

Definition 5 A *non-intensional quantifier* Q is a function which when given a domain M and two sets $U, V \subseteq M$ gives an evaluation of true or false. We will write $Q_M UV$ to denote that this evaluation is true.

For a non-intensional quantifier Q, define its intensional lift $\mathbf{Q}^*$ as follows:

$$\mathbf{Q}^*_m\mathbf{XY} \Leftrightarrow Q_{D(m)}\mathbf{X}^m\mathbf{Y}^m$$

Also, for any set U in domain $D(m)$, the lift $l_m(U)$ is the set of properties $\mathbf{X}$ for which $\mathbf{X}^m = U$.

Theorem 3 *Where Q is a non-intensional quantifier and $\mathbf{Q}^*$ is its lift:*

- $\mathbf{Q}^*$ *satisfies Intensional Conservativity if and only if Q is Conservative*
- $\mathbf{Q}^*$ *satisfies Intensional EXT if and only if Q satisfies EXT*, where EXT* is like EXT but applies for any M, M' such that $A, B \subseteq M$, $A, B \subseteq M'$*
- $\mathbf{Q}^*$ *satisfies Intensional Isomorphism closure if and only if Q satisfies Isomorphism closure*

Proof Conservativity is the easiest. First assume $\mathbf{Q}^*$ satisfies Intensional Conservativity. For a given set M, let m be a model with $D(m) = M$. Then

$$\begin{aligned} Q_M UV &\Leftrightarrow \exists \mathbf{X} \in l_m(U), \mathbf{Y} \in l_m(V) : \mathbf{Q}^*_m\mathbf{XY} && \textit{by construction} \\ &\Leftrightarrow \exists \mathbf{X} \in l_m(U), \mathbf{Y} \in l_m(V) : \mathbf{Q}^*_m(\mathbf{X})(\mathbf{X} \wedge \mathbf{Y}) && \textit{by Intensional Conservativity} \\ &\Leftrightarrow \exists \mathbf{X} \in l_m(U), \mathbf{Z} \in l_m(U \cap V) : \mathbf{Q}^*_m\mathbf{XZ} && \textit{see below} \\ &\Leftrightarrow Q_M U(U \cap V) && \textit{by definition} \end{aligned}$$

For the third step , note that

$$(\mathbf{X} \wedge \mathbf{Y})^m = \mathbf{X}^m \cap \mathbf{Y}^m = U \cap V$$

Therefore $(\mathbf{X} \wedge \mathbf{Y}) \in l_m(U \cap V)$.

Next, assume that Q is (regularly) Conservative, m is some model with $D(m) = M$ and $\mathbf{X}$ and $\mathbf{Y}$ are properties. Then

$$\begin{aligned} \mathbf{Q}^*_m\mathbf{XY} &\Leftrightarrow Q_M\mathbf{X}^m\mathbf{Y}^m && \textit{by definition} \\ &\Leftrightarrow Q_M\mathbf{X}^m(\mathbf{X}^m \cap \mathbf{Y}^m) && \textit{by Conservativity} \\ &\Leftrightarrow Q_M\mathbf{X}^m(\mathbf{X} \wedge \mathbf{Y})^m && \textit{by definition} \\ &\Leftrightarrow \mathbf{Q}^*_m\mathbf{X}(\mathbf{X} \wedge \mathbf{Y}) && \textit{by definition} \end{aligned}$$

For EXT*, let $U, V \subseteq M, M'$, $D(m) = M$, $D(m') = M'$ and let $\mathbf{X}, \mathbf{Y}$ be such that $\mathbf{X}^m = U = \mathbf{X}^{m'}, \mathbf{Y}^m = V = \mathbf{Y}^{m'}$.

First assume $\mathbf{Q}^*$ satisfies Intensional EXT. Then

$$\begin{aligned} Q_M UV &\Leftrightarrow \mathbf{Q}^*_m \mathbf{XY} && \textit{by definition} \\ &\Leftrightarrow \mathbf{Q}^*_{m'} \mathbf{XY} && \textit{by Intensional EXT} \\ &\Leftrightarrow Q_{M'} UV && \textit{by definition} \end{aligned}$$

For the other direction, assume Q satisfies EXT*. Then

$$\begin{aligned} \mathbf{Q}^*_m \mathbf{XY} &\Leftrightarrow Q_M \mathbf{X}^m \mathbf{Y}^m && \textit{by definition} \\ &\Leftrightarrow Q_{M'} \mathbf{X}^{m'} \mathbf{Y}^{m'} && \textit{by EXT*} \\ &\Leftrightarrow \mathbf{Q}^*_{m'} \mathbf{XY} && \textit{by definition} \end{aligned}$$

For Isomorphism closure, let f be a bijection from $D(m)$ to $D(m')$ and let $\mathbf{X}^{m'} = f[\mathbf{X}^m]$, $\mathbf{Y}^{m'} = f[\mathbf{Y}^m]$.

First assume that Q satisfies Isomorphism closure. This yields

$$\begin{aligned} \mathbf{Q}^*_m \mathbf{XY} &\Leftrightarrow Q_{D(m)} \mathbf{X}^m \mathbf{Y}^m && \textit{by definition} \\ &\Leftrightarrow Q_{D(m')} f[\mathbf{X}^m] f[\mathbf{Y}^m] && \textit{by Isomorphism Closure} \\ &\Leftrightarrow Q_{D(m')} \mathbf{X}^{m'} \mathbf{Y}^{m'} && \textit{by condition} \\ &\Leftrightarrow \mathbf{Q}^*_{m'} AB && \textit{by definition} \end{aligned}$$

The other direction is almost trivial: where $U, V \subseteq D$, pick a structure with $D(m) = M$, $\mathbf{X}^m = U$, $\mathbf{Y}^m = V$ and assume $\mathbf{Q}^*$ satisfies Intensional Isomorphism closure to obtain

$$\begin{aligned} Q_{D(m)} UV &\Leftrightarrow \mathbf{Q}^*_m \mathbf{XY} && \textit{by definition} \\ &\Leftrightarrow \mathbf{Q}^*_{m'} \mathbf{XY} && \textit{by Intensional ISOM} \\ &\Leftrightarrow Q_{D(m')} UV && \textit{by definition} \end{aligned}$$

□

A.2 Extensional Intensional Quantifiers

It is a matter of some interest to see under which conditions a given intensional quantifier can be interpreted as a lift of a non-intensional one. As one might expect, the answer is that this is so iff the truth value in a given model depends only on that model and the local extensions there. The following two propositions demonstrate this.

Proposition 1 *If an intensional quantifier* $\mathbf{Q}$ *is such that* $\mathbf{Q}_m \mathbf{XY}$ *is a function of* $\mathbf{X}^m$, $\mathbf{Y}^m$ *and* $D(m)$, *then there is a non-intensional quantifier* Q^2 *such that* $\mathbf{Q}_m \mathbf{XY} \Leftrightarrow (Q^2)^*_m \mathbf{XY}$.

Proof For the proof, define

$$Q^2_M UV \Leftrightarrow \forall m' \textit{with domain } M : \forall \mathbf{X} \in l_{m'}(U), \mathbf{Y} \in l_{m'}(V) : \mathbf{Q}_{m'} \mathbf{XY}$$

This gives

$$\begin{aligned}(Q^2)^*_m\mathbf{XY} &\Leftrightarrow Q^2_M\mathbf{X}^m\mathbf{Y}^m\\ &\Leftrightarrow \forall m' \textit{with domain } M\\ &\quad\ \forall \mathbf{X}' \in l_{m'}(\mathbf{X}^m), \mathbf{Y}' \in l_{m'}(\mathbf{Y}^m) : \mathbf{Q}_{m'}\mathbf{X}'\mathbf{Y}'\\ &\Leftrightarrow \forall \mathbf{X}' \in l_m(\mathbf{X}^m), \mathbf{Y}' \in l_m(\mathbf{Y}^m) : \mathbf{Q}_m\mathbf{X}'\mathbf{Y}'\\ &\Leftrightarrow \mathbf{Q}_m\mathbf{XY}\end{aligned}$$

(In the most important step, we may eliminate "$\forall m'$*with domain M*" because $\mathbf{Q}'_m\mathbf{X}'\mathbf{Y}'$ depends only on the domain and the extensions there and the latter have already been fixed by quantifying over $l_{m'}(\mathbf{X}^m)$, $l_{m'}(\mathbf{Y}^m)$. Similarly, the next universal quantification may be eliminated because by definition all $\mathbf{X}' \in l_m(\mathbf{X}^m)$ have the same extension in m as $\mathbf{X}$ (and the same for $\mathbf{Y}$)).

This covers one direction The other direction is covered by the proposition below, which is trivial enough to require no further proof.

Proposition 2 *For any lift* $\mathbf{Q}^*$ *of a non-intensional quantifier* Q*,* $\mathbf{Q}^*_m\mathbf{XY}$ *is a function of* $\mathbf{X}^m$, $\mathbf{Y}^m$ *and* $D(m)$.

As mentioned before, good readings of 'many' (certainly any reading that avoids the problem mentioned in the introduction while still being Intensionally Conservative) will not be interpretable as a lift of this kind. Such readings will necessarily depend on information beyond what can be drawn from the local extensions and domain, and hence will not be interpretable as a function of only these. □

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